LIST OF POSSIBLE EXAM QUESTIONS

Other questions than those below can certainly appear on the exam, **but:** The last two questions on the exam (10p for each) will be picked among the questions marked by "$\star$" below.

Lecture #2

– Give the definition of a *Dirichlet character* (of period q).
– Define what it means for a Dirichlet character to be principal, real, resp. complex.
– State the defining formula of the Dirichlet L-function $L(s,\chi)$ as a Dirichlet series, and also the Euler product formula for $L(s,\chi)$.
$\star$ State Dirichlet's Theorem on primes in arithmetic progressions; then outline a proof of this theorem. (Note: You do not need to prove anything, instead merely describe the main steps of a proof.)

Lecture #3

– Define what it means for an infinite product to be absolutely convergent.
– Give an example of a sequence of complex numbers $u_1, u_2, u_3, \ldots$ such that $|u_n| < 1$ for all n, and $\prod_{n=1}^{\infty}(1+u_n)$ converges but $\sum_{n=1}^{\infty} u_n$ does *not* converge.
– Prove formulas such as $\sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s} = L(s,\chi)^{-1}$ and $\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)}$, for appropriate $s \in \mathbb{C}$.
– Formulate and prove Euler's product formula for Riemann's zeta function, and also for the Dirichlet L-function.

Lecture #4

– Define $\int_A^B g(x)\, df(x)$ in the case when $g \in C([A,B])$ and f is piecewise constant.
– State the integration-by-parts formula for $\int_A^B g(x)\, df(x)$, when we also assume that $g \in C^1([A,B])$.
– Use the above in concrete problems (e.g.: Obtain an asymptotic formula for $\sum_{p \le X} \frac{1}{p}$ as $X \to \infty$).
– What is a *Dirichlet series?*
– Define the notions of *abscissa of convergence* (σ_c) and *abscissa of absolute convergence* (σ_a). What relation is known to always hold between these two?
– Define von Mangoldt's function $\Lambda(n)$.
– Give the formulas for $\log L(s,\chi)$ and $\dfrac{L'(s,\chi)}{L(s,\chi)}$ as Dirichlet series.

Lecture #6

– Define what it means for a Dirichlet character to be *primitive.*
– Give an example of a primitive, complex Dirichlet character mod 15.
– State and prove a formula for $\sum_{n=1}^{q} \chi(n)$, when χ is a Dirichlet character mod q.
– State and prove a formula for $\sum_{\chi \in X_q} \chi(n)$ for any $q \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$, where X_q is the set of all Dirichlet characters mod q.

Lecture #7

– Define $\vartheta(x)$ and $\psi(x)$ (Chebyshev's auxiliary functions). What is their asymptotic size as $x \to \infty$? Techniques for translating between asymptotics for $\pi(x)$, $\vartheta(x)$ and $\psi(x)$.
– Be able to prove the formula $\log(n!) = \sum_{m=1}^{\infty} \lfloor n/m \rfloor \Lambda(m)$.

Lectures #8,9,10

⋆ Give the statement of the PNT; then outline a proof of this theorem. [More difficult variant: Do same for the PNT *for arithmetical sequences.*]
To be able to carry out the above exercise, you should in particular be able to carry out each of the following:
– Outline the proof that $\zeta(s) \neq 0$ for all $s \in \mathbb{C}$ with $\Re(s) = 1$.
– State (and prove) a formula for $\zeta'(s)/\zeta(s)$ as an integral involving $\psi(x)$ (alt: as an integral involving $\psi_1(x)$).
– State (and outline a proof of) a formula for $\psi_1(x)$ as an integral involving $\zeta'(s)/\zeta(s)$.

Lecture #12

– State the product formula defining $\Gamma(z)$ (for all $z \in \mathbb{C}$), including the definition of γ.
– State the integral formula defining $\Gamma(z)$ for $\mathrm{Re}\, z > 0$.
– State the formula connecting $\Gamma(z)$ and $\Gamma(z+1)$.
– State the formula connecting $\Gamma(z)$ and $\Gamma(1-z)$.
– Explain how the Γ-function provides a meromorphic continuation to all $\mathbb{C}$ of the arithmetic function $n \mapsto n!$.
– You do not need to know *Stirling's formula* by heart, but if you are given the statement of it then you should understand it and be able to use it to solve other problems.

Lecture #13

– State the functional equation for $\zeta(s)$. Also give (outline) the proof.

Lecture #14

– State the explicit formula for $\psi(x)$. Explain where "it comes from".

(Thus: You should be able to state LN Theorem 13.1, and explain that, *formally*, this comes from "$\psi_0(x) = \frac{1}{2\pi i} \int_{(c)} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s}\, ds$". You do *not* need to remember the more precise Theorem 13.2; however you should be able to *use* a result such as Theorem 13.2!)

Lecture #15

– State the result on a zero-free region for $\zeta(s)$, LN Theorem 11.1.
– State the PNT with error term (LN Theorem 13.8).
⋆ You should also be able to *prove* the PNT with error term, if you are given the statement of Theorem 13.2.

Lecture #17

– Define *integral binary quadratic form* and its *discriminant.* Also define *"representation of n by Q",* and what it means for such a representation to be *proper.*
– Define what it means for two quadratic forms to be *equivalent.* Define the notion of *(positive/negative) definite*, *indefinite*, and *primitive.*
– Define the class number $h(d)$ which we studied in the course.

Lecture #18

– State Dirichlet's class number formula for $L(1, (\frac{d}{\cdot}))$. (Do not forget to specify for which values of d the formula is valid.)

Lecture #19

– Define the Jacobi Theta Function $\Theta(z \mid \tau)$.
– State and prove the basic properties of $\Theta(z \mid \tau)$.
(That is: (1) $\Theta(z \mid \tau)$ is holomorphic in $\mathbb{C} \times \mathbb{H}$. (2) $\Theta(z+1 \mid \tau) = \Theta(z \mid \tau)$.

(3) $\Theta(z+\tau \mid \tau) = \Theta(z \mid \tau)e(-z)e(-\frac{1}{2}\tau)$. (4) $\Theta(z \mid \tau) = 0$ when $z = \frac{1}{2}(1+\tau) + m + n\tau$ for any $m, n \in \mathbb{Z}$. (5) $\Theta(z \mid -\tau^{-1}) = \sqrt{\frac{\tau}{i}}\, e(\frac{1}{2}\tau z^2)\Theta(z\tau \mid \tau)$.)

⋆ State the triple product formula for $\Theta(z \mid \tau)$, and outline a proof.

Lecture #20

⋆ State Jacobi's formula for $\#\{\langle x_1, x_2\rangle \in \mathbb{Z}^2 \,:\, x_1^2 + x_2^2 = n\}$, and outline the main steps in a proof of this formula.

⋆ State Jacobi's formula for $\#\{\langle x_1, x_2, x_3, x_4\rangle \in \mathbb{Z}^2 \,:\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = n\}$, and outline the main steps in a proof of this formula.

Lectures #22–23

⋆ Give the statement of Rényi's large sieve bound, and describe the main steps in the proof.

⋆ State the Large Sieve Inequality (LN Theorem 20.4 and Def. 20.1), and describe the main steps in the proof. (In this description you need to give the statement of Theorem 20.2; but you do not need to give the proof of Theorem 20.2.)

Lecture #24

– State the Bombieri–Vinogradov Theorem (LN Theorem 16.7 or 21.1), and explain in which sense it gives "as strong information as GRH, but on average".