Uppsala universitet
Matematiska institutionen
Ernst Dieterich

Kandidatprogrammet i matematik
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Algebraic structures

Seventh sheet of exercises

86. As an application of the theory of finite abelian groups one can prove the following

Theorem. *Every finite subgroup of the unit group of a field is cyclic.*

Use this theorem to show that every finite extension of a finite field is simple.

87. Show that the real number $\alpha = 2 + \sqrt[4]{5}$ is algebraic, and find $\mathrm{irrpol}_{\mathbb{Q}}(\alpha)$.

88. The *n-th cyclotomic polynomial* is defined for all $n \in \mathbb{N} \setminus \{0\}$ as

$$\Phi_n(X) = \mathrm{irrpol}_{\mathbb{Q}}\left(e^{\frac{2\pi}{n}i}\right).$$

Find $\Phi_n(X)$ for all $1 \leq n \leq 8$.

89. Show that the field $\mathbb{A}$ of all algebraic numbers has the following properties.

(a) $[\mathbb{A} : \mathbb{Q}] = \infty$.

(b) $\mathbb{A}$ is not simple over $\mathbb{Q}$.

(c) $\mathbb{A}$ is not finitely generated over $\mathbb{Q}$.

90. A field K is called *algebraically closed* if K is the only algebraic extension of K. Show that $\mathbb{C}$ is algebraically closed.

91. Show that $\mathbb{A}$ is algebraically closed.

92. Show that every field extension $K \subset E$ of degree 2 is Galois, provided that $\mathrm{char}(K) \neq 2$.

93. Prove that $|\mathrm{Gal}(E/K)| = [E : K]$, whenever $K \subset E$ is a finite Galois extension.

(Hint. According to the proof of Proposition 56 there exists a primitive element $\alpha \in E$ such that $E = K(\alpha)$ is a splitting field for $q(X) = \mathrm{irrpol}_K(\alpha)$. Let $R = \{\alpha_1, \ldots, \alpha_n\}$ be the set of all roots of $q(X)$ in E, with $\alpha = \alpha_1$. Show that for each $\alpha_i \in R$ there is a unique $\sigma \in \mathrm{Gal}(E/K)$ such that $\sigma(\alpha) = \alpha_i$.)

Please turn over!

94. Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(a) Determine $\mathrm{Gal}(E/\mathbb{Q})$.

(b) Describe all subgroups of $\mathrm{Gal}(E/\mathbb{Q})$, ordered by inclusion.

(c) Describe all intermediate fields $\mathbb{Q} \subset F \subset E$, ordered by inclusion.

95. Let $E = \mathbb{Q}(\zeta)$, where $\zeta = e^{\frac{2\pi}{13}i}$.

(a) Determine $\mathrm{Gal}(E/\mathbb{Q})$.

(b) Describe all subgroups of $\mathrm{Gal}(E/\mathbb{Q})$, ordered by inclusion.

(c) Describe all intermediate fields $\mathbb{Q} \subset F \subset E$, ordered by inclusion.

96. Let $K \subset E$ be a finite extension of degree n. Show that the inequality

$$\deg(\mathrm{irrpol}_K(\alpha)) \leq n$$

holds for all $\alpha \in E$.

97. Show that every irreducible real polynomial has degree 1 or 2.

98. Find the addition table and the multiplication table of a field of order 8.

99. Find the complex roots of the polynomial $f(X) = 12X^3 - 16X^2 + 3X - 4$.