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Holomorphic Curve Theories in Symplectic Geometry

Lecture I

Georgios Dimitroglou Rizell

Uppsala University

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Plan

1. Course introduction
2. Basic definitions
3. The projective space
4. Gromov's Compactness Theorem

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Outline of course goals

1. Moduli spaces, Gromov compactness & Applications.
2. Floer homology, Symplectic homology & Applications.
3. The Fukaya Category.
4. The surgery formula for the Fukaya category of Weinstein domains (**T. Ekholm**).

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Goal of first lectures

- Today:
  - Basic symplectic definitions
  - Basic example: projective spaces and blow-ups
  - Gromov's compactness theorem for spheres.
- Uniruledness of $\mathbb{C}P^n$ & applications.
- Relative uniruledness of product tori $L(a_1, \ldots, a_n) \subset \mathbb{C}P^n$ & applications.

Symplectic manifolds

Definition

A *symplectic manifold* is a pair (X^{2n}, ω) that consists of a $2n$-dimensional smooth manifold X^{2n} equipped with a closed and non-degenerate two-form $\omega \in \Omega^2(X)$.

Question (Difficult and open)

Which <u>closed</u> manifolds admit a symplectic form?

Necessary topological conditions:

- TX must admit an **almost complex structure**, i.e. $J \in \mathrm{End}(TX)$ for which $J^2 = -\mathrm{Id}$.
- There is some $\alpha \in H^2(X, \mathbb{R})$ such that

$$0 \neq \underbrace{\alpha \smile \alpha \smile \ldots \smile \alpha}_{n} \in H^{2n}(X, \mathbb{R})$$

Example

- The symplectic vector space $\mathbb{R}^{2n} = \mathbb{C}^n$ equipped with its linear symplectic form
$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$
- Surfaces (Σ^2, ω) where ω is an area form.
- Products $(X_1 \times X_2, \omega_1 \oplus \omega_2)$
- The projective space $(\mathbb{C}P^n, \omega_{FS})$ equipped with the Fubini–Study Kähler form. (More details later today).
- Complex subvarieties of $\mathbb{C}P^n$ or, more generallty, Kähler manifolds.
- Non Kähler examples by Thurston [Thu76], Fine–Panov [FP10].

Definition

A *symplectomorphism* is a diffeomorphism $\phi\colon (X_1^{2n},\omega_1) \hookrightarrow (X_2^{2n},\omega_2)$ which satisfies $\phi^*\omega_2 = \omega_1$.

Darboux' theorem: All symplectic manifolds are locally symplectomorphic to an open subset of $(\mathbb{R}^{2n},\omega_0)$.

Definition

A *Hamiltonian isotopy* is a smooth isotopy $\phi^t\colon X^{2n} \xrightarrow{\cong} X^{2n}$ whose generating vector field satisfies the property that

$$\iota_{\frac{d}{dt}\phi^t}\omega = -dH_t$$

is a family of exact one-forms; the function $H\colon X \times \mathbb{R}_t \to \mathbb{R}$ is called the *Hamiltonian*.

Properties:

- By **Cartan's formula** it follows that ϕ^t preserves the symplectic form (it is a symplectic isotopy) whenever
$$\iota_{\frac{d}{dt}\phi^t}\omega \in \Omega^1(X)$$
is closed.
- Conversely, any function $H\colon X \times \mathbb{R} \to \mathbb{R}$ gives rise to a Hamiltonian isotopy
$$\phi_H^t\colon (X,\omega) \xrightarrow{\cong} (X,\omega)$$
via Hamilton's equations
$$\iota_{\frac{d}{dt}\phi^t}\omega = -dH_t.$$
(work it out in Darboux coordinates!).
- The Hamiltonian can be recovered from the Hamiltonian isotopy up to a locally constant time-dependent function.

Almost complex structures

Definition

- An almost complex structure $J \in \mathrm{End}(TX)$ is *tamed* by the symplectic form ω if $\omega(v, Jv) > 0$ whenever $v \neq 0$
- An almost complex structure $J \in \mathrm{End}(TX)$ is *compatible* with the symplectic form ω if $\omega(\cdot, J\cdot)$ is a Riemannian metric.

The space of tame and compatible almost complex structures will be denoted by

$$\mathcal{J}^{\text{tame}}(X,\omega) \ \supset \ \mathcal{J}^{\text{comp}}(X,\omega)$$

Almost complex structures

Example

The standard (integrable) almost complex structure $J_0 \in \mathrm{End}(T\mathbb{R}^{2n})$ given by

$$J_0(\partial_{x_i}) = \partial_{y_i} \text{ and } J_0(\partial_{y_i}) = -\partial_{x_i}$$

is compatible with the standard linear symplectic form ω_0. We can now define $d^c f(\cdot) = df(J_0 \cdot)$ and thus write

$$\omega_0 = \sum_i dx_i \wedge dy_i = -dd^c \sum_i \|z_i\|^2/4 = \sum_i r_i dr_i \wedge d\theta_i$$

in polar coordinates ($d^c f(r) = -rf'(r)d\theta$).

Almost complex structures

Definition

If J is an integrable almost complex structure (isomorphic to J_0 above in suitable local coordinates), then (X, ω, J) is said to be a *Kähler manifold* and $\omega(\cdot, J\cdot)$ is a *Kähler metric*.

Lemma (Gromov [Gro85])

The spaces

$$\mathcal{J}^{\text{tame}}(X, \omega) \text{ and } \mathcal{J}^{\text{comp}}(X, \omega)$$

are both contractible.

The Fubini–Study metric on $\mathbb{C}P^n$

For an almost complex structure J, recall that Recall that

$$d^c f(\cdot) = df(J\cdot),\ \ f \in C^\infty(X,\mathbb{R}),$$

When (X,J) is integrable (J is induced by local hol. coordinates), we have

$$d^c f = \sum_i (\partial_{y_i} f\, dx_i - \partial_{x_i} f\, dy_i) = i(\partial - \overline{\partial})f.$$

More generally

$$d = \partial + \overline{\partial} \text{ and } d^c = i(\partial - \overline{\partial}),$$
$$-dd^c f = 2i\partial\overline{\partial} f,$$

where ∂ and $\overline{\partial}$ are the Dolbeault operators (see [GH94])

$$\partial\colon \Omega^{i,j}(X) \to \Omega^{i+1,j}(X) \text{ and } \overline{\partial}\colon \Omega^{i,j}(X) \to \Omega^{i,j+1}(X),$$
$$\partial^2 = \overline{\partial}^2 = 0,\ \ \partial\overline{\partial} = -\overline{\partial}\partial$$

The Fubini–Study metric on $\mathbb{C}P^n$

The Kähler potential

Consider the following Kähler potential in an affine chart $\mathbf{z} \in \mathbb{C}^n \subset \mathbb{C}P^n$:

$$\rho(\mathbf{z}) = \log\left(1 + \|\mathbf{z}\|^2\right)$$

where $\|\cdot\|$ denotes the Euclidean metric. We define the Fubini–Study symplectic form by

$$\omega_{\mathrm{FS}} := -dd^c\frac{\rho}{4} = \frac{i}{2}\partial\overline{\partial}\rho \in \Omega^{1,1}(\mathbb{C}P^n)$$

in local affine coordinates.

Claim

The two-form $\omega_{\mathrm{FS}} \in \Omega^{1,1}(\mathbb{C}P^n)$ is well-defined, non-degenerate and compatible with J^0. Hence $(\mathbb{C}P^n, \omega_{\mathrm{FS}}, J_0)$ is Kähler.

The Fubini–Study metric on $\mathbb{C}P^n$

The symplectic (Kähler) form

$$\omega_{\mathsf{FS}} = \frac{i}{2}\partial\overline{\partial}\rho, \quad \rho = \log(1 + \|\mathbf{z}\|^2).$$

Proof of well-definedness.

Consider i:th and j:th affine coordinate charts

$$[z_1 : \ldots : z_{i-1} : 1 : z_{i+1} : \ldots : z_{n+1}],$$
$$[w_1 : \ldots : w_{j-1} : 1 : w_{j+1} : \ldots : w_{n+1}]$$

on $\mathbb{C}P^n$, where we set $z_i \equiv 1$ and $w_j \equiv 1$. They are related by the coordinate transformation

$$w_i = z_i / z_j, \quad i = 1, \ldots, n+1, \ \{z_j \neq 0\}.$$

The Fubini–Study metric on $\mathbb{C}P^n$

The symplectic (Kähler) form

$$\omega_{\mathrm{FS}} = \frac{i}{2}\partial\overline{\partial}\rho, \;\; \rho(\mathbf{z}) = \log\left(1 + \|\mathbf{z}\|^2\right).$$

Proof of well-definedness.

We compute

$$\rho(\mathbf{z}) = \log \|z_j\|^2(1 + \|\mathbf{w}\|^2) = \mathfrak{Re}\log z_j^2 + \rho(\mathbf{w}).$$

since $\mathfrak{Re}\log z_j^2$ is pluri-harmonic (real part of holomorphic fcn.), we have $\partial\overline{\partial}\mathfrak{Re}\log z_j^2 = 0$ on $\{z_j = 0\}$ and

$$\frac{i}{2}\partial\overline{\partial}\rho(\mathbf{z}) = \frac{i}{2}\partial\overline{\partial}\rho(\mathbf{w}).$$

The Fubini–Study metric on $\mathbb{C}P^n$

The symplectic (Kähler) form

$$\omega_{\mathrm{FS}} = -dd^c \frac{\rho}{4} = \frac{i}{2}\partial\overline{\partial} \log\left(1 + \|\mathbf{z}\|^2\right)$$

Proof of non-degeneracy.

Easily checked along coordinate $\mathbb{C}$-lines in polar coordinates. Using $dr(J_0 \cdot) = -r d\theta$ we compute

$$\omega_{\mathrm{FS}} = -dd^c \frac{1}{4} \log\left(1 + r^2\right) = \frac{1}{4} d \frac{2r}{1+r^2} r d\theta = \frac{r}{(1+r^2)^2} dr \wedge d\theta.$$

This establishes the non-degeneracy at the origin. For the other points, we can use the invariance of ω_{FS} under the transitive $U(n+1)$-action on $\mathbb{C}P^n$ (see next slide). □

The Fubini–Study metric on $\mathbb{C}P^n$

The symplectic (Kähler) form

With our convention ω_{FS} satisfies

$$\int_{\mathbb{C}P^1} \omega_{\mathrm{FS}} = \pi$$

for any line $\mathbb{C}P^1 \subset \mathbb{C}P^n$.

Proof.

From computation on last slide:

$$\int_{\mathbb{C}P^1} \omega_{\mathrm{FS}} = 2\pi \int_0^\infty \frac{r}{(1+r^2)^2} dr = -\pi \left[\frac{1}{1+r^2} \right]_{r=0}^{+\infty} = \pi.$$

□

The Fubini–Study metric on $\mathbb{C}P^n$

The symplectic (Kähler) form

$$\omega_{\mathrm{FS}} = -dd^c \frac{\rho}{4} = \frac{i}{2}\partial\overline{\partial}\rho = \frac{i}{2}\partial\overline{\partial}\log\left(1 + \|\mathbf{z}\|^2\right)$$

Properties:

- The Fubini–Study form restricts to the Fubini–Study form on any lower dimensional linear embedding $\mathbb{C}P^m \subset \mathbb{C}P^n$.
- The action of $U(n)$ in an affine chart $\mathbb{C}^n \subset \mathbb{C}P^n$ preserves the potential and thus the form ω_{FS} on all of $\mathbb{C}P^n$.
- $U(n)$ fixes the divisor $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$ at infinity setwise, and thus restricts to symplectomorphisms on $(\mathbb{C}P^{n-1}, \omega_{FS})$.
- In particular $SU(2) \subset U(2)$ acts by symplectomorphism on $(\mathbb{C}P^1, \omega_{FS})$, and the latter is the sphere with the "round" area form of total area π.

The Fubini–Study metric on $\mathbb{C}P^n$

The symplectic (Kähler) form

$$\omega_{\mathrm{FS}} = -dd^c\frac{\rho}{4} = \frac{i}{2}\partial\overline{\partial}\rho = \frac{i}{2}\partial\overline{\partial}\log\left(1+\|\mathbf{z}\|^2\right)$$

Properties:

- J_0 is integrable and thus *compatible* with ω_{FS}.
- The primitive $-d^c\frac{\rho}{4}$ of the symplectic form ω_{FS} has a **Liouville vector field** $\zeta \in \Gamma(TX)$ defined by

$$\iota_\zeta\omega_{\mathrm{FS}} = -d^c\frac{\rho}{4} \Leftrightarrow (\iota_\zeta\omega_{\mathrm{FS}})(J_0\cdot) = -d^c\frac{\rho}{4}(J_0\cdot) = d\frac{\rho}{4}.$$

In other words $\omega_{\mathrm{FS}}(\zeta, J_0\cdot) = d\frac{\rho}{4}$ and $\zeta = \nabla\frac{\rho}{4}$ is the gradient.

The Fubini–Study metric on $\mathbb{C}P^n$

The symplectic (Kähler) form

$$\omega_{\mathrm{FS}} = -dd^c\frac{\rho}{4} = \frac{i}{2}\partial\overline{\partial}\rho = \frac{i}{2}\partial\overline{\partial}\log\left(1+\|\mathbf{z}\|^2\right)$$

Properties:

- Since the Liouville vector field ζ corr. to $-d^c\frac{\rho}{4}$ is transverse to the concentric spheres S_r^{2n-1}, it follows that $-d^c\frac{\rho}{4} \in \Omega^1(\mathbb{C}^n)$ restricts to a contact one-form on all $S_r^{2n-1} \subset (\mathbb{C}^n, \omega_{\mathrm{FS}})$; by the $U(n)$-symmetry of $-d^c\frac{\rho}{4}$, these are round contact spheres.
- It is now easy to produce a symplectomorphism

$$\left(\mathbb{C}^n, \omega_{\mathrm{FS}} = -dd^c\frac{\rho}{4}\right) \to \left(B^{2n}, \omega_0 = -dd^c\|\mathbf{z}\|^2/4\right)\right)$$

which *preserves the primitives*, and which takes the sphere

$$S_r^{2n-1} \subset \mathbb{C}^n \text{ to } S^{2n-1}_{\frac{r}{\sqrt{1+r^2}}} \subset B^{2n}.$$

Rulings and line bundles

We will now consider the smooth $\mathbb{C}P^1$-bundles

$$\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k)) \to \mathbb{C}P^{n-1}, \;\; k \geq 0.$$

They are algebraic subvarieties of $(\mathbb{C}P^{n-1} \times \mathbb{C}P^n, \lambda\omega_{\mathrm{FS}} \oplus \omega_{\mathrm{FS}})$, $\lambda > 0$, given as the compactification of

$$\{([Z_0 : \ldots : Z_n], (z_0, \ldots, z_n));\; (z_0, \ldots, z_n) \in \mathbb{C} \cdot (Z_0^k, \ldots, Z_n^k)\}$$
$$\subset$$
$$\mathbb{C}P^{n-1} \times \mathbb{C}^n.$$

Endow them with the restriction of the product symplectic form $\lambda\omega_{\mathrm{FS}} \oplus \omega_{\mathrm{FS}})$.

Rulings and line bundles

The $\mathbb{C}P^1$-bundles

$$\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k)) \to \mathbb{C}P^{n-1}, \ \ k \geq 0,$$

satisfy:

- Projection onto first factor $\mathbb{C}P^{n-1}$ is the bundle projection π.
- $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(0)) = \mathbb{C}P^{n-1} \times \mathbb{C}P^1$.
- $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) = \mathrm{Bl}(\mathbb{C}P^n)$ (the **blow-up** of $\mathbb{C}P^n$)
- $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k)) \setminus (\mathbb{C}P^{n-1} \times \{0\}) = \mathrm{Tot}(\mathcal{O}(k))$ (total space of a positive $\mathbb{C}$-bundle).
  For $k = 1$ we get $\pi \colon \mathbb{C}P^n \setminus \{0\} \to \mathbb{C}P^{n-1}$.
- $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k)) \setminus (\mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}_\infty) = \mathrm{Tot}(\mathcal{O}(-k))$ (total space of a negative $\mathbb{C}$-bundle).
  For $k = 1$ we get the line bundle $\pi \colon \ \mathrm{Bl}_0 \, \mathbb{C}^n \to \mathbb{C}P^{n-1}$.

$\mathcal{O}(-1)$

The exceptional line bundle

$$\mathrm{Bl}_0\,\mathbb{C}^n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \setminus (\mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}_\infty)$$

satisfies:

- The exceptional divisor $E := \mathbb{C}P^{n-1} \times \{0\} \subset \mathrm{Bl}_0\,\mathbb{C}^n$ is the only holomorphic section.
- There exists a *symplectomorphism* between $\mathrm{Bl}_0\,\mathbb{C}^n \setminus E$ and

  $$(\mathbb{C}^n \setminus D^{2n}_{\sqrt{\lambda}}, (\lambda+1)\omega_{\mathrm{FS}}) \subset (\mathbb{C}^n, (\lambda+1)\omega_{\mathrm{FS}})$$

  where

  $$(D^{2n}_{\sqrt{\lambda}}, (1+\lambda)\omega_{\mathrm{FS}}) \cong (D^{2n}_{\sqrt{\frac{\lambda}{1+\lambda}}}, (1+\lambda)\omega_0)$$

- While the complex structure on $\mathrm{Bl}_0\,\mathbb{C}^n \setminus E = \mathbb{C}^n \setminus \{0\}$ extends over $\{0\}$ to all of $\mathbb{C}^n$, it does not extend over $D^{2n}_{\sqrt{\lambda}}$!

The blow-down

The exceptional line bundle

$$\mathrm{Bl}_0\,\mathbb{C}^n = \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1)) \setminus (\mathbb{C}P^{n-1}\times\mathbb{C}P^{n-1}_\infty)$$

satisfies:

- **Symplectic blow-down:** Whenever we see a neighbourhood in (X,ω) symplectomorphic to the above neighbourhood of $(E,\lambda\omega_{\mathrm{FS}})$, we can "simplify" X by *blowing down*: remove E and insert the closed symplectic ball

$$(D^{2n}_{\lambda/\sqrt{1-\lambda^2}},\omega_{\mathrm{FS}}) \cong (D^{2n}_{\sqrt{\lambda}},\omega_0)$$

  with round boundary, where E is equipped with the symplectic form $\lambda\omega_{\mathrm{FS}}$.

The blow-up

The exceptional line bundle

$$\mathrm{Bl}_0\,\mathbb{C}^n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \setminus (\mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}_\infty)$$

satisfies:

- **Symplectic blow-up:** Whenever we see a symplectic disc $D \subset (X, \omega)$ with round boundary parametrised by

$$(D^{2n}_{\lambda/\sqrt{1-\lambda^2}}, \omega_{\mathrm{FS}}) \cong (D^{2n}_{\sqrt{\lambda}}, \omega_0) \hookrightarrow (X^{2n}, \omega)$$

  we can remove it and insert an exceptional divisor E with the symplectic form $\lambda\omega_{\mathrm{FS}}$. Call the result $(\mathrm{Bl}_D\, X^{2n}, \omega_D)$.
- Blow up does not deform the topology when $n = 1$, while in general

$$\mathrm{Bl}_D\, X^{2n} = X^{2n} \sharp \overline{\mathbb{C}P^n}.$$

Pseudoholomorphic spheres

Recall:

- There exists precisely one algebraic curve of degree one (in the homology class $L \in H_2(\mathbb{C}P^n) = \mathbb{Z} \cdot L$) that passes through two given points $P_1 \neq P_2 \in \mathbb{C}P^n$: the complex line

$$\mathbb{C}P^1 \to \mathbb{C}P^n,$$
$$[x_1 : x_2] \mapsto x_1 \cdot P_1 + x_2 \cdot P_2$$

  unique up to reparametrisation.
- The fibres of a $\mathbb{C}P^1$-bundles e.g. $\mathrm{Bl}(\mathbb{C}P^n) \to \mathbb{C}P^n$ and $\mathbb{C}P^n \times \mathbb{C}P^1$ foliate $\mathbb{C}P^1$.

Pseudoholomorphic spheres

We will investigate to what extent this is true for other almost complex structures J_0.
First we need a definition. Let (Σ, j) be a Riemann surface.

Definition

A map $u\colon (\Sigma, j) \to (X, J)$ is said to be *J-holomorphic* (also called *pseudoholomorphic*) if it satisfies the fully non-linear first order PDE

$$\overline{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j) = 0$$

of Cauchy–Riemann type.

Uniruledness

- Tameness of J will be crucial; non-tame J may admit null-homologous but non-constant J-holomorphic spheres.
- For tame J we have $\int_u \omega > 0$ whenever u is a non-constant pseudoholomorphic map.
- The precise statement that we want to show is that: for an arbitrary tame J and two distinct points, there exists a J-holomorphic map $u\colon \mathbb{C}P^1 \to \mathbb{C}P^n$ of degree one which passes through these two points.
- Even better: we want the algebraic count of such curve to be equal to one if one identifies solutions under the action of $\mathrm{Aut}(\mathbb{C}P^1) = \mathbb{P}\,\mathrm{Gl}_2(\mathbb{C})$ by reparametrisation.

Nodal pseudoholomorphic spheres

We also need the concept of a nodal sphere:

Definition

A *nodal pseudoholomorphic sphere* is a continuous map $u_\infty \colon \mathbb{C}P^1 \to (X, J)$ which is pseudoholomorphic for some almost complex structure j_∞ defined on $\mathbb{C}P^1 \setminus \Gamma$, where

- $\Gamma \subset \mathbb{C}P^1$ is an embedded finite union of smooth circles, and
- $(\mathbb{C}P^1 \setminus \Gamma, j_\infty)$ is biholomorphic to a finite union of punctured spheres (i.e. $(\mathbb{C}P^1, j) \setminus \{p_1, \ldots, p_l\}$).

While there is a single almost complex structure on a sphere up to biholomorphism, this is not true for a sphere with points removed.

Nodal pseudoholomorphic spheres

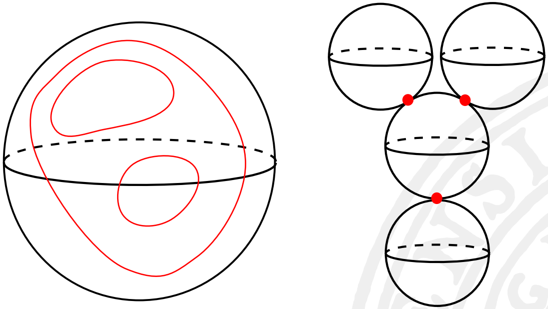


Figure: Left: a nodal sphere. Right: the union of desingularised pseudoholomorphic sphere.

Gromov's Compactness Theorem

- One important feature is that removal of singularities holds in this setting, which gives rise to a union $\{u^1_\infty, \ldots, u^l_\infty\}$ of ordinary pseudoholomorphic spheres from a nodal pseudoholomorphic sphere.
- The **energy** $E(u) = \int_u \omega \geq 0$ of any map depends only on the cohomology class (ω is closed) and can be related to the L^2-norm of du when u is pseudoholomorphic (it is easily seen to be non-negative).
- The energy also makes sense for a nodal pseudoholomorphic sphere, and satisfies

$$E(u_\infty) = E(u^1_\infty) + \ldots + E(u^l_\infty).$$

Nodal pseudoholomorphic spheres

Definition

A nodal pseudoholomorphic sphere is *stable* if all punctured spheres $u_\infty|_{\mathbb{C}P^1 \setminus \Gamma}$ which have less than three punctures are non-constant.

Remark

- Non-constant is equivalent to having positive energy.
- If we remove three or more points from $(\mathbb{C}P^1, j)$, there are only finitely many automorphisms.

Nodal pseudoholomorphic spheres

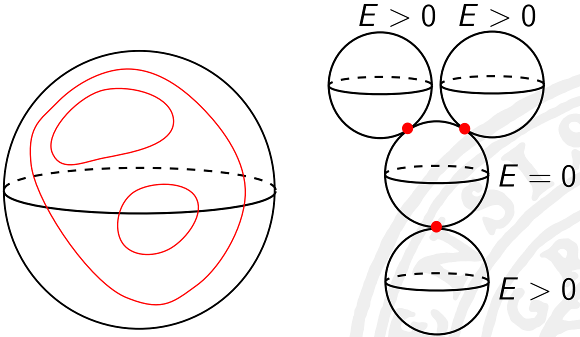


Figure: A *stable* nodal sphere.

Nodal pseudoholomorphic spheres

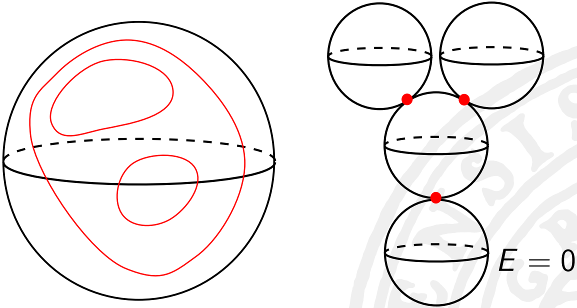


Figure: An *unstable* nodal sphere.

Gromov's Compactness Theorem

Now consider a sequence $\{u_i\}$ of pseudoholomorphic spheres

$$u_i \colon (\mathbb{C}P^1, j) \to (X, J)$$

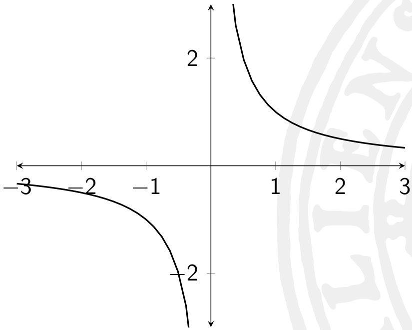
with $i = 1, 2, 3, \ldots$ inside a closed symplectic manifold (X, ω) equipped with a tame almost complex structure J.

Gromov's Compactness Theorem

- A uniform bound of the derivative $\|du\|$ may fail despite the L^2-bound: Consider the family

$$\{z_1 z_2 = t\} \subset \mathbb{C}P^2, \;\; t \to 0$$

of smooth conics which converge to the union $\{z_1 z_2 = 0\}$ of coordinate lines.



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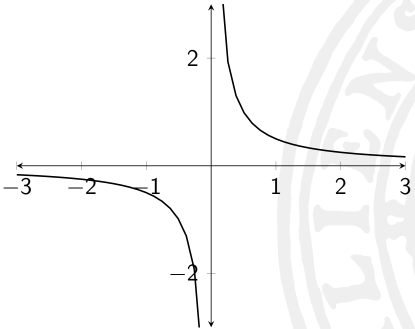

Gromov's Compactness Theorem

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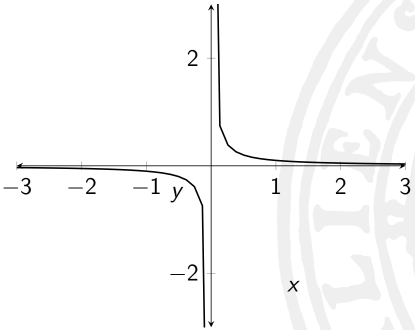

Gromov's Compactness Theorem

- A uniform bound of the derivative $\|du\|$ may fail despite the L^2-bound: Consider the family

$$\{z_1 z_2 = t\} \subset \mathbb{C}P^2, \;\; t \to 0$$

of smooth conics which converge to the union $\{z_1 z_2 = 0\}$ of coordinate lines.
A sequence of parametrisations seen from southern and northern hemispheres

$$[x:1] \mapsto [tx : x^{-1} : 1] \to_{t \to 0} [0 : x^{-1} : 1],$$
$$[1:y] \mapsto [ty^{-1} : y : 1] = [1 : t^{-1}y^2 : t^{-1}y],$$

respectively.

- On the northern hemisphere of $\mathbb{C}P^1$ the map converges to $[1:0:y] = [y^{-1}:0:1]$ only *after the reparametrisation* $y \mapsto ty$.

Gromov's Compactness Theorem

Theorem (Gromov [Gro85])

Assume that $0 < E(u_i) \leq C$ is uniformly bounded. After passing to a subsequence, we may assume that there exists either:

1. *A sequence $\phi_i \in \mathrm{Aut}(\mathbb{C}P^1)$ of reparametrisations that makes $\|d(u_i \circ \phi_i)\|$ uniformly bounded, and the subsequence $\{u_i \circ \phi_i\}$ is C^∞-convergent to a J-holomorphic sphere u_∞.*
2. *A stable nodal pseudoholomorphic sphere u_∞ with at least two nonconstant components, and reparametrisations ϕ_i, such that:*
   - *$(\phi_i)^* j$ is a sequence of complex structures on $\mathbb{C}P^1$ which C^∞_{loc}-converges to the complex structure j_∞ on the nodal sphere;*
   - *$u_i \circ \phi_i$ converges uniformly to u_∞ and C^∞_{loc}-converges on $\mathbb{C}P^1 \setminus \Gamma$ to u_∞.*

Gromov's Compactness Theorem

Corollary

- *The convergent subsequence has the property that the homology class of the* u_i *becomes constantly equal to* $[u_\infty] \in H_2(X)$ *for all* $i \gg 0$;
- *There are only finitely many homology classes inside the possibly infinite subset*

$$\{u \in H_2(X);\ E(u) \leq C\} \subset H_2(X)$$

that admit a pseudoholomorphic sphere when X *is closed.*

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