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Prov i matematik
Funktionalanalys
Kurs: F3B, F4Sy, 1MA283
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SOLUTIONS:

1. The space $\mathcal{P}$ can be identified with the following subspace of l^2:

$$\mathcal{P} = \{x = (\xi_n)_{n\geq 1} : \ \xi_n \neq 0 \text{ for only finitely many } n\}.$$

The norm is therefore induced by the inner product in l^2. Since $\mathcal{P}$ is not a closed subspace of l^2, it is not complete.

2. Let $a \in X \setminus \mathcal{N}(g) = X \setminus \mathcal{N}(f)$. Note that if $x \in X$, then

$$x - \frac{f(x)}{f(a)} a \in \mathcal{N}(f) = \mathcal{N}(g).$$

Hence

$$g(x) = g\left(\frac{f(x)}{f(a)}a + \left(x - \frac{f(x)}{f(a)}a\right)\right) = g\left(\frac{f(x)}{f(a)}a\right) = \frac{g(a)}{f(a)} f(x).$$

3. The series is convergent because

$$\left\| \sum_{n=m}^{m+k} \lambda_n \langle x, u_n \rangle v_n \right\|^2 = \sum_{n=m}^{m+k} |\lambda_n|^2 |\langle x, u_n\rangle|^2 \leq M^2 \sum_{n=m}^{m+k} |\langle x, u_n \rangle|^2 \longrightarrow 0,$$

as $m, k \to \infty$, where $M = \sup_{n\geq 1} |\lambda_n|$. Similarly

$$\|Tx\|^2 \leq M^2 \sum_{n=m}^{\infty} |\langle x, u_n\rangle|^2 = M^2 \|x\|^2.$$

Therefore

$$\|T\| \leq M = \sup_{n\geq 1} \|Tu_n\| \leq \|T\|,$$

because $\|Tu_n\| = |\lambda_n|$.

4. We have that

$$\langle x, T^* x\rangle = \langle Tx, x\rangle = \sum_n \lambda_n \langle x, u_n\rangle \langle v_n, x\rangle = \left\langle x, \sum_n \bar{\lambda}_n \langle x, v_n \rangle u_n \right\rangle.$$

Therefore

$$T^* x = \sum_n \bar{\lambda}_n \langle x, v_n\rangle u_n, \qquad x \in H.$$

Consequently

$$T^*Tx = \sum_n |\lambda_n|^2 \langle x, u_n \rangle u_n$$

and

$$TT^*x = \sum_n |\lambda_n|^2 \langle x, v_n \rangle v_n.$$

This implies the required property.

5. If $a \notin \overline{\mathrm{span}(M)}$, define $Z = \overline{\mathrm{span}\{M, a\}}$ and $g : Z \longrightarrow \mathbf{K}$ as $g(\lambda a + v) = \lambda$ for $\lambda \in \mathbf{K}$ and $v \in \overline{\mathrm{span}(M)}$. By the Hahn-Banach theorem there exists $f \in X'$ such that $f = g$ on Z. In particular, $f(a) = g(a) = 1$ and $\mathcal{N}(f) \supset \overline{\mathrm{span}(M)}$. The converse follows from the definition of the closure of a set and continuity.

6. It suffices to check that $PP = P$ and that P is self-adjoint.

7. In view of the Open Mapping Theorem, T has a continuous inverse. Hence $I = T^{-1}T$ is a compact operator. Consequently, the closed unit ball of X is compact and so X has to be finite dimensional.

8. Let $T_\lambda = T - \lambda I$. By comparing $(T_\lambda x)(t)$ with $(T_\lambda x)(1-t)$ we see that

$$(1 - \lambda^2)x(t) = \lambda y(t) + y(1-t),$$

where $y = T_\lambda x$. Hence, if $\lambda^2 \neq 1$, then

$$(R_\lambda y)(t) = \frac{1}{1-\lambda^2} (\lambda y(t) + y(1-t)), \qquad y \in X, t \in [0,1].$$

The eigenspace corresponding to $\lambda = 1$ consists of all $x \in X$ such that $x(t) = x(1-t)$ for each $t \in [0,1]$. The eigenspace corresponding to $\lambda = -1$ consists of all $x \in X$ such that $x(t) = -x(1-t)$ for each $t \in [0,1]$. Consequently $\sigma(T) = \{1, -1\}$.

9. The inequality $\|P_{E_n}\| \leq 1$ implies (a). Property (b) is true because $P_{E_n} P_{E_m} = P_{E_n}$ if $m > n$. To get (c) note that since $x_n \in E_n$ we have

$$\langle x_{n+k}, x_n \rangle = \langle P_{E_n}(x_{n+k}), x_n \rangle = \|x_n\|^2.$$

Let $c = \lim_{n\to\infty} \|x_n\|$. Then

$$\|x_{n+k} - x_n\|^2 = \|x_{n+k}\|^2 - \|x_n\|^2 \longrightarrow c - c = 0$$

as $n, k \to \infty$. Finally, if $y = \lim x_n$, then by (b) and continuity of projections, $y = \lim_{k\to\infty} P_{E_n}(x_{n+k}) = P_{E_n}(y)$.

10. Note that part of the second sentence in Problem 10 was missing in the exam paper. Nevertheless, marks were awarded for attempts at solving of the incomplete problem. Here is a solution of the complete formulation of the problem (which was meant to check if the graduate students taking the course have read the proof of the Banach-Steinhaus theorem). Let $[a_1, b_1] \subset]a, b[$, where $a_1 < b_1$. Define $F_m = \{t \in [a_1, b_1] : \sup_{n\geq 1} |f_n(t)| \leq m\}$ for $m = 1, 2, \ldots$. Then F_m is closed and $[a_1, b_1]$ is the union of the sets F_m. By Baire's theorem at least one of the sets F_m has non-empty interior. The latter must contain an interval $[\alpha, \beta]$ as required.