**UPPSALA UNIVERSITET**
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Prov i matematik
Funktionalanalys
Kurs: F3B, F4Sy, 1MA283
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**Tillåtna hjälpmedel:** Manuella skrivdon och Kreyszigs bok *Introductory Functional Analysis with Applications.*

**LYCKA TILL!**

**Problems 1 — 8 should be attempted by all students.**
**Graduate students should also try to solve Problems 9 and 10**

**1.** Let $x$ and $y$ be two vectors in a complex inner product space, satisfying $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Is it possible that $x$ is not orthogonal to $y$?

**2.** Let $X = \mathcal{C}([a,b])$ be equipped with the usual norm

$$\|x\| = \sup_{t\in[a,b]} |x(t)|, \qquad x \in X.$$

Let $Y$ denote the subspace of $X$ given by

$$Y = \{x \in X \,:\, x \text{ is continuously differentiable and } x(a) = 0\}.$$

Consider the operator $T : X \longrightarrow Y$ given by the formula

$$(Tx)\,(t) = \int_a^t x(s)ds, \qquad t \in [a,b], x \in X.$$

Show that $T$ is bounded and find its norm. Find $T^{-1}$ and show that it is not bounded.

**3.** Let $X, Y$ be normed spaces and let $T : \mathcal{D}(T) \longrightarrow Y$ be a closed operator. Show that if $K$ is a compact subset of $X$, then $T(K)$ is a closed subset of $Y$.

**4.** Let $M$ be a closed subspace of a Hilbert space $H$ and let $T : H \longrightarrow H$ be a linear operator. Denote by $P$ and $Q$ the orthogonal projections onto $M$ and $M^{\perp}$, respectively. Show that we have $PT = TP$ and $QT = TQ$ if and only if both $T(M) \subset M$ and $T(M^{\perp}) \subset M^{\perp}$.

**5.** Consider the sequence of vectors

$$e_n = (0, \ldots, 0, 1, 0, 0, \ldots), \qquad n = 1, 2, \ldots,$$

where 1 is placed on the $n$-th position. Explain why the sequence $(e_n)_{n\geq 1}$ is weakly convergent in $l^2$ but not in $l^1$.

**6.** Let $(e_n)$ be an orthonormal basis in a Hilbert space $H$ and let $(\lambda_n)$ be a sequence of numbers. Define the operator $T : H \longrightarrow H$ by the formula

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n, \qquad x \in H.$$

Show that if $\lim_{n\to\infty} \lambda_n = 0$, then $T$ is a compact operator.

**7.** Let $T$ be the operator from Problem 6. Show that if $\lim_{n\to\infty} \lambda_n = 1$, then $T$ is not a compact operator.

**8.** Let $T : X \longrightarrow X$ be a bounded operator on a normed space $X$ such that $T \neq 0$ but $T^N = 0$ for some integer $N > 1$. Show that if $\lambda \in \mathbf{C} \setminus \{0\}$, then

$$(T - \lambda I)^{-1} = -\sum_{n=0}^{N-1} \frac{T^n}{\lambda^{n+1}}.$$

Show also that $\sigma(T) = \sigma_p(T) = \{0\}$.

**Additional problems for graduate students:**

**9.** Show that the dual space of $c_0$ is $l^1$.

**10.** Let $(f_n)$ be a sequence in the dual $X'$ of a normed space $X$. Show that if $(f_n)$ is weakly convergent, then it is also weak$*$ convergent.

**GOOD LUCK!**