ON THE LOCATION OF THE ZERO-FREE HALF-PLANE OF A RANDOM EPSTEIN ZETA FUNCTION

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Abstract. In this note we study, for a random lattice L of large dimension n, the supremum of the real parts of the zeros of the Epstein zeta function $E_n(L,s)$ and prove that this random variable scaled by n^{-1} has a limit distribution, which we give explicitly. This limit distribution is studied in some detail; in particular we give an explicit formula for its distribution function. Furthermore, we obtain a limit distribution for the frequency of zeros of $E_n(L,s)$ in vertical strips contained in the half-plane $\Re s > \frac{n}{2}$.

1. Introduction

Let X_n denote the space of all n-dimensional lattices $L \subset \mathbb{R}^n$ of covolume one. For $L \in X_n$ and $\Re s > \frac{n}{2}$, the Epstein zeta function is defined by

$$E_n(L,s) = \sideset{}{'}\sum_{\boldsymbol{v} \in L} |\boldsymbol{v}|^{-2s}, \tag{1.1}$$

where $'$ denotes that the zero vector should be omitted. $E_n(L,s)$ has an analytic continuation to $\mathbb{C}$ except for a simple pole at $s=\frac{n}{2}$ with residue $\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})^{-1}$. Furthermore, $E_n(L,s)$ satisfies the functional equation

$$F_n(L,s) = F_n(L^*, \tfrac{n}{2}-s), \tag{1.2}$$

where $F_n(L,s) := \pi^{-s}\Gamma(s)E_n(L,s)$ and L^* is the dual lattice of L.

The Epstein zeta function is in many ways analogous to the Riemann zeta function. In particular we have the relation

$$E_1(\mathbb{Z},s) = 2\zeta(2s).$$

Because of this analogy and for other related reasons, many studies have been made regarding the location of the zeros of $E_n(L,s)$. From (1.2) it is clear that $E_n(L,s)$ has a "trivial" zero at each point $s=-1,-2,-3,\ldots$, just like $\zeta(2s)$, and the remaining nontrivial zeros of $E_n(L,s)$ are in bijective correspondence with the nontrivial zeros of $E_n(L^*,s)$ under the map $s \mapsto \frac{n}{2}-s$. However, the Riemann hypothesis for $E_n(L,s)$ generally fails: $E_n(L,s)$ typically has many nontrivial zeros which do not lie on the critical line $\Re s = \frac{n}{4}$. Cf. [8], [1], [25], [31], [32], [33], [26].

We denote by $N_L(T)$ the number of nontrivial zeros (counting multiplicity) of $E_n(L,s)$ with $|\Im s| \le T$. Then $N_L(T)$ satisfies the following Riemann-von Mangoldt

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type asymptotics ([26]):

$$N_L(T) = \frac{2T}{\pi} \log \frac{T}{\pi e\, m(L) m(L^*)} + O_L(\log T) \qquad \text{as } T \to \infty, \tag{1.3}$$

where $m(L)$ is the length of the shortest non-zero vector in L.

From the point of view of number theory, the most interesting choices of L are those for which the Gram matrix for some (and thus any) $\mathbb{Z}$-basis of L is proportional to an integer matrix. We call these lattices *rational.* In particular when $n = 2$ many results have been obtained regarding the zeros of $E_2(L, s)$ for rational L corresponding to integral quadratic forms with a fundamental discriminant. It was conjectured by H. L. Montgomery that in this case asymptotically 100% of the nontrivial zeros of $E_2(L, s)$ lie along the critical line $\Re s = \frac{1}{2}$. This was proved conditionally, assuming the Generalized Riemann Hypothesis and a weak form of well-spacing for the zeros of L-functions attached to ideal class characters, by Bombieri and Hejhal in [6]. Furthermore, Selberg has proved unconditionally, in still unpublished work (cf. [12, p. 553] and [5, pp. 225-227]) that a positive proportion of the zeros do lie on the critical line. For related results, see also [15] and [19].

Our main object of study in the present paper is the supremum of the real parts of the zeros of $E_n(L, s)$, i.e.

$$\sigma_L := \sup\big\{\Re\rho \,:\, E_n(L, \rho) = 0\big\}.$$

In other words, σ_L gives the precise location of the zero-free right half-plane of $E_n(L, s)$. One easily shows that σ_L exists and is finite for any given $L \in X_n$; furthermore $\sigma_L \geq \frac{n}{4}$ always holds (cf., e.g., [26, p. 693 and Thm. 1]). Of course, $\sigma_L = \sigma_{L^*} = \frac{n}{4}$ is equivalent with the Riemann hypothesis for $E_n(L, s)$. Note that σ_L is lower semicontinuous (and hence Borel measurable), since any zero $s = s_0$ of $E_n(L_0, s)$ gives rise to a nearby zero for all $E_n(L, s)$ with L in a sufficiently small neighborhood of L_0 (as follows from a standard application of Rouche's theorem using the formula [23, (23)] for $\pi^{-s}\Gamma(s)E_n(L, s)$). We also remark that σ_L takes arbitrarily large values for any given $n \geq 2$ (cf. Remark 2 in Section 2).

For any Dirichlet series $f(s) = \sum_{j=1}^{\infty} e^{-\lambda_j s}$ with exponents $\lambda_1 < \lambda_2 < \ldots$ whose pairwise differences do not satisfy any non-trivial linear relation over $\mathbb{Q}$, the supremum of the real parts of the zeros of $f(s)$ equals the unique number σ for which $e^{-\lambda_1\sigma} = \sum_{j=2}^{\infty} e^{-\lambda_j\sigma}$; cf. Lemma 1 below. This independence condition never holds for $E_n(L, s)$ (e.g. since L contains both $2\boldsymbol{v}$ and $4\boldsymbol{v}$ for any $\boldsymbol{v} \in L$). However, we have

$$E_n(L, s) = 2\zeta(2s) \sum_{\boldsymbol{v} \in \widehat{L}} |\boldsymbol{v}|^{-2s} \qquad (\Re s > \tfrac{n}{2}), \tag{1.4}$$

where $\widehat{L}$ denotes a set containing one representative from each pair $\{\boldsymbol{v}, -\boldsymbol{v}\}$ of *primitive* vectors in L; and it turns out that the Dirichlet series $\sum_{\boldsymbol{v} \in \widehat{L}} |\boldsymbol{v}|^{-2s}$ satisfies the independence condition for μ_n*-almost every* lattice $L \in X_n$, where μ_n is Siegel's measure ([24]) on X_n; see Lemma 2. From this we conclude (cf. Section 2):

Proposition 1. *Let $n \geq 2$. For almost every $L \in X_n$, σ_L equals the unique number $\sigma > \frac{n}{2}$ which satisfies $2m(L)^{-2\sigma} = \frac{1}{2}\zeta(2\sigma)^{-1}E_n(L, \sigma)$. It follows that for almost every $L \in X_n$, $E_n(L, s)$ has infinitely many zeros with $\Re s > \frac{n}{2}$.*

In particular, for small n the formula in Proposition 1 makes it possible to compute σ_L numerically for a given generic $L \in X_n$. We stress, however, that for a lattice L

such that $\sum_{\boldsymbol{v}\in\widehat{L}} |\boldsymbol{v}|^{-2s}$ does *not* satisfy the linear independence condition (e.g. any rational L, cf. Remark 3 in Section 2), the computation of σ_L is in general not an easy task. We mention that Bombieri and Mueller in [7] have shown how to calculate σ_L explicitly for certain examples of rational lattices $L \in X_2$ (with $\sigma_L > 1$), where they also obtained bounds on the asymptotic rate of approach of the zeros of $E_2(L,s)$ to the line $\Re s = \sigma_L$. See also [5] for a related investigation of the supremum of the real parts of the zeros of certain other Dirichlet series.

Our main result concerns the distribution of σ_L for a *random* lattice L in large dimension n. The random element $L \in X_n$ will always be chosen according to Siegel's measure μ_n, normalized to be a probability measure. The present study is motivated by recent investigations [29] of the value distribution of $E_n(L,s)$ for $\Re s > \frac{n}{2}$ and a μ_n-random lattice L of large dimension n, where the following result is established: Let V_n denote the volume of the n-dimensional unit ball. Let $\mathcal{P}$ be a Poisson process on the positive real line with intensity $\frac{1}{2}$ and let $T_1, T_2, T_3, \ldots$ denote the points of $\mathcal{P}$ ordered so that $0 < T_1 < T_2 < T_3 < \cdots$. Then, for any fixed $s \in \mathbb{C}$ with $\Re s > \frac{1}{2}$,

$$V_n^{-2s} E_n(\cdot, ns) \xrightarrow{\mathrm{d}} 2 \sum_{j=1}^{\infty} T_j^{-2s} \qquad \text{as } n \to \infty, \tag{1.5}$$

i.e. the random variable $V_n^{-2s} E_n(\cdot, ns)$ converges in distribution to $2\sum_{j=1}^{\infty} T_j^{-2s}$.

The proof of (1.5) is built on a result [28] which provides the connection between the lengths of lattice vectors appearing in the formula (1.1) and the points of the Poisson process $\mathcal{P}$. Since this result is an important ingredient also in the present investigation we recall it here. Given a lattice $L \in X_n$, we order its non-zero vectors by increasing lengths as $\pm\boldsymbol{v}_1, \pm\boldsymbol{v}_2, \pm\boldsymbol{v}_3, \ldots$, set $\ell_j = |\boldsymbol{v}_j|$ (thus $0 < \ell_1 \leq \ell_2 \leq \ldots$), and define

$$\mathcal{V}_j(L) := V_n \ell_j^n \,, \tag{1.6}$$

so that $\mathcal{V}_j(L)$ is the volume of an n-dimensional ball of radius ℓ_j. The main result in [28] states that, as $n \to \infty$, the volumes $\{\mathcal{V}_j(L)\}_{j=1}^{\infty}$ determined by a random lattice $L \in X_n$ converges in distribution to the points $\{T_j\}_{j=1}^{\infty}$ of the Poisson process $\mathcal{P}$ on the positive real line with constant intensity $\frac{1}{2}$.

In view of the last two paragraphs, together with Proposition 1 and the fact that $\zeta(2\sigma) \to 1$ as $\sigma \to \infty$, it seems reasonable to expect that as $n \to \infty$, $n^{-1}\sigma_L$ should tend in distribution to

$$\sigma_{\{T_j\}} := \Big[\text{the unique } \sigma > \tfrac{1}{2} \text{ satisfying } T_1^{-2\sigma} = \sum_{j=2}^{\infty} T_j^{-2\sigma}\Big]. \tag{1.7}$$

(We will show in Section 3 that $\sigma_{\{T_j\}}$ is a well-defined random variable.) Our first theorem states that this is indeed the case.

Theorem 1. *If L is taken at random in X_n according to μ_n, then*

$$n^{-1}\sigma_L \xrightarrow{\mathrm{d}} \sigma_{\{T_j\}} \qquad \text{as } n \to \infty.$$

By similar techniques we also obtain a limit distribution statement concerning the *frequency* of zeros of $E_n(L,s)$ in arbitrary vertical strips to the right of $\Re s = \frac{n}{2}$. For any $\sigma_1 < \sigma_2$ and $\tau_1 < \tau_2$, let $N_L(\sigma_1, \sigma_2; \tau_1, \tau_2)$ be the number of zeros of $E_n(L,s)$ in the rectangle $s \in (\sigma_1, \sigma_2) \times (\tau_1, \tau_2)$, counting multiplicity. It follows from Jessen

[13, Satz A] that for each $L \in X_n$, and for any fixed numbers $\sigma_1, \sigma_2 \in (\frac{n}{2}, \infty) \setminus \mathfrak{S}_L$, $\sigma_1 < \sigma_2$, where $\mathfrak{S}_L$ is a certain finite or countable set of exceptions, the limit

$$H_L(\sigma_1, \sigma_2) := \lim_{\tau_2 - \tau_1 \to \infty} \frac{N_L(\sigma_1, \sigma_2; \tau_1, \tau_2)}{\tau_2 - \tau_1} \tag{1.8}$$

exists. For L satisfying the independence condition discussed above (recall that this holds for μ_n-almost every $L \in X_n$), $\mathfrak{S}_L$ is empty, i.e. the limit (1.8) exists for *all* $\frac{n}{2} < \sigma_1 < \sigma_2$, and we furthermore have

$$H_L(\sigma_1, \sigma_2) = \int_{\sigma_1}^{\sigma_2} \nu_L(\sigma) \, d\sigma, \tag{1.9}$$

where $\nu_L(\sigma)$ is a continuous function on $(\frac{n}{2}, \infty)$. These statements follow from Jessen's work [14] (cf. Section 4 below). We remark that in recent work by Lee [18] and Gonek-Lee [11], similar asymptotics for the number of zeros of $E_2(L, s)$ are obtained in the more difficult case of a rational $L \in X_2$ corresponding to an integral quadratic form with a fundamental discriminant.

Similarly, for almost any realization of the Poisson process $\mathcal{P}$, there is a continuous function $\nu_{\{T_j\}}$ on $(\frac{1}{2}, \infty)$ such that, if $N(\sigma_1, \sigma_2; \tau_1, \tau_2)$ denotes the number of zeros of the Dirichlet series $f_{\{T_j\}}(s) := \sum_{j=1}^{\infty} T_j^{-2s}$ in the rectangle $s \in (\sigma_1, \sigma_2) \times (\tau_1, \tau_2)$, then for any $\frac{1}{2} < \sigma_1 < \sigma_2$,

$$\lim_{\tau_2 - \tau_1 \to \infty} \frac{N(\sigma_1, \sigma_2; \tau_1, \tau_2)}{\tau_2 - \tau_1} = \int_{\sigma_1}^{\sigma_2} \nu_{\{T_j\}}(\sigma) \, d\sigma. \tag{1.10}$$

Let $C(\frac{1}{2}, \infty)$ be the set of all real-valued continuous functions on $(\frac{1}{2}, \infty)$, provided with the topology of uniform convergence on compacta.

Theorem 2. *If L is taken at random in X_n according to μ_n, then*

$$\big(n^2 \nu_L(n \cdot), n^{-1} \sigma_L\big) \xrightarrow{\mathrm{d}} (\nu_{\{T_j\}}, \sigma_{\{T_j\}}) \qquad \text{as } n \to \infty,$$

in the sense of convergence in distribution for random elements in $C(\frac{1}{2}, \infty) \times \mathbb{R}_{>1/2}$.

Note that Theorem 2 generalizes Theorem 1, and also implies that the random function $\sigma \mapsto n^2 \nu_L(n\sigma)$ in $C(\frac{1}{2}, \infty)$ converges in distribution to $\nu_{\{T_j\}}$. A consequence of the latter fact is that for any fixed $\frac{1}{2} < \sigma_1 < \sigma_2$, the real-valued random variable $n H_L(n\sigma_1, n\sigma_2)$ converges in distribution to $\int_{\sigma_1}^{\sigma_2} \nu_{\{T_j\}}(\sigma) \, d\sigma$.

Returning to our main objects of study, i.e. σ_L and its limit $\sigma_{\{T_j\}}$, we next give an explicit formula for the distribution function of $\sigma_{\{T_j\}}$. Recall that the lower incomplete gamma function $\gamma(s, z)$ is defined by

$$\gamma(s, z) := \int_0^z u^{s-1} e^{-u} \, du \tag{1.11}$$

for $s, z \in \mathbb{C}$ with $\Re s > 0$. In order to make $\gamma(s, z)$ single-valued, we will always keep $z \in \mathbb{C} \setminus \mathbb{R}_{<0}$ (in fact, we will only need to use z with $\Re z \geq 0$), and choose a path of integration in (1.11) which stays inside this cut plane. We agree that $|\arg u| < \pi$ for all $u \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Now the function $\gamma(s, z)$ is extended to all $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, $z \in \mathbb{C} \setminus \mathbb{R}_{<0}$ through the recursion formula

$$\gamma(s, z) = \frac{\gamma(s+1, z) + z^s e^{-z}}{s}. \tag{1.12}$$

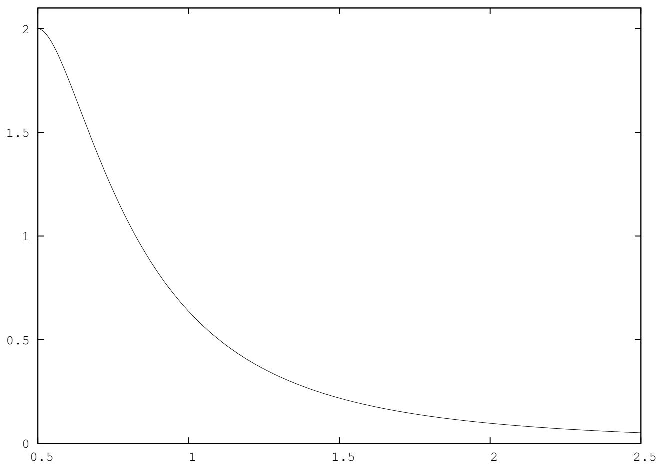

FIGURE 1. A graph of the density function of $\sigma_{\{T_j\}}$. It was computed using the method described in Appendix A (see also [27, numdensity.mpl]).

Theorem 3. *For any $c > \frac{1}{2}$, we have*

$$Prob\big(\sigma_{\{T_j\}} \le c\big) = \frac{1}{2} + \frac{2c}{\pi} \int_0^\infty \Im \left(\frac{e^{(\frac{\pi}{4c} - y)i}}{\gamma(-\frac{1}{2c}, -iy)} \right) y^{-1-\frac{1}{2c}}\, dy.$$

The integral in the right-hand side is absolutely convergent.

Corollary 1. *The random variable $\sigma_{\{T_j\}}$ has a continuous density function $f(c)$ given explicitly in* (6.1) *below. It satisfies*

$$f(c) = 2 - K_1(c - \tfrac{1}{2})^2 + O((c - \tfrac{1}{2})^3) \qquad as\ \ c \to \tfrac{1}{2} \tag{1.13}$$

and

$$f(c) = K_2 c^{-3} + O\left(c^{-4}\right) \qquad as\ \ c \to \infty, \tag{1.14}$$

where $K_1 = 39.47841\ldots$ and $K_2 = 0.822467\ldots$ are positive real numbers given explicitly below in (6.20) *and* (6.14)*, respectively (cf. also* (6.3)*,* (6.13)*,* (6.7) *and* (6.17)*).*

In Appendix A, we also give formulas for the distribution and density functions of $\sigma_{\{T_j\}}$ obtained through the residue theorem, and discuss numerical evaluation. See Figure 1 for a graph of the probability density function of $\sigma_{\{T_j\}}$ generated using the formulas in Appendix A (cf. [27, numdensity.mpl]).

We conclude by remarking that, as is rather clear from the previous discussion, the random variable $\sigma_{\{T_j\}}$ can also be interpreted as the supremum of the real parts of the zeros of the *random Dirichlet series* $f_{\{T_j\}}(s) = \sum_{j=1}^\infty T_j^{-2s}$. Indeed, by the strong law of large numbers the series $f_{\{T_j\}}(s)$ has, with probability one, abscissa of absolute convergence $\sigma_0 = \frac{1}{2}$ and satisfies $\lim_{\sigma \to \frac{1}{2}+} f_{\{T_j\}}(\sigma) > 2T_1^{-1}$; also with probability one the numbers $2(\log T_j - \log T_1)$, $j = 2, 3, \ldots$, are linearly independent over $\mathbb{Q}$; this means that Lemma 1 below applies almost surely and the claim follows. Hence Theorem 3 and Corollary 1 describe explicitly the distribution of the location of the zero-free right half-plane of $f_{\{T_j\}}(s)$. It would be interesting to also seek a more explicit understanding of the random function $\nu_{\{T_j\}}$ (cf. (1.10) and Theorem 2), which describes the density of zeros of $f_{\{T_j\}}(s)$ in any vertical strip to the right of $\Re s = \frac{1}{2}$.

There exists a vast literature on random Dirichlet series; however, we are not aware of many results pertaining to their zeros. Cf., however, Edelman and Kostlan [9, §§3.2.5, 8.2], regarding the zeros of the random Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$, where a_n are independent standard normal random variables.

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2. Proof of Proposition 1

Lemma 1. *Consider any Dirichlet series $f(s) = \sum_{j=1}^{\infty} e^{-\lambda_j s}$ with real exponents $\lambda_1 < \lambda_2 < \ldots$ and abscissa of absolute convergence $\sigma_0 < \infty$. Assume $\lim_{\sigma \to \sigma_0^+} f(\sigma) > 2e^{-\lambda_1 \sigma_0}$. Then the equation $f(\sigma) = 2e^{-\lambda_1 \sigma}$ has exactly one real root $\sigma = \sigma_f > \sigma_0$. If furthermore all the differences $\lambda_j - \lambda_1$ for $j = 2, 3, \ldots$ are linearly independent over $\mathbb{Q}$, then σ_f equals the supremum of the real parts of the zeros of $f(s)$, and the function $f(s)$ has infinitely many zeros in any strip $\sigma_1 < \Re s < \sigma_2$ with $\sigma_0 \leq \sigma_1 < \sigma_2 \leq \sigma_f$.*

Remark 1. The linear independence condition of the lemma is equivalent to the statement that if $c_1, c_2, \ldots$ are any integers all but finitely many vanishing and satisfying $\sum_{n=1}^{\infty} c_n = 0$ and $\sum_{n=1}^{\infty} c_n \lambda_n = 0$, then $c_1 = c_2 = \ldots = 0$. This is also equivalent to the statement that the pairwise differences among $\lambda_1, \lambda_2, \ldots$ do not satisfy any non-trivial linear relation over $\mathbb{Q}$, i.e. if c_{jk} for $1 \leq j < k$ are integers all but finitely many vanishing and satisfying $\sum_{j<k} c_{jk}(\lambda_j - \lambda_k) = 0$, then $\sum_{k=j+1}^{\infty} c_{jk} = \sum_{k=1}^{j-1} c_{kj}$ for all $j \geq 1$.

Proof. The fact that the equation $f(\sigma) = 2e^{-\lambda_1 \sigma}$ has exactly one real root $\sigma = \sigma_f > \sigma_0$ follows since the function $\sigma \mapsto e^{\lambda_1 \sigma} f(\sigma)$ for $\sigma > \sigma_0$ is strictly decreasing, tends to 1 as $\sigma \to \infty$, and by assumption tends to a limit which is greater than 2 as $\sigma \to \sigma_0^+$. For any s with $\Re s > \sigma_f$ we have $|\sum_{j=2}^{\infty} e^{-\lambda_j s}| \leq \sum_{j=2}^{\infty} e^{-\lambda_j \Re s} < e^{-\lambda_1 \Re s} = |e^{-\lambda_1 s}|$, and thus $f(s) \neq 0$. Hence it now only remains to prove that $f(s)$ has infinitely many zeros in any strip $\sigma_1 < \Re s < \sigma_2$ with $\sigma_0 \leq \sigma_1 < \sigma_2 \leq \sigma_f$. Assume the contrary; then there even exist some σ_1, σ_2 with $\sigma_0 \leq \sigma_1 < \sigma_2 \leq \sigma_f$ such that $f(s)$ has *no* zero in the strip $\sigma_1 < \Re s < \sigma_2$. By basic facts in complex analysis, this implies that

$$\inf_{t \in \mathbb{R}} |f(\sigma + it)| > 0 \tag{2.1}$$

for any fixed $\sigma \in (\sigma_1, \sigma_2)$ (cf. [4, §4 (Hilfssatz 3)] and [13, §3 (Hilfssatz 3)]).

On the other hand, for any $\sigma \in (\sigma_1, \sigma_2)$ we have $\sum_{j=2}^{\infty} e^{(\lambda_1 - \lambda_j)\sigma} > 1$, and hence there exist $\zeta_2, \zeta_3, \ldots \in \mathbb{C}$ satisfying $|\zeta_2| = |\zeta_3| = \ldots = 1$ and $\sum_{j=2}^{\infty} e^{(\lambda_1 - \lambda_j)\sigma} \zeta_j = -1$. It follows from Kronecker's density theorem (cf., e.g., [17, Prop. 1.5.1]), using our linear independence assumption, that for any given $J \in \mathbb{Z}_{\geq 2}$ and $\varepsilon > 0$ there exists $t \in \mathbb{R}$ such that $|e^{(\lambda_1 - \lambda_j)it} - \zeta_j| < \varepsilon$ for all $j \in \{2, \ldots, J\}$. Applying this with $J \to \infty$

and $\varepsilon \to 0$, we conclude that

$$\inf_{t\in\mathbb{R}} |f(\sigma+it)| = e^{-\lambda_1\sigma} \inf_{t\in\mathbb{R}} \Big| 1 + \sum_{j=2}^{\infty} e^{(\lambda_1-\lambda_j)(\sigma+it)} \Big| = 0.$$

This contradicts (2.1), and hence the lemma is proved. □

Recall the definition of the set $\widehat{L}$ from just below equation (1.4). We now prove:

Lemma 2. *For each $n \geq 2$ and almost all $L \in X_n$ the following holds: The vector lengths $|\boldsymbol{v}|$ for $\boldsymbol{v} \in \widehat{L}$ are all distinct, and the pairwise differences of their logarithms do not satisfy any non-trivial linear relation over $\mathbb{Q}$.*

Proof. We realize X_n as the homogeneous space $\mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})$, where $\mathrm{SL}(n,\mathbb{Z})g$ corresponds to the lattice $\mathbb{Z}^n g \subset \mathbb{R}^n$. Note that μ_n is the unique probability measure on X_n induced from a Haar measure on $\mathrm{SL}(n,\mathbb{R})$. We will let μ_n denote also the corresponding Haar measure on $\mathrm{SL}(n,\mathbb{R})$. Now the statement of the lemma is equivalent to the following: For almost every matrix $M \in \mathrm{SL}(n,\mathbb{R})$, any finite sequence of primitive vectors $\boldsymbol{u}_1,\ldots,\boldsymbol{u}_N \in \mathbb{Z}^n$ $(N \geq 2)$ with $\boldsymbol{u}_j \neq \pm\boldsymbol{u}_k$ for $j \neq k$, and any $b_1,\ldots,b_N \in \mathbb{Z}\setminus\{0\}$ with $\sum_{j=1}^N b_j = 0$, we have

$$\sum_{j=1}^{N} b_j \log \big|\boldsymbol{u}_j M\big| \neq 0. \tag{2.2}$$

Since there are only countably many possible $2N$-tuples $(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_N,b_1,\ldots,b_N)$ it suffices to prove that for each *fixed* choice of $(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_N,b_1,\ldots,b_N)$ the set of $M \in \mathrm{SL}(n,\mathbb{R})$ satisfying (2.2) has full measure in $\mathrm{SL}(n,\mathbb{R})$. We note that (2.2) is equivalent to $\prod_{j=1}^N \big|\boldsymbol{u}_j M\big|^{2b_j} \neq 1$, i.e.

$$\prod_{\substack{j=1\\(b_j>0)}}^{N} \big|\boldsymbol{u}_j M\big|^{2b_j} - \prod_{\substack{j=1\\(b_j<0)}}^{N} \big|\boldsymbol{u}_j M\big|^{2|b_j|} \neq 0. \tag{2.3}$$

Hence, by the explicit formula for the measure μ_n on $\mathrm{SL}(n,\mathbb{R})$ in terms of the matrix entries (cf., e.g., [34]) and the fact that the left-hand side of (2.3) is homogeneous in M (since $\sum_{j=1}^N b_j = 0$), we find that it is enough to prove that for any fixed choice of $(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_N,b_1,\ldots,b_N)$ as above, the relation (2.3) holds for Lebesgue almost all matrices $M \in \mathrm{Mat}_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

Note that each factor $|\boldsymbol{u}_j M|^2$ in (2.3) is a real polynomial in the n^2 matrix entries of M. Our conditions on $\boldsymbol{u}_1,\ldots,\boldsymbol{u}_N$ imply in particular that $\boldsymbol{u}_1$ is not proportional to any of $\boldsymbol{u}_2,\ldots,\boldsymbol{u}_N$, and thus the set S of vectors in $\mathbb{R}^n$ which are orthogonal to $\boldsymbol{u}_1$ but not orthogonal to any of $\boldsymbol{u}_2,\ldots,\boldsymbol{u}_N$ is non-empty. Now, if we take any $M \in \mathrm{Mat}_{n,n}(\mathbb{R})$ all of whose column vectors lie in S, we note that the left-hand side of (2.3) is non-zero. This proves that the left-hand side of (2.3) is a real, *non-zero* polynomial in the n^2 matrix entries of M. Hence the condition (2.3) is indeed fulfilled for Lebesgue almost all $M \in \mathrm{Mat}_{n,n}(\mathbb{R})$, and the lemma is proved. □

Proof of Proposition 1. Take any $L \in X_n$ such that the vector lengths $|\boldsymbol{v}|$ for $\boldsymbol{v} \in \widehat{L}$ are all distinct, and the pairwise differences of their logarithms do not satisfy any non-trivial linear relation over $\mathbb{Q}$. By Lemma 2 this holds for almost every L. Having fixed any such L, we consider the Dirichlet series $f(s) = \sum_{\boldsymbol{v}\in\widehat{L}} |\boldsymbol{v}|^{-2s}$. The abscissa of (absolute) convergence for $f(s)$ is $\sigma_0 = \frac{n}{2}$, and we have $\lim_{\sigma \to \frac{n}{2}+} f(\sigma) = \infty$; cf. (1.4). Hence, by Lemma 1, the supremum of the real parts of the zeros of $f(s)$ equals the unique real root $\sigma_f > \frac{n}{2}$ of the equation $f(\sigma) = 2|\boldsymbol{v}_1|^{-2\sigma}$, where $\boldsymbol{v}_1$ is the shortest vector in $\widehat{L}$; in fact $f(s)$ has infinitely many zeros in any strip $\sigma_1 < \Re s < \sigma_2$ with $\frac{n}{2} \leq \sigma_1 < \sigma_2 \leq \sigma_f$. Using $|\boldsymbol{v}_1| = m(L)$ and (1.4), we see that the equation for σ_f may equivalently be expressed as $2m(L)^{-2\sigma} = \frac{1}{2}\zeta(2\sigma)^{-1}E_n(L,\sigma)$. Finally, using (1.4) and the fact that $\zeta(s)$ does not have any zeros when $\Re s > 1$, we see that $E_n(L,s)$ has exactly the same zeros (also counting multiplicity) as $f(s)$ in the half-plane $\Re s > \frac{n}{2}$. This completes the proof of Proposition 1. $\square$

Remark 2. Consider the function

$$L \mapsto \tilde{\sigma}_L := \Big[\sigma > \tfrac{n}{2} \text{ such that } 2m(L)^{-2\sigma} = \tfrac{1}{2}\zeta(2\sigma)^{-1}E_n(L,\sigma)\Big].$$

(Thus Proposition 1 says that $\sigma_L = \tilde{\sigma}_L$ almost everywhere.) Let X_n' be the (closed) subset of X_n consisting of those lattices for which $\#\{\boldsymbol{v} \in L \,:\, |\boldsymbol{v}| = m(L)\} > 2$; by [30, Lemma 5.1], X_n' has measure zero. We claim that $\tilde{\sigma}_L$ is a smooth function from $X_n \setminus X_n'$ to $\mathbb{R}_{>n/2}$. This follows by studying the following function on $\mathbb{R}_{>n/2} \times X_n$:

$$\alpha(\sigma, L) = 2m(L)^{-2\sigma} - \tfrac{1}{2}\zeta(2\sigma)^{-1}E_n(L,\sigma) = 2m(L)^{-2\sigma} - \sum_{\boldsymbol{v} \in \widehat{L}} |\boldsymbol{v}|^{-2\sigma}.$$

This function is smooth on all $\mathbb{R}_{>n/2} \times (X_n \setminus X_n')$ and one easily checks that $\frac{\partial}{\partial\sigma}\alpha(\sigma,L) > 0$ at all points $\sigma = \tilde{\sigma}_L$, $L \in X_n \setminus X_n'$. Hence our smoothness claim follows from the implicit function theorem.

On the other hand note that $\tilde{\sigma}_L \to \infty$ whenever $L \to L_0$ for some $L_0 \in X_n'$. In particular this shows, via Proposition 1, that $\sup_{L \in X_n} \sigma_L = \infty$. (But of course, as we remarked in the introduction, σ_L is finite for any fixed $L \in X_n$, in particular for any $L \in X_n'$!)

Remark 3. Note that for any *rational* $L \in X_n$, there occur arbitrarily large multiplicities among the lengths $|\boldsymbol{v}|$ for $\boldsymbol{v} \in \widehat{L}$; in particular, the independence condition in Lemma 2 *fails* for every rational $L \in X_n$. Indeed, for $n \geq 3$ this claim follows easily from the fact that the number of $\boldsymbol{v} \in \widehat{L}$ with $|\boldsymbol{v}| \leq R$ grows like R^n as $R \to \infty$, while the number of possible values of $|\boldsymbol{v}|$ grows at most like R^2 (since L rational implies that there is some $c > 0$ such that $|\boldsymbol{v}|^2 \in c\mathbb{Z}$ for all $\boldsymbol{v} \in L$). The claim also holds for $n = 2$, since in this case the number of possible values of $|\boldsymbol{v}|$ with $|\boldsymbol{v}| \leq R$ is in fact $\ll R^2(\log R)^{-\frac{1}{2}}$ as $R \to \infty$; cf. [2] or [20].

3. Proof of Theorem 1

The sequence $\{T_j\}_{j=1}^{\infty}$ of points of the Poisson process $\mathcal{P}$ belongs to the space

$$\Omega := \Big\{\boldsymbol{x} = \{x_j\}_{j=1}^{\infty} \in (\mathbb{R}_{>0})^{\infty} \,:\, 0 < x_1 < x_2 < x_3 < \ldots\Big\},$$

which we equip with the subspace topology induced from the product topology on $(\mathbb{R}_{>0})^{\infty}$. We denote the distribution of $\mathcal{P}$ on Ω by $\mathbf{P}$; this is a Borel probability measure on Ω.

Recall the definition (1.7) of $\sigma_{\{T_j\}}$; let us prove that this is a well-defined random variable on $(\Omega, \mathbf{P})$. We set

$$\Omega' := \Big\{\boldsymbol{x} = \{x_j\}_{j=1}^{\infty} \in \Omega \,:\, \#\{x_j < X\} \sim \tfrac{1}{2}X \text{ as } X \to \infty\Big\}.$$

This is a Borel subset of Ω and, by the strong law of large numbers, we have $\mathbf{P}\Omega' = 1$. For any $\boldsymbol{x} \in \Omega'$, we have $\sum_{j=1}^{\infty} x_j^{-2\sigma} < \infty$ for all $\sigma > \frac{1}{2}$ and $\sum_{j=1}^{\infty} x_j^{-2\sigma} \to \infty$ as $\sigma \to \frac{1}{2}^{+}$, and thus, by the same argument as in the proof of Lemma 1, there exists a unique $\sigma > \frac{1}{2}$ satisfying $x_1^{-2\sigma} = \sum_{j=2}^{\infty} x_j^{-2\sigma}$. In other words, $\sigma_{\boldsymbol{x}}$ is well-defined for every $\boldsymbol{x} \in \Omega'$. Furthermore, for any $c > \frac{1}{2}$, we have $\{\boldsymbol{x} \in \Omega' : \sigma_{\boldsymbol{x}} < c\} = \{\boldsymbol{x} \in \Omega' : x_1^{-2c} - \sum_{j=2}^{\infty} x_j^{-2c} > 0\}$, which is a Borel set. This proves that the function $\boldsymbol{x} \mapsto \sigma_{\boldsymbol{x}}$ is $\mathbf{P}$-measurable on Ω, i.e. that $\sigma_{\{T_j\}}$ is indeed a well-defined random variable.

For given $n \geq 2$ and $c > \frac{1}{2}$, we let $F_n(L, c)$ be the random variable given by

$$F_n(L,c) := -\mathcal{V}_1(L)^{-2c} + \sum_{j=2}^{\infty} \mathcal{V}_j(L)^{-2c}, \tag{3.1}$$

where as usual L is taken at random in X_n according to μ_n. We also let $F(c)$ be the random variable

$$F(c) := -T_1^{-2c} + \sum_{j=2}^{\infty} T_j^{-2c}. \tag{3.2}$$

Lemma 3. *Let $c > \frac{1}{2}$ be fixed. Then $F_n(L,c)$ converges in distribution to $F(c)$ as $n \to \infty$.*

Proof. The proof is a straightforward adaptation of the proof of [29, Thm. 1] with $m = 1$. □

Lemma 4. *For any given $c > \frac{1}{2}$ and $\tau \in \mathbb{R}$, we have $\mathbf{P}\big\{F(c) = \tau\big\} = 0$.*

Proof. The lemma follows immediately from the calculations in the first paragraph of the proof of Theorem 3 (cf. p. 15 below). □

Proof of Theorem 1. It suffices to prove that for any fixed $c > \frac{1}{2}$, $Prob_{\mu_n}(n^{-1}\sigma_L > c)$ tends to $\mathbf{P}(\sigma_{\{T_j\}} > c)$ as $n \to \infty$. By Proposition 1 and the monotonicity argument at the beginning of the proof of Lemma 1, we have

$$\begin{aligned} Prob_{\mu_n}(n^{-1}\sigma_L > c) &= Prob_{\mu_n}\Big\{L \in X_n \,:\, -2\mathcal{V}_1(L)^{-2c} + \zeta(2cn)^{-1}\sum_{j=1}^{\infty}\mathcal{V}_j(L)^{-2c} > 0\Big\} \\ &= Prob_{\mu_n}\Big\{L \in X_n \,:\, F_n(L,c) > \big(1 - \zeta(2cn)^{-1}\big)\sum_{j=1}^{\infty}\mathcal{V}_j(L)^{-2c}\Big\}. \end{aligned}$$

Now let $\varepsilon > 0$ be given. By Lemma 4 we have $\mathbf{P}\big\{F(c) = 0\big\} = 0$, and hence (using [21, Thm. 1.19(e)]) there exists $\tau > 0$ such that

$$\mathbf{P}\big\{F(c) \in [0,\tau]\big\} < \varepsilon. \tag{3.3}$$

Furthermore, it follows from [29] that there exists $K > 0$ and $N \in \mathbb{Z}^+$ such that

$$Prob_{\mu_n}\Big\{L \in X_n \,:\, \sum_{j=1}^{\infty}\mathcal{V}_j(L)^{-2c} < K\Big\} > 1 - \varepsilon, \qquad \forall n \geq N.$$

After possibly increasing N, we may also assume that $(1 - \zeta(2cn)^{-1})K < \tau$ for all $n \geq N$. It follows that, for all $n \geq N$,

$$Prob_{\mu_n}\big(F_n(L,c) > \tau\big) - \varepsilon \leq Prob_{\mu_n}\big(n^{-1}\sigma_L > c\big) \leq Prob_{\mu_n}\big(F_n(L,c) > 0\big).$$

However, by Lemma 3 and Lemma 4, we have (cf. [3, Thm. 2.1(v)])

$$\lim_{n\to\infty} Prob_{\mu_n}\big(F_n(L,c)>0\big)=\mathbf{P}\big(F(c)>0\big)=\mathbf{P}\big(\sigma_{\{T_j\}}>c\big)$$

and

$$\lim_{n\to\infty} Prob_{\mu_n}\big(F_n(L,c)>\tau\big)=\mathbf{P}\big(F(c)>\tau\big).$$

Furthermore, by (3.3) we have $\mathbf{P}\big(F(c)>\tau\big)>\mathbf{P}\big(F(c)>0\big)-\varepsilon=\mathbf{P}\big(\sigma_{\{T_j\}}>c\big)-\varepsilon$. Hence we obtain

$$\limsup_{n\to\infty} Prob_{\mu_n}\big(n^{-1}\sigma_L>c\big)\le\mathbf{P}\big(\sigma_{\{T_j\}}>c\big)$$

and

$$\liminf_{n\to\infty} Prob_{\mu_n}\big(n^{-1}\sigma_L>c\big)\ge\mathbf{P}\big(\sigma_{\{T_j\}}>c\big)-2\varepsilon.$$

But ε is arbitrary and hence the proof of Theorem 1 is complete. □

4. Proof of Theorem 2

Lemma 5. *Let $f(s)=\sum_{j=1}^\infty e^{-\lambda_j s}$ be a Dirichlet series with real exponents $\lambda_1<\lambda_2<\cdots$ and abscissa of absolute convergence $\sigma_0<\infty$, and set, for $\sigma>\sigma_0$,*

(4.1)
$$\nu(\{\lambda_j\};\sigma):=\frac{1}{2\pi}\sum_{a=1}^\infty\sum_{b=1}^\infty(-1)^{\mathbb{1}(a\neq b)}\lambda_a\lambda_b e^{-(\lambda_a+\lambda_b)\sigma}\int_0^\infty\Big(\prod_{j=1}^\infty J_{[j;a,b]}(e^{-\lambda_j\sigma}r)\Big)\,r\,dr,$$

where $\mathbb{1}(\cdot)$ is the indicator function; $[j;a,b]:=1$ if $j\in\{a,b\}$ and $a\neq b$, otherwise $[j;a,b]:=0$; and $J_\alpha(x)$ is the Bessel function of order $\alpha\in\{0,1\}$. Let $N(\sigma_1,\sigma_2;\tau_1,\tau_2)$ be the number of zeros of $f(s)$ in the rectangle $s\in(\sigma_1,\sigma_2)\times(\tau_1,\tau_2)$, counting multiplicity. If $\lambda_1,\lambda_2,\ldots$ are linearly independent over $\mathbb{Q}$, then, for any fixed $\sigma_1<\sigma_2$ in $\mathbb{R}_{>\sigma_0}$,

$$\lim_{\tau_2-\tau_1\to\infty}\frac{N(\sigma_1,\sigma_2;\tau_1,\tau_2)}{\tau_2-\tau_1}=\int_{\sigma_1}^{\sigma_2}\nu(\{\lambda_j\};\sigma)\,d\sigma. \tag{4.2}$$

Proof. This follows from Jessen [14]; cf. in particular [14, Sec. 28] and the explicit formula for $G(\sigma,z)$ in [14, Sec. 24] (applied with $z=0$; we evaluate the Fourier integral in [14, p. 310 (line -10)] using the explicit formula for Ψ found in the same section). The expression in (4.1) is nicely convergent for any $\sigma>\sigma_0$ and defines a continuous function of σ (cf. [14, Sec. 24] or [35]; more details on convergence also appear in the proof of Lemma 8 below). □

Lemma 6. *Let $\nu(\{\lambda_j\};\sigma)$ be as in Lemma 5. Then $\nu(\{\lambda_j+\alpha\};\sigma)=\nu(\{\lambda_j\};\sigma)$ for any constant $\alpha\in\mathbb{R}$.*

Proof. If $\lambda_1,\lambda_2,\ldots$ are linearly independent over $\mathbb{Q}$, and the same holds for $\lambda_1+\alpha,\lambda_2+\alpha,\ldots$, then the claim follows from (4.2), since $f(s)$ and $e^{-\alpha s}f(s)$ have the same zeros. The general case follows by continuity (cf. [14, end of Sec. 24]). □

Remark 4. Lemma 6 can also be proved directly from (4.1) by using the identity

$$c_a\int_0^\infty\Big(\prod_{j=1}^\infty J_0(c_jr)\Big)\,r\,dr=\sum_{b\neq a}c_b\int_0^\infty J_1(c_ar)J_1(c_br)\Big(\prod_{j\notin\{a,b\}}J_0(c_jr)\Big)\,r\,dr, \tag{4.3}$$

which holds for any $c_1, c_2, \ldots > 0$ with $\sum_j c_j < \infty$, and any $a \in \mathbb{Z}^+$. One proves (4.3) by integration by parts, using $c_a r J_0(c_a r) = \frac{d}{dr}(rJ_1(c_a r))$ and $\frac{d}{dr} J_0(c_b r) = -c_b J_1(c_b r)$.

Now let $L \in X_n$ be any lattice which is generic in the sense of Lemma 2; order the vectors of $\widehat{L}$ by increasing lengths as $\widehat{\boldsymbol{v}}_1, \widehat{\boldsymbol{v}}_2, \ldots$, and set $\lambda_j = 2\log|\widehat{\boldsymbol{v}}_j|$, so that $f(s) = \sum_{\boldsymbol{v}\in\widehat{L}} |\boldsymbol{v}|^{-2s}$ in Lemma 5. The condition of Lemma 2 implies that for Lebesgue almost every $\alpha \in \mathbb{R}$, the numbers $\lambda_1 + \alpha, \lambda_2 + \alpha, \ldots$ are linearly independent over $\mathbb{Q}$. Hence, by (1.4) and Lemmas 5 and 6, relations (1.8) and (1.9) hold for all $\sigma_1 < \sigma_2$ in $(\frac{n}{2}, \infty)$, with $\nu_L(\sigma) = \nu(\{\lambda_j\}; \sigma)$.

Remark 5. For such a lattice $L \in X_n$, it follows from (1.8) and (1.9) that $\nu_L(\sigma) = 0$ for all $\sigma \geq \sigma_L$. Furthermore, by Lemma 1 and Jessen [13, Satz B], $\nu_L(\sigma)$ does not vanish identically on any subinterval of $(\frac{n}{2}, \sigma_L)$. Hence the limit in (1.8) is positive whenever $\sigma_1 < \sigma_L$. This also implies that the support of ν_L in $(\frac{n}{2}, \infty)$ is exactly equal to $(\frac{n}{2}, \sigma_L]$.

Next we set

$$\eta_j = \eta_j(L) := 2\log V_n + n\lambda_j = 2(\log V_n + n\log|\widehat{\boldsymbol{v}}_j|).$$

Then, again by Lemma 6, we have

$$\nu(\{\eta_j\}; \sigma) = n^2 \nu_L(n\sigma). \tag{4.4}$$

Lemma 7. *Fix any $K \in \mathbb{Z}^+$, and take L at random in X_n according to μ_n. Then the random variable $(e^{\eta_1/2}, \ldots, e^{\eta_K/2})$ converges in distribution to $(T_1, \ldots, T_K)$ as $n \to \infty$.*

Proof. This is a simple variant of [28, Thm. 1]. Indeed, with notation as in (1.6), [28, Thm. 1] implies that $\mathcal{V}_K(L) < 2^n \mathcal{V}_1(L)$ holds with probability tending to 1 as $n \to \infty$. Hence, for any such L the lattice vectors $\pm\boldsymbol{v}_1, \ldots, \pm\boldsymbol{v}_K$ are all primitive, so that $e^{\eta_j/2} = \mathcal{V}_j(L)$ for $j = 1, \ldots, K$. $\square$

For any $K \geq 5$, we set

$$\nu^{(K)}(\{\lambda_j\}; \sigma) := \frac{1}{2\pi} \sum_{a=1}^{K} \sum_{b=1}^{K} (-1)^{\mathbb{1}(a\neq b)} \lambda_a \lambda_b e^{-(\lambda_a+\lambda_b)\sigma} \int_0^\infty \left(\prod_{j=1}^{K} J_{[j;a,b]}(e^{-\lambda_j\sigma} r) \right) r\, dr.$$

Given any interval $I \subset \mathbb{R}$, we let $C(I)$ be the space of real-valued continuous functions on I, provided with the supremum norm $\|\cdot\|_{C(I)}$.

Lemma 8. *Let $\eta_j = \eta_j(L) = 2(\log V_n + n\log|\widehat{\boldsymbol{v}}_j|)$ as above. Let I be a compact subinterval of $(\frac{1}{2}, \infty)$, and let $\varepsilon > 0$. Then there exist integers $n_0 \geq 2$ and $K_0 \geq 5$ such that, for all $K \geq K_0$ and $n \geq n_0$,*

$$\mu_n\big(\big\{L \in X_n \;:\; \big\|\nu(\{\eta_j\}; \cdot) - \nu^{(K)}(\{\eta_j\}; \cdot)\big\|_{C(I)} \leq \varepsilon\big\}\big) \geq 1 - \varepsilon.$$

Proof. Fix some c with $\frac{1}{2} < c < \inf I$. By Lemma 7 and [29, Thm. 1], if we take n_0 and A sufficiently large, then for all $n \geq n_0$ we have

$$\mu_n\Big(\Big\{L \in X_n \;:\; -A \leq \eta_1(L) < \eta_5(L) \leq A \;\text{ and }\; \sum_{j=1}^{\infty} e^{-c\eta_j} < A\Big\}\Big) > 1 - \tfrac{1}{3}\varepsilon. \tag{4.5}$$

Note that $|\eta| \ll e^{(\sigma-c)\eta}$ uniformly over all $\eta \geq -A$ and $\sigma \in I$. We set $\overline{J}(x) := \max(|J_0(x)|, |J_1(x)|)$; then $\overline{J}(x) \ll x^{-1/2}$ as $x \to \infty$. Using these facts, we conclude that there is some $B > 0$ such that, for any $n \geq n_0$ and any L in the set in (4.5),

$$\sum_{j=1}^{\infty} |\eta_j| e^{-\eta_j\sigma} < B \quad \text{and} \quad \int_0^\infty \Big| \prod_{j=1}^5 \overline{J}(e^{-\eta_j\sigma} r) \Big| \, r\, dr < B, \qquad \forall \sigma \in I. \tag{4.6}$$

For any L satisfying (4.6), and $\sigma \in I$, we have, since $\overline{J}(x) \leq 1$ for all $x \geq 0$,

$$\begin{aligned} \big|\nu(\{\eta_j\};\sigma) - \nu^{(K)}(\{\eta_j\};\sigma)\big| \leq \frac{B^2}{\pi} \sum_{j>K} |\eta_j| e^{-\eta_j\sigma} \\ + \frac{B^2}{2\pi} \int_0^\infty \Big| 1 - \prod_{j>K} J_0(e^{-\eta_j\sigma} r) \Big| \Big(\prod_{j=1}^5 \overline{J}(e^{-\eta_j\sigma} r) \Big) \, r\, dr. \end{aligned}$$

Hence it now suffices to prove that for any given $\varepsilon' > 0$ and $R > 0$, if we take K and n_0 sufficiently large, then for all $n \geq n_0$ we have both

$$\mu_n\bigg(\Big\{ L \in X_n \, : \, \sup_{\sigma\in I} \sum_{j>K} |\eta_j| e^{-\eta_j\sigma} < \varepsilon' \Big\} \bigg) > 1 - \tfrac{1}{3}\varepsilon \tag{4.7}$$

and

$$\mu_n\bigg(\Big\{ L \in X_n \, : \, \sup_{\sigma\in I} \sup_{r\in[0,R]} \Big| 1 - \prod_{j>K} J_0(e^{-\eta_j\sigma} r) \Big| < \varepsilon' \Big\} \bigg) > 1 - \tfrac{1}{3}\varepsilon. \tag{4.8}$$

Here (4.7) is a consequence of (e.g.) [29, Thm. 5] (applied with $k = 2$, c as above, and δ sufficiently large). To prove (4.8), note that $1 \geq J_0(x) = 1 + O(x^2)$ as $x \to 0$; hence there is a constant $\alpha > 0$ such that $e^{-x} \leq J_0(x) \leq 1$ for all $x \in [0, \alpha]$. It follows that L belongs to the set in the left-hand side of (4.8) whenever

$$\sup_{\sigma\in I} \sum_{j>K} e^{-\eta_j\sigma} < R^{-1} \min(\alpha, |\log(1-\varepsilon')|).$$

Using this observation, (4.8) follows by another application of [29, Thm. 5]. This completes the proof of the lemma. $\square$

Proof of Theorem 2. Consider the random function $\nu_{\{T_j\}}$ (cf. (1.10)). By Lemma 5, $\nu_{\{T_j\}}(\sigma) = \nu(\{2\log T_j\};\sigma)$ (almost surely), and one easily verifies that

$$\nu_{\{T_j\}} = \lim_{K\to\infty} \nu^{(K)}(\{2\log T_j\}; \,\cdot\,) \quad \text{in } C(\tfrac{1}{2}, \infty) \ \ \text{(almost surely)}; \tag{4.9}$$

cf. [14, Sec. 24] or [35], or the proof of Lemma 8. This shows in particular that $\nu_{\{T_j\}}$ is a measurable map from $(\Omega, \mathbf{P})$ to $C(\frac{1}{2}, \infty)$, viz., a random element in $C(\frac{1}{2}, \infty)$.

Now, for any fixed $K \geq 5$, $\nu^{(K)}(\{\lambda_j\}, \,\cdot\,)$ is a continuous function of $(\lambda_1, \ldots, \lambda_K) \in (\mathbb{R}_{>0})^K$ with values in $C(\frac{1}{2}, \infty)$; and by Lemma 7, $(\eta_1, \ldots, \eta_K)$ converges in distribution to $(2\log T_1, \ldots, 2\log T_K)$ as $n \to \infty$. Therefore $\nu^{(K)}(\{\eta_j\}, \,\cdot\,)$ converges in distribution to $\nu^{(K)}(\{2\log T_j\}; \,\cdot\,)$. Using this fact, (4.9) and Lemma 8, it follows that for any fixed compact interval $I \subset (\frac{1}{2}, \infty)$, the restriction of $n^2\nu_L(n\,\cdot\,) = \nu(\{\eta_j\}, \,\cdot\,)$ to I converges in distribution to the restriction of $\nu_{\{T_j\}}$ to I, as random elements in $C(I)$; cf. [16, Thm. 4.28]. Hence, by [16, Prop. 16.6], convergence also holds in $C(\frac{1}{2}, \infty)$, i.e. we have proved that $n^2\nu_L(n\,\cdot\,) \xrightarrow{\text{d}} \nu_{\{T_j\}}$ as $n \to \infty$, in the sense of convergence in distribution for random elements in $C(\frac{1}{2}, \infty)$.

To complete the proof of Theorem 2, it remains to upgrade the result to *joint* convergence of $n^2\nu_L(n\,\cdot\,)$ and $n^{-1}\sigma_L$. Given the previous arguments, this is a standard but somewhat technical exercise: Recalling (3.1) and (3.2), we set

$$F_n^{(K)}(L,c) := -\mathcal{V}_1(L)^{-2c} + \sum_{j=2}^{K} \mathcal{V}_j(L)^{-2c} \quad \text{and} \quad F^{(K)}(c) := -T_1^{-2c} + \sum_{j=2}^{K} T_j^{-2c}.$$

The key fact, now, is that for any fixed $c > \frac{1}{2}$ and $K \geq 5$, the following convergence in distribution of random elements in $C(\frac{1}{2},\infty)$ holds, as $n \to \infty$:

$$\mathbb{1}\big(F_n^{(K)}(L,c) > 0\big)\ \nu^{(K)}(\{\eta_j\},\,\cdot\,) \xrightarrow{\mathrm{d}} \mathbb{1}\big(F^{(K)}(c) > 0\big)\ \nu^{(K)}(\{2\log T_j\};\,\cdot\,).$$

Indeed, by the proof of Lemma 7, away from a set of $L \in X_n$ of measure tending to zero as $n \to \infty$, $F_n^{(K)}(L,c)$ is a continuous function of $(\eta_1,\ldots,\eta_K)$, just as $\nu^{(K)}(\{\eta_j\},\,\cdot\,)$; also $\mathbf{P}(F^{(K)}(c) = 0) = 0$; hence the claim follows from Lemma 7 and the mapping theorem [3, Thm. 2.7]. Furthermore, from the proofs of Theorem 1 and [29, Thm. 1], one extracts the fact that for given c and $\varepsilon > 0$, if K and n are taken sufficiently large and L is picked at random in (X_n,μ_n), then with probability greater than $1-\varepsilon$, the two inequalities $n^{-1}\sigma_L > c$ and $F_n^{(K)}(L,c) > 0$ are both true or both false. Using these facts and previous arguments (in particular Lemma 8), we may again apply [16, Thm. 4.28] to conclude that, for any fixed compact interval $I \subset (\frac{1}{2},\infty)$ and $c > \frac{1}{2}$,

$$\mathbb{1}\big(n^{-1}\sigma_L > c\big)\, n^2\nu_L(n\,\cdot\,)_{|I} \xrightarrow{\mathrm{d}} \mathbb{1}\big(\sigma_{\{T_j\}} > c\big)\, \nu_{\{T_j\}\,|I} \qquad \text{as } n \to \infty$$

(convergence in distribution of random elements in $C(I)$). Finally, Theorem 2 follows by general measure-theoretic arguments (akin to [3, Thms. 2.3, 2.4]). ☐

Remark 6. Although $\sigma_L = \sup(\mathrm{supp}(\nu_L))$ for generic $L \in X_n$ (cf. Remark 5), and $\sigma_{\{T_j\}} = \sup(\mathrm{supp}(\nu_{\{T_j\}}))$ almost surely, it does not seem that the joint convergence of Theorem 2 follows in any automatic way from just knowing $n^2\nu_L(n\,\cdot\,) \xrightarrow{\mathrm{d}} \nu_{\{T_j\}}$. Note in particular that the map $\nu \mapsto \sup(\mathrm{supp}(\nu))$ from $C(\frac{1}{2},\infty)$ to $\mathbb{R}_{>1/2} \cup \{\pm\infty\}$ is far from continuous.

5. Proof of Theorem 3

In this section we prove Theorem 3, which gives an explicit formula for the distribution function of $\sigma_{\{T_j\}}$. Recall that $\sigma_{\{T_j\}} > c$ holds if and only if $\sum_{j=2}^{\infty} T_j^{-2c} > T_1^{-2c}$. Hence our task is to determine the probability

$$\mathbf{P}\big(\sigma_{\{T_j\}} > c\big) = \mathbf{P}\Big(\sum_{j=2}^{\infty} T_j^{-2c} > T_1^{-2c}\Big). \tag{5.1}$$

Let us note that for any fixed $\mu > 0$, the sequence $\mu T_1, \mu T_2, \ldots$ give the points of a Poisson process on the positive real line with intensity $(2\mu)^{-1}$, and we have $\sum_{j=2}^{\infty} T_j^{-2c} > T_1^{-2c}$ if and only if $\sum_{j=2}^{\infty}(\mu T_j)^{-2c} > (\mu T_1)^{-2c}$. Hence we may, in order to make our computations slightly cleaner, *alter our notation* so that from now on, $0 < T_1 < T_2 < \ldots$ denote the points of a Poisson process on the positive real line with constant intensity *one;* the probability in (5.1) remains unchanged by this alteration. Also to make the computations slightly cleaner, we will write

$$a := 2c \in \mathbb{R}_{>1}.$$

As a first step, we consider the *conditional* distribution of the sum $\sum_{j=2}^{\infty} T_j^{-a}$ given the value of T_1. We will see that this distribution is infinitely divisible. For basic facts about infinitely divisible distributions, cf., e.g., [10, Chs. VI.3, IX, XVII]. We formulate the result for a Poisson process having constant intensity 1; it is of course easy to carry this over to the case of an arbitrary constant intensity.

Proposition 2. *Let $0 < T_1 < T_2 < \cdots$ be the points of a Poisson process on the positive real line with constant intensity 1. Then, for any $a > 1$ and $\delta > 0$, the conditional distribution of $\sum_{j=2}^{\infty} T_j^{-a}$, given that $T_1 = \delta$, is an infinitely divisible distribution, the characteristic function of which is given by*

$$\varphi_{a,\delta}(t) = \mathbb{E}\Big(e^{it\sum_{j=2}^{\infty} T_j^{-a}} \mid T_1 = \delta\Big) = \exp\Big\{ - \int_{\delta}^{\infty} \big(1 - e^{itx^{-a}}\big)\, dx \Big\}. \tag{5.2}$$

(Cf. [22, Thm. 1.4.2], where the corresponding fact is proved in the special case $\delta = 0$ but with more general weights in the sum; the resulting distribution is then a stable distribution.)

Proof. Let n be a positive integer and let η be any real number larger than δ. The conditional distribution of $(T_2, \ldots, T_{n+1})$, given that $T_1 = \delta$ and $T_{n+2} = \eta$, is that of the order statistic of n i.i.d. random variables uniformly distributed in the interval (δ, η), and hence the conditional distribution of $\sum_{j=2}^{n+1} T_j^{-a}$, given $T_1 = \delta$ and $T_{n+2} = \eta$, is the same as the distribution of $\sum_{j=1}^{n} (\delta + (\eta - \delta) U_j)^{-a}$, where from now on $U_1, U_2, \ldots$ denotes a sequence of i.i.d. random variables uniformly distributed in $(0, 1)$. It follows that the conditional distribution of $\sum_{j=2}^{n+1} T_j^{-a}$, given only $T_1 = \delta$, is the same as the distribution of

$$X_n := \sum_{j=1}^{n} (\delta + S_{n+1} U_j)^{-a},$$

where S_{n+1} denotes the sum of $n+1$ i.i.d. exponential random variables with mean one, independent from the sequence $\{U_j\}$ (so that S_{n+1} has the same distribution as $T_{n+2} - \delta$ given $T_1 = \delta$).

By the law of large numbers $n^{-1} S_{n+1}$ tends in distribution to 1, i.e. given any $\varepsilon > 0$ there is $N \in \mathbb{Z}_{>0}$ such that for each $n \geq N$, we have $(1 - \varepsilon) n < S_{n+1} < (1 + \varepsilon) n$ with probability $> 1 - \varepsilon$. It follows that if we let

$$Y_n := \sum_{j=1}^{n} (\delta + n U_j)^{-a},$$

then, for each $n \geq N$, we have $(1 + \varepsilon)^{-a} Y_n < X_n < (1 - \varepsilon)^{-a} Y_n$ with probability $> 1 - \varepsilon$. In particular, it now suffices to prove that Y_n tends in distribution to a (non-defective) random variable whose characteristic function is given by the right-hand side of (5.2), since then also X_n must converge in distribution to this random variable, and also it follows from the definition of Y_n that the limit distribution must be infinitely divisible, cf., e.g., [10, Ch IX.5 (see also Ch. XVII.2)].

But Y_n is a sum of n independent random variables, and thus its characteristic function equals

$$\mathbb{E} e^{itY_n} = \Big(\mathbb{E} e^{it(\delta + nU_1)^{-a}}\Big)^n = \Big(\frac{1}{n} \int_{\delta}^{\delta+n} e^{itx^{-a}}\, dx\Big)^n = \Big(1 - \frac{1}{n} \int_{\delta}^{\delta+n} \big(1 - e^{itx^{-a}}\big)\, dx\Big)^n.$$

Note that $|1-e^{itx^{-a}}| \ll |t|x^{-a}$ uniformly for all $x \geq \delta$ and all $t \in \mathbb{R}$. In particular, for each fixed $t \in \mathbb{R}$ the integral $\int_\delta^\infty \big(1-e^{itx^{-a}}\big)\,dx$ is absolutely convergent, and $\mathbb{E}e^{itY_n}$ tends to the expression in the right-hand side of (5.2) as $n \to \infty$. The bound $|1-e^{itx^{-a}}| \ll |t|x^{-a}$ also implies that the function $\varphi_{a,\delta}(t)$ is continuous. Hence Y_n converges in distribution to a (non-defective) random variable whose characteristic function is given by the right-hand side of (5.2), and the proposition is proved. $\square$

Remark 7. Let us note that the integral in (5.2) may be expressed in terms of the incomplete gamma function. Indeed, substituting $x = (iu)^{-\frac{1}{a}}$ and then integrating by parts, we get

$$\begin{aligned}\int_\delta^\infty \big(1-e^{itx^{-a}}\big)\,dx &= -\int_0^{-i\delta^{-a}} \big(1-e^{-tu}\big)\Big(\frac{d}{du}\big((iu)^{-\frac{1}{a}}\big)\Big)\,du \\ &= \delta(e^{it\delta^{-a}}-1) + t\int_0^{-i\delta^{-a}} e^{-tu}(iu)^{-\frac{1}{a}}\,du \\ &= \delta(e^{it\delta^{-a}}-1) + (-it)^{\frac{1}{a}}\gamma\Big(1-\tfrac{1}{a}, -it\delta^{-a}\Big), \end{aligned} \tag{5.3}$$

where for $t \neq 0$ we agree that $\arg(-it) = -(\operatorname{sgn} t)\frac{\pi}{2}$. Hence

$$\varphi_{a,\delta}(t) = \exp\Big\{-\delta(e^{it\delta^{-a}}-1) - (-it)^{\frac{1}{a}}\gamma(1-\tfrac{1}{a}, -it\delta^{-a})\Big\}.$$

Furthermore, using the recursion formula (1.12) together with the formula $\gamma(s,z) = \Gamma(s) - \Gamma(s,z)$, where

$$\Gamma(s,z) := \int_z^\infty u^{s-1}e^{-u}\,du \tag{5.4}$$

is the upper incomplete gamma function, we get the alternative formula

$$\varphi_{a,\delta}(t) = \exp\Big\{\delta - (-it)^{\frac{1}{a}}\Gamma(1-\tfrac{1}{a}) - \tfrac{1}{a}(-it)^{\frac{1}{a}}\Gamma(-\tfrac{1}{a}, -it\delta^{-a})\Big\}. \tag{5.5}$$

Proof of Theorem 3. Note that, for all $z \in \mathbb{C}\setminus\{0\}$ with $\Re z \geq 0$,

$$\big|\Gamma(-\tfrac{1}{a}, z)\big| \leq |z|^{-\frac{1}{a}-1}e^{-\Re z}.$$

Hence, if we denote the exponent in (5.5) by $\psi_{a,\delta}(t)$, we have for $t > 0$,

$$\psi_{a,\delta}(t) = \delta - (-it)^{\frac{1}{a}}\Gamma(1-\tfrac{1}{a}) + O_{a,\delta}\left(t^{-1}\right).$$

Using also $\Re(-it)^{\frac{1}{a}} \gg_a t^{\frac{1}{a}}$, we conclude that $-\Re\psi_{a,\delta}(t) \gg_{a,\delta} t^{\frac{1}{a}}$ as $t \to \infty$. Hence, in view of the symmetry $\varphi_{a,\delta}(-t) = \overline{\varphi_{a,\delta}(t)}$, the function $\varphi_{a,\delta}$ is integrable, and therefore the distribution in Proposition 2 has a density function, which we call $f_{a,\delta}(x)$. Thus

$$f_{a,\delta}(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \varphi_{a,\delta}(t)e^{-itx}\,dt = \frac{1}{\pi}\int_0^\infty \Re\big(\varphi_{a,\delta}(t)e^{-itx}\big)\,dt.$$

It follows that the conditional probability of $\sum_{j=2}^\infty T_j^{-a} > T_1^{-a}$, given that $T_1 = \delta$, is

$$\mathbf{P}\Big(\sum_{j=2}^\infty T_j^{-a} > T_1^{-a} \;\Big|\; T_1 = \delta\Big) = \int_{\delta^{-a}}^\infty f_{a,\delta}(x)\,dx.$$

However, T_1, being the first point of a Poisson process on the positive real line with intensity one, has an exponential distribution of mean one. Hence we conclude:

$$\begin{aligned}\mathbf{P}\big(\sigma_{\{T_j\}} > c\big) = \mathbf{P}\Big(\sum_{j=2}^{\infty} T_j^{-a} > T_1^{-a}\Big) &= \int_0^\infty \int_{\delta^{-a}}^\infty f_{a,\delta}(x)\,dx\,e^{-\delta}\,d\delta \\ &= \int_0^\infty \int_{x^{-\frac{1}{a}}}^\infty f_{a,\delta}(x)\,e^{-\delta}\,d\delta\,dx \\ &= \frac{1}{\pi}\int_0^\infty \int_{x^{-\frac{1}{a}}}^\infty \int_0^\infty \Re\big(\varphi_{a,\delta}(t)e^{-itx-\delta}\big)\,dt\,d\delta\,dx. \qquad (5.6)\end{aligned}$$

Note that the last expression in (5.6) should be viewed as an iterated integral; it is easy to see that $\int_0^\infty \int_{x^{-\frac{1}{a}}}^\infty \int_0^\infty \big|\Re\big(\varphi_{a,\delta}(t)e^{-itx-\delta}\big)\big|\,dt\,d\delta\,dx = \infty$, so that we are not permitted to change order of integration arbitrarily. However, we will prove that the inner double integral is absolutely convergent.

By Proposition 2 we have $e^{-\delta}\varphi_{a,\delta}(t) = \exp\big(-\big(\delta + \int_\delta^\infty (1-e^{itx^{-a}})\,dx\big)\big)$, and here we have, by substituting $x = (u/t)^{-\frac{1}{a}}$ and then integrating by parts,

$$\begin{aligned}\delta + \int_\delta^\infty (1-e^{itx^{-a}})\,dx &= \delta - t^{\frac{1}{a}} \int_0^{t\delta^{-a}} (1-e^{iu})\Big(\frac{d}{du}(u^{-\frac{1}{a}})\Big)\,du \\ &= \delta e^{it\delta^{-a}} - it^{\frac{1}{a}} \int_0^{t\delta^{-a}} e^{iu}u^{-\frac{1}{a}}\,du = t^{\frac{1}{a}}\Phi_a(t\delta^{-a}),\end{aligned}$$

where we have defined

$$\Phi_a(y) := y^{-\frac{1}{a}}e^{iy} - i\int_0^y e^{iu}u^{-\frac{1}{a}}\,du \qquad \text{for } a>1,\ y>0. \tag{5.7}$$

Thus

$$\mathbf{P}\big(\sigma_{\{T_j\}} > c\big) = \frac{1}{\pi}\int_0^\infty \int_{x^{-\frac{1}{a}}}^\infty \int_0^\infty \Re\exp\Big\{-itx - t^{\frac{1}{a}}\Phi_a(t\delta^{-a})\Big\}\,dt\,d\delta\,dx. \tag{5.8}$$

Using $e^{iu} = 1 + O(u)$ for $u \in [0,1]$, we find that $\Phi_a(y) = y^{-\frac{1}{a}}(1+O(y))$ for $0 < y \leq 1$. (Here, and in any "big-O" or "$\ll$" bound below, we allow the implied constant to depend on a.) In particular there exists a positive number κ_1, which may depend on a, such that $\Re\Phi_a(y) \geq \frac{1}{2}y^{-\frac{1}{a}}$ for all $y \in (0,\kappa_1]$. We also note that

$$\Phi_a'(y) = -\frac{1}{a}y^{-1-\frac{1}{a}}e^{iy}. \tag{5.9}$$

In particular $\Re\Phi_a'(y) = -\frac{1}{a}y^{-1-\frac{1}{a}}\cos y$, and this is negative for all $y \in (0,\frac{\pi}{2})$, so that $\Re\Phi_a(y) > \Re\Phi_a(\frac{\pi}{2})$ holds for all $y \in (0,\frac{\pi}{2})$. Furthermore, for all $y \geq \frac{\pi}{2}$ we have $\Re\Phi_a(y) = \Re\Phi_a(\frac{\pi}{2}) - \frac{1}{a}\int_{\frac{\pi}{2}}^y u^{-1-\frac{1}{a}}(\cos u)\,du \geq \Re\Phi_a(\frac{\pi}{2})$. Also note from (5.7) that $\Re\Phi_a(\frac{\pi}{2}) = \int_0^{\frac{\pi}{2}} u^{-\frac{1}{a}}(\sin u)\,du > 0$. Hence we conclude:

$$\Re\Phi_a(y) \geq \kappa_2 := \Re\Phi_a(\tfrac{\pi}{2}) > 0, \qquad \forall y > 0.$$

Using the bounds obtained, we conclude:

$$\int_0^\infty \Big|\exp\Big\{-t^{\frac{1}{a}}\Phi_a(t\delta^{-a})\Big\}\Big|\,dt \leq \int_0^{\kappa_1\delta^a} e^{-\frac{1}{2}\delta}\,dt + \int_{\kappa_1\delta^a}^\infty e^{-\kappa_2 t^{\frac{1}{a}}}\,dt \ll e^{-\kappa_3\delta} \tag{5.10}$$

for all $\delta > 0$, where κ_3 is some positive number which may depend on a. From this estimate we see that the inner double integral in (5.8) is indeed absolutely convergent, in fact even $\int_0^\infty \int_0^\infty \left|\exp(-t^{\frac{1}{a}}\Phi_a(t\delta^{-a}))\right| dt\, d\delta < \infty$. Hence we have

$$\begin{aligned}
\mathbf{P}\big(\sigma_{\{T_j\}} > c\big) &= \frac{1}{\pi}\lim_{X\to\infty} \Re \int_0^X \int_0^\infty \int_{x^{-\frac{1}{a}}}^\infty \exp\Big\{-itx - t^{\frac{1}{a}}\Phi_a(t\delta^{-a})\Big\}\, d\delta\, dt\, dx \\
&= \frac{1}{\pi}\lim_{X\to\infty} \Re \int_0^\infty \int_0^X e^{-itx} \int_{x^{-\frac{1}{a}}}^\infty \exp\Big\{-t^{\frac{1}{a}}\Phi_a(t\delta^{-a})\Big\}\, d\delta\, dx\, dt \\
&= \frac{1}{\pi a}\lim_{X\to\infty} \Re \int_0^\infty t^{\frac{1}{a}} \int_0^X e^{-itx} \int_0^{tx} e^{-t^{\frac{1}{a}}\Phi_a(y)} y^{-1-\frac{1}{a}}\, dy\, dx\, dt.
\end{aligned}$$

Here, for any $t > 0$, we have, by integration by parts:

$$\begin{aligned}
&\int_0^X e^{-itx} \int_0^{tx} e^{-t^{\frac{1}{a}}\Phi_a(y)} y^{-1-\frac{1}{a}}\, dy\, dx \\
&\quad = \frac{ie^{-itX}}{t} \int_0^{tX} e^{-t^{\frac{1}{a}}\Phi_a(y)} y^{-1-\frac{1}{a}}\, dy - \int_0^X ie^{-itx} e^{-t^{\frac{1}{a}}\Phi_a(tx)} (tx)^{-1-\frac{1}{a}}\, dx \\
&\quad = \frac{i}{t} \int_0^{tX} \big(e^{-itX} - e^{-iy}\big) e^{-t^{\frac{1}{a}}\Phi_a(y)} y^{-1-\frac{1}{a}}\, dy.
\end{aligned}$$

Hence

$$\mathbf{P}\big(\sigma_{\{T_j\}} > c\big) = \frac{1}{\pi a}\lim_{X\to\infty} \Im \int_0^\infty t^{\frac{1}{a}-1} \int_0^{tX} \big(e^{-iy} - e^{-itX}\big) e^{-t^{\frac{1}{a}}\Phi_a(y)} y^{-1-\frac{1}{a}}\, dy\, dt.$$

For given $X > 1$, we split the integral over t into two parts, corresponding to $t < X^{-1}$ and $t > X^{-1}$. Regarding the first part, we note that $t < X^{-1}$ and $y < tX$ implies $y < 1$. Thus $\Phi_a(y) = y^{-\frac{1}{a}}(1 + O(y))$, and since also $t < X^{-1} < 1$, we have $e^{-t^{\frac{1}{a}}\Phi_a(y)} = e^{-(t/y)^{\frac{1}{a}}}(1 + O(t^{\frac{1}{a}} y^{1-\frac{1}{a}}))$. Recall that in this case we also have $e^{-iy} = 1 + O(y)$. Hence

$$\begin{aligned}
(5.11)\qquad &\int_0^{X^{-1}} t^{\frac{1}{a}-1} \int_0^{tX} \big(e^{-iy} - e^{-itX}\big) e^{-t^{\frac{1}{a}}\Phi_a(y)} y^{-1-\frac{1}{a}}\, dy\, dt \\
&= \int_0^{X^{-1}} \int_0^{tX} \Big(1 - e^{-itX} + O\Big(y + t^{\frac{1}{a}} y^{1-\frac{1}{a}}\Big)\Big) e^{-(t/y)^{\frac{1}{a}}} t^{\frac{1}{a}-1} y^{-1-\frac{1}{a}}\, dy\, dt \\
&= a \int_0^{X^{-1}} \int_{X^{-\frac{1}{a}}}^\infty \big(1 - e^{-itX} + O\big(tu^{-a} + tu^{1-a}\big)\big) e^{-u} t^{-1}\, du\, dt \\
&= a \int_0^1 \frac{1 - e^{-it}}{t}\, dt \int_{X^{-\frac{1}{a}}}^\infty e^{-u}\, du + O\big(X^{-1}\big) \int_{X^{-\frac{1}{a}}}^\infty (u^{-a} + u^{1-a}) e^{-u}\, du \\
&= a \int_0^1 \frac{1 - e^{-it}}{t}\, dt + O\big(X^{-\frac{1}{a}}\big).
\end{aligned}$$

The remaining part is

$$(5.12)\qquad \int_{X^{-1}}^\infty t^{\frac{1}{a}-1} \int_0^{tX} \big(e^{-iy} - e^{-itX}\big) e^{-t^{\frac{1}{a}}\Phi_a(y)} y^{-1-\frac{1}{a}}\, dy\, dt,$$

and here we have absolute convergence; $\int_{X^{-1}}^\infty t^{\frac{1}{a}-1} \int_0^{tX} e^{-t^{\frac{1}{a}}\Re\Phi_a(y)} y^{-1-\frac{1}{a}}\, dy\, dt < \infty$, as is seen by a similar computation as in (5.10). (The corresponding fact does not hold in (5.11).) We also note that we may replace the range of the inner integral in (5.12) by all of $\mathbb{R}_{>0}$, to the cost of an error which is

$$\ll \int_{X^{-1}}^{\infty} t^{\frac{1}{a}-1} \int_{tX}^{\infty} e^{-\kappa_2 t^{\frac{1}{a}}} y^{-1-\frac{1}{a}}\, dy\, dt \ll X^{-\frac{1}{a}} \int_{X^{-1}}^{\infty} e^{-\kappa_2 t^{\frac{1}{a}}} \frac{dt}{t} \ll X^{-\frac{1}{a}} \log(2X).$$

Collecting the above results, and using the fact that both $X^{-\frac{1}{a}}$ and $X^{-\frac{1}{a}}\log(2X)$ tend to zero as $X \to \infty$, we conclude that

(5.13)
$$\mathbf{P}\big(\sigma_{\{T_j\}} > c\big) = \frac{1}{\pi}\int_0^1 \frac{\sin t}{t}\, dt + \frac{1}{\pi a} \lim_{X\to\infty} \bigg(\int_{X^{-1}}^{\infty} \Im g_1(t)\, dt - \int_{X^{-1}}^{\infty} \Im\big(e^{-iXt} g_0(t)\big)\, dt \bigg),$$

where

$$g_\ell(t) = t^{\frac{1}{a}-1} \int_0^{\infty} e^{-i\ell y - t^{\frac{1}{a}}\Phi_a(y)} y^{-1-\frac{1}{a}}\, dy$$

for $\ell = 0, 1$.

Next, we split $g_\ell(t)$ as $g_\ell(t) = g_{\ell,1}(t) + g_{\ell,2}(t)$, where

$$g_{\ell,1}(t) = t^{\frac{1}{a}-1} \int_0^{1} e^{-i\ell y - t^{\frac{1}{a}}\Phi_a(y)} y^{-1-\frac{1}{a}}\, dy$$

and

$$g_{\ell,2}(t) = t^{\frac{1}{a}-1} \int_1^{\infty} e^{-i\ell y - t^{\frac{1}{a}}\Phi_a(y)} y^{-1-\frac{1}{a}}\, dy.$$

Bounding $\Re\Phi_a(y)$ from below as in (5.10), we see that for all $t > 0$ we have

$$\big|g_{\ell,1}(t)\big| \le t^{\frac{1}{a}-1} \int_0^{1} \big|e^{-t^{\frac{1}{a}}\Phi_a(y)}\big| y^{-1-\frac{1}{a}}\, dy \ll t^{-1} e^{-\kappa_4 t^{\frac{1}{a}}}, \tag{5.14}$$

where κ_4 is (just like $\kappa_1, \kappa_2, \kappa_3$) a positive number which may depend on a, and

$$\big|g_{\ell,2}(t)\big| \le t^{\frac{1}{a}-1} \int_1^{\infty} \big|e^{-t^{\frac{1}{a}}\Phi_a(y)}\big| y^{-1-\frac{1}{a}}\, dy \ll t^{\frac{1}{a}-1} e^{-\kappa_2 t^{\frac{1}{a}}}. \tag{5.15}$$

Note also that for all $t, y \in (0,1]$, we have $e^{-iy - t^{\frac{1}{a}}\Phi_a(y)} = e^{-iy - t^{\frac{1}{a}} y^{-\frac{1}{a}}(1+O(y))} = e^{-t^{\frac{1}{a}} y^{-\frac{1}{a}}}(1 + O(t^{\frac{1}{a}} y^{1-\frac{1}{a}} + y))$, and thus

$$\begin{aligned} t^{\frac{1}{a}-1} \int_0^1 \big|\Im e^{-iy - t^{\frac{1}{a}}\Phi_a(y)}\big|\, y^{-1-\frac{1}{a}}\, dy &\ll t^{\frac{1}{a}-1} \int_0^1 \big(t^{\frac{1}{a}} y^{-\frac{2}{a}} + y^{-\frac{1}{a}}\big) e^{-t^{\frac{1}{a}} y^{-\frac{1}{a}}}\, dy \\ &\ll \int_{t^{\frac{1}{a}}}^{\infty} \big(v^{1-a} + v^{-a}\big) e^{-v}\, dv \ll t^{\frac{1}{a}-1} \end{aligned} \tag{5.16}$$

for all $0 < t \le 1$. Combining this bound with (5.14) and (5.15), we see that

$$\int_0^{\infty} t^{\frac{1}{a}-1} \int_0^{\infty} \big|\Im e^{-iy - t^{\frac{1}{a}}\Phi_a(y)}\big|\, y^{-1-\frac{1}{a}}\, dy\, dt < \infty. \tag{5.17}$$

Hence the contribution from $g_1(t)$ in (5.13) can be treated as follows:

(5.18)
$$\begin{aligned} \frac{1}{\pi a} \lim_{X\to\infty} \int_{X^{-1}}^{\infty} \Im g_1(t)\, dt &= \frac{1}{\pi a} \int_0^{\infty} t^{\frac{1}{a}-1} \int_0^{\infty} \Im e^{-iy - t^{\frac{1}{a}}\Phi_a(y)} y^{-1-\frac{1}{a}}\, dy\, dt \\ &= \frac{1}{\pi a} \int_0^{\infty} \Im\bigg(e^{-iy} \int_0^{\infty} t^{\frac{1}{a}-1} e^{-t^{\frac{1}{a}}\Phi_a(y)}\, dt\bigg) y^{-1-\frac{1}{a}}\, dy = \frac{1}{\pi} \int_0^{\infty} \Im\bigg(\frac{e^{-iy}}{\Phi_a(y)}\bigg) y^{-1-\frac{1}{a}}\, dy. \end{aligned}$$

Finally, we treat the contribution from $g_0(t)$ in (5.13). Note that, by (5.14) and (5.15), the restriction of $g_0(t)$ to $[1,\infty)$ is an L^1-function. Hence, by the Riemann-Lebesgue lemma, $\int_1^\infty e^{-iXt} g_0(t)\,dt$ tends to 0 as $X \to \infty$. Moreover, the restriction of $g_{0,2}(t)$ to $(0,1]$ is in L^1 and hence also $\int_{X^{-1}}^1 e^{-iXt} g_{0,2}(t)\,dt$ tends to 0 as $X \to \infty$. Hence

$$-\frac{1}{\pi a}\lim_{X\to\infty}\int_{X^{-1}}^\infty \Im\big(e^{-iXt}g_0(t)\big)\,dt = -\frac{1}{\pi a}\lim_{X\to\infty}\Im\int_{X^{-1}}^1 e^{-iXt}g_{0,1}(t)\,dt. \tag{5.19}$$

Furthermore, for $0 < t \le 1$, we have

$$\begin{aligned} g_{0,1}(t) = t^{\frac{1}{a}-1}\int_0^1 e^{-t^{\frac{1}{a}}\Phi_a(y)} y^{-1-\frac{1}{a}}\,dy &= t^{\frac{1}{a}-1}\int_0^1 e^{-t^{\frac{1}{a}}y^{-\frac{1}{a}}}\Big(1+O\Big(t^{\frac{1}{a}}y^{1-\frac{1}{a}}\Big)\Big)y^{-1-\frac{1}{a}}\,dy \\ &= \frac{a}{t}\int_{t^{\frac{1}{a}}}^\infty e^{-v}\,dv + O\big(t^{\frac{1}{a}-1}\big) = \frac{a}{t} + O\big(t^{\frac{1}{a}-1}\big), \end{aligned}$$

where we bounded the contribution from the big-O-term in the integral by a similar computation as in (5.16). Thus $g_{0,1}(t) - \frac{a}{t}$ is an L^1-function on $t \in (0,1]$, so that $\int_{X^{-1}}^1 e^{-iXt}(g_{0,1}(t) - \frac{a}{t})\,dt$ tends to 0 as $X \to \infty$. Hence (5.19) equals

$$-\frac{1}{\pi}\lim_{X\to\infty}\Im\int_{X^{-1}}^1 \frac{e^{-iXt}}{t}\,dt = \frac{1}{\pi}\lim_{X\to\infty}\int_1^X \frac{\sin t}{t}\,dt = \frac{1}{\pi}\int_1^\infty \frac{\sin t}{t}\,dt.$$

Collecting our results into (5.13), we obtain, since $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$,

$$\mathbf{P}\big(\sigma_{\{T_j\}} > c\big) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Im\bigg(\frac{e^{-iy}}{\Phi_a(y)}\bigg)y^{-1-\frac{1}{a}}\,dy. \tag{5.20}$$

Let us note that $\Phi_a(y)$ can be expressed in terms of the incomplete gamma function, by substituting $u = iv$ in (5.7) and using formulas (1.11) and (1.12):

$$\Phi_a(y) = y^{-\frac{1}{a}}e^{iy} + e^{-\frac{\pi}{2a}i}\gamma\Big(1-\frac{1}{a},-iy\Big) = -\frac{e^{-\frac{\pi}{2a}i}}{a}\gamma\Big(-\frac{1}{a},-iy\Big). \tag{5.21}$$

Substituting this into (5.20), we obtain the formula stated in Theorem 3. Using $|\Phi_a(y)| \ge \Re\Phi_a(y) \ge \kappa_2 > 0$ for all $y > 0$ and $\Phi_a(y) = y^{-\frac{1}{a}}(1+O(y))$ for $0 < y \le 1$, one immediately sees that the integral in (5.20) is absolutely convergent (this is also clear from the proof, cf. in particular (5.17) and (5.18)). This concludes the proof of Theorem 3. □

Remark 8. It is worth stressing that if we remove the imaginary part in (5.20), then convergence *fails*: We have $\big|\int_{y_0}^1 \frac{e^{-iy}}{\Phi_a(y)}y^{-1-\frac{1}{a}}\,dy\big| \to \infty$ as $y_0 \to 0^+$, since $\Phi_a(y) = y^{-\frac{1}{a}}(1+O(y))$ for $0 < y \le 1$.

6. Proof of Corollary 1

In this section we prove Corollary 1. To begin, note that by formal differentiation under the integral sign in (5.20), we have $Prob\big(\sigma_{\{T_j\}} \le c\big) = \int_{1/2}^c f(c_1)\,dc_1$, where $f : \mathbb{R}_{>\frac{1}{2}} \to \mathbb{R}_{>0}$ is given by

$$f(c) = \frac{2}{\pi}\int_0^\infty \Im\left(\frac{e^{-iy}}{\Phi_a(y)}\left(\frac{\frac{\partial}{\partial a}\Phi_a(y)}{\Phi_a(y)} - \frac{\log y}{a^2}\right)\right)y^{-1-\frac{1}{a}}\,dy. \tag{6.1}$$

Here $a := 2c \in \mathbb{R}_{>1}$ (see Section 5). This manipulation is justified by the fact that the integrand in (6.1) is majorized, uniformly for a in compact subsets of $\mathbb{R}_{>1}$, by an integrable function; this follows from an argument similar to the one that shows that the integral in (5.20) is absolutely convergent, using also that $\frac{\partial}{\partial a}\Phi_a(y) = a^{-2}(\log y)y^{-\frac{1}{a}}(1+O(y))$ for $0<y\leq\frac{1}{2}$ and $\frac{\partial}{\partial a}\Phi_a(y)=O(1)$ for $\frac{1}{2}\leq y<\infty$.

Remark 9. Note in particular that the imaginary part in (6.1) may be taken outside the integral; in fact even $\int_0^\infty \big|\frac{e^{-iy}}{\Phi_a(y)}\big(\frac{\frac{\partial}{\partial a}\Phi_a(y)}{\Phi_a(y)} - \frac{\log y}{a^2}\big)\big| y^{-1-\frac{1}{a}}\,dy<\infty$.

Let us now consider formula (6.1) in the limit as $a\to\infty$. In (5.7), we expand e^{iu} in a power series, change order between summation and integration and then use $(n-a^{-1})^{-1} = n^{-1}\sum_{k=0}^{\infty}(na)^{-k}$ for each $n\in\mathbb{Z}^+$. This gives

$$\Phi_a(y) = y^{-\frac{1}{a}}\Big(1-\sum_{k=1}^{\infty}F_k(y)a^{-k}\Big), \tag{6.2}$$

where

$$F_k(y) := \sum_{n=1}^{\infty}\frac{(iy)^n}{n!n^k}. \tag{6.3}$$

Obviously $|F_k(y)|\leq e^{|y|}-1$ holds for all $y>0$ and all k, and hence we see that given any $y_0>0$ there exists some $a_0=a_0(y_0)>1$ such that $\big|\sum_{k=1}^{\infty}F_k(y)a^{-k}\big|\leq\frac{1}{2}$ holds for all $a\geq a_0$, $y\in(0,y_0]$. We also have $\big|\sum_{k=1}^{\infty}F_k(y)a^{-k}\big|\ll|y|a^{-1}$ for these a,y, and therefore $\Phi_a(y)^{-1} = y^{\frac{1}{a}}(1+F_1(y)a^{-1}+O(|y|a^{-2}))$. The power series in (6.2) may also be differentiated termwise with respect to a. Using these observations, we obtain by a short calculation:

$$\frac{y^{-1-\frac{1}{a}}}{\Phi_a(y)}\bigg(\frac{\frac{\partial}{\partial a}\Phi_a(y)}{\Phi_a(y)}-\frac{\log y}{a^2}\bigg) = \frac{F_1(y)}{ya^2}+\frac{2(F_1(y)^2+F_2(y))}{ya^3}+O\Big(\frac{1+|\log y|}{a^4}\Big), \tag{6.4}$$

uniformly over all $a\geq a_0(y_0)$, $y\in(0,y_0]$ (where we recall that $y_0>0$ is arbitrary).

In order to obtain a similar relation also for large y, we start by setting

$$\xi(a) := \lim_{y\to\infty}\Phi_a(y) = -\frac{e^{-\frac{\pi}{2a}i}\Gamma(-\frac{1}{a})}{a} \tag{6.5}$$

(cf. (5.21)). In view of (5.9), we have $\Phi_a(y) = \xi(a)+a^{-1}\int_y^\infty u^{-1-\frac{1}{a}}e^{iu}\,du$, and integrating by parts twice, we get (for any $a>1$, $y>0$)

$$\Phi_a(y) = \xi(a)+\frac{y^{-\frac{1}{a}}}{a}\Gamma(0,-iy)-\frac{y^{-\frac{1}{a}}}{a^2}\Pi(y)+\frac{1}{a^3}\int_y^\infty u^{-1-\frac{1}{a}}\Pi(u)\,du, \tag{6.6}$$

where

$$\Pi(y) := \int_y^\infty\frac{\Gamma(0,-iu)}{u}\,du. \tag{6.7}$$

We have $|\Gamma(0,-iy)|\ll y^{-1}$ for all $y>0$, and thus also $|\Pi(y)|\ll y^{-1}$ for $y\geq1$. Using this fact together with the trivial observation $-1-\frac{1}{a}<-1$, we bound the integral in (6.6) and get

$$\Phi_a(y) = \xi(a)+\frac{y^{-\frac{1}{a}}}{a}\Gamma(0,-iy)-\frac{y^{-\frac{1}{a}}}{a^2}\Pi(y)+O(a^{-3}y^{-1}), \tag{6.8}$$

uniformly over all $a > 1$, $y \geq 1$. Since also $\xi(a) = 1 + (\gamma - \frac{\pi}{2}i)a^{-1} + O(a^{-2})$ as $a \to \infty$, we see that $\Phi_a(y)/\xi(a)$ is near 1 whenever a and y are large; hence there exist absolute constants $a_1 > 1$ and $y_0 \geq 1$ such that for all $a \geq a_1$ and $y \geq y_0$,

$$\begin{aligned}\frac{1}{\Phi_a(y)} &= \frac{1}{\xi(a)} - \frac{y^{-\frac{1}{a}}}{a\xi(a)^2}\Gamma(0,-iy) + O(a^{-2}y^{-1}) \\ &= \frac{1}{\xi(a)} - \frac{y^{-\frac{1}{a}}}{a}\Gamma(0,-iy) + O(a^{-2}y^{-1}).\end{aligned} \tag{6.9}$$

In order to obtain an asymptotic formula also for $\frac{\partial}{\partial a}\Phi_a(y)$, we note that the right-hand side of (6.6) defines an analytic function of the *complex* variable $w = a^{-1}$ in the region $|w| < 1$ (including $w = 0$). Restricting to $|w| \leq \frac{1}{2}$, we may bound the absolute value of the integral in (6.6) using $|\Pi(u)| \ll u^{-1}$ and $\Re(-1-w) \leq -\frac{1}{2}$. We may then use the Cauchy differentiation formula to obtain an asymptotic formula for the derivative of our analytic function, valid uniformly for $|w| \leq \frac{1}{4}$. In particular,

$$\frac{\partial}{\partial a}\Phi_a(y) = \xi'(a) - \frac{y^{-\frac{1}{a}}}{a^2}\Gamma(0,-iy) + \frac{y^{-\frac{1}{a}}}{a^3}\big(2\Pi(y) + \Gamma(0,-iy)\log y\big) + O\big(a^{-4}y^{-\frac{1}{2}}\big), \tag{6.10}$$

uniformly over all $a \geq 4$ and all $y \geq 1$. Using (6.9) and (6.10), we obtain, via a straightforward computation,

$$\begin{aligned}\frac{y^{-1-\frac{1}{a}}}{\Phi_a(y)}\left(\frac{\frac{\partial}{\partial a}\Phi_a(y)}{\Phi_a(y)} - \frac{\log y}{a^2}\right) &= \frac{\xi'(a)}{\xi(a)^2}y^{-1-\frac{1}{a}} - \frac{(\log y)y^{-1-\frac{1}{a}}}{a^2\xi(a)} - \frac{y^{-1-\frac{2}{a}}\Gamma(0,-iy)}{a^2} \\ &+\frac{y^{-1-\frac{2}{a}}}{a^3}\Big\{\big(4\gamma - 2\pi i + 2\log y + 2y^{-\frac{1}{a}}\Gamma(0,-iy)\big)\Gamma(0,iy) + 2\Pi(y)\Big\} + O\big(a^{-4}y^{-\frac{3}{2}}\big),\end{aligned} \tag{6.11}$$

uniformly over $a \geq \max(a_1, 4)$ and $y \geq y_0$.

We now multiply the relation (6.11) with e^{-iy}, and integrate the result over $y \in [y_0, \infty)$. The contribution from the first term is

$$\frac{\xi'(a)}{\xi(a)^2}\int_{y_0}^{\infty} y^{-1-\frac{1}{a}}e^{-iy}\,dy. \tag{6.12}$$

We split this integral into two parts as $\int_{y_0}^{\exp(a^{1/4})} + \int_{\exp(a^{1/4})}^{\infty}$ (keeping a so large that $\exp(a^{1/4}) > y_0$); then, because of the oscillating character of the integrand, the second integral is $O(\exp(-a^{1/4}))$. In the first integral, we use $y^{-\frac{1}{a}} = 1 - \frac{\log y}{a} + \frac{1}{2}(\frac{\log y}{a})^2 + O((\frac{\log y}{a})^3)$ and $\int_{y_0}^{\exp(a^{1/4})} \frac{(\log y)^3}{y}\,dy \ll a$; then, by a quick computation, we find that (6.12) equals $\frac{\xi'(a)}{\xi(a)^2}\big(\int_{y_0}^{\infty}\frac{e^{-iy}}{y}\,dy - \frac{1}{a}\int_{y_0}^{\infty}\frac{e^{-iy}\log y}{y}\,dy + O(a^{-2})\big)$. The remaining terms in (6.11) can be treated similarly, and using the relations

$$\begin{aligned}&F_1(y) = \tfrac{\pi}{2}i - \gamma - \log y - \Gamma(0,-iy); \\ &F_2(y) = \Pi(y) + \big(\tfrac{1}{24}\pi^2 - \tfrac{1}{2}\gamma^2 + \tfrac{1}{2}\pi i\gamma\big) + \big(\tfrac{1}{2}\pi i - \gamma - \tfrac{1}{2}\log y\big)\log y,\end{aligned} \tag{6.13}$$

the result may be collected as

$$\int_{y_0}^{\infty} \frac{e^{-iy}y^{-1-\frac{1}{a}}}{\Phi_a(y)} \left(\frac{\frac{\partial}{\partial a}\Phi_a(y)}{\Phi_a(y)} - \frac{\log y}{a^2} \right) dy$$
$$= a^{-2} \int_{y_0}^{\infty} \frac{F_1(y)}{y} e^{-iy}\, dy + a^{-3} \int_{y_0}^{\infty} \frac{2(F_1(y)^2 + F_2(y))}{y} e^{-iy}\, dy + O(a^{-4}),$$

for all $a \geq \max(a_1, 4)$. Using also (6.1) and (6.4), we thus obtain an asymptotic formula for $f(c)$ as $c = \frac{1}{2}a \to \infty$. Note, however, that $\Im \int_0^\infty \frac{F_1(y)}{y} e^{-iy}\, dy = \frac{1}{2}\Im \int_{-\infty}^{\infty} \frac{F_1(y)}{y} e^{-iy}\, dy = 0$, where the second equality follows using the Cauchy integral theorem, moving the contour towards infinity in the lower half-plane. Hence the coefficient in front of $a^{-2} = (2c)^{-2}$ in the asymptotic formula vanishes, and we arrive at (1.14), with

$$K_2 = \frac{1}{2\pi} \Im \int_0^\infty \frac{F_1(y)^2 + F_2(y)}{y} e^{-iy}\, dy = 0.822467\ldots. \tag{6.14}$$

(The numerical evaluation of this integral, which is not entirely straightforward, is carried out in [27, constants.mpl].)

We next turn to the study of (6.1) in the limit as $a \to 1$. Our presentation here will be rather brief; we refer to [27, asymptotics.mpl] for further details. The formula (5.7) may be expressed as

$$\Phi_a(y) = y^{-\frac{1}{a}} e^{iy} - ie^{iy} \frac{y^{1-\frac{1}{a}}}{1 - a^{-1}} - \frac{1}{1 - a^{-1}} \int_0^y e^{iu} u^{1-\frac{1}{a}}\, du. \tag{6.15}$$

Now fix $N \in \mathbb{Z}^+$, and let us keep $(a-1)^N \leq y \leq (a-1)^{-N}$, and $a \in (1, 2]$. We split the integral in (6.15) as $\int_0^{(a-1)^{N+1}} + \int_{(a-1)^{N+1}}^{y}$ and bound the first part trivially, while for $u \in [(a-1)^{N+1}, y]$, we use the fact that $u^{1-\frac{1}{a}} = \sum_{k=0}^{N+1} \frac{(1-a^{-1})^k (\log u)^k}{k!}$ $+O((a-1)^{N+2}|\log u|^{N+2})$, where the error is an increasing function of u when $u \geq 1$. This leads to the formula

$$\Phi_a(y) = \frac{-i}{1 - a^{-1}} \left\{ 1 + \sum_{k=1}^{N+1} G_k(y)(1 - a^{-1})^k + O\Big((a-1)^{N+1}\Big(1 + \frac{1}{y}\Big)\Big) \right\}, \tag{6.16}$$

where $G_1(y)$, $G_2(y), \ldots$ are given by

$$G_k(y) := \frac{ie^{iy}(\log y)^{k-1}}{(k-1)!y} + \frac{e^{iy}(\log y)^k}{k!} - \frac{i}{k!} \int_0^y (\log u)^k e^{iu}\, du.$$

Let us now further restrict to the case where $(a-1)^{\frac{1}{2}} \leq y \leq (a-1)^{-N}$. Using $|G_k(y)| \ll_k |\log y|^{k-1} y^{-1} + |\log y|^k + 1$, we see that there is some $a_0 = a_0(N) \in (1, 2]$ such that for all $a \in (1, a_0]$ and all $y \in [(a-1)^{\frac{1}{2}}, (a-1)^{-N}]$ the expression within the brackets in (6.16) lies in $\{z : |z - 1| < \frac{1}{2}\}$, and so we get

$$\frac{1}{\Phi_a(y)} = i(1 - a^{-1}) \left\{ 1 + \sum_{\ell=1}^{N} \left\{ -\sum_{k=1}^{N} G_k(y)(1 - a^{-1})^k \right\}^\ell \right.$$
$$\left. +O\Big(\Big(y^{-1} + |\log y|\Big)^{N+1} (a-1)^{N+1}\Big) \right\}.$$

Working similarly, starting from a differentiated version of (6.15), we also get an asymptotic formula for $\frac{\partial}{\partial a}\Phi_a(y)$, and with further computation, we finally obtain

$$\frac{e^{-iy}y^{-1-\frac{1}{a}}}{\Phi_a(y)}\left(\frac{\frac{\partial}{\partial a}\Phi_a(y)}{\Phi_a(y)}-\frac{\log y}{a^2}\right)=-\frac{ie^{-iy}}{y^2}\Big\{1+\sum_{\ell=1}^{N}H_\ell(y)(a-1)^\ell$$
$$+O\Big(\Big(y^{-1}+|\log y|\Big)^{N+1}(a-1)^{N+1}\Big)\Big\},$$

for all $a\in(1,a_0]$ and $y\in[(a-1)^{\frac{1}{2}},(a-1)^{-N}]$. Here $H_1(y),H_2(y),\ldots$ are certain continuous functions of y satisfying $|H_\ell(y)|\ll_\ell(y^{-1}+|\log y|)^\ell$; in particular we have

$$H_1(y)=2\Big\{i\int_0^y(\log u)e^{iu}\,du-\frac{ie^{iy}}{y}+(1-e^{iy})\log y-1\Big\};$$
$$H_2(y)=3\Big\{\frac{i}{2}\int_0^y(\log u)^2e^{iu}\,du-\Big(\int_0^y(\log u)e^{iu}\,du+i-\frac{i}{2}\log y-\frac{e^{iy}}{y}+ie^{iy}\log y\Big)^2$$
$$-\frac{ie^{iy}\log y}{y}-\log y+\big(\tfrac14-\tfrac12e^{iy}\big)(\log y)^2\Big\}. \tag{6.17}$$

Writing $\widetilde{H}_\ell(y):=-ie^{-iy}y^{-2}H_\ell(y)$, it follows that, for $y\le1$,

$$\Im\widetilde{H}_1(y)=2y^{-2}+\tfrac12-\tfrac{13}{144}y^2+O(y^4);\qquad \Im\widetilde{H}_2(y)=3y^{-4}+\tfrac32y^{-2}-\tfrac{17}{6}+O(y^2).$$

Furthermore, one computes (again for $y\le1$)

$$\Im\widetilde{H}_3(y)=-4y^{-4}-\tfrac{37}{3}y^{-2}+O(1);\qquad \Im\widetilde{H}_4(y)=-5y^{-6}-\tfrac{15}{2}y^{-4}+O(y^{-2});$$
$$\Im\widetilde{H}_5(y)=6y^{-6}+O(y^{-4}).\qquad \Im\widetilde{H}_6(y)=7y^{-8}+O(y^{-6}).$$

Using these relations (taking $N=6$), we obtain

$$\Im\int_{(a-1)^{\frac12}}^{\infty}\frac{e^{-iy}y^{-1-\frac{1}{a}}}{\Phi_a(y)}\left(\frac{\frac{\partial}{\partial a}\Phi_a(y)}{\Phi_a(y)}-\frac{\log y}{a^2}\right)dy=\int_0^\infty\frac{1-\cos y}{y^2}\,dy$$
$$+(a-1)\int_0^\infty\Big(\Im\widetilde{H}_1(y)-2y^{-2}\Big)\,dy+(a-1)^2\int_0^\infty\Big(\Im\widetilde{H}_2(y)-3y^{-4}-\tfrac32y^{-2}\Big)\,dy$$
$$+\Big\{-(a-1)^{-\frac12}+\tfrac52(a-1)^{\frac12}-\tfrac{95}{72}(a-1)^{\frac32}-\tfrac{52759}{5400}(a-1)^{\frac52}\Big\}+O\big((a-1)^3\big). \tag{6.18}$$

(This formula is first derived with each upper integration limit being $(a-1)^{-4}$ (say) in place of ∞; the remaining integrals over $y\in[(a-1)^{-4},\infty)$ are easily seen to be subsumed in the error term.)

To treat the integral over $y\le(a-1)^{\frac12}$, we start with the formula

$$\Phi_a(y)=\frac{y^{-\frac1a}(a-1-iy)}{a-1}\Big\{1-\sum_{k=2}^{N}\frac{i^k(a-1)}{k!(ka-1)(a-1-iy)}y^k+O\Big(y^N\min(a-1,y)\Big)\Big\},$$

which holds uniformly over all $a>1$ and $0<y\le1$, for any fixed $N\in\mathbb{Z}_{\ge2}$; this is proved using (5.7) and the power series expansion of e^{iu}. Note that the sum over k is $O(y\min(a-1,y))$; hence there is an absolute constant $y_0\in(0,1]$ such that for all

$a > 1$ and $0 < y \le y_0$, we have

$$\frac{1}{\Phi_a(y)} = \frac{y^{\frac{1}{a}}(a-1)}{a-1-iy}\Big\{1 + \sum_{1\le\ell\le N/2}\Big(\sum_{k=2}^{N}\frac{i^k(a-1)}{k!(ka-1)(a-1-iy)}y^k\Big)^{\ell} + O\Big(y^N\min(a-1,y)\Big)\Big\}.$$

Using this formula with $N = 5$, together with a similar asymptotic formula for $\frac{\partial}{\partial a}\Phi_a(y)$ deduced from a differentiated version of (5.7), we find after some computation that

$$\frac{e^{-iy}y^{-1-\frac{1}{a}}}{\Phi_a(y)}\Big(\frac{\frac{\partial}{\partial a}\Phi_a(y)}{\Phi_a(y)} - \frac{\log y}{a^2}\Big) = \frac{P_0(a-1,y)+P_1(a-1,y)\log y}{a^2(2a-1)^6(3a-1)^5(4a-1)^4(5a-1)^4(a-1-iy)^6} + O\Big((a-1)y^3\big(1+(a-1)|\log y|\big)\Big),$$

where P_0 and P_1 are explicit polynomials. This formula can now be integrated over y in terms of elementary functions, and we obtain

$$\Im\int_0^{(a-1)^{\frac{1}{2}}}\frac{e^{-iy}y^{-1-\frac{1}{a}}}{\Phi_a(y)}\Big(\frac{\frac{\partial}{\partial a}\Phi_a(y)}{\Phi_a(y)} - \frac{\log y}{a^2}\Big)\,dy = \tfrac{1}{2}\pi - \tfrac{5}{2}\pi(a-1)^2$$
$$+\Big\{(a-1)^{-\frac{1}{2}} - \tfrac{5}{2}(a-1)^{\frac{1}{2}} + \tfrac{95}{72}(a-1)^{\frac{3}{2}} + \tfrac{52759}{5400}(a-1)^{\frac{5}{2}}\Big\} + O\big((a-1)^3\big). \tag{6.19}$$

Finally, we add (6.18) and (6.19), and note that since $H_1(y) = -2(F_1(y) + \frac{ie^{iy}}{y} + 1)$, we have

$$\int_0^{\infty}\big(\Im\widetilde{H}_1(y) - 2y^{-2}\big)\,dy = \int_{-\infty}^{\infty}\frac{\Im\big(ie^{-iy}F_1(y)\big)}{y^2}\,dy - \pi = 0,$$

where the second equality follows by again moving the contour towards infinity in the lower half-plane, noticing the pole at $y = 0$. Hence we arrive at (1.13), with

$$K_1 = 20 + \frac{8}{\pi}\int_0^{\infty}\Big(3y^{-4} + \tfrac{3}{2}y^{-2} - \Im\widetilde{H}_2(y)\Big)\,dy = 39.47841\ldots \tag{6.20}$$

(cf. [27, constants.mpl]). This completes the proof of Corollary 1. □

Remark 10. It appears that by the same method one could obtain asymptotic expansions of $f(c)$, in the limits as $c \to \infty$ and $c \to \frac{1}{2}$, with the error term having any desired power rate of decay.

Appendix A. Residue calculus and numerical computation of the density

In this appendix we discuss the evaluation of the integrals in (5.20) and (6.1) using the residue theorem, resulting in alternative formulas for $\mathbf{P}\big(\sigma_{\{T_j\}} > c\big)$ and the corresponding density. These formulas turn out to be useful for numerical computation, something which we discuss briefly towards the end of the appendix (see also [27, numdensity.mpl]).

We now write z in place of y. By (5.7) we have $\Phi_a(z) = z^{-\frac{1}{a}}(e^{iz} - iz\int_0^1 e^{izt}t^{-\frac{1}{a}}\,dt)$, and here the expression in the parenthesis is clearly an entire function of z. Hence

$$\Psi_a(z) := \frac{e^{-iz}z^{-1-\frac{1}{a}}}{\Phi_a(z)} = \frac{e^{-iz}z^{-1}}{e^{iz} - iz\int_0^1 e^{izt}t^{-\frac{1}{a}}\,dt} \tag{A.1}$$

is a meromorphic function in all of $\mathbb{C}$. In (5.20) we are integrating $\Im\Psi_a(z)$ along the positive real line; using the symmetry $\Psi_a(-z) = -\overline{\Psi_a(\overline{z})}$, we may rewrite this as

$$\mathbf{P}\big(\sigma_{\{T_j\}} > c\big) = \frac{1}{2} + \lim_{r\to 0^+} \frac{1}{2\pi i}\Big(\int_{-\infty}^{-r} \Psi_a(y)\,dy + \int_r^{\infty} \Psi_a(y)\,dy\Big). \tag{A.2}$$

Let C_r' be the semicircle $\{z : |z| = r, \Im z \le 0\}$, oriented in the direction from $-r$ to r, and let C_r be the contour going from $-\infty$ to $-r$ along $\mathbb{R}$, then from $-r$ to r along C_r' and finally from r to $+\infty$ along $\mathbb{R}$. Since $\Psi_a(z)$ has a simple pole at $z = 0$ with residue 1, we have $\int_{C_r'} \Psi_a(z)\,dz = i\pi + O(r)$ as $r \to 0$. Thus (A.2) equals $\lim_{r\to 0^+} \frac{1}{2\pi i}\int_{C_r} \Psi_a(z)\,dz$. However, by Cauchy's integral theorem, $\int_{C_r} \Psi_a(z)\,dz$ is independent of r for all sufficiently small r. Hence

$$\mathbf{P}\big(\sigma_{\{T_j\}} > c\big) = \frac{1}{2\pi i}\int_{C_r} \Psi_a(z)\,dz, \tag{A.3}$$

for any $r > 0$ so small that $\Psi_a(z)$ has no pole in the punctured disk $\{z : 0 < |z| \le r\}$.

We wish to replace C_r in (A.3) by a contour over z's with large negative imaginary part. In order to do so, we first need to understand the poles of $\Psi_a(z)$ in the lower half plane. Numerics indicate that there is exactly one simple pole in the infinite vertical strip $\{z : (2n-1)\pi < \Re z < (2n+1)\pi, \Im z < 0\}$ for each integer n; cf. Figure 2 below. However, for technical reasons it seems easier to prove a corresponding statement instead for certain "curved vertical strips", as follows. For each $n \in \mathbb{Z}^+$, we let Γ_n be the curve in the complex plane given by

$$x \mapsto c_n(x) = x - ix\tan\big((n-\tfrac{1}{4})\pi - \tfrac{1}{2}x\big), \qquad \big(2n-\tfrac{3}{2}\big)\pi < x \le \big(2n-\tfrac{1}{2}\big)\pi. \tag{A.4}$$

One notes that $\Im c_n(x) \to -\infty$ as $x \to (2n-\frac{3}{2})\pi^+$, that $\Im c_n((2n-\frac{1}{2})\pi) = 0$ and that $0 < \arg c_n'(x) < \frac{\pi}{2}$ for all $(2n-\frac{3}{2})\pi < x < (2n-\frac{1}{2})\pi$. Hence Γ_n and Γ_{n+1}, together with the real interval $[(2n-\frac{1}{2})\pi, (2n+\frac{3}{2})\pi]$, bound a curved vertical strip, which we call S_n (we take S_n to be closed). We also let $S_{-n} = \{-\overline{z} : z \in S_n\}$ be the reflection of S_n in the imaginary axis, and we let S_0 be the curved vertical strip bounded by the curves Γ_1, $\{-\overline{z} : z \in \Gamma_1\}$ and $[-\frac{3}{2}\pi, \frac{3}{2}\pi]$. Now the union of all S_n $(n \in \mathbb{Z})$ equals the negative half plane, $\{z : \Im z \le 0\}$, and the S_n's have pairwise disjoint interiors.

Proposition 3. *Let $a > 1$ be given. For each $n \in \mathbb{Z}$, the function $z\Psi_a(z)$ has a unique pole in the strip S_n. This pole is simple, and lies in the interior of S_n.*

For the proof we need the following lemma. We will use the definition (5.4) of $\Gamma(s,z)$ for general $z \in \mathbb{C}\setminus\mathbb{R}_{\le 0}$, the integral being over the infinite ray $u \in z + \mathbb{R}_{>0}$.

Lemma 9. *For any $s \in [1,2]$ and any $z = -x+iy \in \mathbb{C}$, satisfying either $\frac{1}{2}\pi \le |y| \le \frac{1}{2}x$, $\frac{3}{4}\pi \le |y| \le x$ or $[x \ge 0$ and $|y| \ge \pi]$, we have*

$$\big|\Gamma(-s,z)\big| < s^{-1}|z|^{-s}e^{x}. \tag{A.5}$$

Proof. Take s and $z = -x+iy$ satisfying the assumptions. By symmetry, we may assume $y > 0$. We may deform the contour of integration in (5.4) to be the ray $\{z + t(1+ki) : t \ge 0\}$, where k is any fixed non-negative number. This ray intersects the imaginary axis at $(y+kx)i$, and thus $|u| \ge (y+kx)(1+k^2)^{-1/2}$ holds for every point u on the ray, and

$$\big|\Gamma(-s,z)\big| \le \frac{(1+k^2)^{\frac{s+1}{2}}}{(y+kx)^{s+1}}\int_0^{\infty} e^{x-t}(1+k^2)^{\frac{1}{2}}\,dt = \frac{(1+k^2)^{1+\frac{s}{2}}}{(y+kx)^{s+1}}e^{x}. \tag{A.6}$$

Applying this with $k=1$, we see that (A.5) holds whenever $s\big(\frac{\sqrt{2}|z|}{x+y}\big)^s < \frac{x+y}{2}$. But $\frac{\sqrt{2}|z|}{x+y} \geq 1$ for all non-zero z and thus the inequality holds for all $s \in [1,2]$ if and only if it holds for $s=2$, i.e. if and only if $\frac{x^2+y^2}{(x+y)^3} < \frac{1}{8}$. However, it is easily verified that $\frac{x^2+y^2}{(x+y)^3}$ is a decreasing function of $x>0$ for any fixed $y \geq 0$. Hence, if $x \geq y \geq \frac{3}{4}\pi$, then $\frac{x^2+y^2}{(x+y)^3} \leq \frac{1}{4y} \leq \frac{1}{4\cdot\frac{3}{4}\pi} < \frac{1}{8}$; similarly, if $x \geq 2y \geq \pi$, then $\frac{x^2+y^2}{(x+y)^3} \leq \frac{5}{27\cdot\frac{1}{2}\pi} < \frac{1}{8}$, and if $y \geq \pi$ and $x \geq \frac{3}{4}y$, then $\frac{x^2+y^2}{(x+y)^3} \leq \frac{100}{343y} \leq \frac{100}{343\pi} < \frac{1}{8}$. To treat the remaining case, when $y \geq \pi$ and $0 \leq x < \frac{3}{4}y$, we apply (A.6) with $k=0$; from this we see that (A.5) holds whenever $s(|z|/y)^s < y$. However, if $y \geq \pi$ and $0 \leq x < \frac{3}{4}y$, then $s(|z|/y)^s \leq 2(|z|/y)^2 < \frac{25}{8} < \pi \leq y$, and we are done. $\square$

We also record the following bound, which follows from (A.6) with $k=1$:

Lemma 10. *The bound* $\big|\Gamma(-s,z)\big| \ll |z|^{-s-1}e^{-\Re z}$ *holds uniformly for all* $s \in [1,2]$ *and all* $z \in \mathbb{C}$ *with* $\Re z \leq 0$, $\Im z \neq 0$.

Proof of Proposition 3. Let $\eta_a(z) = z^{\frac{1}{a}}\Phi_a(z) = e^{iz} - iz\int_0^1 e^{izt}t^{-\frac{1}{a}}\,dt$ and note that η_a is an entire function. By (A.1), our task is to prove that for each n, $\eta_a(z)$ has a unique zero in S_n, which is simple and lies in the interior of S_n. Using (5.21) and applying the recursion formula $\Gamma(s,z) = e^{-z}z^{s-1} + (s-1)\Gamma(s-1,z)$ twice, we find that for z with $\Re z > 0$, we have

$$\text{(A.7)}\qquad \eta_a(z) = w_1 + w_2 + w_3 \quad \text{with} \quad \begin{cases} w_1 = (-iz)^{\frac{1}{a}}\Gamma(1-a^{-1}) \\ w_2 = a^{-1}(-iz)^{-1}e^{iz} \\ w_3 = -\frac{a+1}{a^2}(-iz)^{\frac{1}{a}}\Gamma\big(-1-a^{-1},-iz\big), \end{cases}$$

wherein $(-iz)^{\frac{1}{a}} = \exp(\frac{1}{a}\log(-iz))$ with the principal branch of the logarithm; $-\pi < \Im\log(-iz) < 0$.

Let $n \in \mathbb{Z}^+$ and $z = x - iy \in \Gamma_n$. We wish to apply Lemma 9 with $s = 1 + a^{-1}$ and with $-iz$ in place of z. In order to justify this application, we have to check that either $x \geq \pi$, $y \geq x \geq \frac{3}{4}\pi$ or $y \geq 2x \geq \pi$; this is clear if $n \geq 2$, since then $x > \pi$, and if $n=1$, then the claim follows using (A.4), $\tan(\frac{1}{4}\pi) = 1$ and $\tan(\frac{3}{8}\pi) > 2$. The conclusion from Lemma 9 is that $|w_3| < |w_2|$ holds in (A.7). We also note that $\arg\big(w_1/w_2\big) \in (1+a^{-1})(-\frac{1}{2}\pi + \arg(z)) - x + 2\pi\mathbb{Z}$, and by (A.4), we have $x \in ((2n-\frac{3}{2})\pi, (2n-\frac{1}{2})\pi]$ and $\arg(z) = -(n-\frac{1}{4})\pi + \frac{1}{2}x \in (-\frac{1}{2}\pi, 0]$; together these imply that $\arg\big(-ize^{-iz}w_1\big)$ lies in $\big[-\frac{1}{2}a^{-1}\pi, (\frac{1}{2}-a^{-1})\pi\big) \subset (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, i.e. that $\Re(w_1/w_2) > 0$. Moreover, $|w_3| < |w_2|$ forces $\Re((w_2+w_3)/w_2) > 0$; hence we conclude that $\Re((w_1+w_2+w_3)/w_2) > 0$, i.e. that

$$\text{(A.8)}\qquad \Re(-ize^{-iz}\eta_a(z)) > 0 \quad \text{for all } z \in \Gamma_n.$$

This shows that $\eta_a(z)$ has no zeros along Γ_n, and also gives a precise control on the variation of $\arg\eta_a(z)$ along Γ_n.

Next, from (A.7) and Lemma 10, we see that for $z = x - iy$ with y large and $x > 0$ bounded, we have $\eta_a(z) = w_1 + w_2 + w_3 = w_2(1 + O(y^{-1}))$, and thus $\arg\eta_a(z) \in \pi + x + O(y^{-1}) + 2\pi\mathbb{Z}$. Also note that $\Re\eta_a(z) > 0$ for all $z \geq 0$, since $\Re\Phi_a(z) > 0$ for all $z > 0$ (as noted previously) and $\eta_a(0) = 1$. Using these facts together with (A.8) (applied both for n and $n+1$), we conclude that for any $n \in \mathbb{Z}^+$ and any sufficiently large $Y > 0$ (depending on both a and n), $\arg\eta_a(z)$ increases by 2π as z travels around the boundary of $S_n \cap \{\Im z \geq -Y\}$ in the positive direction. Hence, by the argument principle, $\eta_a(z)$ has a unique simple zero in the interior of S_n. Using the symmetry $\eta_a(-\overline{z}) = \overline{\eta_a(z)}$, one proves the same fact also for S_0 and any S_n, $n < 0$. This completes the proof of the proposition. $\square$

From now on, we write $\zeta_n = \zeta_n(a)$ for the unique pole of $z\Psi_a(z)$ in S_n $(n \in \mathbb{Z})$. By symmetry we have $\zeta_{-n} = -\overline{\zeta_n}$ for all n, and in particular ζ_0 lies on the negative imaginary axis. Figure 2 below shows the curves traced by $\zeta_0, \ldots, \zeta_4$ as a varies.

The next lemma gives an asymptotic formula for ζ_n $(n > 0)$ with an error which is small whenever at least one of n, a and $(a-1)^{-1}$ is large.

Lemma 11. *We have, uniformly over all $a > 1$ and all $n \in \mathbb{Z}^+$,*

$$\text{(A.9)} \quad \zeta_n = (2n - a^{-1})\pi + (1 + a^{-1}) \arctan\Big(\frac{2\pi n}{Y_n}\Big) - iY_n + O\bigg(\frac{1}{n + \log\big|\Gamma(-a^{-1})\big|}\bigg),$$

where Y_n equals the unique root $y > 0$ of the equation

$$\text{(A.10)} \qquad y - \tfrac{1}{2}(1 + a^{-1}) \log\big((2\pi n)^2 + y^2\big) = \log\big|\Gamma(-a^{-1})\big|.$$

(Regarding the error term in (A.9), we remark that $|\Gamma(-a^{-1})| > 3$, and thus that $\log|\Gamma(-a^{-1})| > 1$, for all $a > 1$.)

Proof. Using (A.7) and Lemma 10, together with the fact that $\eta_a(\zeta_n) = 0$, we get

$$\text{(A.11)} \qquad \Gamma(-a^{-1}) = (-i\zeta_n)^{-1-\frac{1}{a}} e^{i\zeta_n}(1 + O(|\zeta_n|^{-1})), \qquad \forall a > 1,\ n \in \mathbb{Z}^+.$$

Writing $\zeta_n = x_n - iy_n$ $(x_n, y_n > 0)$ and taking absolute values in (A.11), we get

$$\text{(A.12)} \qquad \big|\Gamma(-a^{-1})\big| = |\zeta_n|^{-1-\frac{1}{a}} e^{y_n}(1 + O(|\zeta_n|^{-1})).$$

Now, using the facts that $|\zeta_n| \geq x_n > (2n - \frac{3}{2})\pi \gg n$ and $|\Gamma(-a^{-1})| \to \infty$ as $a \to 1^+$ or $a \to \infty$, we conclude that y_n must be large whenever at least one of n, a and $(a-1)^{-1}$ is large; and due to the form of the error term in (A.9), we may without loss of generality restrict to the case when this holds. Note that also $|\zeta_n|$ must be large, since $|\zeta_n| \geq y_n$.

In more precise terms, we have, considering the logarithm of equation (A.12),

$$\text{(A.13)} \qquad y_n = \tfrac{1}{2}(1 + a^{-1}) \log(x_n^2 + y_n^2) + \log\big|\Gamma(-a^{-1})\big| + O(|\zeta_n|^{-1}).$$

In particular, using $(2n - \frac{3}{2})\pi < x_n < (2n + \frac{3}{2})\pi$ and also $\log(x_n^2 + y_n^2) \leq \frac{1}{2}y_n + 2\log n$ (which holds since y_n is large), we conclude that

$$\text{(A.14)} \quad y_n \asymp \log n + \log|\Gamma(-a^{-1})|; \quad \text{and thus } \ |\zeta_n| \asymp x_n + y_n \asymp n + \log\big|\Gamma(-a^{-1})\big|.$$

(Note: "$\asymp$" means "both $\ll$ and $\gg$".) Now $x_n^2 + y_n^2 = ((2\pi n)^2 + y_n^2)(1 + O(|\zeta_n|^{-1}))$, and thus, in (A.13), we may replace "$\log(x_n^2 + y_n^2)$" by "$\log((2\pi n)^2 + y_n^2)$"; the error from this operation is subsumed in the error term $O(|\zeta_n|^{-1})$. We also note that the expression in the left-hand side of (A.10) is an increasing function of $y > 0$, which is negative for small y and the derivative of which lies in the interval $(1 - (2\pi)^{-1}, 1]$, for all $y > 0$. It follows that Y_n (in the statement of the lemma) is well-defined, and also that

$$\text{(A.15)} \qquad y_n = Y_n + O\big(|\zeta_n|^{-1}\big).$$

Next, taking the argument of both sides of (A.11), we get

$$\text{(A.16)} \quad x_n = (1 - a^{-1})\tfrac{\pi}{2} + (1 + a^{-1}) \arg(\zeta_n) + 2k\pi + O(|\zeta_n|^{-1}) \quad \text{for some } \ k \in \mathbb{Z},$$

where $-\frac{\pi}{2} < \arg(\zeta_n) < 0$. Clearly $(2k-1)\pi - O(|\zeta_n|^{-1}) < x_n < (2k+\frac{1}{2})\pi + O(|\zeta_n|^{-1})$, and in fact, since $-\arg(\zeta_n) \gg y_n|\zeta_n|^{-1}$ and y_n is large, we even have $x_n < (2k+\frac{1}{2})\pi$. But also $x_n > (2n-\frac{3}{2})\pi$; hence $k \geq n$. On the other hand, since ζ_n lies to the left of the curve Γ_{n+1}, we have $x_n < 2\arg(\zeta_n) + (2n+\frac{3}{2})\pi$, and using this fact in (A.16), we get $(1-a^{-1})\arg(\zeta_n) > (2(k-n) - 1 - \frac{1}{2}a^{-1})\pi - O(|\zeta_n|^{-1})$. This forces $k \leq n$, since $\arg(\zeta_n) < 0$ and $|\zeta_n|$ is large. Hence we have proved that $k = n$. Finally, using (A.15) and $(2n-\frac{3}{2})\pi < x_n < (2n+\frac{3}{2})\pi$, we get $\big|\arg(\zeta_n) + \arctan(\frac{Y_n}{2\pi n})\big| \ll Y_n|\zeta_n|^{-2} \ll |\zeta_n|^{-1}$. Now (A.9) follows from (A.15), (A.16) and (A.14). $\square$

We may also remark that Y_n, as defined in Lemma 11, satisfies

$$Y_n = G + \frac{1+a^{-1}}{2}\left(1 + \frac{(1+a^{-1})G}{(2\pi n)^2 + G^2}\right)\log\big((2\pi n)^2 + G^2\big) + O\Big(\frac{1}{n+G}\Big), \tag{A.17}$$

with $G = \log|\Gamma(-a^{-1})|$. This is proved by direct substitution in (A.10), using the properties of the left-hand side in (A.10) noted in the proof of Lemma 11.

We will now change the contour in (A.3). Let $a > 1$ be given, and fix $r > 0$ sufficiently small so that (A.3) holds. For $n \in \mathbb{Z}^+$ and $Y > 0$, we let $z_{n,Y}$ be the unique point where Γ_n intersects $\{\Im z = -Y\}$, and let $C_{n,Y}$ be the contour going from $-\infty$ to $-(2n-\frac{1}{2})\pi$ along $\mathbb{R}$, then along $\Gamma_{-n} := \{-\overline{z} \,:\, z \in \Gamma_n\}$ to $-\overline{z_{n,Y}}$, then along $\{\Im z = -Y\}$ to $z_{n,Y}$, further along Γ_n to $(2n-\frac{1}{2})\pi$, and finally along $\mathbb{R}$ to $+\infty$. By the residue theorem and Proposition 3, for every $n \in \mathbb{Z}^+$ there is some $Y_0 = Y_0(a,n) > 0$ such that for $Y > Y_0$, we have

$$\frac{1}{2\pi i}\int_{C_r} \Psi_a(z)\,dz = \frac{1}{2\pi i}\int_{C_{n,Y}} \Psi_a(z)\,dz - \sum_{m=1-n}^{n-1} \mathrm{Res}_{z=\zeta_m}\,\Psi_a(z). \tag{A.18}$$

Now let w_1, w_2, w_3 be as in (A.7). By Lemma 10 there is an $N = N(a) \in \mathbb{Z}^+$ such that $|w_3| \leq \frac{1}{2}|w_2|$ for all $z \in \Gamma_n$, $n \geq N$. Using also $\Re(w_1/w_2) > 0$ for all $z \in \Gamma_n$, we get $|\eta_a(z)| = |w_1 + w_2 + w_3| \geq \frac{\sqrt{3}}{2}|w_1|$ and thus $|\Psi_a(z)| \ll n^{-1-\frac{1}{a}}e^{\Im z}$ for all $n \geq N$ and $z \in \Gamma_n$. Also, for any fixed a and n, we have $|\eta_a(z)| \gg Y^{-1}e^{Y}$ for all $z \in C_{n,Y} \cap \{\Im z = -Y\}$ (cf. Lemma 10 and (A.7)); thus $|\Psi_a(z)| \ll e^{-2Y}$ for these z. The above bounds imply $\lim_{n\to\infty}\big(\lim_{Y\to\infty}\int_{C_{n,Y}} |\Psi_a(z)|\,|dz|\big) = 0$, and so

$$\mathbf{P}\big(\sigma_{\{T_j\}} > c\big) = \frac{1}{2\pi i}\int_{C_r} \Psi_a(z)\,dz = -\lim_{n\to\infty}\sum_{m=-n}^{n} \mathrm{Res}_{z=\zeta_m}\,\Psi_a(z) = \sum_{n=-\infty}^{\infty} ae^{-2i\zeta_n}. \tag{A.19}$$

Here the last equality follows from an easy calculation using (5.9) and (A.1), noticing that the sum is absolutely convergent, since, by Lemma 11 and (A.17), we have

$$\big|ae^{-2i\zeta_n}\big| \ll ae^{-2G}(|n| + G)^{-2(1+\frac{1}{a})}, \qquad \forall a > 1,\ n \in \mathbb{Z}\setminus\{0\}. \tag{A.20}$$

One also checks that the formula (A.19) may be differentiated termwise with respect to a, yielding

$$f(c) = 2\sum_{n=-\infty}^{\infty} e^{-2i\zeta_n}\Big(2ai\Big(\frac{d}{da}\zeta_n\Big) - 1\Big) \tag{A.21}$$

for the density function (cf. (6.1)).

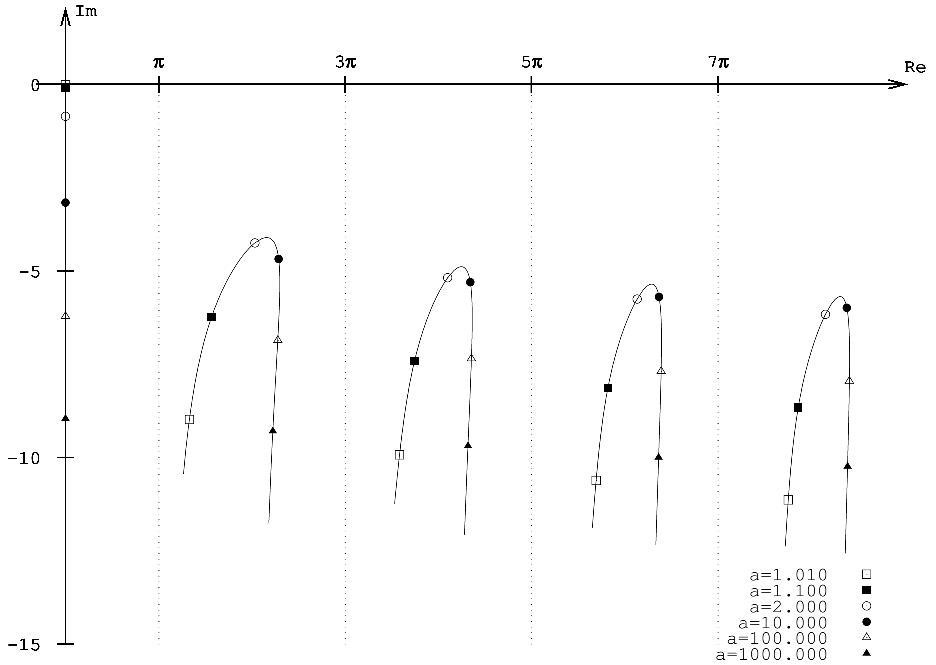

Figure 2. The curves traced by the poles $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ (and ζ_0) as a varies.

For c not too large, the formula (A.21) can be used to compute $f(c)$ numerically to a decent precision. We have implemented this in [27, numdensity.mpl]. Our experiments indicate that for any given $a > 1$ ($a = 2c$) and $n \in \mathbb{Z}^+$, the asymptotic formula in Lemma 11 is sufficiently accurate so that it can be used as the initial value in the Newton iteration algorithm solving for $\Phi_a(z) = 0$, with rapid convergence. Also, $\frac{d}{da}\zeta_n$ is computed using

$$\begin{aligned}\frac{d}{da}\zeta_n &= a\zeta_n^{1+\frac{1}{a}} e^{-i\zeta_n} \Big(\frac{\partial}{\partial a}\Phi_a(z)\Big)_{|z=\zeta_n} \\ &= -a^{-2}\zeta_n^{1+\frac{1}{a}} e^{-i(\zeta_n + \frac{\pi}{2a})} \bigg(\Gamma'(-a^{-1}) - \int_{-i\zeta_n}^{\infty} e^{-u} u^{-1-\frac{1}{a}} (\log u)\, du\bigg),\end{aligned}$$

which most often can be evaluated very quickly via repeated integration by parts; in the remaining cases we use numerical integration.

The data for the graph in Figure 1 can be found in [27, density.dat]; it was assembled by computing $f(c)$ ($c = \frac{1}{2}a$) for $a = 1 + \frac{1}{100}k$, $k = 1, 2, \ldots, 400$. For each a-value we truncated the sum in (A.21) at $|n| \leq 400$ (using also the obvious $n \leftrightarrow -n$ symmetry). It turns out that the terms in (A.21) decay roughly as $n^{-2(1+\frac{1}{a})}$ as $n \to \infty$ (cf. (A.20)). In particular we have slower convergence for larger a and this is seen in the computations: Our numerics indicate that we obtain the first few $f(c)$-values to within an absolute error $\lesssim 10^{-11}$, whereas for a near 5 (where $f(c) \approx 0.05$) the error is $\lesssim 10^{-6}$. Of course the precision can be improved by including more terms in (A.21), again cf. [27, numdensity.mpl].

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