UPPSALA UNIVERSITET
Matematiska institutionen
M. Klimek

Prov i matematik
Funktionalanalys
Kurs: F3B, F4Sy, 1MA283
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Tillåtna hjälpmedel: Manuella skrivdon och Kreyszigs bok *Introductory Functional Analysis with Applications.*

LYCKA TILL!

Problems 1 — 8 should be attempted by all students.
Graduate students should also try to solve Problems 9 and 10

1. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a function. If for every integer n the function f is constant in the interval $[n, n+1)$ we call it a step function. Let $\mathcal{S}$ denote the vector space of all such step functions for which

$$\int_{-\infty}^{\infty} (f(t))^2 \, dt < \infty.$$

Show that $\mathcal{S}$ is a Banach space with the norm given by the formula

$$\|f\| = \sqrt{\int_{-\infty}^{\infty} (f(t))^2 \, dt} < \infty.$$

2. Let α, β be two numbers such that $-1 < \alpha, \beta < 1$. Define $L : l^2 \longrightarrow l^2$ by the formula

$$L((\xi_n)) = \left(\sum_{j=1}^{\infty} \alpha\beta^j \xi_j, \sum_{j=1}^{\infty} \alpha^2\beta^j \xi_j, \sum_{j=1}^{\infty} \alpha^3\beta^j \xi_j, \ldots \right)$$

for all $(\xi_n) \in l^2$. Show that

$$\|L\| = \frac{|\alpha\beta|}{\sqrt{1-\alpha^2}\sqrt{1-\beta^2}}.$$

3. Consider the linear operator $T : l^2 \longrightarrow l^2$ given by the formula

$$T((x_n)) = (y_n) \text{ if and only if } y_n = \begin{cases} \frac{x_n - x_{n+1}}{\sqrt{2}} & \text{, if } n \text{ is odd,} \\ \frac{x_{n-1} + x_n}{\sqrt{2}} & \text{, if } n \text{ is even.} \end{cases}$$

Show that T is an isometry.

4. Let $x_1, x_2, \ldots, x_n$ be a finite sequence of distinct elements in a Hilbert space H and let $K = \operatorname{span}\{x_1, \ldots, x_n\}$. Define the operator $S : K \longrightarrow K$ by the formula

$$S(x) = \sum_{j=1}^{n} \langle x, x_j \rangle x_j, \qquad x \in K.$$

Prove that the operator S is self-adjoint and invertible.

5. With the notation from Problem 4, show that the orthogonal projection of H onto K is given by the formula

$$P_K(x) = \sum_{j=1}^{k} \langle x, S^{-1} x_j \rangle x_j, \qquad x \in H.$$

6. Let X be a normed space. Prove that if a sequence in X' is weakly convergent, then it is also weak* convergent.

7. Consider the vector space $\mathcal{P}$ of all polynomials (of one real veriable and with real coefficients) with the norm

$$\|p\| = \sup\{|p(t)| : 0 \le t \le 1\}, \qquad p \in \mathcal{P}.$$

Define an operator $F : \mathcal{P} \longrightarrow \mathcal{P}$ as follows. If p is a polynomial, then $q = T(p)$ is defined as the polynomial

$$q(t) = tp(0) + t^2 p(1) + t^3 p(2), \qquad t \in \mathbf{R}.$$

Determine the range of the operator T. Is this operator compact?

8. Assume that the space $X = \mathcal{C}[-1, 1]$ is equipped with the usual norm $\|x\| = \sup\{|x(t)| : |t| \le 1\}$. Let a, b be two numbers with $b \ne 0$. Consider the linear operator $T : X \longrightarrow X$ given by the formula

$$T(x) = y, \text{ where } y(t) = ax(t) + bx(1-t) \text{ for any } x \in X.$$

Find explicitly the resolvent operator and describe the spectrum of T.

Additional problems for graduate students:

We will make some common assumptions for Problems 9 and 10. Let X be a Banach space over $\mathbf{K}$, where $\mathbf{K}$ is either $\mathbf{R}$ or $\mathbf{C}$. Let $(x_n)_{n \ge 1}$ be a Schauder basis for X. Then for each $x \in X$ we can find a unique sequence of numbers $(c_n)_{n \ge 1}$ such that $x = \sum_{n=1}^{\infty} c_n x_n$. For each positive integer n we define $f_n(x) = c_n$. In other words,

$f_n(x)$ is defined to be the n-th coefficient of the expansion of x with respect to the given Schauder basis. This way we create a sequence of linear functionals $f_n : X \longrightarrow \mathbf{K}$.

9. Define Y to be the vector space of all sequences of numbers $(c_n)_{n\geq 1}$ such that the series $\sum_{n=1}^{\infty} c_n x_n$ is convergent. Define the norm

$$||(c_n)_{n\geq 1}||_Y = \sup_{k\geq 1} \left\| \sum_{n=1}^{k} c_n x_n \right\|, \qquad (c_n)_{n\geq 1} \in Y.$$

Show that Y is a Banach space with this norm.

10. Let the operator $T : Y \longrightarrow X$ be given by the formula

$$T\left((c_n)_{n\geq 1}\right) = \sum_{n=1}^{\infty} c_n x_n, \qquad (c_n)_{n\geq 1} \in Y.$$

Prove that T is a linear isomorphism and that

$$1 \leq ||x_n||\,||f_n|| \leq 2||T^{-1}||, \qquad n \geq 1.$$

GOOD LUCK!