ON A MEAN VALUE FORMULA FOR MULTIPLE SUMS OVER A LATTICE AND ITS DUAL

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Abstract. We prove a generalized version of Rogers' mean value formula in the space X_n of unimodular lattices in $\mathbb{R}^n$, which gives the mean value of a multiple sum over a lattice L and its dual L^*. As an application, we prove that for L random with respect to the $\mathrm{SL}_n(\mathbb{R})$-invariant probability measure, in the limit of large dimension n, the volumes determined by the lengths of the non-zero vectors $\pm\boldsymbol{x}$ in L on the one hand, and the non-zero vectors $\pm\boldsymbol{x}'$ in L^* on the other hand, converge weakly to two independent Poisson processes on the positive real line, both with intensity $\frac{1}{2}$.

Contents

1. Introduction

Let $n \geq 2$, and set

$$X_n = G/\Gamma, \qquad \text{with } G = \mathrm{SL}_n(\mathbb{R}) \text{ and } \Gamma = \mathrm{SL}_n(\mathbb{Z}).$$

Also let μ be the G-invariant probability measure on X_n. As usual, for any $g \in G$ we identify the point $g\Gamma$ in G/Γ with the lattice $g\mathbb{Z}^n = \{g\boldsymbol{v} : \boldsymbol{v} \in \mathbb{Z}^n\}$ in $\mathbb{R}^n$; in this way X_n becomes the space of all lattices of covolume one in $\mathbb{R}^n$.

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In 1945, Siegel [28] proved the following fundamental integration formula: For any continuous, compactly supported function ρ on $\mathbb{R}^n$,

$$\int_{X_n} \sum_{\boldsymbol{v}\in L} \rho(\boldsymbol{v})\, d\mu(L) = \int_{\mathbb{R}^n} \rho(\boldsymbol{x})\, d\boldsymbol{x} + \rho(\boldsymbol{0}), \tag{1.1}$$

where $d\boldsymbol{x}$ denotes Lebesgue measure on $\mathbb{R}^n$. The integrand in the left-hand side, i.e. the function $L \mapsto \sum_{\boldsymbol{v}\in L} \rho(\boldsymbol{v})$ on X_n, is often called the Siegel transform of the function ρ. Ten years later, Rogers [19][1] proved a generalization of (1.1) which for any $1 \leq k < n$ gives an explicit formula for the average of a k-fold sum over L, i.e.

$$\int_{X_n} \sum_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k\in L} \rho(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k)\, d\mu(L)$$

for any function ρ on $(\mathbb{R}^n)^k$ subject to suitable conditions. See Theorem 2.1 below for a precise statement. Rogers' mean value formula has found a number of applications over the years; see, e.g., [20], [24], [21], [31], [3], [7], [29]. In recent years many variants of Rogers' formula have been developed, with several new applications; see, e.g., [14], [11], [13], [1], [2].

Our main goal in the present paper is to prove a generalization of Rogers' mean value formula, giving an explicit formula for the average of a multiple sum over L as well as the *dual* lattice L^*, i.e.

$$\int_{X_n} \sum_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k_1}\in L}\ \sum_{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_{k_2}\in L^*} \rho(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k_1},\boldsymbol{w}_1,\ldots,\boldsymbol{w}_{k_2})\, d\mu(L).$$

Recall that the dual of a (full-rank) lattice L in $\mathbb{R}^n$ is defined by

$$L^* = \{\boldsymbol{w}\in\mathbb{R}^n \,:\, \boldsymbol{w}\cdot\boldsymbol{v}\in\mathbb{Z}\ \ \forall \boldsymbol{v}\in L\},$$

where the dot denotes the standard scalar product on $\mathbb{R}^n$. The covolume of L^* equals the inverse of that of L: $\mathrm{vol}(\mathbb{R}^n/L^*) = \mathrm{vol}(\mathbb{R}^n/L)^{-1}$; in particular $L^*\in X_n$ for any $L\in X_n$. In fact, under our identification $X_n = G/\Gamma$, the map $L\mapsto L^*$ on X_n is given by $g\Gamma\mapsto g^{-\mathsf{T}}\Gamma$, where $g^{-\mathsf{T}} := (g^{-1})^{\mathsf{T}} = (g^{\mathsf{T}})^{-1}$ (this notation will be used throughout the paper). The study of inequalities relating invariants of L and L^* is a classical topic within the geometry of numbers; see, e.g., [4] and the references therein.

To prepare for the statement of our main result, we introduce some further notation. Throughout the paper, we will identify vectors in $\mathbb{R}^n$ with $n\times 1$ column matrices. More generally, for any $m\geq 1$, we will find it convenient to identify the space $M_{n,m}(\mathbb{R})$ of $n\times m$ matrices with the space $(\mathbb{R}^n)^m$ of m-tuples of vectors in $\mathbb{R}^n$ (by listing the column vectors of any matrix $x\in M_{n,m}(\mathbb{R})$ in order from left to right). For $m\leq n$ and $x\in M_{n,m}(\mathbb{R})$, we write

$$\mathsf{d}(x) := \sqrt{\det(x^{\mathsf{T}}x)}. \tag{1.2}$$

Note that under our identification $M_{n,m}(\mathbb{R}) = (\mathbb{R}^n)^m$, $\mathsf{d}(x)$ equals the m-dimensional volume of the parallelotope in $\mathbb{R}^n$ spanned by the vectors in x. In particular $\mathsf{d}(x) > 0$ if and only if x has rank m. We write

$$U_m := \{x\in M_{n,m}(\mathbb{R}) \,:\, \mathsf{d}(x) > 0\}; \tag{1.3}$$

this is a dense open subset of $M_{n,m}(\mathbb{R})$.

[1] See also the papers [22] and [15] for alternative and corrected proofs.

Given any positive integers m_1, m_2 satisfying $m := m_1 + m_2 < n$, and given any $\beta \in M_{m_1,m_2}(\mathbb{R})$, we set

$$S(\beta) := \{\langle x, y\rangle \in U_{m_1} \times U_{m_2} \,:\, x^{\mathsf{T}} y = \beta\}. \tag{1.4}$$

This is a closed, regular submanifold of $U_{m_1} \times U_{m_2}$ of dimension $mn - m_1 m_2$, since the map $\langle x, y\rangle \mapsto x^{\mathsf{T}} y$ from $U_{m_1} \times U_{m_2}$ to $M_{m_1,m_2}(\mathbb{R})$ has everywhere full rank $m_1 m_2$. We equip $S(\beta)$ with a natural Borel measure η_β as follows. For any $x \in U_{m_1}$, we set

$$S(\beta)'_x := \{y \in M_{n,m_2}(\mathbb{R}) \,:\, x^{\mathsf{T}} y = \beta\} \tag{1.5}$$

and

$$S(\beta)_x := S(\beta)'_x \cap U_{m_2} = \{y \in M_{n,m_2}(\mathbb{R}) \,:\, \langle x, y\rangle \in S(\beta)\}. \tag{1.6}$$

Note that $S(\beta)'_x$ is an affine linear subspace of $M_{n,m_2}(\mathbb{R})$ of dimension $m_2(n - m_1)$; we equip $S(\beta)'_x$ with its structure as Euclidean subspace of $M_{n,m_2}(\mathbb{R}) \cong \mathbb{R}^{nm_2}$ with its standard Euclidean structure, and let $\eta_{\beta,x}$ be the corresponding $m_2(n-m_1)$-dimensional Lebesgue volume measure on $S(\beta)'_x$. Note that $S(\beta)'_x \setminus S(\beta)_x$ has measure zero with respect to $\eta_{\beta,x}$; we write $\eta_{\beta,x}$ also for the restriction of $\eta_{\beta,x}$ to $S(\beta)_x$. Finally we define, for any Borel set $E \subset S(\beta)$:

$$\eta_\beta(E) := \int_{U_{m_1}} \int_{S(\beta)_x} \chi_E(x, y)\, d\eta_{\beta,x}(y)\, \frac{dx}{\mathsf{d}(x)^{m_2}}, \tag{1.7}$$

where dx is the standard nm_1-dimensional Lebesgue measure on $M_{n,m_1}(\mathbb{R})$, restricted to U_{m_1}.

For any $B \in M_{k,m}(\mathbb{R})$, we write

$$V_B := B\mathbb{R}^m = \{B\boldsymbol{v} \,:\, \boldsymbol{v} \in \mathbb{R}^m\}$$

for the image of B viewed as a linear map from $\mathbb{R}^m$ to $\mathbb{R}^k$. Next, for any $k \geq m$, we let $M_{k,m}(\mathbb{Z})^*$ be the set of all $B \in M_{k,m}(\mathbb{Z})$ satisfying $\mathsf{d}(B) > 0$ and $V_B \cap \mathbb{Z}^k = B\mathbb{Z}^m = \{B\boldsymbol{v} \,:\, \boldsymbol{v} \in \mathbb{Z}^m\}$. The group $\mathrm{GL}_m(\mathbb{Z})$ acts on $M_{k,m}(\mathbb{Z})^*$ by multiplication from the right. We fix a subset $A_{k,m} \subset M_{k,m}(\mathbb{Z})^*$ containing exactly one representative from each orbit in $M_{k,m}(\mathbb{Z})^*/\,\mathrm{GL}_m(\mathbb{Z})$. Then, under our identification of $M_{k,m}(\mathbb{R})$ with $(\mathbb{R}^k)^m$, $M_{k,m}(\mathbb{Z})^*$ is the set of all m-tuples of vectors in $\mathbb{Z}^k$ which span a primitive m-dimensional sublattice of $\mathbb{Z}^k$ (a sublattice $L \subset \mathbb{Z}^k$ is said to be primitive if $\mathrm{Span}_{\mathbb{R}}(L) \cap \mathbb{Z}^k = L$). Furthermore, two matrices $B, B' \in M_{k,m}(\mathbb{Z})^*$ lie in the same $\mathrm{GL}_m(\mathbb{Z})$-orbit if and only if $V_B = V_{B'}$. It follows that the map $B \mapsto V_B$ gives a bijection from $A_{k,m}$ onto the set of *rational* m-dimensional linear subspaces of $\mathbb{R}^k$ (a linear subspace $V \subset \mathbb{R}^k$ is said to be rational if $V = \mathrm{Span}_{\mathbb{R}}(V \cap \mathbb{Z}^k)$).

For any $\beta \in M_{m_1,m_2}(\mathbb{Z})$ we define a 'weight' $W(\beta)$ as follows: Let $\mathfrak{W}_{m_1}$ be the set of matrices $A = (a_{ij}) \in M_{m_1,m_1}(\mathbb{Z})$ such that $a_{ij} = 0$ for all $1 \leq j < i \leq m_1$, $a_{ii} > 0$ for all $1 \leq i \leq m_1$, and $0 \leq a_{ij} < a_{jj}$ for all $1 \leq i < j \leq m_1$. Let $\mathfrak{W}_\beta$ be the following subset:

$$\mathfrak{W}_\beta = \{A \in \mathfrak{W}_{m_1} \,:\, A^{-\mathsf{T}}\beta \in M_{m_1,m_2}(\mathbb{Z})\}. \tag{1.8}$$

Finally set:

$$W(\beta) := \frac{\sum_{A \in \mathfrak{W}_\beta} (\det A)^{m_2 - n}}{\zeta(n)\zeta(n-1)\cdots\zeta(n-m_1+1)}. \tag{1.9}$$

We are now ready to state the main result of the paper:

Theorem 1.1. *Let $k_1, k_2 \in \mathbb{Z}^+$, $n > k_1 + k_2$, and let $\rho : (\mathbb{R}^n)^{k_1} \times (\mathbb{R}^n)^{k_2} \to \mathbb{R}_{\geq 0}$ be a non-negative Borel measurable function (which we identify with a function $M_{n,k_1}(\mathbb{R}) \times M_{n,k_2}(\mathbb{R}) \to \mathbb{R}_{\geq 0}$). Then*

(1.10)

$$\int_{X_n} \sum_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k_1} \in L} \sum_{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_{k_2} \in L^*} \rho(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k_1}, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{k_2})\, d\mu(L)$$

$$= \sum_{m_1=1}^{k_1} \sum_{B_1 \in A_{k_1,m_1}} \sum_{m_2=1}^{k_2} \sum_{B_2 \in A_{k_2,m_2}} \sum_{\beta \in M_{m_1,m_2}(\mathbb{Z})} W(\beta) \int_{S(\beta)} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\, d\eta_\beta(x,y)$$

$$+ \sum_{m_1=1}^{k_1} \sum_{B_1 \in A_{k_1,m_1}} \int_{M_{n,m_1}(\mathbb{R})} \rho\big(xB_1^{\mathsf{T}}, 0\big)\, dx + \sum_{m_2=1}^{k_2} \sum_{B_2 \in A_{k_2,m_2}} \int_{M_{n,m_2}(\mathbb{R})} \rho\big(0, yB_2^{\mathsf{T}}\big)\, dy + \rho(0,0),$$

where both sides are finite whenever ρ is bounded and has bounded support.

Remark 1.2. By introducing certain natural conventions concerning "empty matrices of dimensions $a \times 0$ and $0 \times a$" (any $a \geq 0$), and agreeing that for any $k \geq 1$, $A_{k,0}$ is the singleton set containing the empty matrix of dimension $k \times 0$, one may incorporate the last line of (1.10) in the double sum, so that the *whole* of the right-hand side of (1.10) can be expressed as

$$\sum_{m_1=0}^{k_1} \sum_{B_1 \in A_{k_1,m_1}} \sum_{m_2=0}^{k_2} \sum_{B_2 \in A_{k_2,m_2}} \sum_{\beta \in M_{m_1,m_2}(\mathbb{Z})} W(\beta) \int_{S(\beta)} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\, d\eta_\beta(x,y). \tag{1.11}$$

Remark 1.3. It follows from the statement of Theorem 1.1 that the first sum in the right-hand side of (1.10) is independent of the choice of the sets of representatives A_{k_1,m_1} and A_{k_2,m_2}. This fact is also fairly easy to verify directly. Indeed, note that it suffices to verify that for any given $B_1 \in M_{k_1,m_1}(\mathbb{Z})^*$ and $B_2 \in M_{k_2,m_2}(\mathbb{Z})^*$, the sum

$$\sum_{\beta \in M_{m_1,m_2}(\mathbb{Z})} W(\beta) \int_{S(\beta)} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\, d\eta_\beta(x,y)$$

remains the same when B_1 and B_2 are replaced by $B_1\gamma_1$ and $B_2\gamma_2$, respectively, for any $\gamma_1 \in \mathrm{GL}_{m_1}(\mathbb{Z})$ and $\gamma_2 \in \mathrm{GL}_{m_2}(\mathbb{Z})$. This claim follows from Lemma 3.9 below, together with the fact that the map $\beta \mapsto \gamma_1 \beta {\gamma_2}^{\mathsf{T}}$ is a bijection of $M_{m_1,m_2}(\mathbb{Z})$ onto itself which preserves the weight W; cf. Lemma 3.16.

Remark 1.4. The map $L \mapsto L^*$ is a diffeomorphism of X_n onto itself which preserves the measure μ; hence the left-hand side of the formula (1.10) is invariant under replacing $\langle k_1, k_2, \rho \rangle$ by $\langle k_2, k_1, \widetilde{\rho} \rangle$ where $\widetilde{\rho}(x,y) := \rho(y,x)$. In the right-hand side of (1.10) this invariance is directly visible via the following symmetry relations, which we prove in Section 3.4 below: For any $m_1, m_2 \in \mathbb{Z}^+$ and $\beta \in M_{m_1,m_2}(\mathbb{Z})$ we have $W(\beta^{\mathsf{T}}) = W(\beta)$ and $\eta_{\beta^{\mathsf{T}}} = J_* \eta_\beta$, where J is the diffeomorphism $\langle x, y \rangle \mapsto \langle y, x \rangle$ from $S(\beta)$ onto $S(\beta^{\mathsf{T}})$.

The formula in Theorem 1.1 is also valid if either $k_1 = 0$ or $k_2 = 0$, if appropriately interpreted. Indeed, in this case the formula becomes the following slightly reformulated version of Rogers' original formula:

Theorem 1.5. *(Rogers [19].) Let $1 \leq k < n$ and let $\rho : (\mathbb{R}^n)^k \to \mathbb{R}_{\geq 0}$ be a non-negative Borel measurable function. Then*

$$\int_{X_n} \sum_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k \in L} \rho(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k)\, d\mu(L) = \sum_{m=1}^{k} \sum_{B \in A_{k,m}} \int_{M_{n,m}(\mathbb{R})} \rho(xB^{\mathsf{T}})\, dx + \rho(0). \tag{1.12}$$

(We discuss in Section 2 the translation between the statement in [19] and Theorem 1.5.)

Next, we discuss some applications of our main theorem.

Our main motivation for developing the formula in Theorem 1.1 comes from questions concerning the Epstein zeta function of a random lattice $L \in X_n$ as $n \to \infty$; cf. [21, 32, 33, 29]. Recall that for $\operatorname{Re} s > \frac{n}{2}$ and $L \in X_n$ the Epstein zeta function is defined by the absolutely convergent series

$$E_n(L, s) := \sum_{\boldsymbol{m} \in L \setminus \{\boldsymbol{0}\}} |\boldsymbol{m}|^{-2s}.$$

The function $E_n(L,s)$ can be meromorphically continued to $\mathbb{C}$ and satisfies a functional equation of "Riemann type" relating $E_n(L,s)$ and $E_n(L^*, \frac{n}{2} - s)$. An outstanding question from [33] is whether $E_n(L,s)$ for s on or near the central point $s = \frac{n}{4}$, possesses, after appropriate normalization, a limit distribution as $n \to \infty$? It seems clear that Theorem 1.1 should be an important ingredient when seeking to extend the methods of [33] to handle this question. We hope to return to these matters in future work.

Finally, we describe an application of Theorem 1.1 to the limit distribution of the shortest vector lengths of L and L^*, for L random in (X_n, μ), as the dimension n tends to infinity. Given a lattice $L \in X_n$, we order its non-zero vectors by increasing lengths as $\pm\boldsymbol{v}_1, \pm\boldsymbol{v}_2, \pm\boldsymbol{v}_3, \ldots$ and define, for each $j \geq 1$,

$$\mathcal{V}_j(L) := \mathfrak{V}_n |\boldsymbol{v}_j|^n,$$

where $\mathfrak{V}_n$ denotes the volume of the unit ball in $\mathbb{R}^n$. In [31], the second author of the present paper proved that for L random in (X_n, μ), the sequence $\{\mathcal{V}_j(L)\}_{j=1}^{\infty}$ converges in distribution, as $n \to \infty$, to the points of a Poisson process on $\mathbb{R}^+$ with constant intensity $\frac{1}{2}$. Of course the same limit result holds for the sequence $\{\mathcal{V}_j(L^*)\}_{j=1}^{\infty}$, since the map $L \mapsto L^*$ preserves the measure μ_n. The following result generalizes these facts by describing the *joint* limiting distribution of the normalized vector lengths of both L and L^*.

Theorem 1.6. *For L random in (X_n, μ), the two sequences $\{\mathcal{V}_j(L)\}_{j=1}^{\infty}$ and $\{\mathcal{V}_j(L^*)\}_{j=1}^{\infty}$ converge jointly in distribution, as $n \to \infty$, to the sequences $\{T_j\}_{j=1}^{\infty}$ and $\{T'_j\}_{j=1}^{\infty}$, where $0 < T_1 < T_2 < T_3 < \cdots$ and $0 < T'_1 < T'_2 < T'_3 < \cdots$ denote the points of two independent Poisson processes on $\mathbb{R}^+$ with constant intensity $\frac{1}{2}$.*

2. Rogers' formula

In this section we explain the translation between Rogers' original formulation of his mean value formula, [19], and the statement in Theorem 1.5. The following is Rogers' original formulation:

Theorem 2.1. *Let $1 \leq k < n$ and let $\rho : (\mathbb{R}^n)^k \to \mathbb{R}_{\geq 0}$ be a non-negative Borel measurable function. Then*

$$\int_{X_n} \sum_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k \in L} \rho(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k)\,d\mu(L) = \rho(0,\ldots,0) + \sum_D \Big(\frac{e_1}{q}\cdots\frac{e_m}{q}\Big)^n \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n} \rho\Big(\sum_{i=1}^m \frac{d_{i1}}{q}\boldsymbol{x}_i,\ldots,\sum_{i=1}^m \frac{d_{ik}}{q}\boldsymbol{x}_i\Big)\,d\boldsymbol{x}_1\ldots d\boldsymbol{x}_m, \tag{2.1}$$

where the inner sum is over all integer matrices $D = (d_{ij}) \in M_{m,k}(\mathbb{Z})$ with m arbitrary in $\{1,\ldots,k\}$, satisfying the following properties: No column of D vanishes identically; the entries of D have greatest common divisor equal to 1; and finally there exists a division $(\nu;\mu) = (\nu_1,\ldots,\nu_m;\mu_1,\ldots,\mu_{k-m})$ of the numbers $1,\ldots,k$ into two sequences $\nu_1,\ldots,\nu_m$ and $\mu_1,\ldots,\mu_{k-m}$, satisfying

$$\begin{aligned} &1 \leq \nu_1 < \nu_2 < \ldots < \nu_m \leq k,\\ &1 \leq \mu_1 < \mu_2 < \ldots < \mu_{k-m} \leq k,\\ &\nu_i \neq \mu_j, \text{ if } 1 \leq i \leq m,\ 1 \leq j \leq k-m, \end{aligned} \tag{2.2}$$

such that, for some $q = q(D) \in \mathbb{Z}^+$,

$$\begin{aligned} &d_{i\nu_j} = q\delta_{ij}, \quad i = 1,\ldots,m,\ j = 1,\ldots,m,\\ &d_{i\mu_j} = 0, \quad \text{if } \mu_j < \nu_i,\ i = 1,\ldots,m,\ j = 1,\ldots,k-m. \end{aligned} \tag{2.3}$$

Finally $e_i = \gcd(\varepsilon_i, q)$, $i = 1,\ldots,m$, where $\varepsilon_1,\ldots,\varepsilon_m$ are the elementary divisors of the matrix D.

Remark 2.2. Regarding convergence in (2.1), Schmidt in [23] proved that both sides of (2.1) are finite whenever ρ is bounded and has bounded support in $(\mathbb{R}^n)^k$. In fact, Schmidt's proof applies verbatim to any non-negative ρ of sufficiently rapid polynomial decay. It therefore follows that (2.1) holds, with absolute convergence in both sides, for all complex valued ρ of sufficiently rapid polynomial decay, and in particular for any Schwartz function ρ on $(\mathbb{R}^n)^k$.

Remark 2.3. In the formula (2.1) in Theorem 2.1, vector notation is used in both the left and the right-hand side. However in the right-hand side of (1.12) in Theorem 1.5, the identification $M_{n,m}(\mathbb{R}) = (\mathbb{R}^n)^m$ is used to instead view ρ as a function on $M_{n,m}(\mathbb{R})$; in particular the "0" in "$\rho(0)$" stands for the zero matrix in $M_{n,m}(\mathbb{R})$. It should also be noted that the right-hand side of (1.12) is trivially independent of the choice of the set of representatives $A_{k,m}$. Indeed, for any $B \in M_{n,m}(\mathbb{Z})^*$ and any $\gamma \in \mathrm{GL}_m(\mathbb{Z})$, by substituting $x_{new} = x\gamma^{\mathsf{T}}$ we get $\int_{M_{n,m}(\mathbb{R})} \rho(xB^{\mathsf{T}})\,dx = \int_{M_{n,m}(\mathbb{R})} \rho(x(B\gamma)^{\mathsf{T}})\,dx$.

As a preparation for the translation between Theorem 2.1 and Theorem 1.5 we first prove the following lemma, which implies that in Theorem 2.1 we have in fact $e_i = \varepsilon_i$ for all i and all D.

Lemma 2.4. *For any matrix D as in Theorem 2.1, with associated $q = q(D)$ and elementary divisors $\varepsilon_1,\ldots,\varepsilon_m$, we have $\varepsilon_i \mid q$ for all i.*

Proof. Given D, by the Smith normal form theorem, there exist $\gamma_1 \in \mathrm{GL}_m(\mathbb{Z})$ and $\gamma_2 \in \mathrm{GL}_k(\mathbb{Z})$ such that

$$D = \gamma_1 \operatorname{diag}[\varepsilon_1, \ldots, \varepsilon_m]\big(I_m\, 0\big)\gamma_2, \tag{2.4}$$

where $\big(I_m\ 0\big)$ is block notation for an $m \times k$ matrix. By considering only columns number $\nu_1, \ldots, \nu_m$ of the matrix D, it follows that

$$q\gamma_1^{-1} = \operatorname{diag}[\varepsilon_1, \ldots, \varepsilon_m]B, \tag{2.5}$$

where $B \in M_m(\mathbb{Z})$ is the submatrix of γ_2 formed by removing the bottom $k-m$ rows and the columns of indices $\mu_1, \ldots, \mu_{k-m}$. It follows from $\gamma_1^{-1} \in \mathrm{GL}_m(\mathbb{Z})$ that for each $i \in \{1, \ldots, m\}$, the entries in row number i of the matrix $q\gamma_1^{-1}$ have greatest common divisor q. On the other hand, (2.5) implies that all these entries are divisible by ε_i. Hence $\varepsilon_i \mid q$. □

Proof that Theorem 1.5 is a reformulation of Theorem 2.1. For each $m \in \{1, \ldots, k\}$, let $\mathfrak{A}(m)$ be the set of matrices of fixed size $m \times k$ appearing in the sum in (2.1). For each $D \in \mathfrak{A}(m)$, we fix a choice of $\gamma_1 = \gamma_1(D) \in \mathrm{GL}_m(\mathbb{Z})$ and $\gamma_2 = \gamma_2(D) \in \mathrm{GL}_k(\mathbb{Z})$ such that (2.4) holds. It is easy to prove that the map $D \mapsto D^{\mathsf{T}}\mathbb{R}^m$ is a bijection from $\mathfrak{A}(m)$ onto the family of non-zero rational linear subspaces of $\mathbb{R}^k$. Note also that (2.4) implies $D^{\mathsf{T}}\mathbb{R}^m = \gamma_2'\mathbb{R}^m$, where $\gamma_2' = \gamma_2'(D) := \big(\big(I_m\, 0\big)\gamma_2\big)^{\mathsf{T}}$ (in other words, γ_2' is the transpose of the top $m \times k$ submatrix of γ_2); and since $\gamma_2 \in \mathrm{GL}_k(\mathbb{Z})$, we have $\gamma_2' \in M_{k,m}(\mathbb{Z})^*$. It follows that we may choose the set of representatives $A_{k,m}$ (cf. p. 3) to be given by

$$A_{k,m} := \{\gamma_2'(D) \;:\; D \in \mathfrak{A}(m)\}.$$

For each $D \in \mathfrak{A}(m)$, in terms of our matrix notation, the multiple integral appearing in the right-hand side of (2.1) equals

$$\int_{M_{n,m}(\mathbb{R})} \rho(xq^{-1}D)\,dx = q^{nm}\int_{M_{n,m}(\mathbb{R})} \rho(xD)\,dx = \Big(\frac{q}{\varepsilon_1}\cdots\frac{q}{\varepsilon_m}\Big)^n \int_{M_{n,m}(\mathbb{R})} \rho(y\gamma_2'^{\mathsf{T}})\,dy, \tag{2.6}$$

where we used (2.4) and substituted $x = y\gamma_1 \operatorname{diag}[\varepsilon_1, \ldots, \varepsilon_m]$. Hence, using also Lemma 2.4, we conclude that the second line of (2.1) equals

$$\sum_{m=1}^{k} \sum_{B \in A_{k,m}} \int_{M_{n,m}(\mathbb{R})} \rho(yB^{\mathsf{T}})\,dy,$$

and it follows that the right-hand side of (2.1) equals the right-hand side of (1.12). □

Finally, for later reference, we also state a formula which is more basic than Theorem 2.1 and which can be used in the proof of Theorem 2.1. For any lattice $L \in X_n$ and $1 \le k < n$, we write $P_{L,k}$ for the set of primitive k-tuples in L^k.

Theorem 2.5. *Let $1 \le k < n$ and let $\rho : (\mathbb{R}^n)^k \to \mathbb{R}_{\ge 0}$ be a non-negative Borel measurable function. Then*

$$\int_{X_n} \sum_{\langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\rangle \in P_{L,k}} \rho(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)\,d\mu(L) = \frac{1}{\prod_{j=n-k+1}^{n} \zeta(j)} \int_{(\mathbb{R}^n)^k} \rho\,d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_k. \tag{2.7}$$

Proof. See Macbeath and Rogers, [15, Theorem 1 and formula (13)]. □

3. Proof of the main theorem

In this section we prove Theorem 1.1.

3.1. **Initial decompositions.** Recall that for any positive integer m, we identify $(\mathbb{R}^n)^m$ with $M_{n,m}(\mathbb{R})$. In particular, for any lattice $L \in X_n$, the set L^m is identified with a lattice in $M_{n,m}(\mathbb{R})$, and the left-hand side of (1.10) can be expressed as follows:

$$\int_{X_n} \sum_{x \in L^{k_1}} \sum_{y \in (L^*)^{k_2}} \rho(x,y)\, d\mu(L), \tag{3.1}$$

where ρ is now a function on the space

$$\mathcal{M} = \mathcal{M}_{n,k_1,k_2} := M_{n,k_1}(\mathbb{R}) \times M_{n,k_2}(\mathbb{R}).$$

We will study the expression in (3.1) as a functional on the space $C_c(\mathcal{M})$ of continuous functions ρ on $\mathcal{M}$ of compact support. A key observation is that if we let the group $G = \mathrm{SL}_n(\mathbb{R})$ act on $\mathcal{M}$ in the following way:

$$g\langle x,y\rangle := \big\langle gx, g^{-\mathsf{T}}y\big\rangle, \qquad \text{for any } g \in G,\ \langle x,y\rangle \in \mathcal{M}, \tag{3.2}$$

then this functional is G-*invariant:*

Lemma 3.1. *The map $F : C_c(\mathcal{M}) \to \mathbb{C}$ given by*

$$F(\rho) = \int_{X_n} \sum_{x \in L^{k_1}} \sum_{y \in (L^*)^{k_2}} \rho(x,y)\, d\mu(L) \tag{3.3}$$

is a G-invariant positive linear functional on $C_c(\mathcal{M})$.

Proof. We first prove that the functional F is well-defined, by verifying that the expression in (3.3) is absolutely convergent in the sense that $\int_{X_n} \sum_{x \in L^{k_1}} \sum_{y \in (L^*)^{k_2}} |\rho(x,y)|\, d\mu(L) < \infty$ for any $\rho \in C_c(\mathcal{M})$. To do so, it suffices to verify that

$$\int_{X_n} \sum_{x \in L^{k_1}} \sum_{y \in (L^*)^{k_2}} \chi_K(x,y)\, d\mu(L) < \infty \tag{3.4}$$

for any compact subset K of $\mathcal{M}$. To this end, set $k = k_1 + k_2$ and consider the Schwartz function $\varphi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) := \prod_{j=1}^k e^{-\pi\|\boldsymbol{x}_j\|^2}$ on $\mathcal{M}$, where we have identified $\mathcal{M}$ with $(\mathbb{R}^n)^k$ via our identifications $M_{n,k_1}(\mathbb{R}) = (\mathbb{R}^n)^{k_1}$. It follows from Remark 2.2 that

$$\int_{X_n} \sum_{x \in L^{k_1}} \sum_{y \in L^{k_2}} \varphi(x,y)\, d\mu(L) < \infty. \tag{3.5}$$

But for any fixed lattice $L \in X_n$ we have $\sum_{\boldsymbol{x} \in L} e^{-\pi\|\boldsymbol{x}\|^2} = \sum_{\boldsymbol{x} \in L^*} e^{-\pi\|\boldsymbol{x}\|^2}$, by the Poisson summation formula, and since the function $\boldsymbol{x} \mapsto e^{-\pi\|\boldsymbol{x}\|^2}$ ($\boldsymbol{x} \in \mathbb{R}^n$) is its own Fourier transform. Hence also $\sum_{x \in L^{k_1}} \sum_{y \in L^{k_2}} \varphi(x,y) = \sum_{x \in L^{k_1}} \sum_{y \in (L^*)^{k_2}} \varphi(x,y)$ for every $L \in X_n$. Using this fact in (3.5), together with the fact that $\inf_K \varphi > 0$ for any given compact set $K \subset \mathcal{M}$, we obtain (3.4), thus completing the proof of the absolute convergence in (3.3).

Finally we prove that F is G-invariant. Given $\rho \in C_c(\mathcal{M})$ and $g \in G$, let us write $\rho_g(x,y) := \rho(g\langle x,y\rangle)$. For any $L \in X_n$ we have $g^{-\mathsf{T}}L^* = (gL)^*$, and hence

$$\sum_{x \in L^{k_1}} \sum_{y \in (L^*)^{k_2}} \rho_g(x,y) = \sum_{x \in (gL)^{k_1}} \sum_{y \in ((g^{-\mathsf{T}}L)^*)^{k_2}} \rho(x,y) = \sum_{x \in (gL)^{k_1}} \sum_{y \in ((gL)^*)^{k_2}} \rho(x,y).$$

Using this formula in (3.3), and the fact that μ is G-invariant, it follows that $F(\rho_g) = F(\rho)$, which is the desired G-invariance of F. $\square$

Our strategy will now be to combinatorially decompose the above functional as a sum of several "smaller" G-invariant terms, which are in a certain sense more basic and which we will be able to identify in an explicit way. The first decomposition step is standard: In the expression (3.1), we restrict the summation ranges for x and y to the set of *linearly independent* tuples of vectors in L and L^*. (This is in analogy with, e.g., the discussion in [19, between (36) and (42)].) For this we use the following basic fact.

Lemma 3.2. *For any $1 \leq m \leq k \leq n$, the map $\langle \gamma, B \rangle \mapsto \gamma B^{\mathsf{T}}$ is a bijection from $(M_{n,m}(\mathbb{Z}) \cap U_m) \times A_{k,m}$ onto the set of matrices in $M_{n,k}(\mathbb{Z})$ of rank m.*

Proof. Let C be a matrix in $M_{n,k}(\mathbb{Z})$ of rank m. Let V be the column space of C^{T}; this is an m-dimensional subspace of $\mathbb{R}^k$ spanned by integer vectors; hence rational; hence there exists a unique $B \in A_{k,m}$ such that $V = V_B = B\mathbb{R}^m$, that is, $\mathbb{R}^n C = \mathbb{R}^m B^{\mathsf{T}}$. It also follows from $B \in A_{k,m}$ that each vector $\boldsymbol{w}$ in $\mathbb{R}^m B^{\mathsf{T}}$ equals $\boldsymbol{v} B^{\mathsf{T}}$ for a *unique* vector $\boldsymbol{v} \in \mathbb{R}^m$, with $\boldsymbol{v} \in \mathbb{Z}^m$ if and only if $\boldsymbol{w} \in \mathbb{Z}^k$. Applying this fact to each row vector of C, it follows that there exists a unique matrix $\gamma \in M_{n,m}(\mathbb{Z})$ such that $C = \gamma B^{\mathsf{T}}$, and this γ must have full rank m since C has full rank m; in other words $\gamma \in M_{n,m}(\mathbb{Z}) \cap U_m$. □

For any $g \in G$, letting $L := g\mathbb{Z}^n$ in X_n, we have $L^k = (g\mathbb{Z}^n)^k = g \cdot M_{n,k}(\mathbb{Z})$. Hence it follows from Lemma 3.2 that the map $\langle x, B \rangle \mapsto xB^{\mathsf{T}}$ is a bijection from $(L^m \cap U_m) \times A_{k,m}$ onto the set of tuples in L^k whose linear span in $\mathbb{R}^n$ has dimension m. Applying this to both the sums in (3.1), it follows that (3.1) can be rewritten as

$$\sum_{m_1=1}^{k_1} \sum_{B_1 \in A_{k_1,m_1}} \sum_{m_2=1}^{k_2} \sum_{B_2 \in A_{k_2,m_2}} \int_{X_n} \sum_{x \in L^{m_1} \cap U_{m_1}} \sum_{y \in (L^*)^{m_2} \cap U_{m_2}} \rho(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}) \, d\mu(L) \\ + \int_{X_n} \sum_{x \in L^{k_1}} \rho(x, 0) \, d\mu(L) + \int_{X_n} \sum_{y \in (L^*)^{k_2}} \rho(0, y) \, d\mu(L) - \rho(0, 0). \tag{3.6}$$

By applying Theorem 1.5 to the two integrals in the last line (where for the last integral we use the fact that the map $L \mapsto L^*$ is a diffeomorphism of X_n preserving the measure μ), we find that the last line in (3.6) exactly matches the last line in (1.10). Hence it remains to handle the first line in (3.6). Clearly, it will suffice to find an appropriate explicit expression for the following integral, for any given $m_1, m_2 \in \mathbb{Z}^+$ with $m_1 + m_2 < n$ and any given $\rho \in C_c(\mathcal{M})$ (where now $\mathcal{M} = \mathcal{M}_{n,m_1,m_2}$):

$$\int_{X_n} \sum_{x \in L^{m_1} \cap U_{m_1}} \sum_{y \in (L^*)^{m_2} \cap U_{m_2}} \rho(x, y) \, d\mu(L). \tag{3.7}$$

Note that the proof of Lemma 3.1 carries over immediately to show that also the expression in (3.7) is a G-invariant positive linear functional on $C_c(\mathcal{M})$.

The next decomposition step is to note that for each $\beta \in M_{m_1,m_2}(\mathbb{R})$, the G-action in (3.2) *preserves the submanifold $S(\beta)$ of $\mathcal{M}$.* [2] We also note that for any $L \in X_n$ and any $x \in L^{m_1}$ and $y \in (L^*)^{m_2}$ we have $x^{\mathsf{T}} y \in M_{m_1,m_2}(\mathbb{Z})$, i.e. the pair $\langle x, y \rangle$ belongs to $S(\beta)$ for some *integer* matrix $\beta \in M_{m_1,m_2}(\mathbb{Z})$. It is therefore natural to decompose (3.7) according to the values of $\beta = x^{\mathsf{T}} y$, i.e., to rewrite (3.7) as the sum over all $\beta \in M_{m_1,m_2}(\mathbb{Z})$ of the following expression:

$$\int_{X_n} \sum_{x \in L^{m_1} \cap U_{m_1}} \sum_{\substack{y \in (L^*)^{m_2} \cap U_{m_2} \\ (x^{\mathsf{T}} y = \beta)}} \rho(x, y) \, d\mu(L). \tag{3.8}$$

[2]Note that $S(\beta) \subset U_{m_1} \times U_{m_2} \subset \mathcal{M}$.

It turns out that for each fixed β the corresponding term in (3.8) is a G-invariant positive linear functional on $C_c(S(\beta))$ (this follows from (3.10) and Lemma 3.4 below). We will also prove that the action of G on $S(\beta)$ is transitive, and therefore such a G-invariant functional is *unique* up to scalar multiplication. This will allow us to identify the functional in (3.8) in a completely explicit way.

The identification just mentioned will go via one last decomposition step: In (3.8), we restrict the summation over x to *primitive* m_1-tuples in L^{m_1}, i.e. tuples which can be extended to a $\mathbb{Z}$-basis of L. That is, we consider the following expression:

$$F_\beta(\rho) := \int_{X_n} \sum_{x \in P_{L,m_1}} \sum_{\substack{y \in (L^*)^{m_2} \cap U_{m_2} \\ (x^\mathsf{T} y = \beta)}} \rho(x,y)\, d\mu(L), \tag{3.9}$$

where we recall that $P_{L,m_1} \subset L^{m_1} \cap U_{m_1}$ denotes the subset of primitive m_1-tuples in L^{m_1}. The proof that (3.8) can be expressed in terms of (3.9) goes via the following (standard) lemma.

Lemma 3.3. *For any lattice $L \subset \mathbb{R}^n$ and any $1 \le m < n$, the map $\langle x, A \rangle \mapsto xA$ is a bijection from $P_{L,m} \times \mathfrak{W}_m$ onto $L^m \cap U_m$.*

(Here $\mathfrak{W}_m$ is the set introduced just above (1.8), with $m = m_1$.)

Proof. Immediate from [15, Lemma 9]. □

It follows from Lemma 3.3 that for any $\beta \in M_{m_1,m_2}(\mathbb{Z})$ and $\rho \in C_c(S(\beta))$, the expression in (3.8) can be rewritten as

$$\int_{X_n} \sum_{A \in \mathfrak{W}_{m_1}} \sum_{x \in P_{L,m_1}} \sum_{\substack{y \in (L^*)^{m_2} \cap U_{m_2} \\ ((xA)^\mathsf{T} y = \beta)}} \rho(xA, y)\, d\mu(L) = \sum_{A \in \mathfrak{W}_\beta} F_{A^{-\mathsf{T}}\beta}(\rho_A), \tag{3.10}$$

where $\rho_A \in C_c(S(A^{-\mathsf{T}}\beta))$ is defined by $\rho_A(x,y) = \rho(xA, y)$, and F_β is given by (3.9); recall also that $\mathfrak{W}_\beta$ denotes the set of all $A \in \mathfrak{W}_{m_1}$ satisfying $A^{-\mathsf{T}}\beta \in M_{m_1,m_2}(\mathbb{Z})$. The change of order of summation in (3.10) is justified by absolute convergence; indeed the left-hand side of (3.10), with $|\rho|$ in the place of ρ, is majorized by the expression in (3.4) with $K = \operatorname{supp}(\rho)$.

3.2. G-invariant measures on $S(\beta)$.

Lemma 3.4. *For any $\beta \in M_{m_1,m_2}(\mathbb{Z})$, the function F_β defined in (3.9) is a G-invariant positive linear functional on $C_c(S(\beta))$.*

Proof. For any $\rho \in C_c(S(\beta))$, the expression giving $F_\beta(|\rho|)$ is majorized by the expression in (3.4) with $K = \operatorname{supp}(\rho)$; hence F_β is a well-defined positive linear functional on $C_c(S(\beta))$. The G-invariance is verified by the same type of computation as in the proof of Lemma 3.1, the main point being that for any $g \in G$ and $L \in X_n$ we have

$$\sum_{x \in P_{L,m_1}} \sum_{\substack{y \in (L^*)^{m_2} \cap U_{m_2} \\ (x^\mathsf{T} y = \beta)}} \rho\big(gx, g^{-\mathsf{T}} y\big) = \sum_{x \in P_{gL,m_1}} \sum_{\substack{y \in ((gL)^*)^{m_2} \cap U_{m_2} \\ (x^\mathsf{T} y = \beta)}} \rho(x,y).$$

□

Next we will prove that also the measure η_β introduced in (1.7) is G-invariant. For this we will require the following simple auxiliary lemma. We equip $\mathbb{R}^n$ with its standard Euclidean structure; then any non-zero linear subspace V of $\mathbb{R}^n$ inherits a structure as a Euclidean subspace, and we denote by vol_V the corresponding Lebesgue volume measure on V. For any $g \in \mathrm{GL}_n(\mathbb{R})$ we let $\delta(g, V) > 0$ be the volume scaling factor of the linear map $g\big|_V : \boldsymbol{v} \mapsto g\boldsymbol{v}$ from V onto gV, i.e. the number δ such that $\operatorname{vol}_{gV}(gE) = \delta \cdot \operatorname{vol}_V(E)$ for any Borel set $E \subset V$.

Lemma 3.5. *For any $g \in G$ and any non-trivial subspace $V \subset \mathbb{R}^n$, $\delta(g^{\mathsf{T}}, (gV)^{\perp}) = \delta(g, V)^{-1}$.*

Proof. Note that $\delta(k, W) = 1$ for any $k \in O(n)$ and any non-zero subspace $W \subset \mathbb{R}^n$. Using this fact one verifies that for any $k_1, k_2 \in O(n)$ we have $\delta(g, V) = \delta(k_1 g k_2, k_2^{-1} V)$ and $\delta(g^{\mathsf{T}}, (gV)^{\perp}) = \delta(k_2^{-1} g^{\mathsf{T}} k_1^{-1}, k_1(gV)^{\perp})$, and here $k_2^{-1} g^{\mathsf{T}} k_1^{-1} = \tilde{g}^{\mathsf{T}}$ and $k_1(gV)^{\perp} := (\tilde{g}\tilde{V})^{\perp}$ with $\tilde{g} := k_1 g k_2$ and $\tilde{V} := k_2^{-1} V$. Hence it suffices to prove the lemma with $\tilde{g}$ and $\tilde{V}$ in the place of g and V. By choosing k_1, k_2 appropriately, we may assume that $\tilde{V}$ equals the span of the first $d = \dim V$ standard basis vectors of $\mathbb{R}^n$ and also $\tilde{g}\tilde{V} = \tilde{V}$; this means that in block matrix notation we have $\tilde{g} = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \gamma \end{smallmatrix}\right)$ for some $\alpha \in \mathrm{GL}_d(\mathbb{R})$, $\beta \in M_{d,n-d}(\mathbb{R})$, $\gamma \in \mathrm{GL}_{n-d}(\mathbb{R})$. Now $\delta(\tilde{g}, \tilde{V}) = |\det \alpha|$ and $\delta(\tilde{g}^{\mathsf{T}}, (\tilde{g}\tilde{V})^{\perp}) = |\det(\gamma^{\mathsf{T}})| = |\det \gamma|$, and the lemma follows from the fact that $\det \alpha \det \gamma = \det \tilde{g} = \det g = 1$. □

Lemma 3.6. *For any $\beta \in M_{m_1,m_2}(\mathbb{R})$, the measure η_β on $S(\beta)$ is G-invariant.*

Proof. Let $g \in G$ and let E be a Borel subset of $S(\beta)$. Then by (1.7),

$$\eta_\beta(g^{-1}E) = \int_{U_{m_1}} \int_{S(\beta)_x} \chi_E\big(gx, (g^{\mathsf{T}})^{-1} y\big)\, d\eta_{\beta,x}(y)\, \frac{dx}{\mathsf{d}(x)^{m_2}}. \tag{3.11}$$

To rewrite this expression, consider any fixed $x \in U_{m_1}$, and set $V = x\mathbb{R}^{m_1} \subset \mathbb{R}^n$. Then the map $z \mapsto g^{\mathsf{T}} z$ is a diffeomorphism from $S(\beta)_{gx}$ onto $S(\beta)_x$, and since $S(\beta)_{gx}$ and $S(\beta)_x$ are translates of the subspaces $((gV)^{\perp})^{m_2}$ and $(V^{\perp})^{m_2}$, respectively, inside $M_{n,m_2}(\mathbb{R}) = (\mathbb{R}^n)^{m_2}$, we have for any Borel subset $E \subset S(\beta)_{gx}$:

$$\eta_{\beta,x}(g^{\mathsf{T}} E) = \delta(g^{\mathsf{T}}, (gV)^{\perp})^{m_2} \eta_{\beta,gx}(E) = \delta(g, V)^{-m_2} \eta_{\beta,gx}(E),$$

where the last equality holds by Lemma 3.5. Hence

$$\int_{S(\beta)_x} \chi_E\big(gx, (g^{\mathsf{T}})^{-1} y\big)\, d\eta_{\beta,x}(y) = \delta(g, V)^{-m_2} \int_{S(\beta)_{gx}} \chi_E\big(gx, z\big)\, d\eta_{\beta,gx}(z). \tag{3.12}$$

Using this formula in (3.12), together with the fact that $\delta(g, V)\, \mathsf{d}(x) = \mathsf{d}(gx)$, we conclude:

$$\eta_\beta(g^{-1}E) = \int_{U_{m_1}} \int_{S(\beta)_{gx}} \chi_E\big(gx, z\big)\, d\eta_{\beta,gx}(z)\, \frac{dx}{\mathsf{d}(gx)^{m_2}} = \eta_\beta(E),$$

where the last equality follows by substiting $x_{\mathrm{new}} := gx$ and comparing with (1.7). □

Next our goal is to show that *every* G-invariant regular Borel measure on $S(\beta)$ is a constant multiple of η_β. As we explain below, this is a standard consequence of the following lemma.

Lemma 3.7. *For any $\beta \in M_{m_1,m_2}(\mathbb{R})$, the action of G on $S(\beta)$ is transitive.*

Proof. We will use block matrix notation. Fix an arbitrary matrix w_1 in $M_{n-m_1,m_2}(\mathbb{R})$ of full rank, that is, whose row space equals $\mathbb{R}^{m_2}$, and set $z = \begin{pmatrix} I_{m_1} \\ 0 \end{pmatrix} \in M_{n,m_1}(\mathbb{R})$ and $w = \begin{pmatrix} \beta \\ w_1 \end{pmatrix} \in M_{n,m_2}(\mathbb{R})$. Then $\langle z, w \rangle \in S(\beta)$. It now suffices to prove that given an arbitrary element $\langle x, y \rangle \in S(\beta)$, there exists $g \in G$ such that $g\langle x, y \rangle = \langle z, w \rangle$. But the row space of x equals $\mathbb{R}^{m_1}$ since $x \in U_{m_1}$; hence there exists $g \in G$ such that $gx = z$. Hence we may just as well assume $x = z$ from the start. It then follows from $\langle x, y \rangle \in S(\beta)$ that y has the block decomposition $y = \begin{pmatrix} \beta \\ y_2 \end{pmatrix}$ for some $y_2 \in M_{n-m_1,m_2}(\mathbb{R})$. Let $V \subset \mathbb{R}^{m_2}$ be the row space of β; then $y \in U_{m_2}$ means that the row vectors of y_2 together with V span all $\mathbb{R}^{m_2}$. Using also $n - m_1 > m_2$, it follows that there is a way to add appropriate vectors from V to the rows of y_2 so as to obtain a matrix with row space $\mathbb{R}^{m_2}$; in other words, there exists some $a \in M_{n-m_1,m_1}(\mathbb{R})$ such that the row space of $y_2 + a\beta$ equals $\mathbb{R}^{m_2}$. Then there is some $b \in \mathrm{SL}_{n-m_1}(\mathbb{R})$ such that

$y_2 + a\beta = bw_1$. Setting now $g = \begin{pmatrix} I_{m_1} & -a^{\mathsf{T}} \\ 0 & b^{\mathsf{T}} \end{pmatrix} \in G$, we have $gz = z$ and $g^{\mathsf{T}} w = y$ and hence $g\langle x, y\rangle = \langle z, w\rangle$. □

It follows from Lemma 3.7 that for any $p \in S(\beta)$, if $H = H_{\beta,p} = \{g \in G \ : \ gp = p\}$ is the isotropy group at p, then the map $gH \mapsto gp$ is a diffeomorphism from G/H onto $S(\beta)$ [34, Theorem 3.62]. Thus F_β can be viewed as a G-invariant positive linear functional on $C_c(G/H)$, or equivalently a G-invariant regular Borel measure on G/H. Now by a basic result on invariant measures on homogeneous spaces (see e.g. [18, Lemma 1.4]), it follows that F_β is the *unique* G-invariant regular Borel measure on $S(\beta)$, up to a scalar multiple. However, by Lemma 3.6, the same is true for η_β; hence we conclude that F_β is a constant multiple of η_β, for every $\beta \in M_{m_1,m_2}(\mathbb{Z})$.

The next proposition gives the explicit formula for the constant of proportionality; in fact it turns out to be indendent of β:

Proposition 3.8. *For any $\beta \in M_{m_1,m_2}(\mathbb{Z})$ and any $\rho \in C_c(S(\beta))$,*

$$F_\beta(\rho) = \frac{1}{\zeta(n)\zeta(n-1)\cdots\zeta(n-m_1+1)}\,\eta_\beta(\rho).$$

We will prove Proposition 3.8 in Section 3.3. Here we show how the proof of Theorem 1.1 is completed using Proposition 3.8. Let us first note a general transformation formula for the measure η_β.

Lemma 3.9. *For any $T_1 \in \mathrm{GL}_{m_1}(\mathbb{R})$, $T_2 \in \mathrm{GL}_{m_2}(\mathbb{R})$ and $\beta \in M_{m_1,m_2}(\mathbb{R})$, the map $\langle x, y\rangle \mapsto \langle xT_1, yT_2\rangle$ is a diffeomorphism of $S(\beta)$ onto $S\big(T_1^{\mathsf{T}}\beta T_2\big)$, and for any $f \in C_c\big(S(T_1^{\mathsf{T}}\beta T_2)\big)$:*

$$\int_{S(\beta)} f\big(xT_1, yT_2\big)\, d\eta_\beta(x,y) = |\det T_1|^{m_2-n}|\det T_2|^{m_1-n}\int_{S(T_1^{\mathsf{T}}\beta T_2)} f\, d\eta_{T_1^{\mathsf{T}}\beta T_2}. \tag{3.13}$$

Proof. Immediate from the definition of η_β, (1.7). □

It follows from Proposition 3.8 and the formula in (3.10), that for any $\beta \in M_{m_1,m_2}(\mathbb{Z})$ and $\rho \in C_c(S(\beta))$, the expression in (3.8) equals

$$\frac{1}{\zeta(n)\zeta(n-1)\cdots\zeta(n-m_1+1)}\sum_{A\in\mathfrak{W}_\beta}\int_{S(A^{-\mathsf{T}}\beta)} \rho(xA, y)\, d\eta_{A^{-\mathsf{T}}\beta}(x,y). \tag{3.14}$$

Here by Lemma 3.9, the integral over $S(A^{-\mathsf{T}}\beta)$ equals $(\det A)^{m_2-n}\eta_\beta(\rho)$. Recalling also (1.9), it follows that (3.14), and hence also (3.8), equals $W(\beta)\,\eta_\beta(\rho)$. We have also noted that the expression in (3.7) equals (3.8) added over all $\beta \in M_{m_1,m_2}(\mathbb{Z})$. Hence we conclude:

$$\int_{X_n} \sum_{x\in L^{m_1}\cap U_{m_1}} \sum_{y\in (L^*)^{m_2}\cap U_{m_2}} \rho(x,y)\, d\mu(L) = \sum_{\beta\in M_{m_1,m_2}(\mathbb{Z})} W(\beta)\,\eta_\beta(\rho). \tag{3.15}$$

Finally, using the formula (3.15) to evaluate the integral appearing in the first line of (3.6), we finally obtain that the expression in (3.6) equals the right-hand side of (1.10); in other words the equality in (1.10) holds. We have proved this for any $\rho \in C_c(\mathcal{M})$, and it is clear from the proof that all the iterated sums and integrals in (1.10) are absolutely convergent for any such ρ. Hence the equality in (1.10) can be viewed as an equality between two regular Borel measures on $\mathcal{M}$, integrated against the function ρ; and it is then clear that the equality extends to the class of test functions ρ appearing in the statement of Theorem 1.1. □

3.3. **Proof of the explicit formula for F_β.** In this section we prove Proposition 3.8. As we have shown, this will also complete the proof of Theorem 1.1.

We start by giving three lemmas. For L any lattice in $\mathbb{R}^n$ (of full rank, but not necessarily of covolume one) we write $\lambda_i = \lambda_i(L)$ $(i = 1, \ldots, n)$ for its successive minima with respect to the unit ball, i.e.

$$\lambda_i(L) := \min\bigl\{\lambda \in \mathbb{R}_{\geq 0} \,:\, L \text{ contains } i \text{ linearly independent vectors of length} \leq \lambda\bigr\}.$$

Lemma 3.10. *For any lattice in L in $\mathbb{R}^n$ and any $R > 0$,*

$$\#(L \cap \mathcal{B}_R^n) \asymp_n \prod_{i=1}^{n}\Bigl(1 + \frac{R}{\lambda_i(L)}\Bigr).$$

Proof. See [12, Proposition 6].[3] □

Lemma 3.11. *For any lattice L in $\mathbb{R}^n$ and any $\boldsymbol{v} \in \mathbb{R}^n$ and $0 < R_1 \leq R_2$,*

$$\#(L \cap (\mathcal{B}_{R_2}^n + \boldsymbol{v})) \ll_n \Bigl(\frac{R_2}{R_1}\Bigr)^n \cdot \#(L \cap \mathcal{B}_{R_1}^n).$$

Proof. See [12, Propositions 4 and 5]. □

Let us write $\mathfrak{V}_m = \frac{\pi^{\frac{1}{2}m}}{\Gamma(\frac{1}{2}m+1)}$, the volume of the unit ball in $\mathbb{R}^m$.

Lemma 3.12. *For any fixed $\beta \in M_{m_1,m_2}(\mathbb{Z})$, $L \in X_n$ and $x \in P_{L,m_1}$, we have*

$$\lim_{R\to\infty} R^{-(n-m_1)m_2}\#\bigl((L^*)^{m_2} \cap S(\beta)_x \cap (\mathcal{B}_R^n)^{m_2}\bigr) = \frac{\mathfrak{V}_{n-m_1}^{m_2}}{\mathsf{d}(x)^{m_2}}. \tag{3.16}$$

Proof. Set $V = x\mathbb{R}^{m_1} \subset \mathbb{R}^n$. Since the columns of x form a primitive m_1-tuple in L, x can be extended to a matrix $g \in G$ with block decomposition $g = (x \; *)$ such that $L = g\mathbb{Z}^n$. Set $\beta' := \binom{\beta}{0} \in M_{n,m_2}(\mathbb{Z})$ and $y := g^{-\mathsf{T}}\beta'$; then $y \in (L^*)^{m_2} \cap S(\beta)'_x$. It follows that $S(\beta)'_x = y + (V^\perp)^{m_2}$, and so

$$\begin{aligned}\#\bigl((L^*)^{m_2} \cap S(\beta)'_x \cap (\mathcal{B}_R^n)^{m_2}\bigr) &= \#\bigl((y + (L^* \cap V^\perp)^{m_2}) \cap (\mathcal{B}_R^n)^{m_2}\bigr)\\ &= \prod_{j=1}^{m_2} \#\bigl((L^* \cap V^\perp) \cap (\mathcal{B}_R^n - \boldsymbol{y}_j)\bigr),\end{aligned} \tag{3.17}$$

where $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{m_2} \in L^*$ are the column vectors of y.

Next note that $V = gW$ where W is the linear span of the first m_1 standard unit vectors in $\mathbb{R}^n$; hence $L^* \cap V^\perp = g^{-\mathsf{T}}(\mathbb{Z}^n \cap W^\perp)$, and since the lattice $\mathbb{Z}^n \cap W^\perp$ has covolume one in $W^\perp$, it follows that the lattice $L^* \cap V^\perp$ has covolume $\delta(g^{-\mathsf{T}}, W^\perp) = \delta(g, W)$ in $V^\perp$ (the last equality holds by Lemma 3.5). Since $g = (x \; *)$, this covolume equals $\mathsf{d}(x)$. Note also that for each j and for R large, the ball $\mathcal{B}_R^n - \boldsymbol{y}_j$ intersects $V^\perp$ in a $(n - m_1)$-dimensional ball, whose radius is $R\,(1 - o(1))$ as $R \to \infty$. Hence we conclude that $\#\bigl((L^* \cap V^\perp) \cap (\mathcal{B}_R^n - \boldsymbol{y}_j)\bigr) \sim \mathfrak{V}_{n-m_1} R^{n-m_1} \mathsf{d}(x)^{-1}$ as $R \to \infty$, and so

$$\lim_{R\to\infty} R^{-(n-m_1)m_2}\#\bigl((L^*)^{m_2} \cap S(\beta)'_x \cap (\mathcal{B}_R^n)^{m_2}\bigr) = \frac{\mathfrak{V}_{n-m_1}^{m_2}}{\mathsf{d}(x)^{m_2}}. \tag{3.18}$$

Hence the lemma will be proved if we can only prove that the limits in (3.16) and (3.18) are equal. Note that $(L^*)^{m_2} \cap S(\beta)'_x \setminus S(\beta)_x = (y + (L^* \cap V^\perp)^{m_2}) \setminus U_{m_2}$; hence for any given R, the difference between the cardinalities appearing in (3.16) and in (3.18) is bounded above by

[3] Note that there is an obvious misprint in the statement of this proposition; "$\lambda_1 \cdots \lambda_k / M(K)$" should read "$\lambda_1 \cdots \lambda_k M(K)$"; similarly, in the last line of the proof, "$\sim V(K_0)$" should be corrected to "$\sim V(K_0)^{-1}$".

$\sum_{\ell=1}^{m_2} E_\ell(R)$ where $E_\ell(R)$ is the number of matrices $z \in (y + (L^* \cap V^\perp)^{m_2}) \cap (\mathcal{B}_R^n)^{m_2}$ such that the ℓth column of z lies in the ($\mathbb{R}$-)linear span of the other columns. We will in fact prove that $E_\ell(R) = O(R^{(n-m_1)m_2-2})$ for each ℓ; this clearly implies the desired equality of the limits.

Given ℓ, the columns of index $\neq \ell$ of a matrix $z \in (y + (L^* \cap V^\perp)^{m_2}) \cap (\mathcal{B}_R^n)^{m_2}$ can be fixed in $O(R^{(n-m_1)(m_2-1)})$ ways, since the ith column must lie in $(\boldsymbol{y}_i + (L^* \cap V^\perp)) \cap \mathcal{B}_R^n$. Hence it now suffices to prove that for every linear subspace U of $\mathbb{R}^n$ which is spanned by vectors in L^* and has dimension $\leq m_2 - 1$, and for every $R \geq 1$, there exist at most $O(R^{n-m_1-2})$ vectors in $U \cap (\boldsymbol{y}_\ell + (L^* \cap V^\perp)) \cap \mathcal{B}_R^n$, where the implied constant is independent of U and R. In fact we will prove the stronger fact that $\#(L^* \cap U \cap \mathcal{B}_R^n) \ll R^{n-m_1-2}$. Note that $L^* \cap U$ is a full rank lattice in the Euclidean space U, and its successive minima are bounded from below by the successive minima of L^*, i.e. $\lambda_i(L^* \cap U) \geq \lambda_i(L^*)$ for each $i \leq \dim U$. Hence by Lemma 3.10, applied with U in place of $\mathbb{R}^n$, and using $R \geq 1$, we have $\#(L^* \cap U \cap \mathcal{B}_R^n) \ll_L R^{\dim U}$. Also $R^{\dim U} \leq R^{m_2-1} \leq R^{n-m_1-2}$, since $m_1 + m_2 \leq n - 1$. The proof is complete. □

We now give the proof of Proposition 3.8. Let $\beta \in M_{m_1,m_2}(\mathbb{Z})$ be given. As we have noted, it follows from Lemmas 3.4, 3.7 and 3.6 that there exists a constant $c_\beta \geq 0$ such that

$$F_\beta(\rho) = c_\beta \cdot \eta_\beta(\rho) \tag{3.19}$$

for all $\rho \in C_c(S(\beta))$. Also, as we have noted, the above equality can be viewed as an equality between two regular Borel measures on $S(\beta)$; hence the equality in (3.19) holds more generally for any bounded Borel measurable function ρ on $S(\beta)$ of compact support. In particular, given any bounded Borel subset $E \subset U_{m_1}$ and $R > 0$, the formula (3.19) holds for the following function:

$$\rho_R(x,y) := I\big(x \in E\big) \cdot \mathsf{d}(x)^{m_2} \cdot I\big(y \in (\mathcal{B}_R^n)^{m_2}\big). \tag{3.20}$$

We also have

$$\begin{aligned} \eta_\beta(\rho_R) &= \int_{U_{m_1}} \int_{S(\beta)_x} \rho_R(x,y)\, d\eta_{\beta,x}(y)\, \frac{dx}{\mathsf{d}(x)^{m_2}} \\ &= R^{m_2(n-m_1)} \int_{U_{m_1}} \int_{S(R^{-1}\beta)_x} \rho_1(x,y)\, d\eta_{R^{-1}\beta,x}(y)\, \frac{dx}{\mathsf{d}(x)^{m_2}} = R^{m_2(n-m_1)}\, \eta_{R^{-1}\beta}(\rho_1). \end{aligned}$$

(We substituted $y = Ry_{\text{new}}$.) Hence, recalling also (3.9), we have:

$$c_\beta \cdot \eta_{R^{-1}\beta}(\rho_1) = R^{-m_2(n-m_1)} \int_{X_n} \sum_{x \in P_{L,m_1}} \sum_{y \in (L^*)^{m_2} \cap S(\beta)_x} \rho_R(x,y)\, d\mu(L) \tag{3.21}$$

For E fixed, we will let $R \to \infty$ in the above equality. The following lemma will be used to see that we may then change the order of limit and integration. For any $x \in M_{n,m_1}(\mathbb{R})$, let us write $x^\perp$ for the orthogonal complement in $\mathbb{R}^n$ of $x\mathbb{R}^{m_1}$ (viz., the column span of x).

Lemma 3.13. *Define the function* $\mathfrak{M} : X_n \to \mathbb{R}_{\geq 0}$ *by*

$$\mathfrak{M}(L) = \sum_{x \in P_{L,m_1}} \sum_{y \in (L^* \cap x^\perp)^{m_2}} \rho_1(x,y).$$

Then $\int_{X_n} \mathfrak{M}\, d\mu < \infty$, *and for all* $L \in X_n$ *and* $R \geq 1$ *we have*

$$R^{-m_2(n-m_1)} \sum_{x \in P_{L,m_1}} \sum_{y \in (L^*)^{m_2} \cap S(\beta)_x} \rho_R(x,y) \ll_n \mathfrak{M}(L). \tag{3.22}$$

Note that the function $\mathfrak{M}$ depends on our choice of the fixed set E. Note also that the implied constant in (3.22) depends *only* on n; in particular it is independent of L and R.

Proof. Since E is bounded, there is a constant $B > 0$ such that $\mathsf{d}(x)^{m_2} \le B$ for all $x \in E$, and hence $\rho_1(x, y) \le B$ on all $S(\beta)$. Using this, together with the bound in (3.4), applied with $k_1 = m_1$, $k_2 = m_2$ and K as $\overline{E} \times \big(\overline{\mathcal{B}_1^n}\big)^{m_2}$, it follows that $\int_{X_n} \mathfrak{M}\, d\mu$ is finite.

In order to prove (3.22) it suffices to prove

$$R^{-m_2(n-m_1)} \#\big((L^*)^{m_2} \cap S(\beta)_x \cap (\mathcal{B}_R^n)^{m_2}\big) \ll_n \#\big(L^* \cap x^\perp \cap \mathcal{B}_1^n\big)^{m_2} \tag{3.23}$$

for all $L \in X_n$, $x \in P_{L,m_1}$ and $R \ge 1$. Indeed, (3.22) follows by multiplying the inequality in (3.23) by $\mathsf{d}(x)^{m_2}$ and then adding over all $x \in P_{L,m_1} \cap E$.

In order to prove (3.23), recall that $S(\beta)_x \subset S(\beta)'_x$, and as in the proof of Lemma 3.12 there exist vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{m_2} \in \mathbb{R}^n$ (depending on x, β) such that (3.17) holds. Hence it suffices to prove that

$$R^{-(n-m_1)} \#\big(L^* \cap x^\perp \cap (\mathcal{B}_R^n - \boldsymbol{y})\big) \ll_n L^* \cap x^\perp \cap \mathcal{B}_1^n \tag{3.24}$$

for all $L \in X_n$, $x \in P_{L,m_1}$, $R \ge 1$ and $\boldsymbol{y} \in \mathbb{R}^n$.

But $L^* \cap x^\perp$ is a (full rank) lattice in the Euclidean space $x^\perp$, which has dimension $n - m_1$, and for every $\boldsymbol{y} \in \mathbb{R}^n$, $\mathcal{B}_R^n - \boldsymbol{y}$ intersects $x^\perp$ in a ball of radius $\le R$ (or the empty set). Hence (3.24) follows from Lemma 3.11, applied with $x^\perp$ in the place of $\mathbb{R}^n$. □

Now we let $R \to \infty$ in (3.21). In this limit we have

$$\eta_{R^{-1}\beta}(\rho_1) = \int_E \eta_{R^{-1}\beta,x}\big(S(R^{-1}\beta)_x \cap (\mathcal{B}_1^n)^{m_2}\big)\, dx \to \mathfrak{V}_{n-m_1}^{m_2} \operatorname{vol}(E),$$

by the Lebesgue dominated convergence theorem, since for each fixed $x \in U_{m_1}$, the measure $\eta_{R^{-1}\beta,x}\big(S(R^{-1}\beta)_x \cap (\mathcal{B}_1^n)^{m_2}\big)$ tends to $\mathfrak{V}_{n-m_1}^{m_2}$ from below as $R \to \infty$. On the other hand, for each fixed $L \in X_n$ we have, by Lemma 3.12 (using (3.20) and the fact that $P_{L,m_1} \cap E$ is finite, since E is bounded):

$$\lim_{R\to\infty} R^{-m_2(n-m_1)} \sum_{x \in P_{L,m_1}} \sum_{\substack{y \in (L^*)^{m_2} \cap U_{m_2} \\ (x^{\mathsf{T}} y = \beta)}} \rho_R(x, y) = \mathfrak{V}_{n-m_1}^{m_2} \#\big(P_{L,m_1} \cap E\big).$$

Hence, by changing order of the limit and the integration over X_n in (3.21), which is justified by the Lebesgue dominated convergence theorem and Lemma 3.13, we obtain:

$$c_\beta \cdot \mathfrak{V}_{n-m_1}^{m_2} \operatorname{vol}(E) = \mathfrak{V}_{n-m_1}^{m_2} \int_{X_n} \#\big(P_{L,m_1} \cap E\big)\, d\mu(L) = \frac{\mathfrak{V}_{n-m_1}^{m_2} \operatorname{vol}(E)}{\zeta(n) \cdots \zeta(n - m_1 + 1)}, \tag{3.25}$$

where the last equality holds by Theorem 2.5. Hence we obtain $c_\beta = \frac{1}{\zeta(n)\cdots\zeta(n-m_1+1)}$, and thus, via (3.19), Proposition 3.8 is proved. Hence also Theorem 1.1 is proved. □ □ □

3.4. **Symmetries under transposition of β.** We here prove the symmetry relations for $W(\beta)$ and η_β mentioned in Remark 1.4; see Lemma 3.15 and Lemma 3.17.

We first prove an auxiliary lemma:

Lemma 3.14. *For any fixed $m_1, m_2 \in \mathbb{Z}^+$ and $\beta \in M_{m_1,m_2}(\mathbb{Z})$,*

$$\eta_\beta\Big(S(\beta) \cap \big((\mathcal{B}_R^n)^{m_1} \times (\mathcal{B}_R^n)^{m_2}\big)\Big) \sim C(n; m_1, m_2) \cdot R^{n(m_1+m_2) - 2m_1 m_2} \qquad \text{as } R \to \infty, \tag{3.26}$$

where

$$C(n; m_1, m_2) = \frac{\prod_{j=n-m_1+1}^{n} (j\mathfrak{V}_j)}{\prod_{j=n-m_1-m_2+1}^{n-m_2} (j\mathfrak{V}_j)} \mathfrak{V}_{n-m_1}^{m_2} \mathfrak{V}_{n-m_2}^{m_1}. \tag{3.27}$$

(Here $\mathfrak{V}_d^m$ stands for $(\mathfrak{V}_d)^m$, i.e. $\mathfrak{V}_d$ raised to m.)

Proof. Using the definition (1.7) and substituting $y = R\,y_{\text{new}}$ in the inner integral, it follows that the left-hand side of (3.26) equals

$$R^{m_2(n-m_1)} \int_{(\mathcal{B}_R^n)^{m_1}} \eta_{R^{-1}\beta,x}\Big(S(R^{-1}\beta)_x \cap (\mathcal{B}_1^n)^{m_1}\Big)\,\frac{dx}{\mathsf{d}(x)^{m_2}}.$$

Substituting here $x = R\,x_{\text{new}}$, and using the fact that $S(R^{-1}\beta)_{Rx} = S(R^{-2}\beta)_x$ with $\eta_{R^{-1}\beta,Rx} = \eta_{R^{-2}\beta,x}$, the above expression becomes

$$R^{n(m_1+m_2)-2m_1m_2} \int_{(\mathcal{B}_1^n)^{m_1}} \eta_{R^{-2}\beta,x}\Big(S(R^{-2}\beta)_x \cap (\mathcal{B}_1^n)^{m_1}\Big)\,\frac{dx}{\mathsf{d}(x)^{m_2}}.$$

Here for each fixed $x \in U_{m_1}$, the integrand tends to $\eta_{0,x}(S(0)_x \cap (\mathcal{B}_1^n)^{m_1}) \cdot \mathsf{d}(x)^{-m_2} = \mathfrak{V}_{n-m_2}^{m_1}\mathsf{d}(x)^{-m_2}$ from below as $R \to \infty$. Hence by the Monotone Convergence Theorem, we obtain (3.26) with $C(n;m_1,m_2) = \mathfrak{V}_{n-m_2}^{m_1}\int_{(\mathcal{B}_1^n)^{m_1}} \frac{dx}{\mathsf{d}(x)^{m_2}}$. Finally, by applying [29, Lemma 5.2] two times, we obtain (3.27). □

Lemma 3.15. *Let $m_1, m_2 \in \mathbb{Z}^+$ and $\beta \in M_{m_1,m_2}(\mathbb{Z})$, and let J be the diffeomorphism $\langle x, y\rangle \mapsto \langle y, x\rangle$ from $S(\beta)$ onto $S(\beta^{\mathsf{T}})$. Then $\eta_{\beta^{\mathsf{T}}} = J_*\eta_\beta$.*

Proof. Using the fact that the measure η_β on $S(\beta)$ is G-invariant, by Lemma 3.6, together with the relation $J(gp) = g^{-\mathsf{T}}J(p)$ ($\forall g \in G$, $p \in S(\beta)$), it follows that $J_*\eta_\beta$ is a G-invariant measure on $S(\beta^{\mathsf{T}})$. Lemma 3.6 also implies that $\eta_{\beta^{\mathsf{T}}}$ is a G-invariant measure on $S(\beta^{\mathsf{T}})$. Hence as in the discussion below Lemma 3.7, the two measures $J_*\eta_\beta$ and $\eta_{\beta^{\mathsf{T}}}$ are constant multiples of each other, and to complete the proof of the lemma, it now suffices to prove that $J_*\eta_\beta(E_R) \sim \eta_{\beta^{\mathsf{T}}}(E_R)$ as $R \to \infty$, when $E_R := S(\beta^{\mathsf{T}}) \cap \big((\mathcal{B}_R^n)^{m_2} \times (\mathcal{B}_R^n)^{m_1}\big)$. However, by Lemma 3.14 we have

$$J_*\eta_\beta(E_R) = \eta_\beta(J^{-1}E_R) \sim C(n;m_1,m_2)\cdot R^{n(m_1+m_2)-2m_1m_2}$$

and also

$$\eta_{\beta^{\mathsf{T}}}(E_R) \sim C(n;m_2,m_1)\cdot R^{n(m_1+m_2)-2m_1m_2}$$

as $R \to \infty$. Hence the lemma follows from the fact that $C(n;m_1,m_2) = C(n;m_2,m_1)$. □

Next we discuss the weight function $W(\beta)$. Let us first note a basic invariance relation.

Lemma 3.16. *Let $m_1, m_2 \in \mathbb{Z}^+$ and $\beta \in M_{m_1,m_2}(\mathbb{Z})$. Then*

$$W(\gamma_1\beta\gamma_2) = W(\beta) \qquad \forall \gamma_1 \in \mathrm{GL}_{m_1}(\mathbb{Z}),\ \gamma_2 \in \mathrm{GL}_{m_2}(\mathbb{Z}). \tag{3.28}$$

Proof. Let U be the set of non-singular matrices in $M_{m_1,m_1}(\mathbb{Z})$, let $\Delta = \mathrm{GL}_{m_1}(\mathbb{Z}) \subset U$, and let $\Delta\backslash U = \{\Delta A \,:\, A \in U\}$ be the set of all right Δ-cosets contained in U. Note that the set $\mathfrak{W}_{m_1}$ contains exactly one element in each coset $x \in \Delta\backslash U$. Furthermore, for each $x \in \Delta\backslash U$, the relation $A^{-\mathsf{T}}\beta \in M_{m_1,m_2}(\mathbb{Z})$ either holds for *all* or *no* element $A \in x$, and we denote by $(\Delta\backslash U)_\beta$ the subset $\{x \in \Delta\backslash U \,:\, A^{-\mathsf{T}}\beta \in M_{m_1,m_2}(\mathbb{Z})\ \forall A \in x\}$. We also write $|\det x|$ for the common value of the absolute determinant $|\det A|$ for all $A \in x$. With this notation, the formula in (1.9) may be expressed as

$$W(\beta) = \frac{\sum_{x\in(\Delta\backslash U)_\beta} |\det x|^{m_2-n}}{\prod_{j=n-m_1+1}^{n}\zeta(j)}.$$

Now Δ acts on $\Delta\backslash U$ by right multiplication; $(\Delta A)\gamma := \Delta(A\gamma)$ for any $A \in U$ and $\gamma \in \Delta$. One verifies that $(\Delta\backslash U)_{\gamma_1\beta\gamma_2} = \{x\gamma_1^{\mathsf{T}} \,:\, x \in (\Delta\backslash U)_\beta\}$ for any $\gamma_1 \in \Delta$ and $\gamma_2 \in \mathrm{GL}_{m_2}(\mathbb{Z})$. Thus

$$\sum_{x\in(\Delta\backslash U)_{\gamma_1\beta\gamma_2}} |\det x|^{m_2-n} = \sum_{x\in(\Delta\backslash U)_\beta} |\det(x\gamma_1^{\mathsf{T}})|^{m_2-n} = \sum_{x\in(\Delta\backslash U)_\beta} |\det x|^{m_2-n}.$$

Hence $W(\gamma_1\beta\gamma_2) = W(\beta)$. □

Lemma 3.17. *For any $m_1, m_2 \in \mathbb{Z}^+$ and $\beta \in M_{m_1,m_2}(\mathbb{Z})$, $W(\beta^{\mathsf{T}}) = W(\beta)$.*

Proof. Replacing β by β^{T} if necessary, we may assume that $m_1 \le m_2$. By the Smith Normal Form Theorem, there exist $\gamma_1 \in \mathrm{GL}_{m_1}(\mathbb{Z})$ and $\gamma_2 \in \mathrm{GL}_{m_2}(\mathbb{Z})$ such that, in block matrix notation, $\gamma_1\beta\gamma_2 = \big(D\ 0\big)$ for some diagonal matrix $D \in M_{m_1,m_1}(\mathbb{Z})$. If $m_1 = m_2$ then $\gamma_1\beta\gamma_2 = D = D^{\mathsf{T}} = \gamma_2^{\mathsf{T}}\beta^{\mathsf{T}}\gamma_1^{\mathsf{T}}$, and so $W(\beta^{\mathsf{T}}) = W(\beta)$ by Lemma 3.16.

Hence from now on we assume $m_1 < m_2$. Transposing the relation $\gamma_1\beta\gamma_2 = \big(D\ 0\big)$ and multiplying from the left by an appropriate permutation matrix, it follows that there exists $\gamma_3 \in \mathrm{GL}_{m_2}(\mathbb{Z})$ such that $\gamma_3\beta^{\mathsf{T}}\gamma_1^{\mathsf{T}} = \begin{pmatrix} 0 \\ D \end{pmatrix} =: D'$ in $M_{m_2,m_1}(\mathbb{Z})$, so that $W(\beta^{\mathsf{T}}) = W(D')$ by Lemma 3.16. Now for any $A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \in \mathfrak{W}_{m_2}$ (with blocks A_1, A_{12}, A_2 of sizes $(m_2-m_1)\times(m_2-m_1)$, $(m_2-m_1)\times m_1$ and $m_1 \times m_1$, respectively), one verifies that $A \in \mathfrak{W}_{D'}$ if and only if $A_2 \in \mathfrak{W}_D$; furthermore, for any given $A_2 \in \mathfrak{W}_D$ there are exactly $(\det A_2)^{m_2-m_1}$ possibilities for A_{12} which, together with A_1 freely chosen in $\mathfrak{W}_{m_2-m_1}$, makes $A \in \mathfrak{W}_{D'}$. Hence

$$\sum_{A\in\mathfrak{W}_{D'}} (\det A)^{m_1-n} = \sum_{A_2\in\mathfrak{W}_D} (\det A_2)^{m_2-m_1} \sum_{A_1\in\mathfrak{W}_{m_2-m_1}} \big(\det A_1\ \det A_2\big)^{m_1-n}$$
$$= \sum_{A_2\in\mathfrak{W}_D} (\det A_2)^{m_2-n} \prod_{j=n-m_2+1}^{n-m_1} \zeta(j).$$

Using this, and the fact that $\mathfrak{W}_{(D\,0)} = \mathfrak{W}_D$, we get

$$W(\beta^{\mathsf{T}}) = W(D') = \frac{\sum_{A\in\mathfrak{W}_{D'}} (\det A)^{m_1-n}}{\prod_{j=n-m_2+1}^{n} \zeta(j)} = \frac{\sum_{A\in\mathfrak{W}_{D}} (\det A)^{m_2-n}}{\prod_{j=n-m_1+1}^{n} \zeta(j)} = W\big((D\ \ 0)\big) = W(\beta).$$

□

4. Outline of an alternative proof using Poisson summation

In this section we give an outline of an alternative proof of our main result, Theorem 1.1.

The main idea of the proof is to apply the Poisson summation formula in both sides of (1.10). The manipulations which we will describe can be justified for any function $\rho : (\mathbb{R}^n)^{k_1} \times (\mathbb{R}^n)^{k_2} \to \mathbb{C}$ such that ρ and all its derivatives up to a sufficiently high order are of rapid polynomial decay (cf. Remark 2.2). Once the equality in (1.10) has been proved for this family of test functions, it follows that we have a corresponding equality of regular Borel measures on $(\mathbb{R}^n)^{k_1} \times (\mathbb{R}^n)^{k_2}$ (cf. Lemma 3.1), and in particular (1.10) also holds for the class of functions appearing in the statement of Theorem 1.1.

In the left-hand side of (1.10), we apply Poisson summation to the sums over L^*, to obtain

$$\int_{X_n} \sum_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k\in L} \widetilde{\rho}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k)\, d\mu(L), \tag{4.1}$$

with

$$\widetilde{\rho}(x,v) := \int_{M_{n,k_2}(\mathbb{R})} \rho(x,y) e(-\operatorname{Tr}(v^{\mathsf{T}} y))\, dy. \tag{4.2}$$

The integral in (4.1) may be evaluated using Rogers' mean value formula, Theorem 1.5. The main work of the proof now consists in applying Poisson summation to the sum over β in the right-hand side of (1.10), and seeking to match the resulting expressions.

Let us fix an arbitrary choice of m_1, B_1, m_2, B_2 appearing in (1.10). In the following we will assume $m_1 > 0$ and $0 < m_2 < k_2$, but with appropriate interpretations, the argument which we present can be seen to also cover the remaining cases (cf. also Remark 1.2). Using the definitions of η_β and $W(\beta)$ (cf. (1.9) and (1.7)), and substituting $\beta = A^{\mathsf{T}}\gamma$ (with $A \in \mathfrak{W}$ and $\gamma \in M_{m_1,m_2}(\mathbb{Z})$), we find that the sum over β in (1.10) can be rewritten as

$$\sum_{A\in\mathfrak{W}} \sum_{\gamma\in M_{m_1,m_2}(\mathbb{Z})} \frac{(\det A)^{m_2-n}}{\zeta(n)\zeta(n-1)\cdots\zeta(n-m_1+1)} \int_{U_{m_1}} F_x(A^{\mathsf{T}}\gamma)\, \frac{dx}{\mathsf{d}(x)^{m_2}}, \tag{4.3}$$

where, for any $x \in U_{m_1}$, the function F_x is given by

$$F_x : M_{m_1,m_2}(\mathbb{R}) \to \mathbb{C}; \qquad F_x(\beta) = \int_{S(\beta)_x} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\, d\eta_{\beta,x}(y). \tag{4.4}$$

In (4.3) we move the sum over γ inside the integral, and apply the Poisson summation formula to the resulting inner sum, $\sum_\gamma F_x(A^{\mathsf{T}}\gamma)$. For this we need to compute the Fourier transform of F_x,

$$\widehat{F}_x(\xi) := \int_{M_{m_1,m_2}(\mathbb{R})} F_x(\beta) e(-\operatorname{Tr}(\xi^{\mathsf{T}}\beta))\, d\beta \qquad (\xi \in M_{m_1,m_2}(\mathbb{R})).$$

Noticing that $S(\beta)'_x = x(x^{\mathsf{T}}x)^{-1}\beta + (x^{\perp})^{m_2}$ we may rewrite the definition of F_x in (4.4) as an integral over $y' \in x^{\perp}$, with the integrand being $\rho(xB_1^{\mathsf{T}}, (x(x^{\mathsf{T}}x)^{-1}\beta + y')B_2^{\mathsf{T}})$. One verifies that the map taking $\langle\beta, y'\rangle$ to $(x(x^{\mathsf{T}}x)^{-1}\beta + y')B_2{}^{\mathsf{T}}$ is a linear bijection from $M_{m_1,m_2}(\mathbb{R}) \oplus (x^{\perp})^{m_2}$ onto the subspace $(V_{B_2}^n)^{\mathsf{T}}$ of $M_{n,k_2}(\mathbb{R})$, and we thus obtain

$$\widehat{F}_x(\xi) := \mathsf{d}(x)^{m_2}\mathsf{d}(B_2)^{-n} \int_{(V_{B_2}^n)^{\mathsf{T}}} \rho\big(xB_1^{\mathsf{T}}, w\big)\, e\big(-\operatorname{Tr}\big(\xi^{\mathsf{T}}x^{\mathsf{T}}wB_2(B_2^{\mathsf{T}}B_2)^{-1}\big)\big)\, dw.$$

Here note that $\operatorname{Tr}\big(\xi^{\mathsf{T}}x^{\mathsf{T}}wB_2(B_2^{\mathsf{T}}B_2)^{-1}\big) = \operatorname{Tr}\big(\big(x\xi(B_2^{\mathsf{T}}B_2)^{-1}B_2^{\mathsf{T}}\big)^{\mathsf{T}}w\big)$; hence the last integral is the Fourier transform of the function $\rho\big(xB_1^{\mathsf{T}}, \cdot\big)$ restricted to $(V_{B_2}^n)^{\mathsf{T}}$, evaluated at the point $x\xi(B_2^{\mathsf{T}}B_2)^{-1}B_2^{\mathsf{T}}$. In view of [9, Thm. 4.42 and the ensuing remark], this can be re-expressed as an integral over a translate of the orthogonal complement of $(V_{B_2}^n)^{\mathsf{T}}$ in $M_{n,k_2}(\mathbb{R})$, viz., $((V_{B_2}^{\perp})^n)^{\mathsf{T}}$, of the 'full' Fourier transform $\widetilde{\rho}$ (cf. (4.2)):

$$\widehat{F}_x(\xi) := \mathsf{d}(x)^{m_2}\mathsf{d}(B_2)^{-n} \int_{((V_{B_2}^{\perp})^n)^{\mathsf{T}}} \widetilde{\rho}\big(xB_1^{\mathsf{T}}, x\xi(B_2^{\mathsf{T}}B_2)^{-1}B_2^{\mathsf{T}} + u\big)\, du. \tag{4.5}$$

Hence, by the Poisson summation formula, we conclude that (4.3) equals:

$$\sum_{A\in\mathfrak{W}} \frac{(\det A)^{-n}}{\zeta(n)\zeta(n-1)\cdots\zeta(n-m_1+1)}\, \mathsf{d}(B_2)^{-n} \tag{4.6}$$
$$\times \int_{M_{n,m_1}(\mathbb{R})} \sum_{\xi\in M_{m_1,m_2}(\mathbb{Z})} \int_{((V_{B_2}^{\perp})^n)^{\mathsf{T}}} \widetilde{\rho}\Big(xB_1^{\mathsf{T}}, xA^{-1}\,\xi(B_2^{\mathsf{T}}B_2)^{-1}B_2^{\mathsf{T}} + u\Big)\, du\, dx.$$

Next, in (4.6), we substitute $\xi := A\xi'$, and then move the summation over ξ' to the leftmost position. Then ξ' runs through $M_{m_1,m_2}(\mathbb{Q})$, and for each given $\xi' \in M_{m_1,m_2}(\mathbb{Q})$, A runs through all $A \in \mathfrak{W}$ satisfying $A\xi' \in M_{m_1,m_2}(\mathbb{Z})$. For any $\xi \in M_{m_1,m_2}(\mathbb{Q})$ we write $q_\xi \in \mathbb{Z}^+$ for the least common denominator of the entries of ξ, that is, the smallest positive integer q such that $q\xi \in M_{m_1,m_2}(\mathbb{Z})$. Then the last relation, $A\xi' \in M_{m_1,m_2}(\mathbb{Z})$, is equivalent with $A\, q_{\xi'}\, \xi' \equiv 0 \bmod q_{\xi'}$. We will need the following lemma, applied with $\theta := q_{\xi'}\, \xi' \bmod q_{\xi'}$.

Lemma 4.1. *Let q be a positive integer and let $\theta \in M_{m_1,m_2}(\mathbb{Z}/q\mathbb{Z})$. Then*

$$\sum_{\substack{A \in \mathfrak{W} \\ A\theta \equiv 0 \bmod q}} \frac{(\det A)^{-n}}{\zeta(n)\zeta(n-1)\cdots\zeta(n-m_1+1)} = \Big(\frac{N}{q^{m_1}}\Big)^n, \tag{4.7}$$

where

$$N = \#\big\{\boldsymbol{a} \in (\mathbb{Z}/q\mathbb{Z})^{m_1} \,:\, \theta^{\mathsf{T}}\boldsymbol{a} = \boldsymbol{0} \text{ in } (\mathbb{Z}/q\mathbb{Z})^{m_2}\big\}. \tag{4.8}$$

Proof. Let R be the ring $\mathbb{Z}/q\mathbb{Z}$. The lemma follows from a study of the solution set

$$S := \big\{\boldsymbol{a} \in R^{m_1} \,:\, \boldsymbol{a}\theta = \boldsymbol{0} \text{ in } R^{m_2}\big\}.$$

(Note that the equation $\boldsymbol{a}\theta = \boldsymbol{0}$ is equivalent with the equation $\theta^{\mathsf{T}}\boldsymbol{a} = \boldsymbol{0}$ appearing in (4.8), up to switching between row and column vectors; $\boldsymbol{a} \leftrightarrow \boldsymbol{a}^{\mathsf{T}}$.) For $j \in \{1,\ldots,m_1\}$, set

$$I_j := \big\{a_j \in R \,:\, \exists a_{j+1},\ldots,a_{m_1} \in R \text{ such that } (0 \cdots 0\, a_j\, a_{j+1} \cdots a_{m_1}) \in S\big\}.$$

Then I_j is a subgroup of $\langle R,+\rangle$; hence there is some $d_j \mid q$ such that $I_j = d_jR$. For any $d \mid q$ let us write $A_d := \{0,1,\ldots,d-1\} \subset R$. It is now straightforward to verify that

$$\begin{cases} \text{for any } j \in \{1,\ldots,m_1\}, \text{ any } b_j \in d_jR \text{ and any } b_{j+1},\ldots,b_{m_1} \in R, \text{ there is a } \textit{unique} \\ \text{solution } \boldsymbol{a} \in S \text{ in the 'box' } \{0\}^{j-1} \times \{b_j\} \times (b_{j+1} + A_{d_{j+1}}) \times \cdots \times (b_{m_1} + A_{d_{m_1}}). \end{cases} \tag{4.9}$$

In particular, noticing that R^{m_1-1} can be partitioned into $\prod_{j=2}^{m_1}(q/d_j)$ boxes of the form $(b_2 + A_{d_2}) \times \cdots \times (b_{m_1} + A_{d_{m_1}})$, it follows from (4.9) applied with $j = 1$ that

$$N = \#S = \#(d_jR) \cdot \prod_{j=2}^{m_1}(q/d_j) = \frac{q^{m_1}}{\prod_{j=1}^{m_1} d_j}. \tag{4.10}$$

Next we consider the sum in (4.7). Note that the equation $A\theta \equiv 0 \bmod q$ means that for each j, row number j of A satisfies the equation appearing in the definition of I_j. It follows that for any matrix A appearing in the sum in (4.7), the diagonal entries of A must be $d_1c_1, d_2c_2, \ldots, d_{m_1}c_{m_1}$ for some positive integers $c_1,\ldots,c_{m_1}$, and having fixed this diagonal, it follows via (4.9) that for each j, the jth row of A can be choosen in $\prod_{i=j+1}^{m_1} c_j$ ways, and so the total number of matrices A with this diagonal is $\prod_{j=1}^{m_1}\prod_{i=j+1}^{m_1} c_j = \prod_{j=2}^{m_1} c_j^{j-1}$. Hence the left-hand side of (4.7) equals

$$\sum_{c_1=1}^{\infty} \cdots \sum_{c_{m_1}=1}^{\infty} \Big(\prod_{j=2}^{m_1} c_j^{j-1}\Big) \times \frac{\prod_{j=1}^{m_1}(c_jd_j)^{-n}}{\zeta(n)\zeta(n-1)\cdots\zeta(n-m_1+1)} = \Big(\prod_{j=1}^{m_1} d_j\Big)^{-n}.$$

Combining this with (4.10) we obtain (4.7). □

By Lemma 4.1 and the discussion preceding it, (4.6) equals:

$$\sum_{\xi \in M_{m_1,m_2}(\mathbb{Q})} \Big(\frac{N_\xi}{q_\xi^{m_1}}\Big)^n \mathsf{d}(B_2)^{-n} \int_{M_{n,m_1}(\mathbb{R})} \int_{((V_{B_2}^{\perp})^n)^{\mathsf{T}}} \widetilde{\rho}\Big(xB_1^{\mathsf{T}}, x\xi(B_2^{\mathsf{T}}B_2)^{-1}B_2^{\mathsf{T}} + u\Big)\, du\, dx, \tag{4.11}$$

where

$$N_\xi := \#\{\boldsymbol{a} \in (\mathbb{Z}/q_\xi\mathbb{Z})^{m_1} \,:\, q_\xi\xi^{\mathsf{T}}\boldsymbol{a} \equiv \boldsymbol{0} \bmod q_\xi\}. \tag{4.12}$$

Set $m_2' := k_2 - m_2$. Recall that we are assuming $0 < m_2 < k_2$; thus $0 < m_2' < k_2$. Using the fact that the map $V \mapsto V^{\perp}$ gives a bijection from the family of m_2-dimensional rational subspaces of $\mathbb{R}^{k_2}$ onto the family of m_2'-dimensional rational subspaces of $\mathbb{R}^{k_2}$, it follows that for every $B_2 \in A_{k_2,m_2}$ there exists a unique matrix $\widetilde{B}_2 \in A_{k_2,m_2'}$ satisfying $V_{B_2}^{\perp} = V_{\widetilde{B}_2}$, and the map $B_2 \mapsto \widetilde{B}_2$ is a bijection from A_{k_2,m_2} onto $A_{k_2,m_2'}$. With this notation, the map $y \mapsto y\widetilde{B}_2{}^{\mathsf{T}}$

is a linear bijection from $M_{n,m_2'}(\mathbb{R})$ onto $((V_{\widetilde{B}_2}^{\perp})^n)^{\mathsf{T}}$, which scales volume by a factor $\mathsf{d}(\widetilde{B}_2)^n$, and by [26, Sec. 2 (Corollary)] we have $\mathsf{d}(\widetilde{B}_2)=\mathsf{d}(B_2)$. Hence (4.11) equals

$$\sum_{\xi\in M_{m_1,m_2}(\mathbb{Q})}\left(\frac{N_\xi}{q_\xi^{m_1}}\right)^n\int_{M_{n,m_1}(\mathbb{R})}\int_{M_{n,m_2'}(\mathbb{R})}\widetilde{\rho}\Big(xB_1^{\mathsf{T}},x\xi(B_2^{\mathsf{T}}B_2)^{-1}B_2^{\mathsf{T}}+y\widetilde{B}_2^{\mathsf{T}}\Big)\,dy\,dx, \tag{4.13}$$

Set $k:=k_1+k_2$, and view $\widetilde{\rho}$ as a function on $M_{n,k}(\mathbb{R})$ by identifying any pair $\langle u,v\rangle$ in $M_{n,k_1}(\mathbb{R})\times M_{n,k_2}(\mathbb{R})$ with the block matrix $\big(u\,v\big)$ in $M_{n,k}(\mathbb{R})$. We also set

$$\alpha:=(B_2^{\mathsf{T}}B_2)^{-1}\xi^{\mathsf{T}}\in M_{m_2,m_1}(\mathbb{Q}). \tag{4.14}$$

Using this notation, (4.13) equals

$$\sum_{\xi\in M_{m_1,m_2}(\mathbb{Q})}\left(\frac{N_\xi}{q_\xi^{m_1}}\right)^n\int_{M_{n,m_1+m_2'}(\mathbb{R})}\widetilde{\rho}\left(z\begin{pmatrix}B_1&0\\B_2\alpha&\widetilde{B}_2\end{pmatrix}^{\mathsf{T}}\right)dz. \tag{4.15}$$

To sum up, we have prove that the sum over β in (1.10) equals (4.15). Next, as in (1.10), this expression should be added over all $B_1\in A_{k_1,m_1}$ and $B_2\in A_{k_2,m_2}$. The following lemma will allow us to rearrange in a convenient way the triple sum over B_1,B_2,ξ which then appears. Let us set $P_1:=\big(I_{k_1}\ 0\big)\in M_{k_1,k}(\mathbb{Z})$, so that $P_1(v_1,\ldots,v_k)^{\mathsf{T}}=(v_1,\ldots,v_{k_1})^{\mathsf{T}}$ for all $\boldsymbol{v}=(v_1,\ldots,v_k)^{\mathsf{T}}\in\mathbb{R}^k$.

Lemma 4.2. *Given any $B_1\in A_{k_1,m_1}$, $B_2\in A_{k_2,m_2}$ and $\alpha\in M_{m_2,m_1}(\mathbb{Q})$, there exist unique matrices $B_3\in A_{k,m_1+m_2'}$ and $J\in\mathrm{GL}_{m_1+m_2'}(\mathbb{R})$ satisfying*

$$\begin{pmatrix}B_1&0\\B_2\alpha&\widetilde{B}_2\end{pmatrix}=B_3J, \tag{4.16}$$

and the map $\langle B_1,B_2,\alpha\rangle\mapsto B_3$ defined by this relation is a bijection from $A_{k_1,m_1}\times A_{k_2,m_2}\times M_{m_2,m_1}(\mathbb{Q})$ onto the set of all $B_3\in A_{k,m_1+m_2'}$ satisfying $\dim P_1V_{B_3}=m_1$. *Furthermore, when the above relation holds, we also have*

$$|\det J|=N_\xi/q_\xi^{m_1}\qquad\textit{with}\ \ \xi=\alpha^{\mathsf{T}}B_2{}^{\mathsf{T}}B_2. \tag{4.17}$$

(Note that the formula for ξ in (4.17) is equivalent with (4.14).)

Proof (outline). If D is the matrix in the left-hand side of (4.16), then $D\mathbb{R}^{m_1+m_2'}$ is a rational linear subspace of $\mathbb{R}^k$ of dimension m_1+m_2', and thus there exists a unique matrix $B_3\in A_{k,m_1+m_2'}$ satisfying $D\mathbb{R}^{m_1+m_2'}=V_{B_3}$, and then also the unique existence of J follows. Next, when (4.16) holds, we also have $P_1B_3=P_1DJ^{-1}=\big(B_1\ \ 0\big)J^{-1}$ and hence $P_1V_{B_3}=V_{B_1}$; in particular $\dim P_1V_{B_3}=m_1$.

To complete the proof of the bijectivity statement, it now suffices to prove that if V is an arbitrary rational (m_1+m_2')-dimensional linear subspaces $V\subset\mathbb{R}^k$ satisfying $\dim P_1V=m_1$, then there exists unique triple $\langle B_1,B_2,\alpha\rangle$ in $A_{k_1,m_1}\times A_{k_2,m_2}\times M_{m_2,m_1}(\mathbb{Q})$ such that $D\mathbb{R}^{m_1+m_2'}=V$, where $D=D(B_1,B_2,\alpha)$ is the matrix in the left-hand side of (4.16). Given V, we set $K:=\ker P_1\big|_V$; then there exists a uniquely determined matrix $C\in M_{k_2,k_1}(\mathbb{Q})$ such that $W:=K^\perp\cap V=\left\{\begin{pmatrix}\boldsymbol{u}\\C\boldsymbol{u}\end{pmatrix}\ :\ \boldsymbol{u}\in P_1V\right\}$. On the other hand, for any $D=D(B_1,B_2,\alpha)$ as above we have

$$D\mathbb{R}^{m_1+m_2'}=\left\{\begin{pmatrix}B_1\boldsymbol{x}\\B_2\alpha\boldsymbol{x}\end{pmatrix}\ :\ \boldsymbol{x}\in\mathbb{R}^{m_1}\right\}\oplus\left\{\begin{pmatrix}\boldsymbol{0}\\\widetilde{B}_2\boldsymbol{y}\end{pmatrix}\ :\ \boldsymbol{y}\in\mathbb{R}^{m_2'}\right\}, \tag{4.18}$$

and this is in fact an orthogonal direct sum, and the second term is equal to $\ker P_1\big|_{D\mathbb{R}^{m_1+m_2'}}$. Hence we conclude that $D\mathbb{R}^{m_1+m_2'}=V$ holds if and only if the two terms in (4.18) are equal to W and K, respectively, and this, in turn, holds if and only if $V_{B_1} = P_1 V$, $V_{\widetilde{B}_2}$ equals the image of $K \subset \{\mathbf{0}\} \oplus \mathbb{R}^{k_2}$ in $\mathbb{R}^{k_2}$, and $B_2\alpha = CB_1$. The first two of these relations are satisfied for unique choices of $B_1 \in A_{k_1,m_1}$ and $B_2 \in A_{k_2,m_2}$, respectively, and then the third relation holds for a unique choice of $\alpha \in M_{m_2,m_1}(\mathbb{Q})$.

It remains to prove (4.17). When (4.16) holds, one verifies by a direct computation that

$$J^{-1}\mathbb{Z}^{m_1+m_2'} = \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \,:\, \boldsymbol{x} \in \mathbb{Z}^{m_1}, \boldsymbol{y} \in \mathbb{R}^{m_2'},\ B_2\alpha\boldsymbol{x} + \widetilde{B}_2\boldsymbol{y} \in \mathbb{Z}^{k_2} \right\}. \tag{4.19}$$

Let L be the image of $J^{-1}\mathbb{Z}^{m_1+m_2'}$ under the projection onto the first m_1 coordinates. Using $\widetilde{B}_2\mathbb{R}^{m_2'} = V_{B_2}^{\perp} = \{\boldsymbol{w} \in \mathbb{R}^{k_2} \,:\, B_2^{\mathsf{T}}\boldsymbol{w} = \mathbf{0}\}$, it follows that L equals the set of those $\boldsymbol{x} \in \mathbb{Z}^{m_1}$ for which there exists some $\boldsymbol{u} \in \mathbb{Z}^{k_2}$ satisfying $B_2^{\mathsf{T}}(\boldsymbol{u} - B_2\alpha\boldsymbol{x}) = 0$. In other words, L is the set of $\boldsymbol{x} \in \mathbb{Z}^{m_1}$ satisfying $\xi^{\mathsf{T}}\boldsymbol{x} \in B_2^{\mathsf{T}}\mathbb{Z}^{k_2}$, with ξ as in (4.17). Next, using the fact that B_2 can be extended to a matrix $B_2' \in \mathrm{GL}_{k_2}(\mathbb{Z})$ with block decomposition $B_2' = (B_2 \ *)$, it follows that $B_2^{\mathsf{T}}\mathbb{Z}^{k_2} = \mathbb{Z}^{m_2}$. Hence $\xi^{\mathsf{T}}\boldsymbol{x} \in B_2^{\mathsf{T}}\mathbb{Z}^{k_2}$ is equivalent with $\xi^{\mathsf{T}}\boldsymbol{x} \in \mathbb{Z}^{k_2}$, which in turn is equivalent with $q_\xi\xi^{\mathsf{T}}\boldsymbol{x} \equiv \mathbf{0} \bmod q_\xi$. It follows that L is a full-dimensional sub-lattice of $\mathbb{Z}^{m_1}$ of index $q_\xi^{m_1}/N_\xi$ (cf. (4.12)). Furthermore, given any $\boldsymbol{x} \in \mathbb{Z}^{m_1}$ and $\boldsymbol{y}_0 \in \mathbb{R}^{m_2'}$ such that $B_2\alpha\boldsymbol{x} + \widetilde{B}_2\boldsymbol{y}_0 \in \mathbb{Z}^{k_2}$, we have $\{\boldsymbol{y} \in \mathbb{R}^{m_2'} \,:\, B_2\alpha\boldsymbol{x} + \widetilde{B}_2\boldsymbol{y} \in \mathbb{Z}^{k_2}\} = \boldsymbol{y}_0 + \mathbb{Z}^{m_2'}$. Using these facts, it follows that

$$\left|\det J^{-1}\right| = \mathrm{vol}(\mathbb{R}^{m_1+m_2'}/J^{-1}\mathbb{Z}^{m_1+m_2'}) = \mathrm{vol}(\mathbb{R}^{m_1}/L) = q_\xi^{m_1}/N_\xi, \tag{4.20}$$

and the formula (4.17) is proved. □

Recall that we are considering a fixed choice of $m_1 > 0$ and $0 < m_2 < k_2$. Using Lemma 4.2, and noticing that when (4.16) holds then the integral in (4.15) equals $\int \widetilde{\rho}(zJ^{\mathsf{T}}B_3^{\mathsf{T}})\,dz = \left(N_\xi/q_\xi^{m_1}\right)^{-n} \int \widetilde{\rho}(wB_3^{\mathsf{T}})\,dw$, it follows that the sum over all $B_1 \in A_{k_1,m_1}$ and $B_2 \in A_{k_2,m_2}$ of the expression in (4.15) equals

$$\sum_{\substack{B \in A_{k,m_1+m_2'} \\ (\dim P_1 V_B = m_1)}} \int_{M_{n,m_1+m_2'}(\mathbb{R})} \widetilde{\rho}\big(wB^{\mathsf{T}}\big)\,dw \tag{4.21}$$

By appropriate modifications of the previous argument, the identity just stated can be shown to hold also in the remaining cases when $m_1 = 0$ and/or $m_2 \in \{0, k_2\}$. (In the special case $m_1 = 0$, $m_2 = k_2$, the expression in (4.21) should be interpreted to mean $\widetilde{\rho}(0)$.) Finally, adding the expression in (4.21) over all combinations of $m_1 \in \{0, \ldots, k_1\}$ and $m_2 \in \{0, \ldots, k_2\}$ we arrive at the conclusion that the right-hand side of (1.10) equals

$$\sum_{m=1}^{k} \sum_{B \in A_{k,m}} \int_{M_{n,m}(\mathbb{R})} \widetilde{\rho}(xB^{\mathsf{T}})\,dx + \widetilde{\rho}(0).$$

By Rogers' formula, Theorem 1.5, this equals (4.1), i.e. the left-hand side of (1.10), and Theorem 1.1 is proved. □ □ □

5. Application to the joint limit distribution of $\{\mathcal{V}_j(L)\}_{j=1}^{\infty}$ and $\{\mathcal{V}_j(L^*)\}_{j=1}^{\infty}$

5.1. **Joint moments of counting functions** $N_j(L)$ **and** $\widetilde{N}_j(L^*)$**.** Given any numbers $0 < V_1 \le V_2 \le \cdots \le V_{k_1}$ and $0 < W_1 \le W_2 \le \cdots \le W_{k_2}$, for any $L \in X_n$ we denote by $N_j(L)$ the number of non-zero lattice points of L in the open n-ball of volume V_j centered at the origin, and we denote by $\widetilde{N}_j(L^*)$ the number of non-zero lattice points of L^* in the n-ball of volume W_j centered at the origin. The main step in the proof of Theorem 1.6 is the following theorem concerning the joint moments of the counting functions $N_j(L)$ and $\widetilde{N}_j(L^*)$, for L random in (X_n, μ).

For any $k \in \mathbb{Z}_{\geq 0}$ we denote by $\mathcal{P}(k)$ the set of partitions of the set $\{1, \ldots, k\}$. Thus, for example, $\mathcal{P}(1) = \{\{\{1\}\}\}$ and $\mathcal{P}(2) = \{\{\{1\},\{2\}\},\{\{1,2\}\}\}$, while we agree that $\mathcal{P}(0) = \{\emptyset\}$. For any non-empty subset $B \subset \mathbb{Z}^+$ we write $\mathsf{m}_B := \min_{j \in B} j$.

Theorem 5.1. *Let $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ and fix numbers $0 < V_1 \leq V_2 \leq \cdots \leq V_{k_1}$ and $0 < W_1 \leq W_2 \leq \cdots \leq W_{k_2}$. Let L be a random in X_n with respect to μ. Then*

$$\mathbb{E}\bigg(\prod_{j=1}^{k_1} N_j(L) \prod_{j=1}^{k_2} \widetilde{N}_j(L^*)\bigg) \to \bigg(\sum_{P \in \mathcal{P}(k_1)} 2^{k_1 - \#P} \prod_{B \in P} V_{\mathsf{m}_B}\bigg)\bigg(\sum_{P \in \mathcal{P}(k_2)} 2^{k_2 - \#P} \prod_{B \in P} W_{\mathsf{m}_B}\bigg) \tag{5.1}$$

as $n \to \infty$.

Remark 5.2. In (5.1), it should be noted that an empty product equals 1 by convention; hence (using also $\mathcal{P}(0) = \{\emptyset\}$), for $k_2 = 0$, (5.1) says

$$\mathbb{E}\bigg(\prod_{j=1}^{k_1} N_j(L)\bigg) \to \sum_{P \in \mathcal{P}(k_1)} 2^{k_1 - \#P} \prod_{B \in P} V_{\mathsf{m}_B}. \tag{5.2}$$

This limit relation is already known from [31, Thm. 3, eq. (10) and Lemma 3]. Of course, by the $L \leftrightarrow L^*$ symmetry (see Remark 1.4), also the case $k_1 = 0$ is covered by the same reference.

We now embark on the proof of Theorem 5.1. Because of Remark 5.2, we may from start assume that both k_1 and k_2 are positive. In the following we will work with an arbitrary, fixed, dimension $n > k_1 + k_2$. Let ρ_j, $1 \leq j \leq k_1$, be the characteristic function of the open n-ball of volume V_j centered at the origin, with the origin removed. Then

$$N_j(L) = \sum_{\boldsymbol{v} \in L} \rho_j(\boldsymbol{v}).$$

Similarly let $\widetilde{\rho}_j$, $1 \leq j \leq k_2$, be the characteristic function of the open n-ball of volume W_j centered at the origin, with the origin removed. Then

$$\widetilde{N}_j(L) = \sum_{\boldsymbol{v} \in L^*} \widetilde{\rho}_j(\boldsymbol{v}).$$

Also define $\rho : (\mathbb{R}^n)^{k_1} \times (\mathbb{R}^n)^{k_2} \to \mathbb{R}_{\geq 0}$ through

$$\rho(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k_1}, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{k_2}) := \prod_{j=1}^{k_1} \rho_j(\boldsymbol{v}_j) \prod_{j=1}^{k_2} \widetilde{\rho}_j(\boldsymbol{w}_j). \tag{5.3}$$

Then by construction,

$$\mathbb{E}\bigg(\prod_{j=1}^{k_1} N_j(L) \prod_{j=1}^{k_2} \widetilde{N}_j(L^*)\bigg) = \int_{X_n} \sum_{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k_1} \in L} \sum_{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{k_2} \in L^*} \rho(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k_1}, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{k_2})\, d\mu(L).$$

By Theorem 1.1, this equals

$$\sum_{m_1=1}^{k_1} \sum_{B_1 \in A_{k_1, m_1}} \sum_{m_2=1}^{k_2} \sum_{B_2 \in A_{k_2, m_2}} \sum_{\beta \in M_{m_1, m_2}(\mathbb{Z})} W(\beta) \int_{S(\beta)} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\, d\eta_\beta(x, y). \tag{5.4}$$

Indeed, note that all the terms in the last line of (1.10) vanish, since by definition we have $\rho_j(\mathbf{0}) = 0$ and $\widetilde{\rho}_j(\mathbf{0}) = 0$ for all j.

5.2. **The main contribution.** We will start by considering the sums over B_1 and B_2 in (5.4) restricted to certain subsets of A_{k_1,m_1} and A_{k_2,m_2}, respectively; it will turn out that these restricted sums give the main term contribution as $n \to \infty$.

For any $k \geq m$, we let $\mathcal{M}_{k,m}$ be the set of all $k \times m$ matrices which have all entries in $\{-1, 0, 1\}$, exactly one non-zero entry in each row, and at least one non-zero entry in each column. One verifies that $\mathcal{M}_{k,m} \subset M_{k,m}(\mathbb{Z})^*$. Let $\mathcal{M}'_{k,m}$ be the subset of those matrices $B = (b_{ij}) \in \mathcal{M}_{k,m}$ which satisfy the following condition: There exist some $1 = \nu_1 < \nu_2 < \cdots < \nu_m \leq k$ such that for each $j \in \{1, \ldots, m\}$ we have $b_{\nu_j,j} = 1$ and $b_{ij} = 0$ for all $i < \nu_j$. Clearly the indices $\nu_1, \ldots, \nu_m$ are uniquely determined for every $B \in \mathcal{M}'_{k,m}$, and we will denote these by $\nu_1(B), \ldots, \nu_m(B)$.

Let $\mathfrak{S}_m \subset \mathrm{GL}_m(\mathbb{Z})$ be the group of $m \times m$ signed permutation matrices, i.e. matrices which have all entries in $\{-1, 0, 1\}$ and exactly one non-zero entry in each row and in each column. One verifies that $\mathfrak{S}_m$ acts freely on $\mathcal{M}_{k,m}$ by multiplication from the right; in particular each orbit for this action has size $\#\mathfrak{S}_m = 2^m m!$. Furthermore, whenever the relation $B_1\gamma = B_2$ holds for some $B_1, B_2 \in \mathcal{M}_{k,m}$ and $\gamma \in \mathrm{GL}_m(\mathbb{Z})$, γ must in fact belong to $\mathfrak{S}_m$. Using these observations, one easily verifies that $\mathcal{M}'_{k,m}$ contains exactly one representative from every $\mathrm{GL}_m(\mathbb{Z})$-orbit in $M_{k,m}(\mathbb{Z})^*$ which intersects $\mathcal{M}_{k,m}$. Hence we may, without loss of generality, from now on assume that the set of representatives $A_{k,m}$ has been chosen in such a way that

$$\mathcal{M}'_{k,m} \subset A_{k,m}.$$

Let us now consider the contribution in (5.4) from a fixed choice of $m_1 \in \{1, \ldots, k_1\}$, $B_1 \in \mathcal{M}'_{k_1,m_1}$, $m_2 \in \{1, \ldots, k_2\}$, $B_2 \in \mathcal{M}'_{k_2,m_2}$. That is, we consider the following sum:

$$\sum_{\beta \in M_{m_1,m_2}(\mathbb{Z})} W(\beta) \int_{S(\beta)} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\, d\eta_\beta(x, y). \tag{5.5}$$

It is immediate from the definition in (1.9) that, for any $\beta \in M_{m_1,m_2}(\mathbb{Z})$,

$$\frac{1}{\zeta(n)\zeta(n-1)\cdots\zeta(n-m_1+1)} \leq W(\beta) \leq \frac{\zeta(n-m_2)\cdots\zeta(n-m_2-m_1+1)}{\zeta(n)\zeta(n-1)\cdots\zeta(n-m_1+1)}, \tag{5.6}$$

with the second relation being an equality when $\beta = 0$. In particular we have

$$W(\beta) = 1 + O_m(2^{-n}), \tag{5.7}$$

where

$$m := m_1 + m_2.$$

Since $\rho \geq 0$, it follows that (5.5) equals

$$\big(1 + O_m(2^{-n})\big) \sum_{\beta \in M_{m_1,m_2}(\mathbb{Z})} \int_{S(\beta)} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\, d\eta_\beta(x, y). \tag{5.8}$$

Let us here write ρ as

$$\rho(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k_1}, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{k_2}) = \rho'(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k_1})\widetilde{\rho}'(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{k_2}) \tag{5.9}$$

with

$$\rho'(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k_1}) := \prod_{j=1}^{k_1} \rho_j(\boldsymbol{v}_j) \quad \text{and} \quad \widetilde{\rho}'(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{k_2}) := \prod_{j=1}^{k_2} \widetilde{\rho}_j(\boldsymbol{w}_j) \tag{5.10}$$

(cf. (5.3)). Then (5.8) can be expressed as follows, using also (1.7):

$$\big(1+O_m(2^{-n})\big)\sum_{\beta\in M_{m_1,m_2}(\mathbb{Z})}\int_{U_{m_1}}\int_{S(\beta)_x}\rho'\big(xB_1^{\mathsf T}\big)\widetilde{\rho}'\big(yB_2^{\mathsf T}\big)\,d\eta_{\beta,x}(y)\,\frac{dx}{\mathsf{d}(x)^{m_2}}$$

$$=\big(1+O_m(2^{-n})\big)\int_{U_{m_1}}\rho'\big(xB_1^{\mathsf T}\big)\sum_{\beta\in M_{m_1,m_2}(\mathbb{Z})}\int_{S(\beta)_x}\widetilde{\rho}'\big(yB_2^{\mathsf T}\big)\,d\eta_{\beta,x}(y)\,\frac{dx}{\mathsf{d}(x)^{m_2}}. \tag{5.11}$$

Let the column vectors of x be $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{m_1}$, and let the column vectors of y be $\boldsymbol{y}_1,\ldots,\boldsymbol{y}_{m_2}$. Set $\nu_{j,1}:=\nu_j(B_1)$ and $\nu_{j,2}:=\nu_j(B_2)$. It then follows from our definitions, and in particular the assumption that $V_1\leq\cdots\leq V_{k_1}$ and $W_1\leq\cdots\leq W_{k_2}$, that

$$\rho'\big(xB_1^{\mathsf T}\big)=\prod_{j=1}^{m_1}\rho_{\nu_{j,1}}(\boldsymbol{x}_j)\qquad\text{and}\qquad\widetilde{\rho}'\big(yB_2^{\mathsf T}\big)=\prod_{j=1}^{m_2}\widetilde{\rho}_{\nu_{j,2}}(\boldsymbol{y}_j). \tag{5.12}$$

For $x\in U_{m_1}$ we set

$$b_x:=x(x^{\mathsf T}x)^{-1}\in U_{m_1}. \tag{5.13}$$

Then for any $y\in M_{n,m_2}(\mathbb{R})$ we note that $y\in S(\beta)'_x$ holds if and only if $x^{\mathsf T}y=\beta=x^{\mathsf T}b_x\beta$, viz., if and only if $y-b_x\beta\in(x^\perp)^{m_2}$, where we recall that $x^\perp$ denotes the orthogonal complement in $\mathbb{R}^n$ of the column span of x. In other words, $y\in S(\beta)'_x$ holds if and only if $\boldsymbol{y}_j\in b_x\boldsymbol{\beta}_j+x^\perp$ for all $j\in\{1,\ldots,m_2\}$, where $\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_{m_2}$ are the column vectors of β. Hence

$$\int_{S(\beta)_x}\widetilde{\rho}'\big(yB_2^{\mathsf T}\big)\,d\eta_{\beta,x}(y)=\prod_{j=1}^{m_2}\int_{b_x\boldsymbol{\beta}_j+x^\perp}\widetilde{\rho}_{\nu_{j,2}}(\boldsymbol{y}_j)\,d\boldsymbol{y}_j, \tag{5.14}$$

where $d\boldsymbol{y}_j$ denotes the $(n-m_1)$-dimensional Lebesgue measure on the affine subspace $b_x\boldsymbol{\beta}_j+x^\perp$. But $\widetilde{\rho}_{\nu_{j,2}}$ is the characteristic function of the open n-ball of radius $R_j:=(W_{\nu_{j,2}}/\mathfrak{V}_n)^{1/n}$ centered at the origin, with the origin removed. Hence the expression in (5.14) equals $\prod_{j=1}^{m_2}f_j\big(\|b_x\boldsymbol{\beta}_j\|\big)$ where $f_j:\mathbb{R}_{\geq0}\to\mathbb{R}_{\geq0}$ is given by

$$f_j(r)=\begin{cases}\mathfrak{V}_{n-m_1}\cdot(R_j^2-r^2)^{(n-m_1)/2} & \text{if } r\leq R_j\\ 0 & \text{if } r\geq R_j.\end{cases}$$

It follows that, for every $x\in U_{m_1}$,

$$\sum_{\beta\in M_{m_1,m_2}(\mathbb{Z})}\int_{S(\beta)_x}\widetilde{\rho}'\big(yB_2^{\mathsf T}\big)\,d\eta_{\beta,x}(y)=\sum_{\beta\in M_{m_1,m_2}(\mathbb{Z})}\prod_{j=1}^{m_2}f_j\big(\|b_x\boldsymbol{\beta}_j\|\big)=\prod_{j=1}^{m_2}\sum_{\boldsymbol{\beta}_j\in\mathbb{Z}^{m_1}}f_j\big(\|b_x\boldsymbol{\beta}_j\|\big). \tag{5.15}$$

In the last product, we will approximate each sum $\sum_{\boldsymbol{\beta}_j\in\mathbb{Z}^{m_1}}f_j\big(\|b_x\boldsymbol{\beta}_j\|\big)$ by the corresponding *integral*,

$$\int_{\boldsymbol{\beta}\in\mathbb{R}^{m_1}}f_j\big(\|b_x\boldsymbol{\beta}\|\big)\,d\boldsymbol{\beta}=\mathsf{d}(b_x)^{-1}\int_{\boldsymbol{z}\in\mathbb{R}^{m_1}}f_j\big(\|\boldsymbol{z}\|\big)\,d\boldsymbol{z}=\mathsf{d}(b_x)^{-1}\mathfrak{V}_nR_j^n=\mathsf{d}(x)W_{\nu_{j,2}}, \tag{5.16}$$

where in the last equality we used the fact that $\mathsf{d}(b_x)^{-1}=\mathsf{d}(x)$. The following lemma gives a bound on the error in this approximation. From now on we will write $C:=\big[-\frac12,\frac12\big]^{m_1}$; a closed unit cube in $\mathbb{R}^{m_1}$. We also define:

$$\ell_x:=\sup_{\boldsymbol{u}\in C}\|b_x\boldsymbol{u}\|.$$

Lemma 5.3. *Fix $\varepsilon > 0$ and $j \in \{1, \ldots, m_2\}$. Assume that n is so large that $n \geq m_1 + 3$ and $\varepsilon^n \leq W_{\nu_{j,2}} \leq \varepsilon^{-n}$. Then for every $x \in U_{m_1}$ satisfying $\ell_x \leq 1$, we have*

$$\left| \sum_{\boldsymbol{\beta} \in \mathbb{Z}^{m_1}} f_j\big(\|b_x \boldsymbol{\beta}\|\big) - \mathsf{d}(x) W_{\nu_{j,2}} \right| \ll_{m_1,\varepsilon} W_{\nu_{j,2}} \mathsf{d}(x) \ell_x. \tag{5.17}$$

Proof. We split the integral in (5.16) by tesselating $\mathbb{R}^{m_1}$ by translates of the unit cube C; this gives:

$$\mathsf{d}(x) W_{\nu_{j,2}} = \sum_{\boldsymbol{\beta} \in \mathbb{Z}^{m_1}} \int_{\boldsymbol{\beta}+C} f_j\big(\|b_x \boldsymbol{u}\|\big)\, d\boldsymbol{u}. \tag{5.18}$$

Using this equality, and the triangle inequality, it follows that the left-hand side of (5.17) is bounded above by

$$\sum_{\boldsymbol{\beta} \in \mathbb{Z}^{m_1}} \int_C \Big| f_j\big(\|b_x \boldsymbol{\beta}\|\big) - f_j\big(\|b_x(\boldsymbol{\beta}+\boldsymbol{u})\|\big) \Big|\, d\boldsymbol{u}.$$

Since $n \geq m_1 + 3$, the function $f_j(r)$ is C^1, and we have

$$\big|f_j(a) - f_j(b)\big| \leq (b-a) \sup_{s \in [a,b]} |f_j'(s)| \qquad \text{for all } 0 \leq a \leq b. \tag{5.19}$$

Hence, recalling the definition of ℓ_x, it follows that the integrand in (5.18) is everywhere bounded above by $\ell_x \cdot F_j\big(\|b_x \boldsymbol{\beta}\|\big)$, where

$$F_j(r) := \sup\Big\{ |f_j'(s)| \ : \ s \in \big[\max(0, r - \ell_x), r + \ell_x\big] \Big\}.$$

Hence (5.19) (and thus also the left-hand side of (5.17)) is bounded above by

$$\ell_x \sum_{\boldsymbol{\beta} \in \mathbb{Z}^{m_1}} F_j\big(\|b_x \boldsymbol{\beta}\|\big) = -\ell_x \int_0^\infty F_j'(r) A(r)\, dr \tag{5.20}$$

where

$$A(r) := \#\{\boldsymbol{\beta} \in \mathbb{Z}^{m_1} \ : \ \|b_x \boldsymbol{\beta}\| \leq r\}.$$

(The equality in (5.20) holds by integration by parts; note that $F_j(r) = 0$ for all $r \geq R_j + \ell_x$, so that the range of integration can just as well be taken to be $[0, R_j + \ell_x]$.)

We compute

$$\big|f_j'(s)\big| = \mathfrak{V}_{n-m_1}(n - m_1)\big(R_j^2 - s^2\big)^{\frac{n-m_1}{2} - 1} s \qquad (0 \leq s \leq R_j).$$

and this function is verified to be increasing for $0 \leq s \leq \kappa R_j$ and decreasing for $\kappa R_j \leq s \leq R_j$, where

$$\kappa := (n - m_1 - 1)^{-\frac{1}{2}}.$$

(Note that $0 < \kappa \leq 2^{-\frac{1}{2}}$.) Hence the function $F_j(r)$ is increasing for $0 \leq r \leq \kappa R_j + \ell_x$ (in fact it is constant on the interval $\max(0, \kappa R_j - \ell_x) \leq r \leq \kappa R_j + \ell_x$), and decreasing for $r \geq \kappa R_j + \ell_x$. Note also that for $r \geq \kappa R_j + \ell_x$ we have $F_j(r) = |f_j'(r - \ell_x)|$, and thus for all $r \in \big[\kappa R_j, R_j\big)$ we have:

$$F_j'(r + \ell_x) = \frac{d}{dr} |f_j'(r)| = \mathfrak{V}_{n-m_1}(n - m_1)\big(R_j^2 - r^2\big)^{\frac{n-m_1}{2} - 2}\Big(R_j^2 - (n - m_1 - 1) r^2\Big)$$

It follows that (5.20) (and thus also the left-hand side of (5.17)) is bounded above by

$$
\begin{aligned}
&-\ell_x \int_{\kappa R_j+\ell_x}^{R_j+\ell_x} F_j'(r)A(r)\,dr \\
&= \mathfrak{V}_{n-m_1}(n-m_1)\ell_x \int_{\kappa R_j}^{R_j} \big(R_j^2-r^2\big)^{\frac{n-m_1}{2}-2}\Big((n-m_1-1)r^2-R_j^2\Big)\,A(r+\ell_x)\,dr. \qquad (5.21)
\end{aligned}
$$

In order to bound $A(r)$, note that for every $\boldsymbol{\beta}\in\mathbb{Z}^{m_1}$ with $\|b_x\boldsymbol{\beta}\|\leq r$, the parallelogram $b_x(\boldsymbol{\beta}+C)$ is contained in $\mathcal{B}^n_{r+\ell_x}\cap V_x$ (recall that $V_x=x\mathbb{R}^{m_1}=\operatorname{Span}_{\mathbb{R}}\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{m_1}\}$). Furthermore these parallelograms are pairwise disjoint, and each of them has volume $\mathsf{d}(b_x)=\mathsf{d}(x)^{-1}$, whereas the ball $\mathcal{B}^n_{r+\ell_x}\cap V_x$ has volume $\mathfrak{V}_{m_1}(r+\ell_x)^{m_1}$. (Here "volume" refers to the m_1-dimensional Lebesgue measure on V_x.) Hence:

$$
A(r)\leq \mathfrak{V}_{m_1}\,\mathsf{d}(x)\,(r+\ell_x)^{m_1},\qquad \forall r\geq 0.
$$

Using this bound in (5.21), we conclude that the left-hand side of (5.17) is bounded above by

$$
\begin{aligned}
&\mathfrak{V}_{n-m_1}\mathfrak{V}_{m_1}(n-m_1)\mathsf{d}(x)\ell_x\int_{\kappa R_j}^{R_j}\big(R_j^2-r^2\big)^{\frac{n-m_1}{2}-2}\Big((n-m_1-1)r^2-R_j^2\Big)\,(r+2\ell_x)^{m_1}\,dr. \\
&\ll_{m_1} \mathfrak{V}_{n-m_1}n^2\mathsf{d}(x)\ell_x\int_{\kappa R_j}^{R_j}\big(R_j^2-r^2\big)^{\frac{n-m_1}{2}-2}r^2\,(r+2\ell_x)^{m_1}\,dr. \qquad (5.22)
\end{aligned}
$$

Here the integral is

$$
\begin{aligned}
&\ll_{m_1}\sum_{a=0}^{m_1}\ell_x^{m_1-a}\int_0^{R_j}\big(R_j^2-r^2\big)^{\frac{n-m_1}{2}-2}r^{2+a}\,dr=\frac12\sum_{a=0}^{m_1}\ell_x^{m_1-a}R_j^{n-m_1-1+a}\frac{\Gamma\big(\frac{n-m_1}{2}-1\big)\Gamma\big(\frac{a+3}{2}\big)}{\Gamma\big(\frac{n-m_1+a+1}{2}\big)} \\
&\ll_{m_1}\sum_{a=0}^{m_1}\ell_x^{m_1-a}R_j^{n-m_1-1+a}n^{-\frac{a+3}{2}}\ll_{m_1,\varepsilon}R_j^n\cdot n^{-\frac{m_1}{2}-2}, \qquad (5.23)
\end{aligned}
$$

where in the last step we used $\ell_x\leq 1$, and we also used $R_j=(W_{\nu_{j,2}}/\mathfrak{V}_n)^{1/n}$ and $\varepsilon^n\leq W_{\nu_{j,2}}\leq\varepsilon^{-n}$ to see that $R_j\asymp_\varepsilon\sqrt{n}$. Combining (5.22) and (5.23), and using $\mathfrak{V}_{n-m_1}\asymp_{m_1}\mathfrak{V}_n n^{m_1/2}$ and $\mathfrak{V}_nR_j^n=W_{\nu_{j,2}}$, we obtain the first bound in (5.17). ☐

We complement Lemma 5.3 with a cruder bound which is valid regardless of the size of ℓ_x.

Lemma 5.4. *Fix $\varepsilon>0$ and $j\in\{1,\ldots,m_2\}$. Assume that n is so large that $n\geq m_1+3$ and $\varepsilon^n\leq V_{k_1},W_{\nu_{j,2}}\leq\varepsilon^{-n}$. Then for every $x\in U_{m_1}$ with $\rho'(xB_1^{\mathsf{T}})=1$, we have*

$$
\sum_{\boldsymbol{\beta}\in\mathbb{Z}^{m_1}} f_j\big(\|b_x\boldsymbol{\beta}\|\big)\ll_{m_1,\varepsilon}W_{\nu_{j,2}}\,n^{m_1}\quad\textit{and}\quad \mathsf{d}(x)W_{\nu_{j,2}}\ll_{m_1,\varepsilon}W_{\nu_{j,2}}\,n^{m_1}. \qquad (5.24)
$$

Proof. It follows from $\rho'(xB_1^{\mathsf{T}})=1$ that $\|\boldsymbol{x}_j\|<(V_{k_1}/\mathfrak{V}_n)^{1/n}\ll_\varepsilon\sqrt{n}$ for each j. Hence $\mathsf{d}(x)\ll_{m_1,\varepsilon}n^{m_1}$, and thus the second bound in (5.24) holds.

In order to prove the first bound in (5.24), we start by noticing

$$
\sum_{\boldsymbol{\beta}\in\mathbb{Z}^{m_1}} f_j\big(\|b_x\boldsymbol{\beta}\|\big)\leq\mathfrak{V}_{n-m_1}R_j^{n-m_1}\cdot\#\big\{\boldsymbol{\beta}\in\mathbb{Z}^{m_1}\,:\,\|b_x\boldsymbol{\beta}\|<R_j\big\}, \qquad (5.25)
$$

since $f_j(r)\leq\mathfrak{V}_{n-m_1}R_j^{n-m_1}$ for all $r\geq 0$. Now note that for all $\boldsymbol{\beta}\in\mathbb{Z}^{m_1}\setminus\{\mathbf{0}\}$ we have $\|x^{\mathsf{T}}b_x\boldsymbol{\beta}\|=\|\boldsymbol{\beta}\|\geq 1$, and combining this with the fact that $\|\boldsymbol{x}_j\|\ll_\varepsilon\sqrt{n}$ for each $j\in\{1,\ldots,m_1\}$, it follows that $\|b_x\boldsymbol{\beta}\|\gg_{m_1,\varepsilon}n^{-\frac12}$. It follows that the distance between any two vectors $b_x\boldsymbol{\beta}$ and $b_x\boldsymbol{\beta}'$ for $\boldsymbol{\beta}\neq\boldsymbol{\beta}'\in\mathbb{Z}^{m_1}$ is $\gg_{m_1,\varepsilon}n^{-\frac12}$, and hence there exists a number $r\gg_{m_1,\varepsilon}n^{-\frac12}$ such that the balls $\mathcal{B}^n_r+b_x\boldsymbol{\beta}$ are pairwise disjoint as $\boldsymbol{\beta}$ runs through $\mathbb{Z}^{m_1}$. We may also require that $r \leq n^{-\frac{1}{2}}$ (indeed otherwise replace r by $n^{-\frac{1}{2}}$). Then it follows that for every $\boldsymbol{\beta} \in \mathbb{Z}^{m_1}$ with $\|b_x \boldsymbol{\beta}\| < R_j$, the ball $\mathcal{B}_r^n + b_x\boldsymbol{\beta}$ is contained in $\mathcal{B}^n_{R_j+n^{-1/2}}$. Intersecting these balls with V_x and considering the volume, it follows that

$$\#\{\boldsymbol{\beta} \in \mathbb{Z}^{m_1} \,:\, \|b_x\boldsymbol{\beta}\| < R_j\} \leq \frac{\mathfrak{V}_{m_1}(R_j + n^{-1/2})^{m_1}}{\mathfrak{V}_{m_1} r^{m_1}} \ll_{m_1,\varepsilon} R_j^{m_1} r^{-m_1} \ll_{m_1,\varepsilon} R_j^{m_1} n^{\frac{m_1}{2}}. \tag{5.26}$$

(Here we used $\varepsilon^n \leq W_{\nu_{j,2}} \leq \varepsilon^{-n}$ to conclude that $n^{-1/2} \leq n^{1/2} \ll_\varepsilon R_j$.) Combining (5.25) and (5.26), and using $\mathfrak{V}_{n-m_1} \asymp_{m_1} \mathfrak{V}_n n^{\frac{m_1}{2}}$, we obtain the first bound in (5.24). □

We will now apply Lemma 5.3 and Lemma 5.4 to estimate the expression in (5.15). Let us pick $\varepsilon = \frac{1}{2}$ in both the lemmas; thus in order for the lemmas to apply, from now on we assume that n is so large that $n \geq m_1 + 3$ and so that all the V_is and all the W_is lie in the interval $[2^{-n}, 2^n]$. Recall that we also assume $n > k_1 + k_2 \geq m_1 + m_2$. Using Lemma 5.3, we conclude that for every $x \in U_{m_1}$ satisfying $\ell_x \leq 1$, the expression in (5.15) equals

$$\Big(\prod_{j=1}^{m_2} W_{\nu_{j,2}}\Big) \mathsf{d}(x)^{m_2} \Big(1 + O_m(\ell_x)\Big).$$

Using Lemma 5.4, we conclude that for every $x \in U_{m_1}$ satisfying $\rho'(xB_1^\mathsf{T}) = 1$, the expression in (5.15) equals

$$\Big(\prod_{j=1}^{m_2} W_{\nu_{j,2}}\Big) \mathsf{d}(x)^{m_2} \prod_{j=1}^{m_2}\Big(1 + O_{m_1}\Big(\frac{n^{m_1}}{\mathsf{d}(x)}\Big)\Big) = \Big(\prod_{j=1}^{m_2} W_{\nu_{j,2}}\Big) \mathsf{d}(x)^{m_2} \Big(1 + O_m\Big(\frac{n^{m_1 m_2}}{\mathsf{d}(x)^{m_2}}\Big)\Big). \tag{5.27}$$

We will use the first of these estimates when $\ell_x \leq c$ and the second when $\ell_x > c$, where c is a parameter in the interval $(0,1]$ which we will choose later. Let χ_c be the characteristic function of $\ell_x \leq c$ and let $\widetilde{\chi}_c = 1 - \chi_c$ be the characteristic function of $\ell_x > c$. Then we conclude that (5.11) (and thus also (5.5)) equals

$$\begin{aligned}
&\big(1 + O_m(2^{-n})\big)\Big(\prod_{j=1}^{m_2} W_{\nu_{j,2}}\Big)\Big(\int_{U_{m_1}} \rho'(xB_1^\mathsf{T})\,dx + O_m\Big(\int_{U_{m_1}} \rho'(xB_1^\mathsf{T})\,\chi_c(x)\,\ell_x\,dx \\
&\qquad\qquad + n^{m_1 m_2}\int_{U_{m_1}} \rho'(xB_1^\mathsf{T})\,\widetilde{\chi}_c(x)\,\frac{dx}{\mathsf{d}(x)^{m_2}}\Big)\Big)
\end{aligned} \tag{5.28}$$

Here we have, by (5.12),

$$\int_{U_{m_1}} \rho'(xB_1^\mathsf{T})\,dx = \prod_{j=1}^{m_1} V_{\nu_{j,1}}.$$

Furthermore, the first error term is

$$\int_{U_{m_1}} \rho'(xB_1^\mathsf{T})\,\chi_c(x)\,\ell_x\,dx \leq c\int_{U_{m_1}} \rho'(xB_1^\mathsf{T})\,dx = c\prod_{j=1}^{m_1} V_{\nu_{j,1}}.$$

In order to bound the second error term in (5.28), we apply [29, Lemma 5.2]; combined with (5.12) this gives that

$$\begin{aligned}
&\int_{U_{m_1}} \rho'(xB_1^\mathsf{T})\,\widetilde{\chi}_c(x)\,\frac{dx}{\mathsf{d}(x)^{m_2}} \\
&\qquad = \frac{\prod_{j=n-m_1+1}^{n}(j\mathfrak{V}_j)}{\prod_{j=1}^{m_1}(j\mathfrak{V}_j)}\int_{U_{m_1}\cap(\mathbb{R}^{m_1}\times\{\mathbf{0}\})^{m_1}} \rho'(xB_1^\mathsf{T})\,\widetilde{\chi}_c(x)\,\mathsf{d}(x)^{n-m_1-m_2}\,dx,
\end{aligned} \tag{5.29}$$

where in the second line, $\mathbb{R}^{m_1} \times \{\mathbf{0}\}$ denotes the subspace $\{\boldsymbol{x} \in \mathbb{R}^n : x_{m_1+1} = \cdots = x_n = 0\}$ of $\mathbb{R}^n$, and dx denotes the m_1^2-dimensional Lebesgue measure on $(\mathbb{R}^{m_1} \times \{\mathbf{0}\})^{m_1}$ (whereas in the first line it is the $m_1 n$-dimensional Lebesgue measure on $(\mathbb{R}^n)^{m_1}$). We will now use the following lemma:

Lemma 5.5. *For any $x \in U_{m_1}$ with $\rho'(xB_1^{\mathsf{T}})\widetilde{\chi}_c(x) = 1$, we have $\mathsf{d}(x) \ll_{m_1} c^{-1} n^{\frac{m_1-1}{2}}$.*

Proof. Assume that $x \in U_{m_1}$ and $\rho'(xB_1^{\mathsf{T}})\widetilde{\chi}_c(x) = 1$. Then

$$c < \ell_x \leq \frac{1}{2} \sum_{j=1}^{m_1} \|b_x \boldsymbol{e}_j\|,$$

and hence there exists some $j \in \{1, \ldots, m_1\}$ for which $\|b_x \boldsymbol{e}_j\| > (2/m_1)c$. We have $x b_x \boldsymbol{e}_j = \boldsymbol{e}_j$, which implies that $\boldsymbol{x}_j \cdot b_x \boldsymbol{e}_j = 1$ while $b_x \boldsymbol{e}_j$ is orthogonal to the subspace

$$V_{x,j} := \mathrm{Span}_{\mathbb{R}}\{\boldsymbol{x}_i : i \in \{1, \ldots, m_1\} \setminus \{j\}\}.$$

It follows that the orthogonal projection of $\boldsymbol{x}_j$ onto $V_{x,j}^{\perp}$ has length

$$\boldsymbol{x}_j \cdot \frac{b_x \boldsymbol{e}_j}{\|b_x \boldsymbol{e}_j\|} = \|b_x \boldsymbol{e}_j\|^{-1}.$$

Hence $\mathsf{d}(x) = \|b_x \boldsymbol{e}_j\|^{-1} \mathsf{d}'$ where d' is the (m_1-1)-dimensional volume of the parallelotope in $\mathbb{R}^n$ spanned by the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{m_1}$ except $\boldsymbol{x}_j$. Clearly $\mathsf{d}' \leq \prod_{\substack{i=1 \\ (i \neq j)}}^{m_1} \|\boldsymbol{x}_i\|$, and $\rho'(xB_1^{\mathsf{T}})\widetilde{\chi}_c(x) = 1$ implies that $\|\boldsymbol{x}_i\| < (V_{k_1}/\mathfrak{V}_n)^{1/n} \leq (2^n/\mathfrak{V}_n)^{1/n} \ll n^{1/2}$ for each i. Hence:

$$\mathsf{d}(x) \ll_{m_1} \|b_x \boldsymbol{e}_j\|^{-1} n^{\frac{m_1-1}{2}} \ll_{m_1} c^{-1} n^{\frac{m_1-1}{2}}.$$

$\square$

Let $K = K(m_1)$ be the implied constant in the bound in Lemma 5.5. It follows that the expression in (5.29) is

$$\begin{aligned}
&\leq \frac{\prod_{j=n-m_1+1}^{n}(j\mathfrak{V}_j)}{\prod_{j=1}^{m_1}(j\mathfrak{V}_j)} \int_{U_{m_1} \cap (\mathbb{R}^{m_1} \times \{\mathbf{0}\})^{m_1}} \rho'(xB_1^{\mathsf{T}}) \cdot \big(Kc^{-1} n^{\frac{m_1-1}{2}}\big)^{n-m_1-m_2} \, dx \\
&= \frac{\prod_{j=n-m_1+1}^{n}(j\mathfrak{V}_j)}{\prod_{j=1}^{m_1}(j\mathfrak{V}_j)} \big(Kc^{-1} n^{\frac{m_1-1}{2}}\big)^{n-m_1-m_2} \prod_{j=1}^{m_1} \bigg(\mathfrak{V}_{m_1} \Big(\frac{V_{\nu_{j,1}}}{\mathfrak{V}_n}\Big)^{m_1/n}\bigg) \\
&\ll_m \Bigg(\prod_{j=0}^{m_1-1} \bigg(\Big(\frac{2\pi e}{n}\Big)^{\frac{n}{2}} n^{\frac{j+1}{2}}\bigg)\Bigg) K^n c^{m-n} n^{\frac{m_1-1}{2}(n-m)} \prod_{j=1}^{m_1} n^{m_1/2} \\
&= \bigg(\frac{(2\pi e)^{m_1/2} K}{c\sqrt{n}}\bigg)^n n^{\frac{m_1^2}{4} + \frac{m_1}{4} - \frac{m_1 m_2}{2} + \frac{m}{2}} c^m.
\end{aligned}$$

Using this bound, together with the fact that $\prod_{j=1}^{m_1} V_{\nu_{j,1}}^{-1} \leq 2^{m_1 n}$, we conclude that (5.28) (and thus also (5.5)) equals

$$\big(1 + O_m(2^{-n})\big)\bigg(\prod_{j=1}^{m_1} V_{\nu_{j,1}}\bigg)\bigg(\prod_{j=1}^{m_2} W_{\nu_{j,2}}\bigg)\Bigg(1 + O_m\bigg(c + \Big(\frac{2^{m_1}(2\pi e)^{m_1/2} K}{c\sqrt{n}}\Big)^n n^{\frac{m_1^2}{4} + \frac{m_1}{4} + \frac{m_1 m_2}{2} + \frac{m}{2}} c^m\bigg)\Bigg).$$

Choosing now $c := 2^{m_1+2}(2\pi e)^{m_1/2} K n^{-\frac{1}{2}}$, this becomes

$$\bigg(\prod_{j=1}^{m_1} V_{\nu_{j,1}}\bigg)\bigg(\prod_{j=1}^{m_2} W_{\nu_{j,2}}\bigg)\Big(1 + O_m\big(n^{-\frac{1}{2}}\big)\Big). \tag{5.30}$$

To sum up, we have proved that the contribution to (5.4) from any fixed choice of $m_1 \in \{1,\ldots,k_1\}$, $B_1 \in \mathcal{M}'_{k_1,m_1}$, $m_2 \in \{1,\ldots,k_2\}$, $B_2 \in \mathcal{M}'_{k_2,m_2}$ equals (5.30) whenever n is sufficiently large. Recall here that $\nu_{j,1} := \nu_j(B_1)$ and $\nu_{j,2} := \nu_j(B_2)$. Hence, since each set $\mathcal{M}'_{k,m}$ is finite, it follows that

$$\sum_{m_1=1}^{k_1} \sum_{B_1 \in \mathcal{M}'_{k_1,m_1}} \sum_{m_2=1}^{k_2} \sum_{B_2 \in \mathcal{M}'_{k_2,m_2}} \sum_{\beta \in M_{m_1,m_2}(\mathbb{Z})} W(\beta) \int_{S(\beta)} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\, d\eta_\beta(x,y)$$
$$\to \Big(\sum_{m_1=1}^{k_1} \sum_{B_1 \in \mathcal{M}'_{k_1,m_1}} \prod_{j=1}^{m_1} V_{\nu_j(B_1)}\Big) \cdot \Big(\sum_{m_2=1}^{k_2} \sum_{B_2 \in \mathcal{M}'_{k_2,m_2}} \prod_{j=1}^{m_2} W_{\nu_j(B_2)}\Big), \qquad \text{as } n \to \infty.$$

Finally we note that for any $k \in \mathbb{Z}^+$, the matrices in $\sqcup_{m=1}^k \mathcal{M}'_{k,m}$ are exactly the transposes of the matrices considered in [31, Thm. 3], with the "ν" being the same as ours. Hence by [31, Lemma 3], for any given $1 = \nu_1 < \cdots < \nu_m \le k$, the number of matrices $B \in \mathcal{M}'_{k,m}$ having $\nu_j(B) = \nu_j$ for all j equals 2^{k-m} times[4] the number of partitions $P = \{B_1,\ldots,B_{\#P}\}$ in $\mathcal{P}(k)$ satisfying $\#P = m$ and $\{\mathsf{m}_{B_1},\ldots,\mathsf{m}_{B_m}\} = \{\nu_1,\ldots,\nu_m\}$. Therefore the previous limit relation can be re-expressed as

$$\sum_{m_1=1}^{k_1} \sum_{B_1 \in \mathcal{M}'_{k_1,m_1}} \sum_{m_2=1}^{k_2} \sum_{B_2 \in \mathcal{M}'_{k_2,m_2}} \sum_{\beta \in M_{m_1,m_2}(\mathbb{Z})} W(\beta) \int_{S(\beta)} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\, d\eta_\beta(x,y)$$
$$\to \Big(\sum_{P \in \mathcal{P}(k_1)} 2^{k_1-\#P} \prod_{B \in P} V_{\mathsf{m}_B}\Big)\Big(\sum_{P \in \mathcal{P}(k_2)} 2^{k_2-\#P} \prod_{B \in P} W_{\mathsf{m}_B}\Big), \qquad \text{as } n \to \infty. \tag{5.31}$$

5.3. **Bounding the remaining terms.** In view of (5.31), in order to complete the proof of Theorem 5.1, it remains to prove that the total contribution from all terms in (5.4) with $B_1 \notin \mathcal{M}'_{k_1,m_1}$ or $B_2 \notin \mathcal{M}'_{k_2,m_2}$ tends to zero as $n \to \infty$.

To start with, we will prove a bound on the contribution in (5.4) from any given $m_1 \in \{1,\ldots,k_1\}$, $B_1 \in A_{k_1,m_1}$, $m_2 \in \{1,\ldots,k_2\}$, $B_2 \in A_{k_2,m_2}$, that is, on the expression

$$\sum_{\beta \in M_{m_1,m_2}(\mathbb{Z})} W(\beta) \int_{S(\beta)} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\, d\eta_\beta(x,y). \tag{5.32}$$

Throughout our discussion, we will assume that n is so large that

$$\max(V_{k_1}, W_{k_2}) \le 2^n. \tag{5.33}$$

Recall that we write $m := m_1 + m_2$. Let us now also set $k := k_1 + k_2$.

Lemma 5.6. *The number of $\beta \in M_{m_1,m_2}(\mathbb{Z})$ for which $\int_{S(\beta)} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\, d\eta_\beta(x,y) > 0$, is $\ll_k n^{k_1k_2}$.*

Proof. If $\beta \in M_{m_1,m_2}(\mathbb{Z})$ satisfies $\int_{S(\beta)} \rho(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}})\, d\eta_\beta(x,y) > 0$ then there exists some $\langle x,y\rangle \in S(\beta)$ for which $\rho(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}) > 0$. By the definition of ρ, and using (5.33), this implies that all the column vectors of xB_1^{T} and of yB_2^{T} have lengths $\le (2^n/\mathfrak{V}_n)^{1/n} \ll \sqrt{n}$. But $\langle x,y\rangle \in S(\beta)$ means that $x^{\mathsf{T}}y = \beta$, and so $B_1\beta B_2^{\mathsf{T}} = B_1x^{\mathsf{T}}yB_2^{\mathsf{T}} = (xB_1^{\mathsf{T}})^{\mathsf{T}}(yB_2^{\mathsf{T}})$. This means that every matrix entry of $B_1\beta B_2^{\mathsf{T}}$ is equals the scalar product of a column vector of xB_1^{T} and a column vector of yB_2^{T}. Hence, by the Cauchy–Schwarz inequality, all the matrix entries of $B_1\beta B_2^{\mathsf{T}}$ are $\ll n$. It follows that the number of possibilities for the matrix $B_1\beta B_2^{\mathsf{T}} \in M_{k_1,k_2}(\mathbb{Z})$

[4] This factor 2^{k-m} comes from the number of ways to choose the *signs* of the matrix entries.

is $\ll_k n^{k_1k_2}$. The proof of the lemma is now concluded by noticing that the map $\beta \mapsto B_1\beta B_2^{\mathsf{T}}$ is injective, since $\mathsf{d}(B_1) > 0$ and $\mathsf{d}(B_2) > 0$. □

Lemma 5.7. *For every $\beta \in M_{m_1,m_2}(\mathbb{R})$, we have*

$$\begin{aligned}\int_{S(\beta)} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\, d\eta_\beta(x,y) &\leq \int_{S(0)} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\, d\eta_0(x,y) \\ &= \frac{\prod_{j=n-m_1+1}^{n}(j\mathfrak{V}_j)}{\prod_{j=n-m_1-m_2+1}^{n-m_2}(j\mathfrak{V}_j)} \int_{(\mathbb{R}^{n-m_2}\times\{\mathbf{0}\})^{m_1}} \rho'(xB_1^{\mathsf{T}})\, dx \int_{(\mathbb{R}^{n-m_1}\times\{\mathbf{0}\})^{m_2}} \widetilde{\rho}'(xB_2^{\mathsf{T}})\, dx,\end{aligned} \tag{5.34}$$

where in the last two integrals, ρ' and $\widetilde{\rho}'$ are as in (5.10)*, and for any $0 < \ell < n$, $\mathbb{R}^\ell \times \{\mathbf{0}\}$ denotes the subspace $\{\boldsymbol{x} \in \mathbb{R}^n : x_{\ell+1} = \cdots = x_n = 0\}$ of $\mathbb{R}^n$, and "dx" stands for the natural Lebesgue measure in each case.*

Proof. By (1.7) we have

$$\int_{S(\beta)} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\, d\eta_\beta(x,y) = \int_{U_{m_1}} \rho'(xB_1^{\mathsf{T}}) \int_{S(\beta)_x} \widetilde{\rho}'\big(yB_2^{\mathsf{T}}\big)\, d\eta_{\beta,x}(y)\, \frac{dx}{\mathsf{d}(x)^{m_2}}.$$

Given $x \in U_{m_1}$ we set $b_x := x(x^{\mathsf{T}}x)^{-1} \in U_{m_1}$ as in (5.13), and then we have $S(\beta)'_x = b_x\beta + (x^\perp)^{k_2}$, and so

$$\int_{S(\beta)_x} \widetilde{\rho}'\big(yB_2^{\mathsf{T}}\big)\, d\eta_{\beta,x}(y) = \int_{(x^\perp)^{k_2}} \widetilde{\rho}'\big((b_x\beta + y')B_2^{\mathsf{T}}\big)\, dy',$$

where dy' is the $m_2(n-m_1)$-dimensional Lebesgue measure on $(x^\perp)^{k_2}$. By (5.10), for any $w \in M_{n,k_2}(\mathbb{R})$ we have $\widetilde{\rho}'(w) = 1$ if and only if the jth column vector of w is non-zero and has length $< (W_j/\mathfrak{V}_n)^{1/n}$, for each $j \in \{1, \ldots, k_2\}$; otherwise $\widetilde{\rho}'(w) = 0$. However, for every $y' \in (x^\perp)^{k_2}$ we have $x^{\mathsf{T}}y' = 0$ and hence $(b_x\beta)^{\mathsf{T}}y' = 0$, which means that every column vector of $b_x\beta$ is orthogonal against every column vector of y'. Hence also every column vector of $b_x\beta {B_2}^{\mathsf{T}}$ is orthogonal against every column vector of $y'{B_2}^{\mathsf{T}}$, and therefore, for each $j \in \{1, \ldots, k_2\}$, the jth column vector of $(b_x\beta + y'){B_2}^{\mathsf{T}}$ is at least as long as the jth column vector of $y'{B_2}^{\mathsf{T}}$. It follows that for every $y' \in (x^\perp)^{k_2}$ we have $\widetilde{\rho}'\big((b_x\beta + y')B_2^{\mathsf{T}}\big) \leq \widetilde{\rho}'\big(y'B_2^{\mathsf{T}}\big)$. Hence:

$$\int_{S(\beta)_x} \widetilde{\rho}'\big(yB_2^{\mathsf{T}}\big)\, d\eta_{\beta,x}(y) \leq \int_{(x^\perp)^{k_2}} \widetilde{\rho}'\big(y'B_2^{\mathsf{T}}\big)\, dy' = \int_{S(0)_x} \widetilde{\rho}'\big(yB_2^{\mathsf{T}}\big)\, d\eta_{0,x}(y).$$

Since this holds for every $x \in U_{m_1}$, we conclude that the first relation (the inequality) in (5.34) holds.

Next, to prove the equality in (5.34), we first note that, using the fact that $\widetilde{\rho}'$ is rotationally invariant in the sense that $\widetilde{\rho}'(\kappa z) = \widetilde{\rho}'(z)$ for all $\kappa \in O(n)$ and $z \in M_{n,k_2}$, we have

$$\int_{(x^\perp)^{k_2}} \widetilde{\rho}'\big(y'B_2^{\mathsf{T}}\big)\, dy' = \int_{(\mathbb{R}^{n-m_1})^{m_2}} \widetilde{\rho}'\big(y'B_2^{\mathsf{T}}\big)\, dy'$$

for each $x \in U_{m_1}$. Note that the latter integral is independent of x. Hence we get

$$\int_{S(0)} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\, d\eta_0(x,y) = \int_{U_{m_1}} \rho'(xB_1^{\mathsf{T}})\, \frac{dx}{\mathsf{d}(x)^{m_2}} \int_{(\mathbb{R}^{n-m_1})^{m_2}} \widetilde{\rho}'\big(y'B_2^{\mathsf{T}}\big)\, dy'.$$

Finally, by using the corresponding rotational invariance of ρ' and applying [29, Lemma 5.2] twice, we have:

$$\int_{U_{m_1}} \rho'(xB_1^{\mathsf{T}}) \frac{dx}{\mathsf{d}(x)^{m_2}} = \frac{\prod_{j=n-m_1+1}^{n}(j\mathfrak{V}_j)}{\prod_{j=1}^{m_1}(j\mathfrak{V}_j)} \int_{(\mathbb{R}^{m_1}\times\{\mathbf{0}\})^{m_1}} \rho'(xB_1^{\mathsf{T}})\,\mathsf{d}(x)^{n-m_1-m_2}\,dx$$
$$= \frac{\prod_{j=n-m_1+1}^{n}(j\mathfrak{V}_j)}{\prod_{j=n-m_1-m_2+1}^{n-m_2}(j\mathfrak{V}_j)} \int_{(\mathbb{R}^{n-m_2}\times\{\mathbf{0}\})^{m_1}} \rho'(xB_1^{\mathsf{T}})\,dx.$$

Hence we obtain the equality in (5.34). □

Regarding the product of unit ball volumes appearing in Lemma 5.7, we note that

$$\frac{\prod_{j=n-m_1+1}^{n}(j\mathfrak{V}_j)}{\prod_{j=n-m_1-m_2+1}^{n-m_2}(j\mathfrak{V}_j)} \asymp_m n^{-m_1m_2/2} \ll 1.$$

(We can use this wasteful bound since it will turn out below that our integrals are exponentially decreasing with respect to n.) Hence it follows from (5.7) and Lemmas 5.6 and 5.7 that

$$\sum_{\beta\in M_{m_1,m_2}(\mathbb{Z})} W(\beta)\int_{S(\beta)} \rho\big(xB_1^{\mathsf{T}}, yB_2^{\mathsf{T}}\big)\,d\eta_\beta(x,y)$$
$$\ll_k n^{k_1k_2}\cdot \int_{(\mathbb{R}^{n-m_2}\times\{\mathbf{0}\})^{m_1}} \rho'(xB_1^{\mathsf{T}})\,dx \int_{(\mathbb{R}^{n-m_1}\times\{\mathbf{0}\})^{m_2}} \widetilde{\rho}'(xB_2^{\mathsf{T}})\,dx.$$

Note that the subspace $(\mathbb{R}^{n-m_2}\times\{\mathbf{0}\})^{m_1}$ is mapped into $(\mathbb{R}^{n-m_2}\times\{\mathbf{0}\})^{k_1}$ by right multiplication by B_1^{T}. Now let σ' be the function on $(\mathbb{R}^{n-m_2})^{k_1}$ given by the restriction of ρ' to $(\mathbb{R}^{n-m_2}\times\{\mathbf{0}\})^{k_1}$ composed with the obvious isomorphism $(\mathbb{R}^{n-m_2}\times\{\mathbf{0}\})^{k_1} = (\mathbb{R}^{n-m_2})^{k_1}$. We also let $\widetilde{\sigma}'$ be the corresponding function on $(\mathbb{R}^{n-m_1})^{k_2}$ obtained from $\widetilde{\rho}'$. Then the last bound can be written:

$$n^{k_1k_2}\cdot \int_{(\mathbb{R}^{n-m_2})^{m_1}} \sigma'(xB_1^{\mathsf{T}})\,dx \int_{(\mathbb{R}^{n-m_1})^{m_2}} \widetilde{\sigma}'(xB_2^{\mathsf{T}})\,dx.$$

Using the above bound, we conclude that the total contribution from all terms in (5.4) with $B_1 \notin \mathcal{M}'_{k_1,m_1}$ or $B_2 \notin \mathcal{M}'_{k_2,m_2}$ is

$$\ll_k n^{k_1k_2} \sum_{m_1=1}^{k_1}\sum_{m_2=1}^{k_2} \sum_{\langle B_1,B_2\rangle\in(A_{k_1,m_1}\times A_{k_2,m_2})\smallsetminus(\mathcal{M}'_{k_1,m_1}\times\mathcal{M}'_{k_2,m_2})} \int_{(\mathbb{R}^{n-m_2})^{m_1}} \sigma'(xB_1^{\mathsf{T}})\,dx$$
$$\times \int_{(\mathbb{R}^{n-m_1})^{m_2}} \widetilde{\sigma}'(xB_2^{\mathsf{T}})\,dx. \tag{5.35}$$

We will now translate this sum into the notation used in Rogers' original formulation of his mean value formula (see Theorem 2.1). Let us denote by $\mathfrak{A}(k;m)$ the set of matrices of fixed size $m\times k$ appearing in the sum in (2.1).[5] From the proof in Section 2 of the fact that Theorem 1.5 is a reformulation of Theorem 2.1, we recall the definition of the map $\gamma_2' : \mathfrak{A}(k;m)\to M_{k,m}(\mathbb{Z})^*$, and the fact that an admissible choice of the set of representatives $A_{k,m}$ (cf. p. 3) is given by

$$A_{k,m} := \{\gamma_2'(D) \,:\, D\in\mathfrak{A}(k;m)\}.$$

Let us set

$$\widetilde{\mathcal{M}}'_{k,m} := \{B^{\mathsf{T}} \,:\, B\in\mathcal{M}'_{k,m}\},$$

[5] This set was called $\mathfrak{A}(m)$ in Section 2, since there we were working with a single, fixed k.

and note that $\widetilde{\mathcal{M}}'_{k,m} \subset \mathfrak{A}(k;m)$; in fact the matrices in $\widetilde{\mathcal{M}}'_{k,m}$ are exactly the matrices considered in [31, Thm. 3], as we noted near the end of Section 5.2. We claim that the map γ_2' can be chosen so that

$$\gamma_2'(D) = D^{\mathsf{T}}, \qquad \forall D \in \widetilde{\mathcal{M}}'_{k,m}. \tag{5.36}$$

Indeed, given $D \in \widetilde{\mathcal{M}}'_{k,m}$ one easily verifies that there exists a matrix $\tau = (\tau_{i,j}) \in \mathrm{GL}_k(\mathbb{Z})$ satisfying $D\tau = \big(I_m\, 0\big)$ and $\tau_{i,j} = \delta_{i,\nu_j(D^{\mathsf{T}})}$ for all $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,m\}$. Hence the relation (2.4) holds for our matrix D (which has elementary divisors $\varepsilon_1 = \cdots = \varepsilon_m = 1$) with $\gamma_1 = I$ in $\mathrm{GL}_m(\mathbb{Z})$ and $\gamma_2 = \tau^{-1}$ in $\mathrm{GL}_k(\mathbb{Z})$. Thus: We may choose $\gamma_1(D) = I$ and $\gamma_2(D) = \tau^{-1}$. Now by definition, $\gamma_2'(D) = \big((I_m\, 0)\gamma_2\big)^{\mathsf{T}}$, and it follows from $D\tau = \big(I_m\, 0\big)$ that $\big(I_m\, 0\big)\gamma_2 = D$; therefore (5.36) holds.

From now on we assume that for any $1 \le m \le k$, the map γ_2' has been chosen so that (5.36) holds. It then follows from the above discussion, combined with the identity in (2.6), that our bound in (5.35) can be equivalently expressed as

$$n^{k_1k_2} \sum_{m_1=1}^{k_1} \sum_{m_2=1}^{k_2} \sum_{\langle D_1,D_2\rangle \in (\mathfrak{A}(k_1,m_1)\times\mathfrak{A}(k_2,m_2))\smallsetminus(\widetilde{\mathcal{M}}'_{k_1,m_1}\times\widetilde{\mathcal{M}}'_{k_2,m_2})} J_{n-m_2}(\sigma';D_1)\, J_{n-m_1}(\widetilde{\sigma}';D_2), \tag{5.37}$$

where for any $1 \le m \le k < \widetilde{n}$, $D \in \mathfrak{A}(k,m)$ and (measurable) function $\sigma : (\mathbb{R}^{\widetilde{n}})^m \to \mathbb{R}_{\ge 0}$, $J_{\widetilde{n}}(\sigma;D)$ is defined by

$$J_{\widetilde{n}}(\sigma;D) := \Big(\frac{e_1}{q}\cdots\frac{e_m}{q}\Big)^{\widetilde{n}} \int_{(\mathbb{R}^{\widetilde{n}})^m} \sigma(xq^{-1}D)\,dx,$$

where $q = q(D)$ is as in Theorem 2.1, and $e_1,\ldots,e_m$ are the elementary divisors of D. Note here that for any $1 \le m_1 \le k_1$ and $1 \le m_2 \le k_2$, the sum over $\langle D_1,D_2\rangle$ appearing in (5.37) can be decomposed as the sum of the three products

$$\Bigg(\sum_{D_1\in\mathfrak{A}(k_1,m_1)\smallsetminus\widetilde{\mathcal{M}}'_{k_1,m_1}} J_{n-m_2}(\sigma';D_1)\Bigg)\Bigg(\sum_{D_2\in\mathfrak{A}(k_2,m_2)\smallsetminus\widetilde{\mathcal{M}}'_{k_2,m_2}} J_{n-m_1}(\widetilde{\sigma}';D_2)\Bigg) \tag{5.38}$$

and

$$\Bigg(\sum_{D_1\in\widetilde{\mathcal{M}}'_{k_1,m_1}} J_{n-m_2}(\sigma';D_1)\Bigg)\Bigg(\sum_{D_2\in\mathfrak{A}(k_2,m_2)\smallsetminus\widetilde{\mathcal{M}}'_{k_2,m_2}} J_{n-m_1}(\widetilde{\sigma}';D_2)\Bigg) \tag{5.39}$$

and

$$\Bigg(\sum_{D_1\in\mathfrak{A}(k_1,m_1)\smallsetminus\widetilde{\mathcal{M}}'_{k_1,m_1}} J_{n-m_2}(\sigma';D_1)\Bigg)\Bigg(\sum_{D_2\in\widetilde{\mathcal{M}}'_{k_2,m_2}} J_{n-m_1}(\widetilde{\sigma}';D_2)\Bigg). \tag{5.40}$$

Now the sum

$$\sum_{D_1\in\mathfrak{A}(k_1,m_1)\smallsetminus\widetilde{\mathcal{M}}'_{k_1,m_1}} J_{n-m_2}(\sigma';D_1) \tag{5.41}$$

which appears as a factor in (5.38) and (5.40), is exactly the sum which is treated in [31, pp. 950(top)–951(middle)] (applied with "n" taken to be $n-m_2$ in our notation). It should here be noted that (cf. (5.10))

$$\sigma'(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k_1}) := \prod_{j=1}^{k_1} \sigma_j(\boldsymbol{v}_j)$$

where for each j, σ_j is the characteristic function of the open $(n-m_2)$-ball of radius $(V_j/\mathfrak{V}_n)^{1/n}$ centered at the origin, with the origin removed. The volume of the ball which is the support of σ_j is therefore

$$\mathfrak{V}_{n-m_2}(V_j/\mathfrak{V}_n)^{(n-m_2)/n} \ll_m V_j \tag{5.42}$$

(in fact the left-hand side tends to $e^{m_2/2}V_j$ as $n \to \infty$). The key points for us is that these volumes stay *bounded* as $n \to \infty$, since the volumes $V_1, \ldots, V_{k_1}$ are fixed. Therefore, it follows from the bounds in [31, pp. 950(top)–951(middle)] that the sum in (5.41) is $\ll_m (3/4)^{\frac{n-m_2}{2}} \ll_m (3/4)^{\frac{n}{2}}$ (the implied constant also depends on the V_js).

In the same way, we also have

$$\sum_{D_2 \in \widetilde{\mathcal{M}}'_{k_2,m_2}} J_{n-m_1}(\widetilde{\sigma}'; D_2) \ll_m (3/4)^{\frac{n}{2}}. \tag{5.43}$$

Furthermore, by the proof of [31, Prop. 1], for each D_1 in the finite set $\widetilde{\mathcal{M}}'_{k_1,m_1}$ we have that $J_{n-m_2}(\sigma'; D_1)$ tends to a finite limit as $n \to \infty$; the same thing of course also holds for $J_{n-m_1}(\widetilde{\sigma}'; D_2)$ for each fixed $D_2 \in \widetilde{\mathcal{M}}'_{k_2,m_2}$.

In view of these bounds, we conclude that the expression in (5.37) tends to zero exponentially rapidly as $n \to \infty$. This completes the proof of Theorem 5.1. □ □ □

5.4. **Proof of Theorem 1.6.** As in the statement of Theorem 1.6, let $0 < T_1 < T_2 < T_3 < \cdots$ and $0 < T'_1 < T'_2 < T'_3 < \cdots$ denote the points of two independent Poisson processes on $\mathbb{R}^+$ with constant intensity $\frac{1}{2}$. For any fixed $V > 0$ we introduce the two integer-valued random variables

$$N(V) := \#\{j \,:\, T_j < V\} \qquad \text{and} \qquad \widetilde{N}(V) := \#\{j \,:\, T'_j < V\}.$$

Now by standard arguments (cf. the proof of [31, Cor. 1]), Theorem 5.1 implies that, for any fixed $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ and any fixed real numbers $0 < V_1 \leq V_2 \leq \cdots \leq V_{k_1}$ and $0 < W_1 \leq W_2 \leq \cdots \leq W_{k_2}$, the random vector

$$\Big(\tfrac{1}{2}N_1(L), \ldots, \tfrac{1}{2}N_{k_1}(L), \tfrac{1}{2}\widetilde{N}_1(L^*), \ldots, \tfrac{1}{2}\widetilde{N}_{k_2}(L^*)\Big)$$

converges in distribution to the random vector

$$\Big(N(V_1), \ldots, N(V_{k_1}), \widetilde{N}(W_1), \ldots, \widetilde{N}(W_{k_2})\Big).$$

Theorem 1.6 is an immediate consequence of this fact. □ □ □

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