Hints / solution sketches to problems

8.1. By Lemma 8.14 we have

$$\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma - 1 - \sum_{n=1}^{\infty}\Big(\frac{1}{1+n} - \frac{1}{n}\Big) = -\gamma - 1 - \frac{1}{2} + 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \ldots = -\gamma.$$

Also $\Gamma(1) = 1$ by Lemma 8.12. Hence $\Gamma'(1) = -\gamma$. □

8.3. For any $a, b \in \mathbb{C}$ with $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$ we have:

$$\Gamma(a)\Gamma(b) = \int_0^\infty \int_0^\infty e^{-t-s} t^{a-1} s^{b-1}\, ds\, dt.$$

Here let us substitute

$$\begin{cases} u = t + s \\ r = s/(t+s) \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} s = ur \\ t = u(1-r). \end{cases}$$

This map is a diffeomorphism between the quadrant

$$\{(s,t) \in \mathbb{R}^2 \,:\, s > 0, t > 0\}$$

and the strip $\{(u,r) \in \mathbb{R}^2 \,:\, u > 0,\ 0 < r < 1\}$, and its Jacobian is

$$\left|\frac{\partial(s,t)}{\partial(u,r)}\right| = \det\begin{pmatrix} r & u \\ 1-r & -u \end{pmatrix} = -ru - u(1-r) = -u.$$

Hence:

$$\begin{aligned}\Gamma(a)\Gamma(b) &= \int_0^\infty \int_0^1 e^{-u}\big(u(1-r)\big)^{a-1}\big(ur\big)^{b-1}\, u\, dr\, du \\ &= \int_0^\infty e^{-u} u^{a+b-1}\, du \int_0^1 (1-r)^{a-1} r^{b-1}\, dr \\ &= \Gamma(a+b)\int_0^1 (1-r)^{a-1} r^{b-1}\, dr.\end{aligned}$$

□

8.4. The most "natural" solution is perhaps to study

$$f(z) = \frac{\Gamma(2z)}{\Gamma(z)\Gamma(z+\frac{1}{2})},$$

which is easily verified to be an entire function with no zeros and no poles. It is now natural to apply Weierstrass factorization, Theorem 8.7[1], to $f(z)$; however then we first need to prove that $f(z)$ is of *finite order*, and this involves some technical work.

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[1]Or, in our specific situation, we could simply apply Lemma 8.1.

Instead let us here work with the *logarithmic derivative!* By Lemma 8.14 we have for every $z \in \mathbb{C} \setminus \{0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots\}$:

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})} = -2\gamma - \frac{1}{z} - \frac{1}{z+\frac{1}{2}} - \sum_{n=1}^{\infty}\Big(\frac{1}{z+n} - \frac{1}{n}\Big) - \sum_{n=1}^{\infty}\Big(\frac{1}{z+\frac{1}{2}+n} - \frac{1}{n}\Big)$$

$$= 2\Big(-\gamma - \frac{1}{2z} - \frac{1}{2z+1} - \sum_{n=1}^{\infty}\Big(\frac{1}{2z+2n} - \frac{1}{2n}\Big) - \sum_{n=1}^{\infty}\Big(\frac{1}{2z+1+2n} - \frac{1}{1+2n}\Big)$$
$$+ \sum_{n=1}^{\infty}\Big(\frac{1}{2n} - \frac{1}{1+2n}\Big)\Big),$$

where the last step is justified since $\sum_{n=1}^{\infty}\big(\frac{1}{2n} - \frac{1}{1+2n}\big)$ is convergent. In fact by formula (236) on page 101 in the Lecture Notes, we have

$$\sum_{n=1}^{\infty}\Big(\frac{1}{2n} - \frac{1}{1+2n}\Big) = 1 + \sum_{m=1}^{\infty} m^{-1}(-1)^m = 1 - \log 2.$$

[Alternative: We have

$$\sum_{n=1}^{\infty}\Big(\frac{1}{2n} - \frac{1}{1+2n}\Big) = \lim_{N\to\infty}\Big(\sum_{n=1}^{N}\frac{1}{2n} - \sum_{n=1}^{N}\frac{1}{1+2n}\Big) = \lim_{N\to\infty}\Big(2\sum_{n=1}^{N}\frac{1}{2n} - \sum_{n=1}^{N}\frac{1}{1+2n} - \sum_{n=1}^{N}\frac{1}{2n}\Big)$$
$$= \lim_{N\to\infty}\Big(\sum_{n=1}^{N}\frac{1}{n} - \sum_{m=2}^{2N+1}\frac{1}{m}\Big),$$

and using Lemma 8.13 this is

$$= \lim_{N\to\infty}\Big(\gamma + \log N - (\gamma - 1 + \log(2N+1))\Big) = 1 - \log 2.]$$

Hence from our previous computation we conclude

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})} = 2\Big(-\gamma - \frac{1}{2z} - \frac{1}{2z+1} - \sum_{m=2}^{\infty}\Big(\frac{1}{2z+m} - \frac{1}{m}\Big) + 1 - \log 2\Big)$$
$$= 2\Big(-\gamma - \log 2 - \frac{1}{2z} - \sum_{m=1}^{\infty}\Big(\frac{1}{2z+m} - \frac{1}{m}\Big)\Big)$$
$$= 2\Big(-\log 2 + \frac{\Gamma'(2z)}{\Gamma(2z)}\Big)$$

Hence we have proved (cf. Definition 8.3 and let's keep $z \in \mathbb{C}\setminus(-\infty, 0]$)

$$\frac{d}{dz}\Big(\log\Gamma(z) + \log\Gamma(z+\tfrac{1}{2}) - \log\Gamma(2z) + 2(\log 2)z\Big) = 0.$$

Thus the function inside the parenthesis is *constant* throughout $z \in \mathbb{C} \setminus (-\infty, 0]$; exponentiating we conclude that $\Gamma(z)\Gamma(z+\frac{1}{2})\Gamma(2z)^{-1}2^{2z}$ is also constant throughout $z \in \mathbb{C} \setminus (-\infty, 0]$. We can compute the constant e.g. by taking $z = \frac{1}{2}$ (and using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$); this gives that the constant is $= 2\sqrt{\pi}$. Hence $\Gamma(z)\Gamma(z+\frac{1}{2})\Gamma(2z)^{-1}2^{2z} = 2\sqrt{\pi}$ for all $z \in \mathbb{C} \setminus (-\infty, 0]$, and by continuity this must in fact hold for all $z \in \mathbb{C} \setminus \{0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots\}$. This proves the claimed formula. $\square$

Remark: One can get an even quicker solution by working with the *derivative* of the logarithmic derivative of $\Gamma(z)$; indeed, we have the very nice formula

$$\frac{d}{dz}\Big(\frac{\Gamma'(z)}{\Gamma(z)}\Big) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

(See Ahlfors [1, p. 200].)

8.5. By Stirling's formula (Theorem 8.17) we have, when $x \in [a, b]$ and $y \geq 1$,

$$\begin{aligned}
&\log\big|\Gamma(x \pm iy)\big| = \text{Re } \log \Gamma(x+iy) \\
&\quad = \text{Re } \big((x - \tfrac{1}{2} + iy)\log(x+iy)\big) - x + \log\sqrt{2\pi} + O\big(y^{-1}\big) \\
&\quad = (x - \tfrac{1}{2})\log|x+iy| - y\arg(x+iy) - x + \log\sqrt{2\pi} + O\big(y^{-1}\big) \\
&\quad = (x - \tfrac{1}{2})\tfrac{1}{2}\Big(\log(y^2) + \log\Big(1 + \tfrac{x^2}{y^2}\Big)\Big) - y\Big(\tfrac{\pi}{2} - \arctan\tfrac{x}{y}\Big) - x + \log\sqrt{2\pi} + O\big(y^{-1}\big) \\
&\quad = (x - \tfrac{1}{2})\log y + (x - \tfrac{1}{2})\tfrac{1}{2}\cdot O\big(y^{-2}\big) - \tfrac{\pi}{2}y + y\big(\tfrac{x}{y} + O(y^{-2})\big) - x + \log\sqrt{2\pi} + O\big(y^{-1}\big) \\
&\quad = (x - \tfrac{1}{2})\log y - \tfrac{\pi}{2}y + \log\sqrt{2\pi} + O\big(y^{-1}\big)
\end{aligned}$$

Exponentiation of this gives the stated formula (since $e^{O(y^{-1})} = 1 + O(y^{-1})$ for $y \geq 1$). $\square$

8.6. By Stirling's formula, Theorem 8.17, we have

(1)
$$\log\Gamma(z+\alpha) = \big(z+\alpha-\tfrac{1}{2}\big)\log(z+\alpha) - (z+\alpha) + \log\sqrt{2\pi} + O\big(|z+\alpha|^{-1}\big),$$

for all z with $|z+\alpha| \geq 1$ and $\big|\arg(z+\alpha)\big| \leq \pi - \varepsilon$. Here and below, for definiteness, we consider the argument function to take its values in $(-\pi,\pi]$, i.e. $\arg : \mathbb{C}\setminus\{0\} \to (-\pi,\pi]$.

Let us fix a constant $C > 1$ so large that $\big|\arg(1+w)\big| < \frac{1}{2}\varepsilon$ for all $w \in \mathbb{C}$ with $|w| \leq C$. Then note that if $|z| \geq C|\alpha|$ and $|z| \geq 1$ then $\arg(z+\alpha) = \arg(z(1+\alpha/z)) \equiv \arg(z) + \arg(1+\alpha/z) \pmod{2\pi}$ together with $\big|\arg(z+\alpha)\big| \leq \pi-\varepsilon$ and $\big|\arg(1+\alpha/z)\big| < \frac{1}{2}\varepsilon$ and imply that $\big|\arg(z)\big| \leq \pi - \frac{1}{2}\varepsilon$ and $\arg(z+\alpha) = \arg(z) + \arg(1+\alpha/z)$. Hence

$$\log(z+\alpha) = \log z + \log\Big(1+\frac{\alpha}{z}\Big),$$

where in all three places we use the principal branch of the logarithm function. Since $|\alpha/z| \leq C^{-1} < 1$ we can continue:

$$\log(z+\alpha) = \log z + \frac{\alpha}{z} + O\Big(\frac{\alpha^2}{z^2}\Big) = \log z + \frac{\alpha}{z} + O\big(|z|^{-2}\big)$$

(since we allow the implied constant to depend on α). Using this in (1) we get

$$\begin{aligned}\log\Gamma(z+\alpha) &= \big(z+\alpha-\tfrac{1}{2}\big)\Big(\log z + \frac{\alpha}{z} + O\big(|z|^{-2}\big)\Big) - (z+\alpha) + \log\sqrt{2\pi} + O\big(|z+\alpha|^{-1}\big)\\ &= \big(z+\alpha-\tfrac{1}{2}\big)\log z - z + \log\sqrt{2\pi} + O\big(|z|^{-1}\big) + O\big(|z+\alpha|^{-1}\big)\\ &= \big(z+\alpha-\tfrac{1}{2}\big)\log z - z + \log\sqrt{2\pi} + O\big(|z|^{-1}\big),\end{aligned}$$

where in the last step we used the fact that $|z+\alpha| \geq |z| - |\alpha| = |z|\big(1-|\alpha/z|\big) \geq (1-C^{-1})|z| \gg |z|$. Hence we have proved the desired formula for all z satisfying $|z| \geq 1$, $|z+\alpha| \geq 1$, $|\arg(z+\alpha)| \leq \pi-\varepsilon$ and $|z| \geq C|\alpha|$.

It remains to treat z satisfying $|z| \geq 1$, $|z+\alpha| \geq 1$, $|\arg(z+\alpha)| \leq \pi-\varepsilon$ and $|z| \leq C|\alpha|$. This is trivial: These set of such z is *compact* and $\log\Gamma(z+\alpha) - \big(z+\alpha-\frac{1}{2}\big)\log z + z - \log\sqrt{2\pi}$ is continuous on this set, hence bounded. Also $|z|$ is bounded on the set; hence $|z|^{-1}$ is bounded from below. Hence by adjusting the implied constant we have $\log\Gamma(z+\alpha) - \big(z+\alpha-\frac{1}{2}\big)\log z + z - \log\sqrt{2\pi} = O\big(|z|^{-1}\big)$ for all z in our compact set, as desired. $\square$

9.1. If we set $\tau = i/x$ (with $x \in \mathbb{R}_{>0}$) in the formula that we want to prove, it becomes:

$$\Theta\big(z \mid ix\big) = \sqrt{\frac{1}{x}} e^{-\pi z^2/x} \Theta\big(iz/x \mid i/x\big), \tag{2}$$

viz.,

$$\sum_{n\in\mathbb{Z}} e^{2\pi i n z} e^{-\pi n^2 x} = \sqrt{\frac{1}{x}} e^{-\pi z^2/x} \sum_{n\in\mathbb{Z}} e^{-2\pi n z/x} e^{-\pi n^2/x}.$$

Multiplying by $\sqrt{x}$, we see that the above formula is equivalent with

$$\sqrt{x} \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x + 2\pi i n z} = \sum_{n\in\mathbb{Z}} e^{-(n+z)^2\pi/x},$$

which is exactly the formula in Theorem 9.2 (after replacing z by α)! Hence (since we worked with equivalences), we have proved that (2) holds for all $x > 0$ and all $z \in \mathbb{C}$. In other words, the formula that we want to prove,

$$\Theta\Big(z \mid -\frac{1}{\tau}\Big) = \sqrt{\frac{\tau}{i}} e^{\pi i z^2 \tau} \Theta\big(z\tau \mid \tau\big), \tag{3}$$

holds for *all* τ *along the positive imaginary axis,* and all $z \in \mathbb{C}$. Hence, by analyticity, (3) in fact holds for all $\tau \in \mathbf{H}$ and $z \in \mathbb{C}$.

(Details for the last step: *Fix* an arbitrary $z \in \mathbb{C}$, and set

$$f(\tau) := \Theta\Big(z \mid -\frac{1}{\tau}\Big) - \sqrt{\frac{\tau}{i}} e^{\pi i z^2 \tau} \Theta\big(z\tau \mid \tau\big).$$

This is a holomorphic function of τ in $\mathbf{H}$, and $f(\tau) = 0$ for all τ along the positive imaginary axis. Hence by [4, Theorem 10.18], $f(\tau) = 0$ for all $\tau \in \mathbf{H}$.) $\square$

9.2 (a). Writing out the relation $\Lambda(s) = \Lambda(1-s)$ from Theorem 9.1 we have:

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s) = \pi^{-\frac{1}{2}+\frac{1}{2}s}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\zeta(1-s).$$

This identity, as well as those below, is an equality between two functions meromorphic in the whole complex plane. It follows that

$$\zeta(1-s) = \pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}\zeta(s).$$

But we have $\Gamma(\frac{1}{2}-\frac{1}{2}s)\Gamma(\frac{1}{2}+\frac{1}{2}s) = \frac{\pi}{\sin(\pi(\frac{1}{2}-\frac{1}{2}s))} = \frac{\pi}{\cos(\frac{\pi}{2}s)}$, by (319) with $z = \frac{1}{2} - \frac{1}{2}s$. Hence

$$\zeta(1-s) = \pi^{-\frac{1}{2}-s}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\cos(\tfrac{\pi}{2}s)\zeta(s).$$

Finally using Legendre's duplication formula (cf. Problem 8.4; use this with $z = \frac{1}{2}s$) we get

$$\zeta(1-s) = \pi^{-s}2^{1-s}\Gamma(s)\cos(\tfrac{\pi}{2}s)\zeta(s).$$

□

9.5. (a). (See, e.g., Ingham, [2, Theorem 14].) ***OUTLINE:*** For any $s \in \mathbb{C}$ with $\Re(s) > 1$ we have

$$\Gamma(s)\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \int_0^{\infty} u^{s-1} e^{-u}\, du = \sum_{n=1}^{\infty} \int_0^{\infty} x^{s-1} e^{-nx}\, dx, \tag{4}$$

where in the last step we substituted $t = nx$ in the integral. Using the fact that we have absolute convergence (writing $\Re(s) = \sigma$):

$$\sum_{n=1}^{\infty} \int_0^{\infty} \left| x^{s-1} e^{-nx} \right| dx = \sum_{n=1}^{\infty} \int_0^{\infty} x^{\sigma-1} e^{-nx}\, dx = \sum_{n=1}^{\infty} n^{-\sigma} \int_0^{\infty} u^{\sigma-1} e^{-u}\, du = \zeta(\sigma)\Gamma(\sigma) < \infty.$$

Hence we may change order of integration and summation in (4), obtaining:

$$\Gamma(s)\zeta(s) = \int_0^{\infty} \sum_{n=1}^{\infty} x^{s-1} e^{-nx}\, dx = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx. \tag{5}$$

We will compute the integral in (5) by first instead considering *another* (related) integral: For $0 < \varepsilon < 1$, let

$$I(s) := \int_{C_\varepsilon} \frac{z^{s-1}}{e^{-z} - 1}\, dz, \tag{6}$$

where $C_\varepsilon = C_{1,\varepsilon} + C_{2,\varepsilon} + C_{3,\varepsilon}$, with $C_{1,\varepsilon}$ being the contour from $-\infty$ to $-\varepsilon$ along the negative real axis, $C_{2,\varepsilon}$ being contour from $-\varepsilon$ back to $-\varepsilon$ along the circle of radius ε around the origin in positive direction, and finally $C_{3,\varepsilon}$ being the contour from $-\varepsilon$ to $-\infty$ along the negative real axis. In order to have a consistent branch of the function $z^{s-1} = e^{(s-1)\log z}$, we take $\arg(z) = -\pi$ along $C_{1,\varepsilon}$, then $\arg(z)$ increasing from $-\pi$ to π along the circle $C_{2,\varepsilon}$, and finally $\arg(z) = \pi$ along $C_{3,\varepsilon}$. Note that $I(s)$ is independent of the choice of ε, by Cauchy's integral theorem. Parametrizing the negative real axis as $z = -x$, $x \in \mathbb{R}_{>0}$, we compute that the contribution from $C_{1,\varepsilon}$ to $I(s)$ is

$$\int_{C_{1,\varepsilon}} \frac{z^{s-1}}{e^{-z} - 1}\, dz = \int_{\infty}^{\varepsilon} \frac{e^{-\pi i(s-1)} x^{s-1}}{e^x - 1} \big(-dx\big) = e^{-\pi i(s-1)} \int_{\varepsilon}^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx,$$

and the contribution from $C_{2,\varepsilon}$ is

$$\int_{C_{2,\varepsilon}} \frac{z^{s-1}}{e^{-z} - 1}\, dz = \int_{\varepsilon}^{\infty} \frac{e^{\pi i(s-1)} x^{s-1}}{e^x - 1} \big(-dx\big) = -e^{\pi i(s-1)} \int_{\varepsilon}^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx.$$

When $\varepsilon \to 0$, the sum of these two contributions tend to, by (5):

$$\big(e^{-\pi i(s-1)} - e^{\pi i(s-1)}\big) \cdot \Gamma(s)\zeta(s) = 2i \cdot \sin(\pi s) \cdot \Gamma(s)\zeta(s).$$

Furthermore, for all $z \in C_{2,\varepsilon}$ we have, at least if ε is sufficiently small: $|e^{-z}-1| > \frac{1}{2}|z| = \frac{1}{2}\varepsilon$ and $|z^{s-1}| = |z|^{\Re(s)-1} e^{-\Im(s-1)\cdot \arg(z)} \leq \varepsilon^{\sigma-1} e^{|\Im(s)|\pi}$. Hence, by the triangle inequality,

$$\left|\int_{C_{2,\varepsilon}} \frac{z^{s-1}}{e^{-z}-1}\,dz\right| \leq \int_{C_{2,\varepsilon}} \frac{|z^{s-1}|}{|e^{-z}-1|}\,|dz| \leq \int_{C_{2,\varepsilon}} \frac{\varepsilon^{\sigma-1}e^{|\Im(s)|\pi}}{\frac{1}{2}\varepsilon}\,|dz| = 2e^{|\Im(s)|\pi}\varepsilon^{\sigma-2}\cdot 2\pi\varepsilon$$
$$= 4\pi e^{|\Im(s)|\pi}\varepsilon^{\sigma-1},$$

which tends to 0 when $\varepsilon \to 0$. Hence, since $I(s)$ is independent of ε, we conclude that $I(s) = 2i \cdot \sin(\pi s)\cdot \Gamma(s)\zeta(s)$. Equivalently, using $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, we have:

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i}\cdot I(s). \tag{7}$$

The formula (7) has been proved for s with $\sigma > 1$, but the integral $I(s) = \int_{C_\varepsilon} \frac{z^{s-1}}{e^{-z}-1}\,dz$ (for any fixed $0 < \varepsilon < 1$) is easily verified to be an *entire* function of s. Hence the formula (7) provides the meromorphic extension of $\zeta(s)$ to all $s \in \mathbb{C}$!

Next we compute $I(s)$ in a different way, for s belonging to a certain region in the complex plane: For $0 < \varepsilon < 1$ and $R > 1$, let $C_{R,\varepsilon}$ be the *finite* contour obtained by replacing $-\infty$ by $-R$ in the definition of C_ε; then clearly $I(s) = \lim_{R\to+\infty} \int_{C_{R,\varepsilon}} \frac{z^{s-1}}{e^{-z}-1}\,dz$, for every $s \in \mathbb{C}$. Also let D_R be the contour from $-R$ back to $-R$ along the circle of ratius R around the origin in negative direction. Then $C_{R,\varepsilon} + D_R$ is a closed curve in the complex plane. Note that the poles of the function $z \mapsto \frac{z^{s-1}}{e^{-z}-1}$ (in our cut plane) are the points $z = k\cdot 2\pi i$ for $k \in \mathbb{Z}\setminus\{0\}$. Hence if we take $R = (n+\frac{1}{2})2\pi$ for some positive integer n, then by the Cauchy Residue Theorem,

$$\frac{1}{2\pi i}\int_{C_{R,\varepsilon}+D_R} \frac{z^{s-1}}{e^{-z}-1}\,dz = -\sum_{1\leq|k|\leq n} \operatorname*{Res}_{z=k\cdot 2\pi i}\left(\frac{z^{s-1}}{e^{-z}-1}\right) = \sum_{1\leq|k|\leq n} (k\cdot 2\pi i)^{s-1}$$

$$= \sum_{k=1}^{n} (2\pi k)^{s-1}\cdot\left(e^{\frac{\pi}{2}i(s-1)} + e^{-\frac{\pi}{2}i(s-1)}\right) = (2\pi)^{s-1}\cdot 2\cdot\cos\left(\tfrac{\pi}{2}(s-1)\right)\cdot\sum_{k=1}^{n} k^{s-1}. \tag{8}$$

Now assume that s lies in the half place $\sigma < 0$! Then one verifies that $\int_{D_R}\left|\frac{z^{s-1}}{e^{-z}-1}\right|\,|dz| \to 0$ as $n \to +\infty$, $R = (n+\frac{1}{2})2\pi$, and hence, by (8):

$$\frac{1}{2\pi i} I(s) = 2(2\pi)^{s-1}\cos\left(\tfrac{\pi}{2}(s-1)\right)\sum_{k=1}^{\infty} k^{s-1} = 2(2\pi)^{s-1}\cos\left(\tfrac{\pi}{2}(s-1)\right)\zeta(1-s).$$

Combining this with (7) (which as we discussed is valid for all $s \in \mathbb{C}$), we conclude that for s with $\sigma < 0$ we have:

$$\zeta(s) = \Gamma(1-s)\cdot 2(2\pi)^{s-1}\cos\left(\tfrac{\pi}{2}(s-1)\right)\zeta(1-s). \tag{9}$$

Hence by meromorphicity, the formula (9) in fact holds for all $s \in \mathbb{C}$ (away from poles). Finally note that after replacing s by $1-s$, (9) agrees with the formula (362) in Problem 9.2(a); and this formula was proved to be equivalent with the functional equation in Theorem 9.1. $\square$

(b). Recall that we have proved that the formula (7) is valid for all $s \in \mathbb{C}$ (away from poles). Let us apply that formula for $s = -n$ where n is a nonnegative integer. In this case, the integrand in $I(s) = \int_{C_\varepsilon} \frac{z^{s-1}}{e^{-z}-1}\,dz = \int_{C_\varepsilon} \frac{z^{-n-1}}{e^{-z}-1}\,dz$ is a meromorphic function in the whole complex plane, i.e. we do *not* need to cut the plane along the negative real axis! This implies that $\int_{C_{1,\varepsilon}+C_{3,\varepsilon}} \frac{z^{-n-1}}{e^{-z}-1}\,dz = 0$, and hence

$$I(-n) = \int_{C_{2,\varepsilon}} \frac{z^{-n-1}}{e^{-z}-1}\,dz = 2\pi i \cdot \operatorname*{Res}_{z=0}\left(\frac{z^{-n-1}}{e^{-z}-1}\right) = 2\pi i \cdot (-1)^n \cdot \operatorname*{Res}_{z=0}\left(\frac{z^{-n-1}}{e^{z}-1}\right).$$

In the above computation, the second equality holds by the Cauchy Residue Theorem, and the last equality is proved by writing $f(z) = \frac{z^{-n-1}}{e^{-z}-1}$, and then noticing that $\operatorname{Res}_{z=0} f(z) = -\operatorname{Res}_{z=0} f(-z)$ (true for an arbitrary meromorphic function), and also $f(-z) = (-1)^{n-1}\frac{z^{-n-1}}{e^{z}-1}$ ($\forall z$).

Combining the above with (7), we get:

$$\zeta(-n) = \Gamma(1+n)\cdot(-1)^n \cdot \operatorname*{Res}_{z=0}\left(\frac{z^{-n-1}}{e^{z}-1}\right) = (-1)^n n!\ \operatorname*{Res}_{z=0}\left(\frac{z^{-n-1}}{e^{z}-1}\right).$$

$\square$

(c). We take the definition of the Bernoulli polynomials to be the generating series $\frac{ze^{rz}}{e^z-1} = \sum_{n=0}^\infty \frac{B_n(r)}{n!} z^n$ ($z, r \in \mathbb{C}$, $|z|$ small). Recall that the Bernoulli numbers are given by $B_n := B_n(0)$; hence, setting $r = 0$ in the previous relation, we have the generating series

$$\frac{z}{e^z-1} = \sum_{m=0}^\infty \frac{B_m}{m!} z^m,$$

for $z \in \mathbb{C}$ with $|z|$ small. (In fact the above relation is valid for all z with $|z| < 2\pi$, since the function $\frac{z}{e^z-1}$ is holomorphic in this disc, after noticing that the singularity at $z = 0$ is removable.)

It follows that for any nonnegative integer n, we have the Laurant series

$$\frac{z^{-n-1}}{e^z-1} = z^{-n-2}\sum_{m=0}^\infty \frac{B_m}{m!} z^m = \sum_{m=0}^\infty \frac{B_m}{m!} z^{m-n-2}.$$

Here the coefficient in front of z^{-1} is $B_{n+1}/(n+1)!$, viz.,

$$\operatorname*{Res}_{z=0}\left(\frac{z^{-n-1}}{e^z-1}\right)=\frac{B_{n+1}}{(n+1)!}.$$

Combining this formula with part (b), we obtain

$$\zeta(-n)=(-1)^n\frac{B_{n+1}}{n+1}.$$

□

Remark: Recall that $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_3=0$ and $B_4=-\frac{1}{30}$; and in fact $B_n=0$ for all odd integers $n\geq 3$. Hence the formula proved above gives that $\zeta(0)=-\frac{1}{2}$, $\zeta(-1)=-\frac{1}{12}$, $\zeta(-3)=\frac{1}{120}$, and $\zeta(-n)=0$ for all even integers $n\geq 2$.

(d). Recall the formula (362) in Problem 9.2(a):

$$\zeta(1-s)=2(2\pi)^{-s}\cos\bigl(\tfrac{\pi}{2}s\bigr)\Gamma(s)\zeta(s).$$

Setting $s=2m$ (with $m\in\mathbb{Z}^+$) in this formula gives:

$$\zeta(1-2m)=2(2\pi)^{-2m}(-1)^m(2m-1)!\cdot\zeta(2m).$$

Here by part (c) we have $\zeta(1-2m)=-\frac{B_{2m}}{2m}$; hence:

$$\zeta(2m)=\frac{1}{2}(2\pi)^{2m}(-1)^m\frac{1}{(2m-1)!}\cdot\left(-\frac{B_{2m}}{2m}\right)=2^{2m-1}\pi^{2m}\frac{(-1)^{m+1}B_{2m}}{(2m)!}.$$

The last formula, together with the fact that $\zeta(2m)>0$, implies that $(-1)^{m+1}B_{2m}>0$ [2]; hence $(-1)^{m+1}B_{2m}=|B_{2m}|$, and we obtain the formula stated in the problem formulation. □

[2] This can of course be proved in many other ways as well.

15.1. Clearly (b) implies (a) and hence *we will only give a proof of (b).* (For another proof of the weaker bound (a), cf., e.g. [3, Problems 1.3.4-5].)

Recall $\phi(q) = q\prod_{p|q}(1-p^{-1})$; hence our task is to prove

$$\prod_{p|q}(1-p^{-1}) \gg \frac{1}{\log\log q} \tag{10}$$

for all $q \geq 3$. By taking the logarithm we see that this is equivalent to proving

$$\sum_{p|q}\log(1-p^{-1}) \geq -\log\log\log q - O(1) \tag{11}$$

for all $q \geq 3$. We know from the Taylor expansion of $\log(1+x)$ that there is a constant $C > 0$ such that $\big|\log(1+x) - x\big| \leq Cx^2$ for all $|x| \leq \frac{1}{2}$. Hence the left hand side of (11) differs from $-\sum_{p|q} p^{-1}$ by

$$\leq \sum_{p|q} Cp^{-2} \leq C\sum_{n=1}^{\infty} n^{-2} = O(1).$$

Hence our task is equivalent with the task of proving

$$\sum_{p|q} p^{-1} \leq \log\log\log q + O(1). \tag{12}$$

We will first treat the special case when $q = \prod_{p\leq x} p$ for some $x \geq 2$. In this case we have

$$\sum_{p|q} p^{-1} = \sum_{p\leq x} p^{-1} = \log\log x + O(1), \tag{13}$$

by Mertens' Proposition 6.5, and also

$$\log q = \log\Big(\prod_{p\leq x} p\Big) = \sum_{p\leq x}\log p = \vartheta(x) \sim x \qquad \text{as } x\to\infty,$$

by the prime number theorem (cf. Theorem 7.1 and Proposition 6.2), so that

$$\log\log q = \log x + o(1) \qquad \text{as } x\to\infty \tag{14}$$

and

$$\log\log\log q = \log\log x + o(1) \qquad \text{as } x\to\infty,$$

and in particular $\log\log\log q = \log\log x + O(1)$ for all $x \geq 3$. The last relation together with (13) implies that (12) holds.

From this it is easy to prove that (12) also holds for a *general* $q \geq 3$: Let q be an arbitrary integer ≥ 3. Suppose that q contains exactly n

distinct primes $p'_1 < p'_2 < \ldots < p'_n$ in its prime factorization, and let $p_1 < p_2 < \ldots < p_n$ be the *smallest* n primes. Then $p_j \le p'_j$ for each j, so that

$$\sum_{p|q} p^{-1} = \sum_{j=1}^{n} {p'_j}^{-1} \le \sum_{j=1}^{n} {p_j}^{-1},$$

and using the fact that (12) holds with q replaced by $\prod_{j=1}^n p_j$ we can continue:

$$\le \log\log\log\Big(\prod_{j=1}^{n} p_j\Big) + O(1) \le \log\log\log\Big(\prod_{j=1}^{n} p'_j\Big) + O(1) \le \log\log\log q + O(1),$$

i.e. (12) holds for our q. □

Remark 1. Using the full strength of Mertens' Proposition 6.5 together with Proposition 6.6 we actually obtain $\prod_{p\le x}(1-p^{-1}) \sim \frac{e^{-\gamma}}{\log x}$ as $x \to \infty$ (cf. [2, Thm. 7 (24)]). Combining this with (14) we get

$$\phi(q) \sim e^{-\gamma}\frac{q}{\log\log q} \qquad \text{as } q = \prod_{p\le x} p, \quad x \to \infty. \tag{15}$$

In particular this shows that the lower bound given in (b) is the best possible. In fact from the proof of (b) we also see that $\phi(q) \ge e^{-\gamma}\frac{q}{\log\log q}(1-o(1))$ as $q \to \infty$ through *all* integers, and thus

$$\liminf_{q\to\infty} \phi(q)\frac{\log\log q}{q} = e^{-\gamma}. \tag{16}$$

16.3. By Theorem 15.4, to prove (554) we only have to prove that if $1 \le q \le x^{\frac12}(\log)^{-A-2}$ then $x^{\frac12}\log^2 x \ll \frac{x}{\phi(q)}(\log x)^{-A}$, where the implied constant is absolute. In other words we wish to prove $\phi(q) \ll x^{\frac12}(\log x)^{-A-2}$. This is clear since $\phi(q) \le q \le x^{\frac12}(\log)^{-A-2}$.

To prove the second statement we assume $x^{\frac12}(\log)^{-A-2} \ll q \le x$, and then wish to prove that (554) implies (524) apart from an extra factor $\log\log x$ in the big-O-term; in other words we wish to prove that $\frac{x}{\phi(q)}(\log x)^{-A} \ll x^{\frac12}(\log x)^2(\log\log x)$. Equivalently, we wish to prove $\phi(q) \gg x^{\frac12}(\log x)^{-A-2}(\log\log x)^{-1}$. This is clear since, using Problem 15.1(b) (if $q \ge 3$) $\phi(q) \gg \frac{q}{\log\log q} \ge \frac{q}{\log\log x} \gg x^{\frac12}(\log x)^{-A-2}(\log\log x)^{-1}$. □

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