PROBLEM SUGGESTIONS FOR THE COURSE "SELECTED TOPICS IN DYNAMICAL SYSTEMS"

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The basic condition for passing the course is to hand in (acceptable) solutions for problems for a total of at least "100 pt" (see the list below). I will try to suggest several relevant problems; however *students are most welcome to suggest problems of their own;* please just discuss with me!

I hope to discuss problems individually with each student. For example, I have often refrained from providing hints even when this could easily be done; instead I hope that you will frequently ask me for hints and suggestions on the problems which you are working on. Another good reason to ask me is that it is quite possible that I've made some mistake in the formulation of the problem.

I have marked with an "(E)" problems which I believe are "easy" and suitable for students who feel inexperienced in the field. I recommend more experienced students to still take a quick look at these problems and think through if you know how to solve them.

Problem 1. (E). Let $f : I \to I$ be an IET (Interval exchange map; see [19, Sec. 1] for the definition); let $\mathcal{B}$ be the standard Borel σ-algebra of I and let m be Lebesgue measure on I. Prove that $(I, \mathcal{B}, f, m)$ is an mpt. (5 pt.)

(Comments: I took this statement as "clear" in my Lecture #1; similarly Viana takes it as clear; cf. [19, p. 22(bottom)]. A quick proof could look as follows: "We have to prove that for every $E \in \mathcal{B}$, we have $f^{-1}(E) \in \mathcal{B}$ and $m(f^{-1}(E)) = m(E)$. This is clear by splitting E appropriately and then using the fact that $\mathcal{B}$ and m are invariant under translations." However, please give a slightly more detailed solution, in particular please be a bit more specific about how to "split E appropriately".)

Problem 2. (E). **Atoms of ergodic measures.** Let $(X, \mathcal{B}, \mu, T)$ be an ergodic ppt, and assume that every point in X is measurable, i.e. $\{x\} \in \mathcal{B}$ for every $x \in X$. Assume that there is a point $x \in X$ with $\mu(\{x\}) > 0$. Prove that x is a periodic point, and that if its period is n then μ is the normalized uniform measure on the finite set $\{x, T(x), \ldots, T^{n-1}(x)\}$, i.e. $\mu = n^{-1} \sum_{k=0}^{n-1} \delta_{T^k(x)}$. (7 pt.)

Problem 3. (E). **Invariance of ergodicity and mixing under completion and isomorphism.**

(a) Let $(X, \mathcal{B}, \mu, T)$ be an mpt and let $\mathcal{B}^\mu$ be the μ-completion of $\mathcal{B}$; then we know that also $(X, \mathcal{B}^\mu, \mu, T)$ is an mpt (cf. my notes to [16, Def. 1.2]). Prove that $(X, \mathcal{B}, \mu, T)$ is ergodic iff $(X, \mathcal{B}^\mu, \mu, T)$ is ergodic.

(b) Similarly, for $(X, \mathcal{B}, \mu, T)$ any ppt, prove that $(X, \mathcal{B}, \mu, T)$ is mixing[1] iff $(X, \mathcal{B}^\mu, \mu, T)$ is mixing.

(c) Let $(X_i, \mathcal{B}_i, \mu_i, T_i)$ for $i = 1, 2$ be two mpt's which are isomorphic and such that $\mathcal{B}_i$ is μ_i-complete. Prove that if $(X_1, \mathcal{B}_1, \mu_1, T_1)$ is ergodic, then so is $(X_2, \mathcal{B}_2, \mu_2, T_2)$.

(d) Let $(X_i, \mathcal{B}_i, \mu_i, T_i)$ for $i = 1, 2$ be two ppt's which are isomorphic, and such that $\mathcal{B}_i$ is μ_i-complete. Prove that if $(X_1, \mathcal{B}_1, \mu_1, T_1)$ is mixing, then so is $(X_2, \mathcal{B}_2, \mu_2, T_2)$.

(10 pt.)

(Comments: Recall the definition of two mpt's being "isomorphic"; cf. [16, Def. 1.3] and my notes about it. Please be careful in your treatment of the various null sets appearing in the discussion.)

Problem 4. (E). Prove [16, Thm. 1.1], i.e. Poincaré's Recurrence Theorem. (This is [16, Probl. 1.3].)

(5 pt.)

Problem 5. (E). Let X be a *finite* set, provided with the σ-algebra $\mathcal{B} = \mathcal{P}(X)$ (the power set of X), and let T be a given map from X to X. Give a classification of the set of T-invariant probability measures on $(X, \mathcal{B})$. Also give a classification of the subset of *ergodic* T-invariant probability measures on $(X, \mathcal{B})$.

(8 pt.)

Problem 6. Let X be an lcscH space (or if you prefer: let $X = \mathbb{R}^n$) and $\mu, \mu_1, \mu_2, \ldots \in P(X)$. Prove that if $\mu_n(f) \to \mu(f)$ for all $f \in C_c(X)$, then $\mu_n \to \mu$ in $P(X)$ (weak convergence).

(8 pt.)

(Hint: You may use the fact that for any $\mu \in P(X)$ and any $\varepsilon > 0$ there is some compact set $K \subset X$ such that $\mu(K) > 1 - \varepsilon$; cf., e.g., [14, Thm. 2.18][2]. You may also use the fact that for any compact subset $K \subset X$, there is some $h \in C_c(X)$ satisfying $0 \leq h \leq 1$ and $h_{|K} = 1$; cf., e.g., [14, Lemma 2.12].)

Problem 7. (E). Let A be any $n \times n$ matrix with integer entries and nonzero determinant. Let $\mathbb{T}^n$ be the torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$.

(a) Prove that the map $x \mapsto Ax$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ induces a well-defined and smooth map $T : \mathbb{T}^n \to \mathbb{T}^n$.

(b) Determine the number of preimages of any point, i.e. determine $\#T^{-1}(p)$ for any $p \in \mathbb{T}^n$.

(c) Prove that T preserves the Lebesgue volume measure on $\mathbb{T}^n$.

(10 pt.)

[1] Recall that "mixing" is synonymous with "strongly mixing".

[2] noticing that every open subset of an lcscH space is σ-compact.

Problem 8. (a) Solve [12, pp. 8–9, Exercise 10]! In other words, prove that $\mathrm{SL}(2,\mathbb{R})$ acts on the hyperbolic upper half plane $\mathfrak{H}$ by isometries and that this leads to an identification of

$$G = \mathrm{PSL}(2,\mathbb{R}) := \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$$

with the unit tangent bundle $T^1\mathfrak{H}$, and that under this identification, the geodesic flow and the horocycle flow correspond to flows on G generated by certain explicit 1-parameter subgroups.

(b) Solve [12, p. 9, Exercise 11]! In other words, given any compact hyperbolic surface X, find a natural identification of T^1X with $\Gamma\backslash G$ for some discrete subgroup $\Gamma < G = \mathrm{PSL}(2,\mathbb{R})$, and find the counterparts of the geodesic and horocycle flows on T^1X under this correspondence. (15 pt.)

(One good reference for this is [10], and I can also guide you to other appropriate references. Your solution does not have to be *too* long; you can focus on presenting the main points of the arguments.)

Problem 9. (This problem is related to [16, Sec. 1.3]; "The probabilistic point of view".)

Let $(X, \mathcal{B}, \mu, T)$ be a ppt, where we assume that $(X, \mathcal{B})$ is a standard Borel space. Let $X^{\mathbb{N}} = \prod_{n\geq 0} X$ be the space of all sequences $(x_n)_{n\geq 0}$ with $x_0, x_1, \ldots \in X$, provided with its product σ-algebra $\otimes_{n\geq 0}\mathcal{B}$ (cf., e.g., [6, Ch. 1 (just before Lemma 1.2)] or [3, Ch. 1 (just before Prop. 1.3)]). Let σ be the shift map on $X^{\mathbb{N}}$, and let $J : X \to X^{\mathbb{N}}$ be the map

$$J(x) = \big(x, T(x), T^2(x), T^3(x), \ldots\big).$$

Prove that $(X^{\mathbb{N}}, \otimes_{n\geq 0}\mathcal{B}, J_*(\mu), \sigma)$ is a ppt, and that the two ppt's $(X, \mathcal{B}, \mu, T)$ and $(X^{\mathbb{N}}, \otimes_{n\geq 0}\mathcal{B}, J_*(\mu), \sigma)$ are isomorphic. (10 pt.)

Problem 10. Let $I = [0, 1)$, provided with its standard Borel σ-algebra $\mathcal{B}$ and Lebesgue measure m. Let $T : I \to I$ be an IET which is uniquely ergodic (viz., $P^T(I) = \{m\}$). Prove or disprove: For every $x \in I$, the orbit $\{T^k(x)\}_{k=0}^{\infty}$ is equidistributed in X with respect to m. (10 pt.)

(Note: We proved such a result in lecture #2 for any *continuous* map on a compact metric space; however the issue here is that an IET has discontinuities. Also there is an issue that I is not compact.)

Problem 11. Try to make precise sense of the "category of mpt's", along the lines of wikipedia. We want the definition of "morphism" in the category to correspond (in an appropriate sense) to the definition of "factor map" in [16, Def. 1.14][3] and we want the "category theoretical isomorphisms" to correspond (in an appropriate sense) to the "isomorphisms" as defined in [16, Def. 1.3] (cf. also my notes to Sarig's Def. 1.3). (15 pt.)

(Comment: It seems that we have to define a "morphism" from an mpt $(X, \mathcal{B}, \mu, T)$ to an mpt $(Y, \mathcal{C}, \nu, S)$ to be an appropriate *equivalence class* of maps...)

Problem 12. (a) Give a classification of the orbit closures for a translation map $x \mapsto x + \alpha$ (any fixed $\alpha \in \mathbb{R}^n$) on the n-dimensional torus $\mathbb{T}^n$! (In the same spirit as the classification of orbit closures for a linear flow on $\mathbb{T}^n$ given in [12, Example (1.1.1)].)
(b) Also classify the ergodic measures for such a map. (15 pt.)

(Comment: To a large extent it should be possible to mimic the classification of orbit closures for a linear *flow* on $\mathbb{T}^n$, which I work out in detail in my notes to [12, p. 8, Exercise 5].)

Problem 13. Digest the proof of the Isomorphism Theorem for standard Borel spaces (cf., e.g., [8, Thm. 15.6]), write your own summary (or detailed presentation) of it, and be prepared to discuss it with your teacher!

(10-20 pt?)

Problem 14. Let S be an lcscH space (or if you prefer: let $S = \mathbb{R}^n$) and let $M(S)$ be the space of locally finite Borel measures on S, provided with the vague topology.
(a) Prove that $M(S)$ is Polish.
(b) Prove that a set $A \subset M(S)$ is relatively compact iff $\sup_{\mu \in A} \mu(K) < \infty$ for every compact set $K \subset S$. (10 pt.)

(Comment: See Kallenberg, [6, Thm. A2.3(i), (ii)] – but I would like you to include some more details in your proof! Note that for Kallenberg's proof of Thm. A2.3(i) to work, it seems we need to require that his countable subset $B = \{f_1, f_2, \ldots\}$ satisfies the following condition: For every $f \in C_c(S)$, $f \geq 0$, there exists a compact set $K \subset S$ and a sequence $(f_{k_j})_{j=1}^{\infty}$ of elements in B such that $\operatorname{supp} f_{k_j} \subset K$ for all j and $f_{k_j} \to f$ in the uniform metric as $j \to \infty$.)

Problem 15. Digest (from [2, Sec. 4.4.3] or directly from [4]) Furstenberg's proof of Weyl's result on equidistribution mod 1 of the integer values of a polynomial with at least one irrational coefficient. Write your own summary (or detailed presentation) of this proof, and be prepared to discuss it with your teacher! (10 pt.)

[3]except I believe that the requirement "onto" in Sarig's definition should be removed; cf. my notes to [16, Def. 1.14].

Problem 16. (E). Let $(X, \mathcal{B}, \mu)$ be a probability space and let $B_1, B_2, \ldots$ be a finite or countable list of sets in $\mathcal{B}$ which partition X, i.e. $X = \sqcup_{n \geq 1} B_n$.[4] Let $\mathcal{F} = \sigma(\{B_1, B_2, \ldots\})$, the σ-algebra generated by $B_1, B_2, \ldots$. Let $f \in L^1(X, \mathcal{B}, \mu)$. Prove that for every j with $\mu(B_j) > 0$,

$$\mathbb{E}(f \mid \mathcal{F})(x) = \frac{\int_{B_j} f \, d\mu}{\mu(B_j)}, \qquad \forall x \in B_j.$$

Also prove that for every set $A \in \mathcal{B}$, and every j with $\mu(B_j) > 0$,

$$\mu(A \mid \mathcal{F})(x) = \frac{\mu(A \cap B_j)}{\mu(B_j)}, \qquad \forall x \in B_j.$$

(5 pt.)

Problem 17. Let $\mathcal{B}_n$ be the standard Borel σ-algebra of $\mathbb{R}^n$. Let μ be an absolutely continuous probability measure on $\mathbb{R}^2$; thus we can write $d\mu = \delta \cdot dm$ where m is Lebesgue measure on $\mathbb{R}^2$ and $\delta \in L^1(\mathbb{R}^2, \mathcal{B}_2, m)$, $\delta \geq 0$. Let $\mathcal{F}$ be the sub-σ-algebra of $\mathcal{B}_2$ given by

$$\mathcal{F} = \{B \times \mathbb{R} \, : \, B \in \mathcal{B}_1\}.$$

Prove that for any $A \in \mathcal{B}_1$,

$$\mu(\mathbb{R} \times A \mid \mathcal{F})(x_1, x_2) = \frac{\int_A \delta(x_1, t) \, dt}{\int_{\mathbb{R}} \delta(x_1, t) \, dt} \qquad \text{for } \mu\text{-a.e. } (x_1, x_2) \in \mathbb{R}^2.$$

(8 pt.)

Problem 18. Digest how to complete the proof of Birkhoff's PET (Pointwise Ergodic Theorem), [16, Thm. 2.2], to the case of L^1-functions, by studying [16, Sec. 2.4] (specializing to $d = 1$, if you like). Write your own summary (or detailed presentation) of this proof, and be prepared to discuss it with your teacher!

Alternatively, carry out the corresponding task for some other proof of the PET, e.g. one of the proofs in Einsiedler & Ward, [2, Thm. 2.30], the proof in Katok & Hasselblatt, [7, p. 136, Thm 4.1.2], or the proof in Kallenberg, [6, Thm 10.6].

(15 pt.)

Problem 19. (E?) Let $A_1, A_2, \ldots$ be subsets of a set X, and let $\mathcal{F}$ be the countably generated σ-algebra $\mathcal{F} = \sigma(\{A_1, A_2, \ldots\})$. Recall that for each $x \in X$, the *atom* of x is given by

$$[x]_{\mathcal{F}} := \bigcap_{\substack{A \in \mathcal{F} \\ (x \in A)}} A = \bigcap_{j=1}^{\infty} \left\{ \begin{matrix} A_j & \text{if } x \in A_j \\ X \setminus A_j & \text{if } x \notin A_j \end{matrix} \right\}. \tag{1}$$

Prove the second equality in (1), and explain why this implies that $[x]_{\mathcal{F}} \in \mathcal{F}$ for every $x \in X$. Prove also that the atoms $\{[x]_{\mathcal{F}} \, : \, x \in X\}$ form a partition of X.

(8 pt.)

[4] "$\sqcup$" stands for disjoint union; thus "$X = \sqcup_{n \geq 1} B_n$" means that $X = \cup_{n \geq 1} B_n$ and $B_n \cap B_m = \emptyset$ for all $n \neq m$.

Problem 20. (E) *Alternative def. of "uniquely ergodic"; cf. Lecture #1.* Let $(X, \mathcal{B})$ be a standard Borel space and let $T : X \to X$ be a measurable map. Prove that if there is exactly one $\mu \in P(X)$ which is T-invariant and ergodic, then there is exactly one $\mu \in P(X)$ which is T-invariant.

(5 pt.)

(Hint: Use ergodic decomposition.)

Problem 21. Let G be a second countable locally compact group and let Γ be a discrete subgroup of G. Prove that there exists a Borel set $F \subset G$ which is a fundamental domain for $\Gamma \backslash G$.

(10 pt.)

Problem 22. (E?). Let G be a second countable locally compact group, let Γ be a discrete subgroup of G, set $X = \Gamma \backslash G$, and let $\pi : G \to X$ be the projection map; $\pi(g) = \Gamma g$. Let μ be a left Haar measure on G. Let $F \subset G$ be a Borel set which is a fundamental domain for $\Gamma \backslash G$.

(a) Prove that we obtain a Borel measure μ_X on X by setting $\mu_X(E) := \mu(\pi^{-1}(E) \cap F)$ for every Borel subset $E \subset X$.

(b) Prove that μ_X is independent of the choice of F.

(c) Prove that μ can be expressed in terms of μ_X by the formula

$$\int_G f \, d\mu = \int_X \sum_{g \in \pi^{-1}(x)} f(g) \, d\mu_X(x), \qquad \forall f \in L^1(G, \mu).$$

In particular $\mu(E) = \int_X \#(\pi^{-1}(x) \cap E) \, d\mu_X(x)$ for every Borel set $E \subset G$.

(12 pt.)

Problem 23. *PET for semi-flows.* Let $\{\Phi_t\}$ be a measure preserving *semi-flow* on a probability space $(X, \mathcal{B}, \mu)$, i.e. $\langle t, x \rangle \mapsto \Phi_t(x)$ is a measurable map from $\mathbb{R}_{\geq 0} \times X$ to X, $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for all $t, s \geq 0$, and $\Phi_{t*}(\mu) = \mu$ for all $t \geq 0$. Given $f \in \mathcal{L}^1(X, \mu)$ [5], set

$$A_T^f(x) := \frac{1}{T} \int_0^T f(\Phi_t(x)) \, dt \qquad (T > 0).$$

Then as $T \to \infty$ (through all positive real numbers), A_T^f converges μ-a.e. and in L^1 to some $\overline{f} \in \mathcal{L}^1(X, \mu)$, which is $\{\Phi_t\}$-invariant μ-a.e. (viz., $\mu(\{x \in X : f(\Phi_t(x)) \neq f(x)\}) = 0$ for every $t \geq 0$).

(10 pt.)

(Hint: One approach is to first apply the PET for the function $g := A_1^f$ and the *map* Φ_1.)

[5] Recall the def of $f \in \mathcal{L}^1(X, \mu) = \mathcal{L}^1(X, \mathcal{B}, \mu)$ from the beginning of Lecture #4.

Problem 24. *Asymptotic equidistribution of pieces of closed horocycles.* As in Theorem 4 in Lecture #5, let $G = \mathrm{SL}_2(\mathbb{R})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $X = \Gamma\backslash G$. For any $y > 0$ and real numbers $\alpha < \beta$, define $h_y^{[\alpha,\beta]} \in P(X)$ by

$$h_y^{[\alpha,\beta]}(f) = \frac{1}{\beta-\alpha}\int_\alpha^\beta f\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}\right)dx \qquad (f \in C_c(X)).$$

(In particular $h_y^{[\alpha,\alpha+1]} = h_y$ for any $y > 0$, $\alpha \in \mathbb{R}$.)

Prove that if $\{y_j\} \subset \mathbb{R}_{>0}$ and (α_j), $(\beta_j) \subset \mathbb{R}$ are any sequences subject to

$$\alpha_j < \beta_j, \quad \lim_{j\to\infty} y_j = 0, \quad \text{and} \quad \lim_{j\to\infty}(\beta_j - \alpha_j)/\sqrt{y_j} = +\infty,$$

then $h_{y_j}^{[\alpha_j,\beta_j]} \to \mu_X$ in $P(X)$ as $j \to \infty$. (10 pt.)

(Comments: This can be proved by fairly straight-forward modifications of the proof of Theorem 4 in Lecture #5; it suffices if you explain what has to be modified. See also [17].)

Problem 25. Let $X = \Gamma\backslash G$, with G a Lie group and Γ a lattice. Let ν be a *homogeneous* probability measure on X, as in Ratner's Theorem 3 in Lecture #5. Work out the details proving the explicit statements on the lower half of p. 4 in Lecture #5. (See also my notes to that lecture. In particular, digest the proofs of [13, Thm. 1.13] and [1, p. 81, Exc. 1].)

(15 pt.)

Problem 26. (E). Let $(X, \mathcal{B}, \mu, T)$ be an *invertible* ppt and let $A : X \to \mathrm{GL}_d(\mathbb{R})$ be a measurable map. Show how to extend the definition of $A_n : X \to \mathrm{GL}_d(\mathbb{R})$ (cf. Lecture #7) from $n \in \mathbb{Z}^+$ to $n \in \mathbb{Z}$, and prove that the cocycle identity holds for all $n, m \in \mathbb{Z}$. (5 pt.)

Problem 27. *The Lyapunov exponents of a linear map.*
Consider the Multiplicative Ergodic Theorem (Thms. 1 & 2 in Lecture #7) applied to the one-point ppt $X = \{x\}$ and $A(x) = A \in \mathrm{GL}_d(\mathbb{R})$. Prove that the Lyapunov exponents $\chi_1 < \cdots < \chi_s$ and the associated Oseledets decomposition $\mathbb{R}^d = H^1 \oplus \cdots \oplus H^s$ are determined as follows. Let $\mathrm{Sp}(A) \subset \mathbb{C}$ be the set of eigenvalues of A and for each $\lambda \in \mathrm{Sp}(A)$ let $E_\lambda \subset \mathbb{C}^d$ be the associated *generalized eigenspace,* i.e.

$$E_\lambda := \{v \in \mathbb{C}^d \,:\, (A - \lambda)^n v = 0 \text{ for some } n \in \mathbb{Z}^+\}.$$

(Then it is known that $\mathbb{C}^d = \oplus_{\lambda \in \mathrm{Sp}(A)} E_\lambda$.) Then $\chi_1 < \cdots < \chi_s$ are exactly the numbers appearing in the set $\{\log |\lambda| \,:\, \lambda \in \mathrm{Sp}(A)\}$, and for each $i \in \{1, \ldots, s\}$,

$$H^i = \Big(\bigoplus_{\lambda \in \mathrm{Sp}(A) \,:\, \log|\lambda| = \chi_i} E_\lambda \Big) \cap \mathbb{R}^d.$$

(10 pt.)

(Hint: Use the Jordan decomposition of A, and the fact that any two norms on $\mathbb{R}^d$ are equivalent.)

Problem 28. *Oseledets Theorem in the setting of a general vector bundle.*
Let $(X, \mathcal{B}, \mu, T)$ be a ppt. Give a definition of a "(finite-dimensional) vector bundle $\mathcal{E}$ over X", and of a "linear cocycle on $\mathcal{E}$ over T"; also provide the vector space $\mathcal{E}_x$ with a norm for each $x \in X$. Prove that, under appropriate assumptions, the Oseledets Theorems (= Theorems 1 and 2 of Lecture #7) extend to this situation.

Also prove that your setting includes the case of the linear cocycle $df : T(M) \to T(M)$, where M is a compact Riemannian manifold and $f : M \to M$ is an immersion (and $T_x(M)$ is endowed with the norm coming from the Riemannian metric, for every $x \in M$). (15 pt.)

(Comments: I have not myself worked out a detailed solution (yet). It seems that it should be possible to provide a nice general definition which ensures that M can be partitioned by Borel sets which each trivialize $\mathcal{E}$, and in this way reduce to the case of $X \times \mathbb{R}^d$, i.e. the case of Thms. 1,2 in Lecture #7. Cf. [7, pp. 663(bot)–664(mid)] and [20, p. 16] for some inspiration.)

Problem 29. *General first-return map of an IET.*
Let $f : I \to I$ be an IET and let $J = [a, b)$ be any subinterval of I. Prove that the first return map to J is again an IET.

(10 pt.)

Problem 30. (E). Let A be any $n \times n$ matrix with integer entries and non-zero determinant, and let $f : \mathbb{T}^n \to \mathbb{T}^n$ be the map induced by the linear map $x \mapsto Ax$ on $\mathbb{R}^n$ (cf. Problem 7). Prove that under the "obvious" trivialization of $T(\mathbb{T}^n)$, the differential $df : T(\mathbb{T}^n) \to T(\mathbb{T}^n)$ equals the linear cocycle defined by the *constant* map $\mathbb{T}^n \to \mathrm{GL}_n(\mathbb{R})$, $x \mapsto A$. Also prove that if A is *hyperbolic*, i.e. A does not have any eigenvalues of absolute value one, then for every $x \in \mathbb{T}^n$ there is a direct sum decomposition $T_x(\mathbb{T}^n) = E_x^s \oplus E_x^u$ such that
(a) For every $x \in \mathbb{T}^n$, $df(E_x^s) = E_{f(x)}^s$ and $df(E_x^u) = E_{f(x)}^u$;
(b) there exist $C > 0$ and $\lambda \in (0,1)$ such that for any $x \in \mathbb{T}^n$, $\boldsymbol{v} \in E_x^s$, $\boldsymbol{w} \in E_x^u$ and $m \geq 1$, $\|df^m(\boldsymbol{v})\| \leq C\lambda^m \|\boldsymbol{v}\|$ and $\|df^{-m}(\boldsymbol{w})\| \leq C\lambda^m \|\boldsymbol{w}\|$.
(10 pt.)

(Comments: If A is hyperbolic and furthermore $\det A = \pm 1$, so that f is a diffeomorphism, then the existence of such a decomposition of $T(\mathbb{T}^n)$ means that f is an *Anosov diffeomorphism;* cf., e.g., [7, Def. 6.4.2] or [21, Ex. 2.10].)

Problem 31. (E). In the notes to Lecture #7 we explain how to give $\mathrm{Gr}(d,l)$ the structure of a C^∞ manifold via "$\mathrm{Gr}(d,l) = G/H$" where $G = \mathrm{GL}_d(\mathbb{R})$ and H is the stabilizer of any fixed point $V_0 \in \mathrm{Gr}(d,l)$. Following this construction but instead starting with $G' = O(d)$ we get an identification $\mathrm{Gr}(d,l) = G'/H'$ where H' is a closed subgroup of $O(d)$. Prove that these two constructions endow $\mathrm{Gr}(d,l)$ with the *same* C^∞-manifold structure. Also compute H and H' for the special choice $V_0 = \mathrm{Span}_{\mathbb{R}}\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_l\}$, where $\boldsymbol{e}_j = (0, \cdots, 0, 1, 0, \cdots, 0)^t$ is the jth standard unit vector (the "1" appears at position j), and use this to explain the identity

$$\text{“}\mathrm{Gr}(d,l) = O(d)/(O(l) \times O(d-l))\text{”}.$$

(10 pt.)

Problem 32. Prove the formula

$$\#\{0 \leq j < r_{\pi,\lambda}^n(I_\alpha^n) \,:\, f^j(I_\alpha^n) \subset I_\beta\} = (\Theta_{\pi,\lambda}^n)_{\alpha,\beta}$$

stated in Lecture #18. (10 pt.)

(Cf. Viana, [20, Prop. 4.3 (and Lemma 4.2)]; it suffices if you write out his proof in your own words. But you may find it more rewarding to work it out on your own!)

Problem 33. (E). Let $\{a_n\}$ be a sequence of complex numbers. Assume that

$$\forall \delta > 0 :\ \exists N \geq 1 :\ \forall n \geq N :\ |a_n| \leq 10\,\delta^{-10}\,e^{\delta n}.$$

Prove that

$$\forall \delta > 0 :\ \exists N \geq 1 :\ \forall n \geq N :\ |a_n| \leq \frac{1}{10} e^{\delta n}.$$

(5 pt.)

(Comment: The purpose of the problem is to illustrate how large factors outside the exponential can be "swallowed in the $e^{o(n)}$ factor"; this is used repeatedly in the proof of the Multiplicative Ergodic Theorem.)

Problem 34. (Basics about the Hausdorff metric) Let (M, d) be a metric space. We define the *Hausdorff distance* between any two non-empty subsets $X, Y \subset M$ by

$$d_H(X,Y) = \max\Big(\sup_{x\in X} \inf_{y\in Y} d(x,y), \sup_{y\in Y} \inf_{x\in X} d(x,y)\Big) \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

Let $F(M)$ be the set of all non-empty compact subsets of M.
(a) Prove that d_H is a metric on $F(M)$.
(b) Prove that if M is complete then $F(M)$ is complete.
(c) Prove that if M is compact then $F(M)$ is compact.

(10 pt.)

(Comments: Cf. wikipedia.)

Problem 35. Let $0 < l < d$; recall that $\mathrm{Gr}(d, l)$ is the set of all l-dimensional linear subspaces of $\mathbb{R}^d$. Let S_1^{d-1} be the unit sphere in $\mathbb{R}^d$,

$$\mathrm{S}_1^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d \,:\, \|\boldsymbol{x}\| = 1\},$$

provided with the Euclidean metric coming from $\mathbb{R}^d$ (viz., the distance between two points $\boldsymbol{x}, \boldsymbol{x}' \in \mathrm{S}_1^{d-1}$ is $\|\boldsymbol{x} - \boldsymbol{x}'\|$). For any $U, V \in \mathrm{Gr}(d, l)$, let $\delta(U, V)$ be the Hausdorff distance (cf. Problem 34) between $U \cap \mathrm{S}_1^{d-1}$ and $V \cap \mathrm{S}_1^{d-1}$:

$$\delta(U, V) := d_H(U \cap \mathrm{S}_1^{d-1}, V \cap \mathrm{S}_1^{d-1}).$$

(a) Prove that δ is a metric on the set $\mathrm{Gr}(d, l)$ and that $\mathrm{Gr}(d, l)$ is compact in this metric.
(b) Recall from the notes to Lecture #7 (cf. also Problem 31) that $\mathrm{Gr}(d, l)$ has a standard structure of a connected C^∞-manifold. Prove that δ is a metrization of the corresponding topology on $\mathrm{Gr}(d, l)$.

(10 pt.)

(Comments: For part (b), you may take for granted the following fact: Points $V, V_1, V_2, \ldots \in \mathrm{Gr}(d, l)$ satisfy $\lim_{j\to\infty} V_j = V$ wrt the standard manifold structure on $\mathrm{Gr}(d, l)$ iff there exist ON-bases $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_l\}$ of V and $\{\boldsymbol{b}_1^{(j)}, \ldots, \boldsymbol{b}_l^{(j)}\}$ of each V_j, such that $\lim_{j\to\infty} \boldsymbol{b}_k^{(j)} = \boldsymbol{b}_k$ in $\mathbb{R}^d$ for each $k \in \{1, \ldots, l\}$.)

Problem 36. Let $0 < l < d$, and let δ be the metric on $\mathrm{Gr}(d,l)$ introduced in Problem 35. Prove that there exists an increasing function $F : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ satisfying $\lim_{\delta \to 0^+} F(\delta) = 0$, such that for all $U, V \in \mathrm{Gr}(d,l)$,

$$\delta(U,V) \leq F(\cos \angle(U, V^{\perp}))$$

(10 pt.)

(Comments: Taken together with Problem 35, this result shows that the intuitive argument on p. 6 in Lecture #8 for the existence of the limit space V^r, can be made rigorous.)

Problem 37. Prove the Martingale Convergence Theorem. (10 pt.)

(See Sarig, [16, Problem 2.8] and/or Einsiedler-Ward, [2, Thm. 5.5].)

Problem 38. [16, Problem 4.8] Let $(X, \mathcal{B}, \mu, T)$ be a ppt and let α, β be partitions of X (as always α, β are assumed to be measurable, and finite or countable). Prove that

$$\big| h_\mu(T,\alpha) - h_\mu(T,\beta) \big| \leq H_\mu(\alpha|\beta) + H_\mu(\beta|\alpha).$$

(7 pt.)

Problem 39. [16, Problem 4.9] Let $(X, \mathcal{B}, \mu, T)$ be a ppt. Recall that

$$h_\mu(T) := \sup\big\{ h_\mu(T,\alpha) \ : \ \alpha \text{ is partition of } X \text{ which is measurable, countable or finite, and } H_\mu(\alpha) < \infty \big\}$$

Prove that

$$h_\mu(T) := \sup\big\{ h_\mu(T,\alpha) \ : \ \alpha \text{ is partition of } X \text{ which is measurable and finite} \big\}.$$

(7 pt.)

(Hint: You may e.g. use Problem 38.)

Problem 40. (E). We say that a ppt $(X, \mathcal{B}, \mu, T)$ is a *factor* of the ppt $(Y, \mathcal{C}, \nu, S)$ if there is a measurable set $Y' \subset Y$ of full measure satisfying $S(Y') \subset Y'$, and a measurable map $\pi : Y' \to X$ such that $\mu = \pi_* \nu$ and $\pi \circ S = T \circ \pi$ on Y'. (Cf. Problem 11, and my notes to Sarig's Def. 1.14.) Prove that when this holds, the entropies of the two ppt's satisfy

$$h_\mu(T) \leq h_\nu(S).$$

(7 pt.)

Problem 41. Let $\sigma : \Sigma_A^+ \to \Sigma_A^+$ be a subshift of finite type with alphabet S and transition matrix A; let P be a stochastic matrix compatible with A, let $\underline{p} = (p_i)_{i \in S}$ be a stationary probability vector wrt P (viz., $\underline{p}P = \underline{p}$), and let μ be the σ-invariant Markov chain measure on Σ_A^+ determined by $P, \underline{p}$.

Prove that μ is ergodic (for σ) iff P is irreducible, and μ is mixing iff P is irreducible and periodic.

(10 pt.)

(Cf. Sarig [16, Thm. 1.2, Cor. 1.1].)

Problem 42. (Coding of a toral automorphism.) If X, Y are compact metric spaces, and $T : X \to X$ and $S : Y \to Y$ are continuous maps, then T is said to be a *topological factor* of S if there is a surjective continuous map $\pi : Y \to X$ such that $\pi \circ S = T \circ \pi$.

Now let $X = \mathbb{T}^2$ and let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be the map induced by the linear map $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ (cf. Problem 7). Prove that T is a topological factor of a certain subshift of finite type.

(10 pt.)

(Hint: See Katok & Hasselblatt, [7, pp. 84–86].)

Problem 43. Let $(X, \mathcal{B}, \mu)$ be a probability space and let α and β be partitions of X (as always, all partitions are assumed to be measurable and countable or finite). Recall that we have defined $\alpha = \beta$ to hold iff [$\alpha \leq \beta$ and $\beta \leq \alpha$], i.e. iff [$\alpha \underset{\mu}{\subset} \sigma(\beta)$ and $\beta \underset{\mu}{\subset} \sigma(\alpha)$]. Prove that $\alpha = \beta$ holds iff

$$\forall A \in \alpha \ : \ \exists B \in \beta \cup \{\emptyset\} \ : \ \mu(A \Delta B) = 0.$$

(7 pt.)

Problem 44. (E). Suppose μ is a shift invariant Markov measure with transition matrix $P = (p_{ij})$ and probability vector $\underline{p} = (p_i)$ on the subshift Σ_A^+ with (finite) alphabet S and transition matrix A. Let α be the partition $\alpha = \{[a] \ : \ a \in S\}$ of Σ_A^+. Prove that for any $n \in \mathbb{Z}^+$,

$$H_\mu(\alpha_0^n) = n \sum_{i,j \in S} p_i p_{ij} \big(-\log p_{ij} \big) + \sum_{i \in S} p_i \big(-\log p_i \big).$$

(7 pt.)

(Hint: Sarig proves this in the computation on [16, p. 108]. Please make sure that you understand each step in the computation, or else ask me to discuss.)

Problem 45. Digest the proof of the formula $\boxed{h_\mu(T,\alpha) = H_\mu(\alpha|\alpha_1^\infty)}$ cf. Sarig [16, Theorem 4.2]. Write up the proof in your own words, and be prepared to discuss it with your teacher!

(10 pt.)

Problem 46. Digest the proof of the Shannon-McMillan-Breiman Theorem, cf. Sarig [16, Theorem 4.3]. Write up the proof in your own words, and be prepared to discuss it with your teacher!

(10 pt.)

Problem 46'. (Shannon-McMillan-Breiman in the non-ergodic case.)

Let $(X, \mathcal{B}, \mu, T)$ be a ppt and α a countable measurable partition of X with $H_\mu(\alpha) < \infty$. Prove that $\lim_{n\to\infty} \frac{1}{n} I_\mu(\alpha_0^{n-1})$ exists μ-a.e., and that this limit is μ-a.e. T-invariant. Prove also that

$$h_\mu(T,\alpha) = \int_X \left(\lim_{n\to\infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}) \right) d\mu$$

(Cf. Mañé, [9, Ch. IV.1–2].)

(10 pt.)

Problem 47. Digest the material in Sarig, [7, Sec. 4.6], i.e. the two equivalent definitions of topological entropy and the proof of the Variational Principle. Write your own summary (or detailed presentation), and be prepared to discuss it with your teacher!

(10-20 pt?)

(Comments: As with all problems, you are also welcome to discuss with your teacher if you get stuck during the work.)

Problem 48. (A basic fact used in Ruelle, [15, p. 86].)

(a) Prove that for any linear map $A : \mathbb{R}^d \to \mathbb{R}^d$, the image of the unit ball under A is an ellipsoid, whose semi-axes have lengths equal to the eigenvalues of $\sqrt{A^t A}$, with multiplicity.

(b) Prove that there is a constant $K > 0$ which only depends on d such that the following holds: For any open subset $U \subset \mathbb{R}^d$, any C^1 map $g : U \to \mathbb{R}^d$, and any compact subset $C \subset U$, there is $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0]$, any $x \in C$, and any cube $F \subset \mathbb{R}^d$ of side-length η containing x, we have $F \subset U$ and $g(F)$ is contained in a rectangular parallelepiped with sides

$$K \max(\lambda_j, 1)\, \eta \qquad (j = 1, \ldots, d),$$

where $\lambda_1, \ldots, \lambda_d \in \mathbb{R}_{\geq 0}$ are the eigenvalues of $\sqrt{(dg_x)^t (dg_x)}$, with multiplicity.

(10 pt)

Problem 49. [16, p. 117, Problem 4.4] Prove that for any ppt $(X, \mathcal{B}, \mu, T)$ and any $n \in \mathbb{Z}^+$, $h_\mu(T^n) = n h_\mu(T)$.

(7 pt)

Problem 50. (Entropy is affine.) [16, p. 117, Problem 4.6]

(10 pt)

Problem 51. Let $(\pi, \lambda) \in \Sigma_{\mathcal{A}} \times \mathbb{R}_+^{\mathcal{A}}$ and assume that for some $n \geq 0$, $\hat{R}^n(\pi, \lambda)$ represents an IET which is a circle rotation. Prove that for any $\tau \in T_\pi^+$, $M(\pi, \lambda, \tau)$ is a torus.

(10 pt)

(Hint: Using some fundamental facts which have been mentioned in the course, we can reduce to $n = 0$. One may then prove that $M(\pi, \lambda, \tau)$ has genus 1 by using the computational scheme from [19, Sec. 14], which I tried to explain in Lecture #14.)

Problem 52. Let $f : M \to M$ be a measurable map, let μ and ν be σ-finite (not necessarily finite!) measures on M which are f-invariant, and assume that $\mu \ll \nu$ and that (f, ν) is ergodic and recurrent. Then $\mu = c \cdot \nu$ for some $c \geq 0$.

(10 pt)

(Comment: This type of fact is used for the uniqueness part of the proof of [19, Thm. 7.2]; cf. [19, end of proof of Cor. 27.2]. Viana seems to consider this fact to be obvious; and also Veech in [18, p. 237, line 9]; however I spent some hours being confused about it!)

Hint: First verify that (f, μ) must *also* be ergodic and recurrent. Then start comparing the restriction of μ and ν to any subset E with $\mu(E)$ and $\nu(E)$ *finite;* here one can use [19, Lemma 25.2 and Remark 25.3], as well as Cor. 1 in my Lecture #2.

Problem 53. Digest the proof of $\hat{m}_1(\hat{\mathcal{S}}) < \infty$ in Viana, [19, Sec. 23–24]. Write up the proof in your own words, and be prepared to discuss it with your teacher!

(15 pt.)

Problem 54. Digest the proof of the fact that almost every IET is uniquely ergodic (using the existence of the invariant ergodic measure ν on $C \times \Lambda_{\mathcal{A}}$); cf. [19, Sec. 28–29]. Write up the proof in your own words, and be prepared to discuss it with your teacher!

(15 pt.)

Problem 55. (E). Let Q be a rational polygon having k vertices with angles $\pi m_i/n_i$, $i = 1, \ldots, k$. Here $m_i, n_i \in \mathbb{Z}^+$ and $\gcd(m_i, n_i) = 1$. Let M be the translation surface obtained by unfolding the billiard flow in Q (cf. Lecture #16 and [11, Sec. 1.5]). Express the number of singular points and their conical angles in terms of $(m_i)_1^k$ and $(n_i)_1^k$. Use your result together with the formula [20, (18)] (wherein "m_i" isn't the same as here) to verify that the genus of M is

$$g = 1 + \frac{N}{2}\Big(k - 2 - \sum \frac{1}{n_i}\Big) \qquad (N = \gcd(n_1, \ldots, n_k)),$$

in agreement with [11, Lemma 1.2].

(8 pt.)

Problem 56. (E). Prove that for any translation surface M and any $R > 0$, there are only finitely many saddle connections of length $\leq R$ on M. (Hence the set of *all* saddle connections on M is countable.)

(8 pt.)

Problem 57. Let M be a translation surface. Prove that if there does not exist any *closed* vertical geodesic on M, then *every* vertical geodesic ray on M which does not end in a singular point is *dense* in M.

(10 pt.)

(Comments: I believe that it might be fun to try to solve this problem without help! Alternatively you may see [11, Thm. 1.8]; but then please be careful; the authors use the fact that their set A (and also ∂A) is "invariant under the flow F_θ" – how should this be interpreted in order to be correct? The issue is that the flow F_θ is discontinuous...)

Problem 58. Let M be a translation surface having κ singularities, with conical angles $2\pi(m_i + 1)$, $i = 1, \ldots, \kappa$. Prove that M has a triangulation such that each face is isometric to a triangle in $\mathbb{R}^2$ and each singular point of M is a vertex of the triangulation. Also prove that if F, E, V are the number of faces, edges and vertices of such a triangulation, then

$$F - E + V = -\sum_{i=1}^{\kappa} m_i.$$

(Hence, using $F - E + V = \chi(M) = 2 - 2g$, we have proved the formula $2g - 2 = \sum_{i=1}^{\kappa} m_i$.)

(10 pt.)

(Hint: ... is available upon request – as always...)

Problem 59. Prove that for M path connected, the definition of $H_1(M)$ given in Lecture #17 agrees with the standard definition ([5, Ch. 2.1]).

(10 pt.)

(Hint: This uses to a large extent the same ingredients as the proof of the fact that $H_1(M)$ can be identified with the abelianization of $\pi_1(M)$; [5, Thm. 2A.1].)

Problem 60. (The following is stated in [20, Sec. 1.2.5] without proof.) Let $M = M(\pi, \lambda, \tau)$. As in Lecture # 17, for each $\beta \in \mathcal{A}$, let $[v_\beta] \in H_1(M, \mathbb{Z})$ be the homology class represented by a vertical segment crossing from bottom to top the rectangle $R^0_\beta = R^1_\beta$, with its endpoints joined by a horizontal segment inside I. Prove that $\{[v_\beta] \, : \, \beta \in \mathcal{A}\}$ spans $H_1(M, \mathbb{R})$. Prove also that the kernel of the map

$$\Phi : \mathbb{R}^{\mathcal{A}} \to H_1(M, \mathbb{R}), \qquad \Phi(\tau) := \sum_{\beta \in \mathcal{A}} \tau_\beta [v_\beta],$$

equals the kernel of Ω_π.

(10 pt.)

(Hint: One approach is to use the explicit formula for the intersection form [20, (35)], together with the known fact that $\dim H_1(M, \mathbb{R}) = 2g$ and the basic linear algebra facts from [20, Sec. 1.1.3] $\approx$ [19, Sec. 10]. You may take the formula [20, (35)] as known – but you may also enjoy trying to prove it, at least in some special cases.)

Problem 61. In the setting of Problem 60, prove that $\{[v_\beta] \, : \, \beta \in \mathcal{A}\}$ generate $H_1(M, \mathbb{Z})$.

(10 pt.)

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