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Prov i matematik
Algebraiska strukturer
2009-04-15

Skrivtid: 8.00-13.00. Inga hjälpmedel förutom skrivdon. Lösningarna skall åtföljas av förklarande text. Varje uppgift ger maximalt 5 poäng.

1. (a) Define the *dihedral* group D_n for any natural number $n \geq 2$.

(b) Show that every non-trivial morphism $\varphi : \mathrm{D}_2 \to \mathrm{D}_3$ has the property $|\ker \varphi| = 2 = |\mathrm{im}\varphi|$.

(c) Use (b) to find the number of morphisms from D_2 to D_3.

2. (a) The *centre* of a group G is a subset $\mathrm{Z}(G) \subset G$. Which one? Reproduce its definition!

(b) Show that $\mathrm{Z}(G)$ always is a normal subgroup of G.

(c) What can you say about $\mathrm{Z}(G)$ in case G has prime squared order? Motivate your answer!

3. Classify all groups of order 529.

4. (a) What is a *subring* of a ring R? Reproduce the definition!

(b) Show that $\mathbb{H} = \left\{ \begin{pmatrix} w & -z \\ \overline{z} & \overline{w} \end{pmatrix} \middle| w, z \in \mathbb{C} \right\}$ is a subring of the ring $\mathbb{C}^{2\times 2}$ of all complex 2×2-matrices.

5. (a) Reproduce the definition of a *unit* (also called *invertible element*) of a ring R, and show that the set R^ι of all units in R is a multiplicative group.

(b) Determine the unit group R^ι for each of the following rings R, and motivate your description: (i) $R = \mathbb{Z}$, (ii) $R = \mathbb{Q}[X]$, (iii) $R = \mathbb{R}^{3\times 3}$, (iv) $R = \mathbb{H}$, the ring defined in problem 4(b).

6. (a) Reproduce the *cubic formula*, expressing the roots of a complex cubic $f(X) = X^3 + qX + r$ in terms of its coefficients q and r.

(b) Express the roots of the cubic $f(X) = X^3 + 3X + 2$ in terms of its coefficients 3 and 2.

PLEASE TURN OVER!

7. Let $E = \mathbb{Q}(\zeta)$, where $\zeta = e^{\frac{2\pi}{19} i}$.

(a) When is a field extension called *separable*, when is it called *normal*, and when is it called *Galois*? Reproduce the definitions!

(b) Explain why $\mathbb{Q} \subset E$ is a finite Galois extension.

(c) Determine $\mathrm{Gal}(E/\mathbb{Q})$, up to isomorphism.

(d) Describe all subgroups of $\mathrm{Gal}(E/\mathbb{Q})$, ordered by inclusion.

(e) Describe all intermediate fields $\mathbb{Q} \subset F \subset E$, ordered by inclusion.

8. Let $f(X) = X^5 - 4X + 2 \in \mathbb{Q}[X]$. Let $R = \{\alpha_1, \ldots, \alpha_5\}$ be the set of all complex roots of $f(X)$, set $E = \mathbb{Q}(\alpha_1, \ldots, \alpha_5)$, and denote for every $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ by σ_R the permutation of R induced by σ. Give reasons for each of the following statements.

(a) The polynomial $f(X)$ is irreducible in $\mathbb{Q}[X]$.

(b) All the roots α_i are simple.

(c) There is a $\tau \in \mathrm{Gal}(E/\mathbb{Q})$ such that τ_R is a transposition.

(d) There is a $\gamma \in \mathrm{Gal}(E/\mathbb{Q})$ such that γ_R is a 5-cycle.

(e) The equation $f(X) = 0$ is not solvable by radicals.

GOOD LUCK!