Assignment 1: Some answers, comments and references

Problem 1. One obtains this formula by applying the Euler-MacLaurin summation formula (Theorem 1.19 in the course notes) with $f(x) = \log(x+z)$, $h = 2$ and $A = 0$, $B = N$. One must note, however, that the left hand side in Theorem 1.19 is "$\sum_{A<n\leqslant B} f(n)$", so that the above choices give "$\sum_{n=1}^{N} \log(n+z)$"; therefore one has to add "$\log(z)$" to both sides in order to obtain the formula in the problem formulation. (Alternatively one may apply the Euler-MacLaurin summation formula with $A < 0$ close to 0, and then let $A \to 0^-$.)

Problem 2. Writing $N(x) := \#\{\omega_n \leqslant x\}$, and letting T_0 be a fixed number in the interval $1 < T_0 < \omega_1$, one has

$$\log \prod_{\omega_n < T} (1 - \omega_n^{-1}) = \int_{T_0}^{T-0} \log(1 - x^{-1})\, dN(x)$$

(where the "$T - 0$" indicates that the right hand side should be understood to mean $\lim_{T' \to T-} \int_{T_0}^{T'} \log(1 - x^{-1})\, dN(x)$). After integrating by parts, the contribution which is most difficult(?) to handle is

$$-\int_{T_0}^{T} \frac{N(x)}{x(x-1)}\, dx. \tag{1}$$

This can be written as $-\int_{T_0}^{T} \frac{cx + O(x^{1/2})}{x(x-1)}\, dx$, and here the contribution from the error term is: $\int_{T_0}^{T} \frac{O(x^{1/2})}{x(x-1)}\, dx = O(1)$; note that this is the best possible bound on $\int_{T_0}^{T} \frac{O(x^{1/2})}{x(x-1)}\, dx$ as $T \to \infty$! However, it *is* possible to obtain a sharper asymptotic formula for (1) by rewriting $\int_{T_0}^{T}$ as $\int_{T_0}^{\infty} - \int_{T}^{\infty}$ [1]; namely:

$$\begin{aligned} -\int_{T_0}^{T} \frac{N(x)}{x(x-1)}\, dx &= -\int_{T_0}^{T} \frac{cx}{x(x-1)}\, dx - \int_{T_0}^{T} \frac{N(x) - cx}{x(x-1)}\, dx \\ &= -\int_{T_0}^{T} \frac{c}{x-1}\, dx - \int_{T_0}^{\infty} \frac{N(x) - cx}{x(x-1)}\, dx + \int_{T}^{\infty} \frac{N(x) - cx}{x(x-1)}\, dx. \end{aligned}$$

[1] This method was also used on the last slide of lecture #1.

The above is justified by the fact that the integral $\int_{T_0}^{\infty} \frac{N(x)-cx}{x(x-1)}\,dx$ is absolutely convergent, since $|N(x) - cx| \ll x^{1/2}$. Now one can contine:

$$= -c\log\Big(\frac{T-1}{T_0-1}\Big) - \int_{T_0}^{\infty} \frac{N(x)-cx}{x(x-1)}\,dx + \int_{T}^{\infty} \frac{O(x^{1/2})}{x(x-1)}\,dx.$$

$$= -c\log T + O(T^{-1}) + c\log(T_0-1) - \int_{T_0}^{\infty} \frac{N(x)-cx}{x(x-1)}\,dx + O(T^{-1/2}).$$

Combining this with the other term from the integration by parts, we conclude:

$$\log \prod_{\omega_n<T} (1-\omega_n^{-1}) = -c\log T + \beta + O(T^{-1/2}), \tag{2}$$

where

$$\beta := -c + c\log(T_0-1) - \int_{T_0}^{\infty} \frac{N(x)-cx}{x(x-1)}\,dx. \tag{3}$$

Hence, exponentiating (using $\exp(O(T^{-1/2})) = 1 + O(T^{-1/2})$ for T large), we obtain:

Answer: $\displaystyle\prod_{\omega_n<T} (1-\omega_n^{-1}) = e^{\beta}\cdot T^{-c} + O(T^{-c-\frac{1}{2}}) \qquad \text{as } T\to\infty.$

Remark 1: If one works on using the less precise "$O(1)$" bound discussed below (1), one obtains instead of (2) the less precise estimate "$\log\prod_{\omega_n<T}(1-\omega_n^{-1}) = -c\log T + O(1)$", which after exponentiating gives the following less precise final answer:
"$\prod_{\omega_n<T}(1-\omega_n^{-1}) \asymp T^{-c}$ as $T\to\infty$".

Remark 2: Of course the right hand side of (3) is independent of the choice of the number $T_0 \in (1,\omega_1)$; this is clear from the proof but also easy to verify aposteriori, using the fact that $N(x) = 0$ for $x < \omega_1$.

Problem 3. This problem is Folland, Exc. 2.28, mildly modified.

Problem 4. This problem is Folland, Exc. 2.36.

Problem 5. I took this problem from Folland, Exc. 6.15. It is a Cauchy version of the Vitali Convergence Theorem.

Problem 6. Part (a) and (b) are borrowed from Folland, Exc. 7.24.

I admit that I had missed that giving (a) and (b) together was perhaps a bit "boring", since the sequence $\mu_n = \delta_{-n}$ (the Dirac measure at the point $-n$) works in *both* (a) and (b)! In hindsight, I would have preferred to instead give the following version of part (a): "Find an example of a sequence (μ_n) in $M([0,1])$ such that $\mu_n \to 0$ vaguely, but $\|\mu_n\| \nrightarrow 0$." (See the following footnote for an example: [2])

Problem 7. (a) $c_{\alpha,\beta} = (-1)^{|\alpha|} \prod_{j=1}^n \big(\beta_j(\beta_j+1)\cdots(\beta_j+\alpha_j-1)\big)$.

Problem 8. (a) $\int_{\mathbb{R}} |f_n|\,dx = \int_{\mathbb{R}} f_n\,dx = 1$.
(b) $\text{supp}(f_n) = [0, a_1 + a_2 + \cdots + a_n]$.
(c) One may compute f_2 explicitly and then verify that $f_2 \in C(\mathbb{R})$. From this, one may prove $f_n \in C^{n-2}(\mathbb{R})$ by induction, where the key step is to note (using $f_n = f_{n-1} * g_{a_n}$) that

$$\forall n \geqslant 3: \ \forall x \in \mathbb{R}: \quad f_n'(x) = \frac{1}{a_n}\big(f_{n-1}(x) - f_{n-1}(x - a_n)\big).$$

On the other hand, by induction one may also prove that

$$\forall n \geqslant 2: \ \forall x \in [0, \min(a_1, \ldots, a_n)]: \quad f_n(x) = \frac{x^{n-1}}{(n-1)! \prod_{j=1}^n a_j},$$

while $f_n(x) = 0$ for $x < 0$, and from this it is easy to verify that the $(n-1)$st derivative $f_n^{(n-1)}(x)$ does not exist at $x = 0$; hence $f_n \notin C^{n-1}(\mathbb{R})$.
(d) By (b) we have $f_n(x) = 0$ for $x \leqslant 0$ and for $x \geqslant \sum_{j=1}^n a_j$. One may also verify that f_n is symmetric about the point $\frac{1}{2}s_n$ where $s_n := \sum_{j=1}^n a_j$, i.e. $f_n(s_n - x) \equiv f_n(x)$; also the function $f_n(x)$ is increasing for $x \in [0, \frac{1}{2}s_n]$ and (hence) decreasing for $x \in [\frac{1}{2}s_n, s_n]$. Now for any fixed point $x' > 0$, for every n so large that $x' < \frac{1}{2}s_n$, we have

$$1 = \int_0^{s_n} f_n(x)\,dx \geqslant \int_{x'}^{\frac{1}{2}s_n} f_n(x)\,dx \geqslant \big(\tfrac{1}{2}s_n - x'\big) f_n(x').$$

But as $n \to \infty$ we have $\frac{1}{2}s_n - x' \to +\infty$ and hence the above inequality (together with the fact that $f_n(x') \geqslant 0$) implies that $f_n(x') \to 0$.

[2] One may e.g. take $\mu_n = \delta_0 - \delta_{1/n}$.

Problem 8 – the "even more challenging tasks": For the case $\sum_{n=1}^{\infty} a_n < \infty$, cf., e.g., Theorem 1.3.5 in Hörmander, "The Analysis of Linear Partial Differential Operators I" (1990).

We now turn to the question about uniform convergence to 0. We will outline a proof that

$$(4) \qquad f_n \textit{ tends uniformly to } 0 \textit{ if and only if } \sum_{n=1}^{\infty} a_n^2 = \infty.$$

(Note that $\sum_{n=1}^{\infty} a_n^2 = \infty \Rightarrow \sum_{n=1}^{\infty} a_n = \infty$, but the converse is not true.)

In the first few paragraphs we consider *arbitrary* positive numbers $a_1, a_2, \ldots$. Let us start by centering the functions f_n: Write $s_n = a_1 + \cdots + a_n$ and set $F_n := \tau_{-s_n/2} f_n$; then from the properties of f_n mentioned in part (d) above, it follows that for each n, F_n *is even*, and F_n is *increasing on* $(-\infty, 0]$ *and (thus) decreasing on* $[0, +\infty)$. In particular F_n attains a global maximum at $x = 0$, and since F_n is nonnegative it follows that f_n tends uniformly to 0 if and only if $\lim_{n\to\infty} F_n(0) = 0$.

Note that

$$F_n = \tau_{-s_n/2} f_n = \tau_{-s_n/2}(g_{a_1} * \cdots * g_{a_n}) = G_{a_1} * \cdots * G_{a_n},$$

where we have defined

$$G_a := \tau_{-a/2} g_a = a^{-1} \cdot \chi_{-(a/2,a/2)}.$$

We obviously have $G_a \in \mathrm{L}^1(\mathbb{R})$, and its Fourier transform is:

$$\widehat{G}_a(\xi) = a^{-1} \int_{-a/2}^{a/2} e^{-2\pi i \xi x}\, dx = \frac{e^{\pi i a \xi} - e^{-\pi i a \xi}}{2\pi i a \xi} = \frac{\sin(\pi a \xi)}{\pi a \xi} = \mathrm{sinc}(\pi a \xi).$$

(Recall that the sinc function, $\mathrm{sinc}(z)$, is given by $\mathrm{sinc}(z) = \frac{\sin z}{z}$ for all $z \in \mathbb{C} \setminus \{0\}$ and $\mathrm{sinc}(0) = 1$. It is an entire function. The above computation is only valid for $\xi \neq 0$, but one also verifies that $\widehat{G}_a(0) = 1 = \mathrm{sinc}(0)$; hence the final formula, $\widehat{G}_a(\xi) = \mathrm{sinc}(\pi a \xi)$, is valid for *all* $\xi \in \mathbb{R}$.)

It follows that for every $n \in \mathbb{Z}^+$ we have $F_n \in \mathrm{L}^1(\mathbb{R})$ and

$$(5) \qquad \widehat{F}_n(\xi) = \prod_{j=1}^{n} \widehat{G}_{a_j}(\xi) = \prod_{j=1}^{n} \mathrm{sinc}(\pi a_j \xi).$$

Using $|\mathrm{sinc}(z)| \leqslant \min(1, |z|^{-1})$ $(\forall z \in \mathbb{R})$ we see that if $n \geqslant 2$ then $|\widehat{F}_n(\xi)| \leqslant \min(1, \pi^{-2}(a_1 a_2)^{-1} |\xi|^{-2})$, which implies that $\widehat{F}_n \in \mathrm{L}^1(\mathbb{R})$. Hence for every $n \geqslant 2$, the Fourier Inversion formula applies to F_n, i.e.

we have

$$F_n(x) = \int_{\mathbb{R}} \widehat{F}_n(\xi) e^{2\pi i x\xi}\, d\xi, \qquad \forall x \in \mathbb{R},\ n \geqslant 2. \tag{6}$$

Next, it is an easy consequence of (5) and $-\frac{1}{6} < \operatorname{sinc}(z) \leqslant 1$ ($\forall z \in \mathbb{R}$) that the following pointwise limit exists for every $\xi \in \mathbb{R}$:

$$H(\xi) := \lim_{n\to\infty} \widehat{F}_n(\xi). \quad ^3 \tag{7}$$

Hence by (6) and the Dominated Convergence Theorem (using the majorant function $\xi \mapsto \min(1, \pi^{-2}(a_1 a_2)^{-1}|\xi|^{-2})$), we have $H \in \mathrm{L}^1(\mathbb{R})$ and for each fixed $x \in \mathbb{R}$:

$$\begin{aligned} \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \int_{\mathbb{R}} \widehat{F}_n(\xi) e^{2\pi i x\xi}\, d\xi &= \int_{\mathbb{R}} \lim_{n\to\infty} \widehat{F}_n(\xi) e^{2\pi i x\xi}\, d\xi \\ &= \int_{\mathbb{R}} H(\xi) e^{2\pi i x\xi}\, d\xi = \check{H}(x). \end{aligned} \tag{8}$$

Hence: *f_n tends uniformly to zero iff $H = 0$ a.e.* (Indeed, if $H = 0$ a.e. then $\lim_{n\to\infty} F_n(0) = \check{H}(0) = 0$ which as we noted above implies that f_n tends uniformly to zero; conversely if f_n tends uniformly to zero then $\check{H}(x) = \lim_{n\to\infty} F_n(x) = 0$ for all x and hence $H = 0$ a.e., cf. Folland's Cor. 8.27.)

Finally, we now prove in two steps that $H = 0$ a.e. (in fact $H(\xi) = 0$ for all $\xi \in \mathbb{R}\setminus\{0\}$) holds iff $\sum_{n=1}^{\infty} a_n^2 = \infty$:

Step 1: If $a_n \nrightarrow 0$ as $n \to \infty$ then $H(\xi) = 0$ for all $\xi \in \mathbb{R}\setminus\{0\}$.

[Proof: Assume $a_n \nrightarrow 0$ as $n \to \infty$; this means that there exist $\delta > 0$ and an infinite sequence $1 \leqslant n_1 < n_2 < \cdots$ such that $a_{n_k} \geqslant \delta$ for all k. Now let $\xi \in \mathbb{R}\setminus\{0\}$ be given. Then $\eta := \sup_{|x| \geqslant \pi\delta|\xi|} |\operatorname{sinc}(x)|$ is a number strictly between 0 and 1, and we have $|\widehat{G}_{a_{n_k}}(\xi)| = |\operatorname{sinc}(\pi a_{n_k} \xi)| \leqslant \eta$ for all k, and also $|\widehat{G}_{a_n}(\xi)| \leqslant 1$ for all n. Hence for every $k \in \mathbb{Z}^+$ and every $n \geqslant n_k$, by (5) we have $|\widehat{F}_n(\xi)| \leqslant \eta^k$. Hence $H(\xi) = \lim_{n\to\infty} \widehat{F}_n(\xi) = 0$.]

Step 2: If $a_n \to 0$ as $n \to \infty$ then $\big[H(\xi) = 0$ for all $\xi \in \mathbb{R}\setminus\{0\}\big] \Leftrightarrow [H = 0$ a.e.$] \Leftrightarrow \big[\sum_{n=1}^{\infty} a_n^2 = \infty\big]$.

[Proof: Assume that $a_n \to 0$ as $n \to \infty$. By Taylor's formula,

$$\log(\operatorname{sinc}(x)) = \log\Big(1 - \frac{x^2}{6} + O(x^4)\Big) = -\frac{x^2}{6} + O(x^4) \qquad \text{as } x \to 0;$$

[3] Indeed, using only $|\operatorname{sinc}(z)| \leqslant 1$ it follows that $|\widehat{F}_1(\xi)| \geqslant |\widehat{F}_2(\xi)| \geqslant \cdots$ and hence $\lim_{n\to\infty} |\widehat{F}_n(\xi)|$ exists; and if this limit is non-zero, then using $\operatorname{sinc}(z) > -\frac{1}{6}$ one shows that $\widehat{F}_n(\xi)$ has constant sign for n large, so that (7) exists.

hence there exists a constant $\delta > 0$ such that

$$\operatorname{sinc}(x) > 0 \text{ and } -\frac{x^2}{5} \leqslant \log(\operatorname{sinc}(x)) \leqslant -\frac{x^2}{7} \qquad \forall x \in [-\delta, \delta].$$

Now let $\xi_0 > 0$ be given. Then there exists $N \in \mathbb{Z}^+$ such that $|\pi a_n \xi| < \delta$ for all $n \geqslant N$ and all $\xi \in [-\xi_0, \xi_0]$. It follows that for all $M \geqslant N$ and all $\xi \in [-\xi_0, \xi_0]$ we have

$$(9) \qquad -\frac{1}{5}(\pi\xi)^2 \sum_{n=N}^{M} a_n^2 \leqslant \log\left(\prod_{n=N}^{M} \operatorname{sinc}(\pi a_n \xi)\right) \leqslant -\frac{1}{7}(\pi\xi)^2 \sum_{n=N}^{M} a_n^2.$$

Now if $\sum_{n=1}^{\infty} a_n^2 = +\infty$ then for every $\xi \in [-\xi_0, \xi_0] \setminus \{0\}$ we have, by the right inequality in (9): $\lim_{M\to\infty} \log\left(\prod_{n=N}^{M} \operatorname{sinc}(\pi a_n \xi)\right) = -\infty$, and hence $H(\xi) = \lim_{M\to\infty} \prod_{n=1}^{M} \operatorname{sinc}(\pi a_n \xi) = 0$.

On the other hand, if $\sum_{n=1}^{\infty} a_n^2 < +\infty$, then for every $\xi \in [-\xi_0, \xi_0] \setminus \{0\}$ it follows by using the left inequality in (9) and the fact that $\log(\operatorname{sinc}(\pi a_n \xi)) < 0$ $(\forall n \geqslant N)$, that the limit $\lim_{M\to\infty} \prod_{n=N}^{M} \operatorname{sinc}(\pi a_n \xi)$ exists and is a number strictly between 0 and 1. Hence also the limit $H(\xi) = \lim_{M\to\infty} \prod_{n=1}^{M} \operatorname{sinc}(\pi a_n \xi)$ exists for every $\xi \in [-\xi_0, \xi_0]$, and is zero only at those finitely many $\xi \in [-\xi_0, \xi_0]$ where $\prod_{n=1}^{N-1} \operatorname{sinc}(\pi a_n \xi) = 0$.

In both cases, since $\xi_0 > 0$ was arbitrary, we obtain the statement of Step 2.]

□

Remark: One can alternatively obtain the result in (4) as a consequence of fairly standard results in probability theory, namely appropriate versions of the *Central Limit Theorem* (CLT).[4] Indeed, let $X_1, X_2, \ldots$ be independent real-valued random variables, with X_k having a uniform distribution between $-\frac{a_k}{2}$ and $\frac{a_k}{2}$; then the probability density function of X_k is the function G_{a_k} discussed above, and the probability density function of $X_1 + \cdots + X_n$ is $F_n = G_{a_1} * \cdots * G_{a_n}$. Note that $\operatorname{Var}(X_k) = \frac{a_k^2}{12}$. Now if $\sum_{k=1}^{\infty} a_k^2 = \infty$ then the *Lindeberg CLT* implies that the distribution of the normalized sum $\left(\sum_{k=1}^{n} \frac{a_k^2}{12}\right)^{-1}(X_1 + \cdots + X_n)$ tends, as $n \to \infty$, to a *normal* distribution with zero expectation and unit variance. Indeed, this is exactly the case discussed in Feller, "An Introduction to Probability Theory and Its Applications, Vol. II", Ch. VIII.4, Ex. (d) (but with a_k in place of $a_k/2$). Applying this fact instead to $\left(\sum_{k=2}^{n} \frac{a_k^2}{12}\right)^{-1}(X_2 + \cdots + X_n)$, one fairly easily deduces

[4] This is perhaps not an "alternative proof", but rather an "alternative viewpoint" – since one way to prove the CLT is by working via the Fourier transform, just as we did above.

that $F_n(0) \to 0$ as $n \to \infty$,[5] meaning that f_n tends uniformly to zero. Alternatively, one may prove $F_n(0) \to 0$ by applying an appropriate CLT for *densities*[6]; cf. e.g. Exc. XV.9.28[7] in the same book by Feller.

On the other hand, if $\sum_{k=1}^{\infty} a_k^2 < \infty$, then by Ch. VIII.5 in the same book, the distribution of the *unnormalized* sum $X_1 + \cdots + X_n$ tends, as $n \to \infty$, to a probability distribution with zero expectation and variance $\frac{1}{12}\sum_{k=1}^{\infty} a_k^2$.[8] This implies that f_n does *not* tend uniformly to zero. (Indeed, assume the opposite, i.e. that $f_n \to 0$ uniformly. Then also $F_n \to 0$ uniformly, and this implies that $\int_{-\infty}^{\infty} F_n(x)\phi(x)\,dx \to 0$ for any $\phi \in C_0(X)$, meaning that the distribution of $X_1 + \cdots + X_n$ tends vaguely to the *zero measure* on $\mathbb{R}$, i.e. we have "escape of mass", and the distribution of $X_1 + \cdots + X_n$ does *not* tend to the distribution of some random variable on $\mathbb{R}$.)

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[5]This argument was pointed out to me by Benjamin Meco. To give some details, set $V_n := \frac{1}{12}\sum_{k=2}^{n} a_k^2$. The probability density function of $X_2 + \cdots + X_n$ (for $n \geqslant 2$) is $\widetilde{G}_n := G_{a_2} * \cdots * G_{a_n}$, and we have $F_n = G_{a_1} * \widetilde{G}_n$; therefore

$$F_n(0) = a_1^{-1}\int_{-a_1/2}^{a_1/2} \widetilde{G}_n(x)\,dx = a_1^{-1}\,\mathrm{Prob}\big[|X_2 + \cdots + X_n| \leqslant a_1/2\big]$$
$$= a_1^{-1}\,\mathrm{Prob}\big[|V_n^{-1}(X_2 + \cdots + X_n)| \leqslant a_1/(2V_n)\big]. \qquad (*)$$

But the application of the Lindeberg CLT gives that the distribution of $V_n^{-1}(X_2 + \cdots + X_n)$ tends to a standard normal $N(0,1)$ distribution, and this implies that the probability in $(*)$ tends to zero as $n \to \infty$, essentially since $a_1/(2V_n) \to 0$, and so, if Z is an $N(0,1)$-distributed random variable then $\lim_{n\to\infty} \mathrm{Prob}[|Z| \leqslant a_1/(2V_n)] = 0$.

[6]also called a local limit theorem.

[7]To apply this result, one has to group together the terms appropriately, e.g. as $(X_1 + X_2 + X_3) + (X_4 + X_5 + X_6) + \cdots$, so that the probability densities of the individual terms satisfy the required bounds.

[8]An alternative to Feller's proof of this fact is to note that

$$E\big[(X_n + X_{n+1} + \cdots + X_m)^2\big] \to 0 \qquad \text{as } n, m \to \infty \ (n \leqslant m),$$

meaning that "Cauchy's criterion for mean square convergence" applies and gives the result.