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Prov i matematik
Funktionalanalys
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SOLUTIONS:

1. If $x = (1/n)_{n\geq 1}$ and $x_N = (1, 1/2, \ldots, 1/N, 0, 0, \ldots)$, then $x \in l^2 \setminus c_{00}$, $x_N \in c_{00}$ for all N and $x_N \longrightarrow x$ as $N \to \infty$. Thus c_{00} is not a closed subset of l^2 and hence it is not complete.

2. Linearity of T follows from the linearity of real inner products with respect to the second variable. By the Cauchy-Schwarz inequality $\|f_z\| \leq \|z\|$. But $|f_z(z/\|z\|)| = \|z\|$. Hence $\|f_z\| = \|z\|$. Therefore T is an isometry. If X was complex, T would not be linear. For example we would have $T(ix) = -iT(x)$.

3. $y \perp \mathcal{R}(A) \Leftrightarrow \langle Ax, y\rangle = 0$ for all $x \in H_1$. The last condition is equivalent to saying that $\langle x, A^*y\rangle = 0$ for all x. This is the same as to say that $y \in \mathcal{N}(A^*)$. The second conclusion follows from the first one because for any subspace $S \subset H_2$, we have $\overline{S} = S^{\perp\perp}$.

4. We know that, the coefficients $\langle x, u\rangle$ and $\langle x, v\rangle$ should satisfy the equations

$$\langle x,u\rangle = \frac{\begin{vmatrix} \langle x,u\rangle & \langle v,u\rangle \\ \langle x,v\rangle & \langle v,v\rangle \end{vmatrix}}{\begin{vmatrix} \langle u,u\rangle & \langle v,u\rangle \\ \langle u,v\rangle & \langle v,v\rangle \end{vmatrix}} \quad \text{and} \quad \langle x,v\rangle = \frac{\begin{vmatrix} \langle u,u\rangle & \langle x,u\rangle \\ \langle u,v\rangle & \langle x,v\rangle \end{vmatrix}}{\begin{vmatrix} \langle u,u\rangle & \langle v,u\rangle \\ \langle u,v\rangle & \langle v,v\rangle \end{vmatrix}}$$

for all x. In particular, by substituting x equal to u (resp. v), we conclude that $\|u\| = \|v\| = 1$. Since $u - Pu$ must be orthogonal to v, we have

$$\langle u - \langle u,u\rangle u + \langle u,v\rangle v, v\rangle = \langle u,v\rangle - \langle u,v\rangle - \langle u,v\rangle = 0$$

and so $u \perp v$.

5. Suppose that S is not bounded. Then there exists a sequence $(x_n) \subset S$ such that $\lim_{n\to\infty} \|x_n\| = \infty$. Define $\delta_n \in X''$ by the formula $\delta_n(f) = f(x_n)$. Then $\|\delta_n\| = \|x_n\|$ and for each $f \in X'$ the sequence $(\delta_n(f))$ is bounded. This contradicts the Banach-Steinhaus Theorem.

6. The identity mapping $I : (X, \|\ \ \|_1) \longrightarrow (X, \|\ \ \|_2)$ is bounded (with the operator norm $\|I\|_{1,2} \leq M$). By the Open Mapping Theorem its inverse (which is also I) is continuous as a mapping $I : (X, \|\ \ \|_2) \longrightarrow (X, \|\ \ \|_1)$. It is enough to take $m = 1/\|I\|_{2,1}$.

7. Let $T_N(x) = \sum_{n=1}^{N} \frac{\langle x, e_{n+1}\rangle}{n+1} e_n$. Then by Bessel's inequality $\|Tx - T_N x\|^2 \leq \|x\|^2/(N+1)^2$ and hence $\|T - T_N\| \leq 1/(N+1)$. Thus T is compact as a limit of a sequence of bounded operators with finite dimensional range. Furthermore

$$\langle Tx, y\rangle = \sum_{n=1}^{\infty} \frac{\langle x, e_{n+1}\rangle}{n+1} \langle e_n, y\rangle = \left\langle x, \sum_{n=1}^{\infty} \frac{\langle y, e_n\rangle}{n+1} e_{n+1} \right\rangle.$$

Hence

$$T^*y = \sum_{n=1}^{\infty} \frac{\langle y, e_n \rangle}{n+1} e_{n+1}.$$

8. Note that $(Kx)(t) = ct$, where the constant c depends on x. Hence $\mathcal{R}(K) = \text{span}\{t\}$. Consequently. if K has a non-zero eigenvalue λ it must correspond to the eigenvector $x(t) = t$. Thus $(Kx)(t) = t/3 = \lambda x = \lambda t$ and so $\lambda = 1/3$.

9. By comparing the Fourier coefficients of Lx and λx we conclude that λ is an eigenvalue only with an eigenvector of the form $x_\lambda = (1, \lambda, \lambda^2, \lambda^3, \ldots)$ (or its multiple) provided that $x_\lambda \in l^2$. The latter is true only if $|\lambda| < 1$. Hence $\sigma_p(L) = \{\lambda \in \mathbf{C} : \ |\lambda| < 1\}$. Since $\sigma(L)$ is closed and contained in the disc $|\lambda| \le \|L\| = 1$, we must have $\sigma(L) = \{\lambda \in \mathbf{C} : \ |\lambda| \le 1\}$.

10. Let $p_i \in (l^\infty)'$ be given by the formula $p_i(x) = \xi_i$, where $x = (\xi_i) \in l^\infty$. Then $p_i \circ A \in (c_0)' = l^1$.Therefore p_i corresponds to a vector $(\alpha_{ij})_{j \ge 1} \in l^1$ and $\|p_i\| = \sum_{j=1}^{\infty} |\alpha_{ij}|$. Hence, in calculation of the norm "$\le$" is obvious. The equality is approximated on elements of c_0 with only finitely non-zero entries of the form $|\alpha_{ij}|/\alpha_{ij}$