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Prov i matematik
Algebraiska strukturer
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Skrivtid: 9.00-14.00. Inga hjälpmedel förutom skrivdon. Lösningarna skall åtföljas av förklarande text! Varje uppgift ger maximalt 5 poäng.

1. (a) Define Klein's four-group V_4 and the symmetric group S_3 respectively.

(b) Show that the groups $\mathrm{Aut}(\mathrm{V}_4)$ and S_3 are isomorphic.

2. Classify all abelian groups of order 2475, up to isomorphism.

3. (a) Reproduce the statements of the three Sylow theorems.

(b) Show that every group of order 30 has a nontrivial proper normal subgroup.

4. A ring is called *simple* if it has precisely two (two-sided) ideals. Decide for each of the following rings R whether they are simple or not, and prove your statements.

(a) $R = \{0\}$;

(b) $R = \mathbb{Z}$;

(c) $R = K^{n \times n}$, where K is a field and $n \in \mathbb{N} \setminus \{0\}$.

5. Let R be a commutative ring.

(a) What is meant by the universal property of the polynomial ring $R[X]$? Reproduce the statement!

(b) Show that the substitution map

$$\sigma_a : R[X] \to R[X],\ \sigma_a(f(X)) = f(X + a)$$

is an automorphism of the polynomial ring $R[X]$, for each $a \in R$.

(c) Find a subgroup $H < \mathrm{Aut}(R[X])$ together with an isomorphism of groups

$$\varphi : (R, +) \tilde{\to} H.$$

Please turn over!

6. Find the addition table and the multiplication table of a field of order 8.

7. Let $E = \mathbb{Q}(\zeta)$, where $\zeta = e^{\frac{2\pi}{7}i}$.

(a) When is a field extension called separable, when is it called normal, and when is it called Galois? Reproduce the definitions!

(b) Explain why $\mathbb{Q} \subset E$ is a finite Galois extension.

(c) Determine $\mathrm{Gal}(E/\mathbb{Q})$, up to isomorphism.

(d) Describe all subgroups of $\mathrm{Gal}(E/\mathbb{Q})$, ordered by inclusion.

(e) Describe all intermediate fields $\mathbb{Q} \subset F \subset E$, ordered by inclusion.

8. Let $f(X) = X^5 - 6X + 3 \in \mathbb{Q}[X]$. Let $R = \{\alpha_1, \ldots, \alpha_5\}$ be the set of all complex roots of $f(X)$, set $E = \mathbb{Q}(\alpha_1, \ldots, \alpha_5)$, and denote for every $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ by σ_R the permutation of R induced by σ. Give reasons for each of the following statements.

(a) The polynomial $f(X)$ is irreducible in $\mathbb{Q}[X]$.

(b) All the roots α_i are simple.

(c) There is a $\tau \in \mathrm{Gal}(E/\mathbb{Q})$ such that τ_R is a transposition.

(d) There is a $\gamma \in \mathrm{Gal}(E/\mathbb{Q})$ such that γ_R is a 5-cycle.

(e) The equation $f(X) = 0$ is not solvable by radicals.

Good luck!