Assignment 2: Some answers, comments and references

Problem 1. This is Excercise 9.5 in Rudin's "Real and Complex Analysis" (but using Folland's notation.)

Problem 2. This is Folland's Exercise 8.39 (corrected).

Problem 3. Here is a solution covering the more general situation of an arbitrary $k > \frac{1}{2}$:

Case 1: $|a| \leq 10$. In this case we have $1 + |x - a| \asymp 1 + |x|$ for *all* $x \in \mathbb{R}$, and hence

$$\int_{-\infty}^{\infty} \frac{dx}{(1+|x-a|)^k(1+|x|)^k} \asymp_k \int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^{2k}} \asymp_k 1.$$

Case 2: $|a| \geq 10$. Note that the integral is symmetric under $a \mapsto -a$; hence in this Case 2 we may in fact assume $a \geq 10$. Then we have $(1+|x-a|)^{-k} \geq (1+|(-x)-a|)^{-k}$ for all $x \geq 0$, and so

$$\int_{-\infty}^{\infty} \frac{dx}{(1+|x-a|)^k(1+|x|)^k} \asymp \int_{0}^{\infty} \frac{dx}{(1+|x-a|)^k(1+|x|)^k}.$$

Next we note that if $0 \leq x \leq a/2$ then $1 + |x-a| \asymp 1 + a \asymp a$; if $a/2 \leq x \leq 3a/2$ then $1 + |x| \asymp a$, and if $3a/2 \leq x$ then $1 + |x - a| \asymp 1 + x \asymp x$. Hence the above integral is

$$\asymp_k \int_0^{a/2} \frac{dx}{a^k(1+x)^k} + \int_{a/2}^{3a/2} \frac{dx}{(1+|x-a|)^k a^k} + \int_{3a/2}^{\infty} \frac{dx}{x^{2k}}.$$

Here each term is easy to compute explicitly. (In particular, regarding the middle term, note that the integrand there is symmetric about the point a; using this and the substitution $x = a + y$ we see that the middle term equals $2a^{-k} \int_0^{a/2} (1+y)^{-k}\,dy$.) We obtain that the above expression is (still assuming $a \geq 10$):

$$\begin{cases} \text{If } k > 1: & \asymp_k a^{-k} + a^{-k} + a^{1-2k} \asymp a^{-k}; \\ \text{If } k = 1: & \asymp a^{-1}\log a + a^{-1}\log a + a^{-1} \asymp a^{-1}\log a; \\ \text{If } \frac{1}{2} < k < 1: & \asymp_k a^{-k+1-k} + a^{-k+1-k} + a^{1-2k} \asymp a^{1-2k}. \end{cases}$$

Combining the above Cases 1 and 2 we obtain the estimate(s) stated in the problem formulation – and when $\frac{1}{2} < k < 1$ we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{(1+|x-a|)^k(1+|x|)^k} \asymp_k (1+|a|)^{1-2k} \qquad (\forall a \in \mathbb{R}).$$

$\square$

Addendum to Problem 3: Let us here also determine a more precise asymptotic formula for the given integral as $a \to +\infty$.

For this task, it is helpful to first use the fact that the integrand is symmetric about the point $x = \frac{a}{2}$, i.e. invariant under $x \mapsto a - x$. This implies that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+|x-a|)^k(1+|x|)^k} = 2\int_{a/2}^{\infty} \frac{dx}{(1+|x-a|)^k(1+x)^k}.$$

We are studying the limit $a \to +\infty$; hence we may assume throughout that $a \geq 10$. Hence we can approximate the integrand using $(1+x)^{-k} \approx x^{-k}$. Namely, by Taylor expansion we have $(1+x)^{-k} = x^{-k}(1+x^{-1})^{-k} = x^{-k}(1 + O_k(x^{-1})) = x^{-k}(1 + O_k(a^{-1}))$ for all $x \geq a/2 \geq 5$, and hence the above expression equals

$$2\big(1 + O_k(a^{-1})\big) \int_{a/2}^{\infty} \frac{1}{(1+|x-a|)^k x^k}\, dx. \tag{1}$$

Next we examine whether we can also get rid of the "1" in "$1+|x-a|$" – or how we should otherwise handle that term. Of course this depends on whether the main contribution comes from the part of the integral where x is near a, or from the part where x is far from a. To study this, it is convenient to move the point a to 0, i.e. we substitute $x = a - y$ (for $x < a$) and $x = a + y$ (for $x > a$), to get:

(2)

$$\int_{a/2}^{\infty} \frac{1}{(1+|x-a|)^k x^k}\, dx = \int_0^{a/2} \frac{dy}{(1+y)^k(a-y)^k} + \int_0^{\infty} \frac{dy}{(1+y)^k(a+y)^k}.$$

Then the question is whether the main contribution in the last two integrals comes from y (fairly) near 0, or from y far from 0. For the first integral, $\int_0^{a/2} \frac{dy}{(1+y)^k(a-y)^k}$, this is easily answered: Indeed, we have $(a-y) \asymp a$ throughout the range of integration, and hence the question is simply whether $\int_0^{a/2} \frac{dy}{(1+y)^k}$ (for a very large) has its main contribution for "y small" or "y large"? Answer: If $k < 1$ then the main contribution is for "y large", while if $k > 1$ then the main contribution is for y small (or: 'not so large') – basically because $\int_0^{\infty} \frac{dy}{(1+y)^k}$ diverges for $k < 1$ but not for $k > 1$. In both cases it makes sense to split $\int_0^{a/2}$ as $\int_0^{\sqrt{a}} + \int_{\sqrt{a}}^{a/2}$. (Note that $a \geq 10$ ensures that $\sqrt{a} < a/2$.)

Case 1: $k>1$. Then the main contribution comes from $0<y<\sqrt{a}$. In this interval we have "$a-y\approx a$", namely: $a-y=a(1+O(y/a))=a(1+O(a^{-1/2}))$, and so $(a-y)^{-k}=a^{-k}(1+O_k(a^{-1/2}))$. Hence:

$$\int_0^{a/2}\frac{dy}{(1+y)^k(a-y)^k}=\int_0^{\sqrt{a}}\frac{dy}{(1+y)^k(a-y)^k}+\int_{\sqrt{a}}^{a/2}\frac{dy}{(1+y)^k(a-y)^k}$$

$$=\big(1+O_k(a^{-1/2})\big)\int_0^{\sqrt{a}}\frac{dy}{(1+y)^ka^k}+O_k\bigg(\int_{\sqrt{a}}^{a/2}\frac{dy}{(1+y)^ka^k}\bigg)$$

$$=\big(1+O_k(a^{-1/2})\big)\cdot a^{-k}\cdot\frac{1}{k-1}\cdot\Big(1-(1+\sqrt{a})^{1-k}\Big)+O_k\big(a^{\frac{1}{2}(1-k)-k}\big)$$

$$=\frac{a^{-k}}{k-1}\cdot\big(1+O_k\big(a^{-\frac{1}{2}}+a^{\frac{1}{2}(1-k)}\big)\big).$$

Note that both the exponents inside the last "$O_k(\cdots)$" are negative, so that the error term tends to zero. The second integral in (2) is handled in a completely similar way, giving:

$$\int_0^{\infty}\frac{dy}{(1+y)^k(a+y)^k}=\int_0^{\sqrt{a}}\frac{dy}{(1+y)^k(a+y)^k}+\int_{\sqrt{a}}^{\infty}\frac{dy}{(1+y)^k(a+y)^k}$$

$$=\frac{a^{-k}}{k-1}\cdot\big(1+O_k\big(a^{-\frac{1}{2}}+a^{\frac{1}{2}(1-k)}\big)\big).$$

Adding these two, and inserting in (1), we finally conclude:

$$(3)\qquad\int_{-\infty}^{\infty}\frac{dx}{(1+|x-a|)^k(1+|x|)^k}=\frac{4a^{-k}}{k-1}\big(1+O_k\big(a^{-\frac{1}{2}}+a^{\frac{1}{2}(1-k)}\big)\big),$$

which in particular implies that

$$(4)\qquad\int_{-\infty}^{\infty}\frac{dx}{(1+|x-a|)^k(1+|x|)^k}\sim\frac{4a^{-k}}{k-1}\qquad\text{as }\ a\to\infty.$$

(Note that (3) is a strictly stronger statement than (4).)

Case 2: $\frac{1}{2} < k < 1$. Then the main contribution comes from $\sqrt{a} < y < a/2$. In this interval we have "$1 + y \approx y$", namely $1+y = y(1+O(y^{-1})) = y(1+O(a^{-1/2}))$, and so $(1+y)^{-k} = y^{-k}(1+O_k(a^{-1/2}))$. Hence:

$$\int_0^{a/2} \frac{dy}{(1+y)^k(a-y)^k} = \int_0^{\sqrt{a}} \frac{dy}{(1+y)^k(a-y)^k} + \int_{\sqrt{a}}^{a/2} \frac{dy}{(1+y)^k(a-y)^k}$$
$$= O_k\Big(a^{-k}\int_0^{\sqrt{a}} \frac{dy}{(1+y)^k}\Big) + \big(1+O_k(a^{-1/2})\big)\int_{\sqrt{a}}^{a/2} \frac{dy}{y^k(a-y)^k}$$
$$= O_k\big(a^{-k+\frac{1}{2}(1-k)}\big) + \big(1+O_k(a^{-1/2})\big)\Big(\int_0^{a/2} \frac{dy}{y^k(a-y)^k} - O_k\Big(\int_0^{\sqrt{a}} \frac{dy}{y^k a^k}\Big)\Big)$$
$$= O_k\big(a^{\frac{1}{2}-\frac{3}{2}k}\big) + \big(1+O_k(a^{-1/2})\big)\int_0^{a/2} \frac{dy}{y^k(a-y)^k}.$$

(The point of the last steps of the above computation was to make the integral "$\int_{\sqrt{a}}^{a/2} \frac{dy}{y^k(a-y)^k}$" cleaner, by replacing the end-point $\sqrt{a}$ by 0 and showing that this causes a total error $O_k(a^{\frac{1}{2}-\frac{3}{2}k})$, i.e. exactly the same error as we got from the integral $\int_0^{\sqrt{a}} \frac{dy}{(1+y)^k(a-y)^k}$ in the step before.) Substituting $y = at$ in the last integral we get:

$$= O_k\big(a^{\frac{1}{2}-\frac{3}{2}k}\big) + \big(1+O_k(a^{-1/2})\big)\cdot a^{1-2k}\cdot \int_0^{1/2} \frac{dt}{t^k(1-t)^k}$$
$$= \Big(\int_0^{1/2} \frac{dt}{t^k(1-t)^k}\Big)\cdot a^{1-2k}\cdot \big(1+O_k(a^{-\frac{1}{2}(1-k)})\big).$$

The second integral in (2) is handled in a completely similar way, giving:

$$\int_0^{\infty} \frac{dy}{(1+y)^k(a+y)^k} = \Big(\int_0^{\infty} \frac{dt}{t^k(1+t)^k}\Big)\cdot a^{1-2k}\cdot \big(1+O_k(a^{-\frac{1}{2}(1-k)})\big).$$

Adding these two, and inserting in (1), we conclude:

$$(5) \qquad \int_{-\infty}^{\infty} \frac{dx}{(1+|x-a|)^k(1+|x|)^k} \sim C_k \cdot a^{1-2k} \qquad \text{as } a \to +\infty,$$

where C_k is the constant $C_k = 2\big(\int_0^{1/2} \frac{dt}{t^k(1-t)^k} + \int_0^{\infty} \frac{dt}{t^k(1+t)^k}\big) = \int_{-\infty}^{\infty} \frac{dt}{|t-1|^k|t|^k}$.

Remark 1: A brief way to describe what happens in the computation in Case 2 ($\frac{1}{2} < k < 1$) is that it turns out that in this case, both the terms "1" can be removed without changing the leading asymptotic behaviour of the integral: $\int_{-\infty}^{\infty} \frac{dx}{(1+|x-a|)^k(1+|x|)^k} \sim \int_{-\infty}^{\infty} \frac{dx}{|x-a|^k|x|^k}$. Substituting $x = at$ in the last integral gives $\big(\int_{-\infty}^{\infty} \frac{dt}{|t-1|^k|t|^k}\big)\cdot a^{1-2k} = C_k a^{1-2k}$.

Remark 2: A closed formula for the constant C_k can be obtained as follows. We have

$$\int_0^{1/2} \frac{dt}{t^k(1-t)^k} = \frac{1}{2}\int_0^1 \frac{dt}{t^k(1-t)^k} = \frac{\Gamma(1-k)^2}{2\Gamma(2-2k)} = \frac{2^{2k-2}\sqrt{\pi}\Gamma(1-k)}{\Gamma(\frac{3}{2}-k)}$$
$$= -\frac{2^{2k-2}}{\sqrt{\pi}}\Gamma(k-\tfrac{1}{2})\Gamma(1-k)\cos(\pi k),$$

where we used (7.7) and then (7.6) and (7.5) from the lecture notes. For the other integral, the substituting $t = s^{-1} - 1$ leads to:

$$\int_0^\infty \frac{dt}{t^k(1+t)^k} = \int_0^1 s^{2k-2}(1-s)^{-k}\,ds = \frac{\Gamma(2k-1)\Gamma(1-k)}{\Gamma(k)}$$
$$= \frac{2^{2k-2}}{\sqrt{\pi}}\Gamma(k-\tfrac{1}{2})\Gamma(1-k),$$

where we used (7.7) and then (7.6) from the lecture notes. Adding these two, we conclude:

$$C_k = \frac{2^{2k-1}}{\sqrt{\pi}}\Gamma(k-\tfrac{1}{2})\Gamma(1-k)\big(1-\cos(\pi k)\big).$$

Remark 3: I encourage you to also sort out the case $k = 1$! In particular, I think that an interesting challenge would be to seek an asymptotic formula which is valid *uniformly* for all k in a neighborhood of 1.

Problem 4. We have
$$\int_1^\infty e^{-bx}x^{ab}\,dx = \int_1^\infty e^{-b(x-a\log x)}\,dx,$$
and for any $a \in [0,1)$ the function
$$x \mapsto x - a\log x$$
is strictly increasing for $x \geq 1$ (since its derivative is $1-\frac{a}{x} \geq 1-a > 0$). Hence for b large, the main contribution to the integral will come from x near 1, and we may hope to get an adequate estimate by using the Taylor expansion of $\log x$ at the point $x = 1$. In fact we can obtain quite simple and precise bounds from above and below by making use of the following inequality, which is inspired by the aforementioned Taylor expansion:
$$(*) \qquad x - 1 - \tfrac{1}{2}(x-1)^2 \leq \log x \leq x - 1, \qquad \forall x \geq 1.$$
(To verify the first inequality, note that the derivative of the function $\log x-(x-1)+\frac{1}{2}(x-1)^2$ equals $x^{-1}+x-2$, which is ≥ 0 for all $x \geq 1$.)

Using the second inequality in (*), we get:
$$\int_1^\infty e^{-bx}x^{ab}\,dx \leq \int_1^\infty e^{-bx+ab(x-1)}\,dx = \frac{e^{-b}}{(1-a)b}.$$
Using the first inequality in (*), we get:
$$\int_1^\infty e^{-bx}x^{ab}\,dx \geq \int_0^\infty e^{-b(u+1)+abu-\frac{1}{2}abu^2}\,du,$$
and since $e^{-y} \geq 1-y$ for all $y \geq 0$, the above is
$$\geq e^{-b}\int_0^\infty e^{-(1-a)bu}\Big(1-\tfrac{1}{2}abu^2\Big)\,du \geq \frac{e^{-b}}{(1-a)b}\bigg(1-\frac{a}{(1-a)^2b}\bigg).$$
In conclusion, we have proved:
$$\frac{e^{-b}}{(1-a)b}\bigg(1-\frac{a}{(1-a)^2b}\bigg) \leq \int_1^\infty e^{-bx}x^{ab}\,dx \leq \frac{e^{-b}}{(1-a)b}.$$
This easily implies the claim in the problem formulation, and in fact gives the stronger information that the required asymptotic relation $\int_1^\infty e^{-bx}x^{ab}\,dx \sim \frac{e^{-b}}{(1-a)b}$ holds uniformly over all $a \in [0, a_0(b))$, where a_0 is any function $\mathbb{R}_{\geq 0} \to [0,1)$ such that $(1-a_0(b)) \cdot b^{1/2} \to \infty$.

(That is, a is allowed to approach 1 as $b \to \infty$, but not too quickly.)
$\square$

Problem 6. Outline of a solution: Set

$$f_x(t) = x_1 t + x_2 t^2 + \cdots + x_n t^n,$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We wish to show that if $x \neq 0$ then $\sup_{t\in[0,1]} |\frac{d^k}{dt^k} f_x(t)| \gg |x|$ for *some* $k \in \{1, \ldots, n\}$, and then use this together with Stein's Prop. 2 (Ch. 8.1).

One way to prove such a lower bound is to partition the unit sphere in a clever way such that the desired bound can be proved with a *specific* k for all x belonging to any given part of the partition. Three students have nicely carried out such a solution (please ask me if you are interested in seeing details on how this can be done). Here, for fun, we give instead a less concrete argument, using compactness. Set

$$g(x) = \max_{k\in\{1,\ldots,n\}} \inf_{t\in[0,1]} \left| \frac{d^k}{dt^k} f_x(t) \right| \qquad (x \in \mathbb{R}^n).$$

Note for all x, k, the infimum over $t \in [0,1]$ is attained for some t, since $\frac{d^k}{dt^k} f_x(t)$ is a continuous function of t and $[0,1]$ is a compact interval; one also proves that for each k this infimum depends continuously on x; hence also $g(x)$ is a continuous function of x. Let us now consider *the infimum of* $g(x)$ *over the unit sphere* $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Since S^{n-1} is compact, this infimum is attained, say at the point $x' \in S^{n-1}$. Clearly $g(x') \geq 0$. Assume $g(x') = 0$. This means that

$$(*) \qquad \inf_{t\in[0,1]} \left| \frac{d^k}{dt^k} f_{x'}(t) \right| = 0, \qquad \forall k \in \{1, \ldots, n\}.$$

But note that $\frac{d^n}{dt^n} f_{x'}(t) = x'_n \cdot n!$ ($\forall t$); hence (*) for $k = n$ implies that $x'_n = 0$. Using $x'_n = 0$ we have $\frac{d^{n-1}}{dt^{n-1}} f_{x'}(t) = x'_{n-1} \cdot (n-1)!$ ($\forall t$), and hence (*) for $k = n-1$ implies that $x'_{n-1} = 0$. Repeating this argument for $k = n-2, n-3, \ldots, 1$ (in this order), we conclude that $x'_n = x'_{n-1} = \cdots = x'_1 = 0$. This is a contradiction against $x' \in S^{n-1}$!

Hence we conclude that

$$c := \inf_{x\in S^{n-1}} g(x) > 0.$$

Using this constant c (which only depends on n), we may now argue as follows, for any non-zero point $x \in \mathbb{R}^n$: Set $\widetilde{x} = |x|^{-1} x \in S^{n-1}$. Then $g(\widetilde{x}) \geq c$, i.e. there is some $k \in \{1, \ldots, n\}$ such that $\left|\frac{d^k}{dt^k} f_{\widetilde{x}}(t)\right| \geq c$ for all $t \in [0,1]$. Note that $f_x(t) = |x| \cdot f_{\widetilde{x}}(t)$; hence it follows that $\left|\frac{d^k}{dt^k} f_x(t)\right| \geq c|x|$ for all $t \in [0,1]$. If $k \geq 2$ then this immediately gives, via Stein's Prop. 2, that

$$\int_0^1 e(f_x(t))\, dt \ll_k (c|x|)^{-1/k}.$$

If $k = 1$ then we have to also satisfy the monotonicity assumption in Stein's Prop. 2; however note that $\frac{d^2}{dt^2} f_x(t)$ is a polynomial of degree $\leq n-2$; hence it can have at most $n-2$ zeros in the interval $[0,1]$;[1] and so we can partition $[0,1]$ into at most $n-1$ subintervals such that $\frac{d}{dt} f_x(t)$ is monotonic on each of these subintervals. Applying Stein's Prop. 2 (with $k = 1$) to the integral over each such subinterval, and adding up, we obtain:

$$\int_0^1 e(f_x(t))\,dt \ll (n-1)\cdot(c|x|)^{-1}.$$

The fact that at least one of the above inequalities must hold implies that if $|x| \geq 1$ then we have

$$\int_0^1 e(f_x(t))\,dt \ll_n \max(|x|^{-1}, |x|^{-1/2}, \ldots, |x|^{-1/n}) = |x|^{-1/n}.$$

But we also have, trivially,

$$\int_0^1 e(f_x(t))\,dt \leq 1, \qquad (\forall x \in \mathbb{R}^n).$$

Hence we conclude that, for all $x \in \mathbb{R}^n$: we have

$$\int_0^1 e(f_x(t))\,dt \ll \min(1, |x|^{-1/n}) \asymp (1+|x|)^{-1/n}.$$

Hence we have proved that the bound in the problem formulation holds with $\alpha = 1/n$. To see that this is the best possible exponent, it suffices to consider points of the form $x = (0, \ldots, 0, x_n)$, with $x_n \to +\infty$. We don't give the details here.

Answer: $\alpha = 1/n$. $\square$

[1] Or it may be the zero polynomial; but then we are done, since this implies that $\frac{d}{dt} f_x(t)$ is (constant and hence) monotonic as a function of t over the whole real axis.

Problem 7. It should be noted that the integrand is *not* periodic with period 2π, unless λ is an integer. Hence there is in general a "non-negligible"[2] contribution from the *end-points.* It is part of the problem to prove that this contribution is $O(\lambda^{-1})$.

[2]By this I mean: Not as small as "$O(\lambda^{-N})$ with arbitrarily large N", but instead quite a bit larger!