Uppsala Universitet
Matematiska Institutionen
Andreas Strömbergsson

Prov i matematik
Funktionalanalys
Kurs: F3B, F4Sy, NVP
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Skrivtid: 9–14
Tillåtna hjälpmedel: Manuella skrivdon, Kreyszigs bok *Introductory Functional Analysis with Applications* och Strömbergssons häfte *Spectral theorem for compact, self-adjoint operators.*

1. Let $\mathcal{P}$ be the space of all polynomials (of one real variable and with real coefficients) with the norm

$$||p|| = \sup\Big\{|p(t)| \ : \ -1 \leqq t \leqq 1\Big\}.$$

Define $f : \mathcal{P} \to \mathbb{R}$ by $f(p) = p(2)$. Prove that f is an unbounded linear functional. Also define $T : \mathcal{P} \to \mathcal{P}$ by $T(p) = p'$ (the derivative of p) and prove that T is an unbounded linear operator.

(6p)

2. Let $Y = \{x = (\xi_j) \in \ell^2 \mid \xi_1 = 0,\ \xi_2 + \xi_3 = 0,\ \xi_1 + \xi_3 + \xi_4 = 0\}$ and let $P : \ell^2 \to \ell^2$ be the orthogonal projection onto $Y^{\perp}$. Determine P.

(6p)

3. Let

$$Y = \big\{(\xi_j) \in \ell^1 \mid \text{at most finitely many } \xi_j \neq 0\big\}.$$

Show that Y is not complete. Determine the closure of Y in ℓ^1.

(5p)

4. Define $T : \ell^\infty \to \ell^1$ by

$$T((\xi_1, \xi_2, \xi_3, \cdots)) = \big(2^{-1}\xi_1,\ 2^{-2}(\xi_1 + \xi_2),\ 2^{-3}(\xi_1 + \xi_2 + \xi_3),\ \cdots\big).$$

Prove that T is a bounded linear operator $T : \ell^\infty \to \ell^1$ and compute the norm $||T||$.

(6p)

5. Let H be a Hilbert space and let $f_1, f_2 \in H'$, $f_1 \neq 0$ and $f_2 \neq 0$. Assume that the following condition holds:

$$(A) \qquad \forall x \in H: \quad |f_1(x)| = ||f_1|| \cdot ||x|| \Longrightarrow f_2(x) = 0.$$

Prove that then the following is also true:

$$(B) \qquad \forall x \in H: \quad |f_2(x)| = ||f_2|| \cdot ||x|| \Longrightarrow f_1(x) = 0.$$

(*Hint:* Apply Riesz's Theorem both to f_1 and f_2 and then try to interpret the condition (A).)

(6p)

6. Let X be a normed space and let $x_1, x_2, x_3, \cdots \subset X$ be a sequence of points in X. Assume that for each $f \in X'$ the sequence of numbers $f(x_1), f(x_2), f(x_3), \cdots$ is bounded. Prove that the sequence $x_1, x_2, x_3, \cdots$ is bounded.

(5p)

7. Let $T : \ell^2 \to \ell^2$ be the operator given by

$$(\eta_j) = T((\xi_k)), \quad \text{where } \eta_j = \sum_{k=1}^{\infty} 2^{-j-k} \xi_k.$$

Prove that T is compact and self-adjoint. Find all the eigenvalues and eigenvectors of T.

(6p)

GOOD LUCK!

Note: If you are taking the 6p-course, please contact me to decide a date for further examination.

Solutions

1. If $a, b \in \mathbb{R}$ and $p, q \in \mathcal{P}$ then $f(ap + bq) = [ap + bq](2) = ap(2) + bq(2) = af(p) + bf(q)$, and $T(ap + bq) = (ap + bq)' = a(p') + b(q') = aT(p) + bT(q)$. This proves that f and T are linear.

Given an integer $n \geqq 1$ we consider the polynomial $p(t) = t^n$. Note that $||p|| = 1$. Furthermore we have $f(p) = p(2) = 2^n$. Hence f is unbounded, since there is no constant C such that $|2^n| \leqq C \cdot 1$ holds for all $n \geqq 1$. We also note $T(p)(t) = nt^{n-1}$ and thus $||T(p)|| = n$. Hence T is unbounded, since there is no constant C such that $n \leqq C \cdot 1$ holds for all $n \geqq 1$.

2. (Very similar to exam 2002-03-01:2.) Note that for $x = (\xi_j) \in \ell^2$ we have $\xi_1 = \langle x, e_1 \rangle$, $\xi_2 + \xi_3 = \langle x, e_2 + e_3 \rangle$ and $\xi_1 + \xi_3 + \xi_4 = \langle x, e_1 + e_3 + e_4 \rangle$. Hence $x = (\xi_j)$ belongs to Y if and only if x is orthogonal to e_1, $e_2 + e_3$ and $e_1 + e_3 + e_4$. That is:

$$Y = \{e_1, e_2 + e_3, e_1 + e_3 + e_4\}^\perp.$$

This can be rewritten as:

$$Y = \big(\text{Span}\{e_1, e_2 + e_3, e_1 + e_3 + e_4\}\big)^\perp = \big(\text{Span}\{e_1, e_2 + e_3, e_3 + e_4\}\big)^\perp.$$

But $\text{Span}\{e_1, e_2 + e_3, e_3 + e_4\}$ is a closed subspace of ℓ^2 since it is finite dimensional (Theorem 2.4-3), and hence by Lemma 3.3-6,

$$Y^\perp = \big(\text{Span}\{e_1, e_2 + e_3, e_3 + e_4\}\big)^{\perp\perp} = \text{Span}\{e_1, e_2 + e_3, e_3 + e_4\}.$$

Using Gram-Schmidt we find an orthogonal basis $f_1 = e_1$, $f_2 = e_2 + e_3$, $f_3 = -e_2 + e_3 + 2e_4$ in $Y^\perp$. The orthogonal projection P onto $Y^\perp$ is then given by

$$\begin{aligned} P((\xi_j)) &= \frac{\langle (\xi_j), f_1 \rangle}{\langle f_1, f_1 \rangle} f_1 + \frac{\langle (\xi_j), f_2 \rangle}{\langle f_2, f_2 \rangle} f_2 + \frac{\langle (\xi_j), f_3 \rangle}{\langle f_3, f_3 \rangle} f_3 \\ &= \frac{1}{3}\Big(3\xi_1,\ 2\xi_2 + \xi_3 - \xi_4,\ \xi_2 + 2\xi_3 + \xi_4,\ -\xi_2 + \xi_3 + 2\xi_4,\ 0,\ 0,\ 0,\ \cdots\Big). \end{aligned}$$

3. Let

$$x_n = (2^{-1}, 2^{-2}, 2^{-3}, \cdots, 2^{-n}, 0, 0, 0, \cdots).$$

Then $x_1, x_2, ... \in Y$. We also define

$$x = (2^{-1}, 2^{-2}, 2^{-3}, \cdots) \in \ell^1.$$

(The fact that $x \in \ell^1$ follows from $\sum_{k=1}^{\infty} 2^{-k} = 1 < \infty$.) We now claim that $x_n \to x$ as $n \to \infty$; this follows from

$$\begin{aligned} ||x - x_n|| &= ||(0, 0, \cdots, 0, 2^{-n-1}, 2^{-n-2}, \cdots)|| \\ &= \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n} \to 0 \quad \text{as } \; n \to \infty. \end{aligned}$$

Note that $x \notin Y$, by the definition of Y. Hence by Theorem 1.4-6(b), Y is not closed (as a subspace of ℓ^1). Hence by Theorem 1.4-7, Y is not complete.

We claim that the closure of Y is all of ℓ^1, i.e. $\overline{Y} = \ell^1$. To prove this, let us fix an arbitrary vector $x = (\xi_1, \xi_2, \cdots) \in \ell^1$. Then form the sequence $x_1, x_2, ...$ where

$$x_n = (\xi_1, \xi_2, \cdots, \xi_n, 0, 0, 0, \cdots).$$

Then $x_1, x_2, \cdots \in Y$ and $x_n \to x$ as $n \to \infty$, since

$$||x - x_n|| = \sum_{k=n+1}^{\infty} |\xi_k| \to 0 \quad \text{as } \; n \to \infty.$$

(Detailed proof of the last statement: Let $S_n = \sum_{k=1}^{n} |\xi_k|$. Then $S = \lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} |\xi_k|$ exists as a real number, since $x = (\xi_n) \in \ell^1$. Hence (S_n) is a Cauchy sequence, and thus for any $\varepsilon > 0$ there is N such that $|S_m - S_n| \leqq \varepsilon$ whenever $m \geqq n \geqq N$, that is, $\sum_{k=n+1}^{m} |\xi_k| \leqq \varepsilon$ whenever $m \geqq n \geqq N$. For fixed n we may let $m \to \infty$ in the last statement to obtain $\sum_{k=n+1}^{\infty} |\xi_k| \leqq \varepsilon$ whenever $n \geqq N$. But such $\varepsilon > 0$ was arbitrary; hence we have proved $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} |\xi_k| = 0$.)

Hence by Theorem 1.4-6(a), $x \in \overline{Y}$. This is true for every $x \in \ell^1$. Hence $\overline{Y} = \ell^1$.

4. T is obviously a linear operator. We have, for all $(\xi_k) \in \ell^\infty$, if $(\eta_k) = T((\xi_k))$,

$$\begin{aligned} ||(\eta_j)||_{\ell^1} = \sum_{j=1}^{\infty} |\eta_j| = \sum_{j=1}^{\infty} \left| 2^{-j} \sum_{k=1}^{j} \xi_k \right| \leqq \sum_{j=1}^{\infty} 2^{-j} \sum_{k=1}^{j} |\xi_k| = \sum_{k=1}^{\infty} |\xi_k| \cdot \sum_{j=k}^{\infty} 2^{-j} \\ = \sum_{k=1}^{\infty} |\xi_k| \cdot 2^{1-k} \leqq ||(\xi_k)||_{\ell^\infty} \sum_{k=1}^{\infty} 2^{1-k} = 2 \cdot ||(\xi_k)||_{\ell^\infty} \end{aligned}$$

Hence T is a bounded, and $||T|| \leqq 2$.

On the other hand, taking $(\xi_j) = (1, 1, 1, \cdots)$ we clearly obtain equality in each step in the above computation, hence $||T((1, 1, 1, \cdots))|| = 2$, and this shows that $||T|| = 2$.

Answer: $||T|| = 2$.

5. By Riesz's Theorem (Theorem 3.8-1 in the book) there are vectors $z_1, z_2 \in H$ such that $f_1(x) = \langle x, z_1 \rangle$ and $f_2(x) = \langle x, z_2 \rangle$ for all $x \in H$. Since $f_1, f_2 \neq 0$ we must have $z_1, z_2 \neq 0$. We also see in that theorem that $||f_1|| = ||z_1||$ and $||f_2|| = ||z_2||$. Now the condition (A) can be written:

$$(A) \qquad \forall x \in H: \quad |\langle x, z_1 \rangle| = ||z_1|| \cdot ||x|| \Longrightarrow \langle x, z_2 \rangle = 0.$$

However we know from Schwarz inequality (Lemma 3.2-1(a)) that $|\langle x, z_1 \rangle| = ||z_1|| \cdot ||x||$ holds if and only if x and z_1 are linearly independent, i.e. if and only if $x = cz_1$ for some $c \in \mathbb{C}$ (here we used the fact $z_1 \neq 0$). Hence condition (A) can be rewritten as:

$$(A) \qquad \forall x \in H: \quad \left[x = cz_1 \text{ for some } c \in \mathbb{C}\right] \Longrightarrow \langle x, z_2 \rangle = 0.$$

This can also be written as:

$$(A) \qquad \forall c \in \mathbb{C}: \quad \langle cz_1, z_2 \rangle = 0.$$

Clearly this holds if and only if $\langle z_1, z_2 \rangle = 0$.

Similarly one proves that condition (B) is equivalent to $\langle z_2, z_1 \rangle = 0$. Hence condition (A) indeed implies condition (B).

6. Define $g_n \in X''$ by $g_n(f) = f(x_n)$ for all $f \in X'$. It follows from Lemma 4.6-1 that each g_n is indeed a bounded linear functional on X', and $||g_n|| = ||x_n||$. Now the assumption in the problem says that for each $f \in X'$ the sequence $g_1(f), g_2(f), g_3(f), \cdots$ is bounded. Now X' is a Banach space by Theorem 2.10-4, and $g_1, g_2, g_3, \cdots$ is a sequence of bounded linear operators $X' \to \mathbb{C}$. Hence by the Uniform Boundedness Theorem (Theorem 4.7-3), there is a constant c such that $||g_n|| \leqq c$ for all n. By what we have noted above, this means $||x_n|| \leqq c$ for all n. In other words, the sequence $x_1, x_2, x_3, \cdots$ is bounded.

7. The quickest solution is probably to compute the spectral decomposition of T on a scratch paper and then use this result to "prove everything at once". But we first have to prove that T is a bounded

linear operator: T is clearly linear. If $(\eta_j) = T((\xi_k))$ then

$$\begin{aligned}||(\eta_j)||^2 &= \sum_{j=1}^{\infty} |\eta_j|^2 = \sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} 2^{-j-k}\xi_k \right|^2 = \sum_{j=1}^{\infty} 2^{-2j} \left| \sum_{k=1}^{\infty} 2^{-k}\xi_k \right|^2 \\ &= \left(\sum_{j=1}^{\infty} 2^{-2j} \right) \left| \sum_{k=1}^{\infty} 2^{-k}\xi_k \right|^2 = \frac{1}{3} \left| \sum_{k=1}^{\infty} 2^{-k}\xi_k \right|^2 \\ &\leqq \frac{1}{3} \left(\sum_{k=1}^{\infty} 2^{-2k} \right) \sum_{k=1}^{\infty} |\xi_k|^2 \leqq \frac{1}{9} ||(\xi_k)||^2.\end{aligned}$$

Hence T is bounded and $||T|| \leqq \frac{1}{3}$. Let $f_1 = \sqrt{3} \cdot (2^{-1}, 2^{-2}, 2^{-3}, \cdots) \in \ell^2$. Note that $||f_1|| = 1$, since

$$||f_1||^2 = 3 \sum_{j=1}^{\infty} 2^{2j} = 3 \cdot \frac{1}{4} \cdot \frac{1}{1 - 1/4} = 1.$$

By the Gram-Schmidt process (recall that ℓ^2 is separable) we can now extend f_1 to a total orthonormal set $f_1, f_2, f_3, ...$ in ℓ^2.

We compute $(\eta_j) = T(f_1)$:

$$\eta_j = 2^{-j} \sum_{k=1}^{\infty} 2^{-k}\xi_k = \sqrt{3} \cdot 2^{-j} \sum_{k=1}^{\infty} 2^{-2k} = \frac{\sqrt{3}}{3} \cdot 2^{-j}.$$

Hence

$$T(f_1) = \tfrac{1}{3} \cdot f_1.$$

Note also that for $n \geqq 2$, writing $f_n = (\xi_j)$, the fact $\langle f_n, f_1 \rangle = 0$ implies

$$\sum_{k=1}^{\infty} 2^{-k}\xi_k = 0$$

Hence $T(f_n) = 0$ for all $n \geqq 2$.

Now we have exactly the situation from Homework assignment 3, problem 4: T is a bounded linear operator on ℓ^2, $\{f_1\}$ is a total orthonormal sequence in ℓ^2, and $T(f_n) = \lambda_n f_n$ with $\lambda_1 = \frac{1}{3}$and $\lambda_2 = \lambda_3 = \cdots = 0$ (thus $\lim_{n\to\infty} \lambda_n = 0$). Hence by that problem, T is compact and self-adjoint. [Note that in the present case the fact that T is compact can also be seen directly using Theorem 8.1-4(a), for by the above formulas we have $T(H) = \text{Span}\{f_1\}$, i.e. the range of T is finite dimensional.]

We also see that the eigenvalues of T are $\frac{1}{3}$ (with eigenspace $=$ Span$\{f_1\}$) and 0 (with eigenspace $=$ Span$\{f_2, f_3, \cdots\} = \{f_1\}^{\perp}$).

[Proof that there are no other eigenvectors and eigenvalues: Assume that $x = \sum_{n=1}^{\infty} a_n f_n \in \ell^2$ is an eigenvector of T with eigenvalue λ. Then $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ and $\lambda x = Tx = \frac{1}{3}a_1 f_1 + 0a_2 f_2 + 0a_3 f_3 + \cdots = \frac{1}{3}a_1 f_1$, hence $\lambda a_1 = \frac{1}{3}a_1$ and for each $n \geqq 2$, $\lambda a_n = 0$. We also know that $x \neq 0$, i.e. there is at least one n for which $a_n \neq 0$. It follows that either $\lambda = \frac{1}{3}$ and $x = a_1 f_1$ or else $\lambda = 0$ and $x = a_2 f_2 + a_3 f_3 + \cdots \in \text{Span}\{f_2, f_3, \cdots\}$, as claimed.]