#8. Positive integrals; the Γ-function

Ex: For $R \geqslant 1$ and $h > 0$ (and $n \geq 2$) bound

$$S = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} (1+|Rk|)^{-\frac{n+1}{2}} (1+|hk|)^{-\frac{n+1}{2}}$$

optimally!

Solution, using dyadic decomposition

Using $1+|x| \asymp \begin{cases} 1 & \text{if } |x| \ll 1 \\ |x| & \text{if } |x| \gg 1, \end{cases}$ we have

$$S \asymp R^{-\frac{n+1}{2}} \sum_{k \neq 0} |k|^{-\frac{n+1}{2}} (1+|hk|)^{-\frac{n+1}{2}}$$

$$\asymp R^{-\frac{n+1}{2}} \sum_{0<|k| \leq h^{-1}} |k|^{-\frac{n+1}{2}} + (Rh)^{-\frac{n+1}{2}} \sum_{|k|>h^{-1}} |k|^{-(n+1)}$$

Hence if $h > 1$: $\quad S \asymp 0 + (Rh)^{-\frac{n+1}{2}}$.

Now assume $0 < h \leq 1$

Decompose $\{k \in \mathbb{Z}^n : 0 < |k| \leq h^{-1}\} \subset A_0 \sqcup A_1 \sqcup \dots \sqcup A_M$

where $A_m := \{k \in \mathbb{Z}^n : 2^m \leq |k| < 2^{m+1}\}$

and M minimal with $2^{M+1} > h^{-1}$ $(M \in \mathbb{Z}_{\geq 0})$

Note $\forall k \in A_m : \quad |k|^{-\frac{n+1}{2}} \leq (2^m)^{-\frac{n+1}{2}}$.

Also $\# A_m \leq \# \{k \in \mathbb{Z}^n : |k| < 2^{m+1}\} \ll (2^{m+1})^n \ll 2^{mn}$.

Hence $$\sum_{0 < |k| \leq h^{-1}} |k|^{-\frac{n+1}{2}} \leq \sum_{m=0}^{M} \# A_m \cdot (2^m)^{-\frac{n+1}{2}} \ll \sum_{m=0}^{M} 2^{mn - \frac{m(n+1)}{2}} = \sum_{m=0}^{M} 2^{\frac{n-1}{2} m}$$

using $n \geq 2$

$$\asymp 2^{\frac{n-1}{2} M} \leq h^{-\frac{n-1}{2}}$$

since $2^M \leq h^{-1}$

Similarly, for $\sum_{|k|>h^{-1}} |k|^{-(n+1)}$ (still assuming $0<h\leq 1$)

Decompose $\{k\in\mathbb{Z}^n : |k|>h^{-1}\} = A_0 \sqcup A_1 \sqcup A_2 \sqcup \cdots$

where $A_m = \{k\in\mathbb{Z}^n : h^{-1}2^m < |k| \leq h^{-1}2^{m+1}\}$,

get:
$$\sum_{|k|>h^{-1}} |k|^{-(n+1)} \leq \sum_{m=0}^{\infty} \# A_m \cdot (h^{-1}2^m)^{-(n+1)} \ll \sum_{m=0}^{\infty} (h^{-1}2^m)^n \cdot (h^{-1}2^m)^{-(n+1)}$$
$$= h\sum_{m=0}^{\infty} 2^{-m} \ll h$$

Hence get
$$S \simeq R^{-\frac{n+1}{2}} \sum_{0<|k|\leq h^{-1}} |k|^{-\frac{n+1}{2}} + (Rh)^{-\frac{n+1}{2}} \sum_{|k|>h^{-1}} |k|^{-(n+1)} \ll R^{-\frac{n+1}{2}} h^{-\frac{n-1}{2}}$$

(assuming $0<h\leq 1$).

Alternative (simpler!)

$$\sum_{0<|k|\leq h^{-1}} |k|^{-\frac{n+1}{2}} \asymp \int_{B^n_{h^{-1}} \setminus B^n_1} |x|^{-\frac{n+1}{2}} dx \asymp \int_1^{h^{-1}} r^{-\frac{n+1}{2}} \cdot r^{n-1} dr \asymp h^{-\frac{n-1}{2}}$$

Using $|x| \asymp |k|$ for all $x \in [-\frac{1}{2}, \frac{1}{2}]^n + k$

polar coordinates

and similarly

$$\sum_{|k|>h^{-1}} |k|^{-(n+1)} \asymp \int_{\mathbb{R}^n \setminus B^n_{h^{-1}}} |x|^{-(n+1)} dx \asymp \int_{h^{-1}}^{\infty} r^{-(n+1)} \cdot r^{n-1} dr \asymp h.$$

Ex: How bound $f(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}$ as $x \to 1^-$?

(A "complete elliptic integral")

Assume $\frac{1}{2} \leq x < 1$. Note $\int_0^{1/2} \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}} \asymp 1$. Hence the crux

or 0,99

is to understand the integrand for t near 1.

Note $(1-t^2)(1-x^2t^2) = (1-t)(1+t)(1-xt)(1+xt) \asymp (1-t)(1-xt)$

Substitute $t = 1-h$.

$\therefore$ $f(x) \asymp 1 + \int_0^{1/2} \frac{dh}{\sqrt{h(1-x+xh)}}$

$$\boxed{\text{For } a,b>0: \quad \underline{\underline{a+b \asymp \max(a,b)}}}$$

Hence $\underline{\underline{1-x+xh}} \asymp \max(1-x, xh) = \begin{cases} 1-x & \text{if } h \leqslant \frac{1-x}{x} \\ xh & \text{if } h \geqslant \frac{1-x}{x} \end{cases} \asymp \underline{\underline{\begin{cases} 1-x & \text{if } h \leqslant 1-x \\ h & \text{if } h \geqslant 1-x \end{cases}}}$

Thus $\underline{\underline{f(x)}} \asymp 1 + \int_0^{1-x} \frac{dh}{\sqrt{h(1-x)}} + \int_{1-x}^{1/2} \frac{dh}{\sqrt{h \cdot h}}$

$$= 1 + \underbrace{2\sqrt{\frac{1-x}{1-x}}}_{=1} + \log\left(\frac{1}{2(1-x)}\right)$$

$$= \left(3 + \log\left(\tfrac{1}{2}\right)\right) + \log\left(\frac{1}{1-x}\right) \quad \underline{\underline{\asymp \log\left(\frac{1}{1-x}\right)}} \quad \underline{\text{as } x \to 1^-}.$$

<u>Answer</u>: $$\underline{\underline{f(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}} \asymp \log\left(\frac{1}{1-x}\right) \quad \text{as } x \to 1^-}}$$

The gamma function (see Lecture notes, Sec. 7)

DEF: $\Gamma(s) := \int_0^\infty e^{-x} x^{s-1}\, dx$ for $s \in \mathbb{C}$ with $Re(s) > 0$.

Using $\Gamma(s+1) \equiv s\Gamma(s)$ one extends the definition of $\Gamma(s)$ to a meromorphic function in all $\mathbb{C}$, with poles (all simple) at $s = 0, -1, -2, -3, \ldots$

Check that $\Gamma(s+1) = s\Gamma(s)$ holds when $Re(s) > 0$:

$$\Gamma(s+1) = \int_0^\infty e^{-x} x^s\, dx = \left[-e^{-x}x^s\right]_0^\infty - \int_0^\infty (-e^{-x}) \cdot s x^{s-1}\, dx = 0 + s\int_0^\infty e^{-x} x^{s-1}\, dx$$

$$= s\,\Gamma(s).$$

Clearly $\Gamma(1) = 1$. Hence $\Gamma(2) = 1$; $\Gamma(3) = 2 \cdot 1$; $\Gamma(4) = 3 \cdot 2 \cdot 1$, etc: $\boxed{\Gamma(n) = (n-1)! \quad \forall n \in \mathbb{Z}^+}$

$$\underline{\underline{\Gamma(\tfrac{1}{2})}} = \int_0^\infty e^{-x} x^{-\frac{1}{2}} dx = \{x = u^2\} = 2\int_0^\infty e^{-u^2} du = \underline{\underline{\sqrt{\pi}}}$$

Hence $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, $\Gamma(\frac{5}{2}) = \frac{1}{2}\cdot\frac{3}{2}\sqrt{\pi}$, etc:

$$\boxed{\begin{gathered}\Gamma(k+\tfrac{1}{2}) = \frac{(2k-1)!!}{2^k}\sqrt{\pi} \\ \forall k \in \mathbb{Z}_{\geq 0}\end{gathered}}$$

Volume of B_1^n: $\underline{\underline{\dfrac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}}}$

$$\underline{\underline{\Gamma(s)\,\Gamma(1-s) \equiv \frac{\pi}{\sin(\pi s)}}}$$

$$\underline{\underline{\Gamma(2s) \equiv \pi^{-\frac{1}{2}} \cdot 2^{2s-1}\,\Gamma(s)\,\Gamma(s+\tfrac{1}{2})}}$$

$$\underline{\underline{\frac{1}{\Gamma(s)} \equiv s \cdot e^{\gamma s} \cdot \prod_{n=1}^{\infty}\left(\left(1+\frac{s}{n}\right)e^{-s/n}\right)}}$$

with $\underline{\gamma = \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \log N\right) = 0,57722\ldots}$

(<u>Euler's constant</u>)

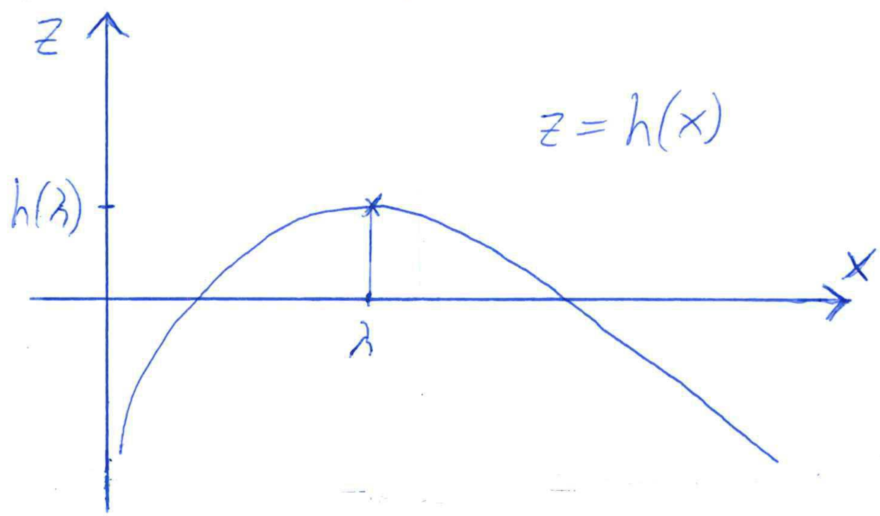
Asymptotic formula for $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\,dx$ as $s \to +\infty$ $(s \in \mathbb{R})$?

A positive integral! Cleaner: Set $\boxed{\lambda := s-1}$, study $\Gamma(\lambda+1) = \int_0^\infty e^{-x} x^{\lambda}\,dx$

Study the integrand: $e^{-x} x^{\lambda} = \exp\big(\underbrace{-x + \lambda \log x}_{= h(x)}\big)$

$$h(x) = -x + \lambda \log x$$

$$h'(x) = -1 + \frac{\lambda}{x}$$



$\to$ Subst. $x = \lambda + y$ $\quad (y > -\lambda)$

$$\Rightarrow h(\lambda+y) = -\lambda - y + \lambda \log(\lambda + y)$$

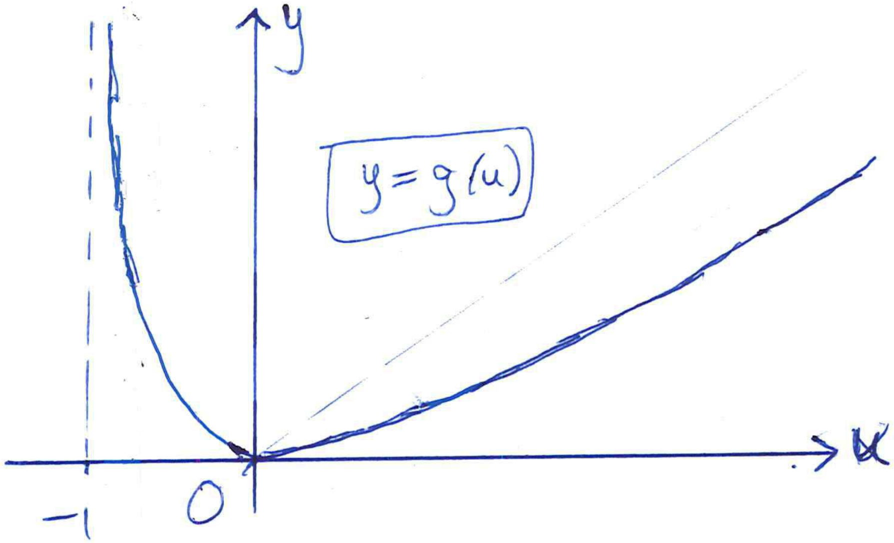
$$= -\lambda\left(1 + \frac{y}{\lambda} - \log(\lambda) - \log\left(1 + \frac{y}{\lambda}\right)\right)$$

$\to$ subst $y = \lambda u$ $\quad (u > -1)$

$$\Rightarrow h(\lambda+\lambda u) = -\lambda(1 - \log(\lambda)) - \lambda\underbrace{\big(u - \log(1+u)\big)}_{g(u)}$$

Get (for $\lambda > 0$): $\Gamma(\lambda+1) = \lambda^{\lambda+1} e^{-\lambda} \int_{-1}^{\infty} e^{-\lambda \cdot g(u)}\, du$,

where $g(u) = u - \log(1+u)$



For u near 0: $g(u) = \frac{1}{2}u^2 + O(|u|^3)$.

Thus

$$e^{-\lambda \cdot g(u)} = \exp\left(-\frac{\lambda}{2}u^2 + O(\lambda|u|^3)\right)$$

must require $\lambda|u|^3 \ll 1$

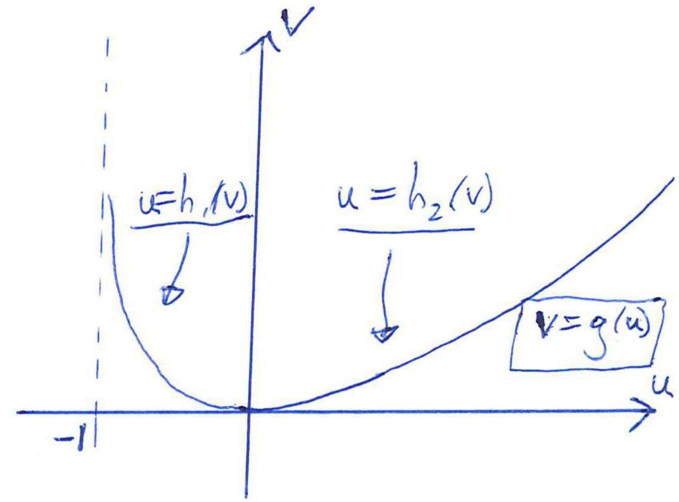
$$= e^{-\frac{\lambda}{2}u^2} \cdot \left(1 + O(\lambda|u|^3)\right)$$

$$\leadsto \Gamma(\lambda+1) = \sqrt{2\pi}\, \lambda^{\lambda+\frac{1}{2}} e^{-\lambda} \left(1 + O(\lambda^{-\frac{1}{2}})\right)$$ as $\lambda \to +\infty$

More convenient method (Laplace):

$$\int_{-1}^{\infty} e^{-\lambda \cdot g(u)}\,du \approx ?$$

Take $v = g(u)$ as new variable of integration!



$$= \int_0^\infty e^{-\lambda v}\cdot\left(-h_1'(v)\right)dv + \int_0^\infty e^{-\lambda v}\, h_2'(v)\,dv$$

We have $-h_1'(v) = \frac{1}{\sqrt{2v}} - \frac{2}{3} + \frac{1}{6\sqrt{2}}\, v^{\frac{1}{2}} + \dots$

Hence, if $c > 0$ is sufficiently small:

$$\int_0^\infty e^{-\lambda v}\left(-h_1'(v)\right)dv = \int_0^c e^{-\lambda v}\left(\frac{1}{\sqrt{2v}} - \frac{2}{3} + \frac{1}{6\sqrt{2}}\sqrt{v} + O(v)\right)dv + \underbrace{\int_c^\infty e^{-\lambda v}\cdot\left(-h_1'(v)\right)dv}_{\ll \int_c^\infty e^{-\lambda v}\,dv \ll e^{-c\lambda}}$$

Remains to understand: $\int_0^c e^{-\lambda v}\cdot v^{\alpha}\,dv$ as $\lambda \to +\infty$,

for $\alpha = -\frac{1}{2}, 0, \frac{1}{2}, \dots$

$\ll \int_c^\infty e^{-\lambda v}\,dv \ll e^{-c\lambda}$

since $|h_1'(v)| = O(v)$ for $v \geq c$

Estimating $\int_0^c e^{-\lambda v} \cdot v^\alpha \, dv$ as $\lambda \to +\infty$ (α fixed, $\alpha > -1$)

Note $\int_0^c e^{-\lambda v} v^\alpha \, dv = \lambda^{-\alpha-1} \cdot \underbrace{\int_0^{c\lambda} e^{-x} x^\alpha \, dx}$

$= $ "$\gamma(\alpha+1, c\lambda)$", an incomplete gamma function

(See lecture notes, Sec. 7.3)

Surely $\int_0^{c\lambda} e^{-x} x^\alpha \, dx \approx \int_0^\infty e^{-x} x^\alpha \, dx = \Gamma(\alpha+1)$, as $\lambda \to +\infty$ with α fixed.

exponentially close!?

Error: $\int_{c\lambda}^\infty e^{-x} x^\alpha \, dx$

If $\alpha \leq 0$: $\int_{c\lambda}^\infty e^{-x} x^\alpha \, dx \leq (c\lambda)^\alpha \int_{c\lambda}^\infty e^{-x} \, dx = (c\lambda)^\alpha e^{-c\lambda}$

If $\alpha > 0$: Integrate by parts!

$$\int_{c\lambda}^{\infty} e^{-x} x^{\alpha} dx = \left[-e^{-x} x^{\alpha}\right]_{c\lambda}^{\infty} + \alpha \int_{c\lambda}^{\infty} e^{-x} x^{\alpha-1} dx = e^{-c\lambda}(c\lambda)^{\alpha} + \alpha \int_{c\lambda}^{\infty} e^{-x} x^{\alpha-1} dx$$

If $\alpha > 1$: Repeat!

$\leadsto$ Conclude: For any fixed $\alpha > -1$: $\int_{c\lambda}^{\infty} e^{-x} x^{\alpha} dx \ll \lambda^{\alpha} e^{-c\lambda}$ as $\lambda \to \infty$.

Hence: $$\int_{0}^{c\lambda} e^{-x} x^{\alpha} dx = \Gamma(\alpha+1) + O\left(\lambda^{\alpha} e^{-c\lambda}\right) \quad \text{as } \lambda \to \infty$$

Hence: $$\Gamma(\lambda+1) = \sqrt{2\pi}\, \lambda^{\lambda+\frac{1}{2}} e^{-\lambda} \cdot \left(1 + \frac{1}{12}\lambda^{-1} + O(\lambda^{-2})\right) \quad \text{as } \lambda \to +\infty.$$