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Holomorphic Curve Theories in Symplectic Geometry

Lecture II

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Goal of lecture

Last time:

- Gromov compactness: sequences of pseudoholomorphic spheres of bounden energy have subsequences that converge to "nodal solutions".
- Crucial feature: $\mathrm{Aut}(\mathbb{C}P^1)$ is a non-compact group (the group of $\dim_{\mathbb{C}} = 3$ of Möbius transformations).

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Goal of lecture

Today:

- The moduli space of pseudoholomorphic spheres and its dimension formula (Fredholm index).
- Computation of first Chern classes.
- Main applications:
  - “Uniruledness” of $(\mathbb{C}P^n, J)$ for any tame J.
  - Restriction of the topology of “symplectic fillings” of the round contact sphere (S^{2n-1}, α_0) (Gromov [Gro85], Eliashberg–Floer–McDuff [McD91b]).

Take-home message

There are J_0-holomorphic lines $\mathbb{C}P^1 \to \mathbb{C}P^n$ through every pair of points, this property remains for all tame J.

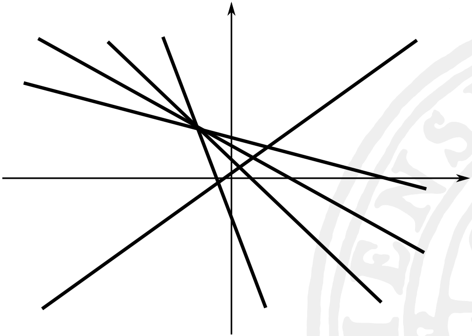

Figure: Lines inside $\mathbb{C}P^2$.

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Plan

Deformation theory

- Gromov's compactness concerns the global topological structure of the space of solutions.
- The local structure of the space of solutions is controlled by ellipticity of the operator $\overline{\partial}_J$.
- The operator

$$\overline{\partial}_J\colon C^\infty(\mathbb{C}P^1, X) \to \Omega^{0,1}(TX^{2n}),$$
$$u \mapsto \frac{1}{2}(du + J \circ du \circ j),$$

has an elliptic linearisation (derivative) $D_u\overline{\partial}_J$ at u is thus *Fredholm* when extended to suitable Banach spaces. [Gro85]

$\Omega^{0,1}(TX) = \Gamma((T^*\mathbb{C}P^1)^{0,1} \otimes u^*TX)$: sections of u^*TX-valued anti-holomorphic one-forms on $\mathbb{C}P^1$, i.e. anti-complex bundle maps $T\mathbb{C}P^1 \to u^*TX$ over $\mathbb{C}P^1$.

The Fredholm property

- The kernel and cokernel of $D_u\overline{\partial}_J$ are both finite dimensional.

The *Fredholm index* of $D_u\overline{\partial}_J$ is indep. of u and J, and is equal to

$$\text{index}\, D_u\overline{\partial}_J = \dim_{\mathbb{R}} \ker D_u\overline{\partial}_J - \dim_{\mathbb{R}} \operatorname{coker} D_u\overline{\partial}_J = n\cdot\chi(\mathbb{C}P^1) + 2\cdot c_1^{TX}[u].$$

(Follows from Riemann–Roch below. Also the Chern class c_1 will be treated below.)

- The index is *even*: the reason is that ker and coker admit complex structures (obvious in the integrable case).
- In favourable cases: choosing J generic makes $\operatorname{coker} D_u\overline{\partial}_J u = 0$ at any solution $\overline{\partial}_J u = 0$.
- The latter solution space $\{\overline{\partial}_J u = 0\}$ is then a *smooth manifold* of dimension equal to the index.

The Fredholm index

The *Fredholm index* of $D_u\overline{\partial}_J$ is equal to

$$\operatorname{index} D_u\overline{\partial}_J = \dim_{\mathbb{R}} \ker D_u\overline{\partial}_J - \dim_{\mathbb{R}} \operatorname{coker} D_u\overline{\partial}_J = n\cdot\chi(\mathbb{C}P^1) + 2\cdot c_1^{TX}[u].$$

The index formula can be derived by using:

- Invariance of the index under deformations by compact operators.
- The classical Riemann–Roch formula for a (sum of) line bundle(s).

The Fredholm index

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More precisely: After homotopy through complex bundles, we may assume that

$$u^*TX \cong \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_n$$

is a sum of holomorphic line bundles $\mathcal{L} \to \mathbb{C}P^1$.
Below we will see that the first Chern class is undeformed by this homotopy, and satisfies

$$c_1^{TX}[u] = \sum_{i=1}^{n} c_1^{\mathcal{L}_i}.$$

The terms on the right are the "Chern numbers" to be defined below.

Riemann–Roch

Recall the Riemann–Roch theorem [GH94] for line bundles on a closed Riemann surface (Σ, j) of genus $g \geq 0$ (today $g = 0$).

- $\mathcal{L} \to \Sigma$ a line bundle, $\mathcal{L}^* \to \Sigma$ its dual e.g. $\mathcal{L} \otimes \mathcal{L}^* \to \Sigma$ is the trivial $\mathbb{C}$-bundle $\Sigma \times \mathbb{C} \to \Sigma$.
- Denote by
$$H^0(\Sigma, \mathcal{L})$$
the finite dim. $\mathbb{C}$-vector space of *holomorphic* sections of a line bundle $\mathcal{L} \to \Sigma$.
- Denote by
$$H^1(\Sigma, \mathcal{L}) = H^0(\Sigma, \mathcal{L} \otimes T^*\Sigma^{0,1})$$
the finite dim. $\mathbb{C}$-vector space of sections of anti-holomorphic $\mathcal{L}^*$-valued forms that solve the $\overline{\partial}$-equation.

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the finite dim. $\mathbb{C}$-vector space of sections of anti-holomorphic $\mathcal{L}$-valued forms that solve the $\overline{\partial}$-equation.
- Serre duality gives us:
$$H^1(\Sigma, \mathcal{L})^* = H^0(\Sigma, \mathcal{L}^* \otimes T^*\Sigma^{1,0}),$$
where $T^*\Sigma^{1,0}$ is the *canonical line-bundle* of holomorphic forms. (Unlike $\mathcal{L} \otimes T^*\Sigma^{0,1}$, $\mathcal{L}^* \otimes T^*\Sigma^{1,0}$ is a holomorphic bundle!)

Riemann–Roch

Recall the Riemann–Roch theorem [GH94] for line bundles on a closed Riemann surface (Σ, j) of genus $g \geq 0$.

Theorem (Riemann–Roch [GH94])

$$\dim_{\mathbb{R}} H^0(\Sigma, \mathcal{L}) - \dim_{\mathbb{R}}(H^1(\Sigma, \mathcal{L})^*) = \chi(\Sigma) + 2c_1^{\mathcal{L}} = 2 - 2g + 2c_1^{\mathcal{L}}$$

- Observe that the space

$$H^1(\Sigma, \mathcal{L})^* = H^0(\Sigma, \mathcal{L} \otimes T^*\Sigma^{0,1})^*$$

  can be identified with the cokernel of

$$\overline{\partial} \colon \Gamma(\mathcal{L}) \to \Omega^{0,1}(\mathcal{L}) = \Gamma(\mathcal{L} \otimes T^*\Sigma^{0,1}).$$

- Riemann–Roch thus gives us the index formula!

The first Chern class [MS74]

Recall: For any complex vector bundle $E \to X$ there is an associated first Chern class

$$c_1^E \in H^2(X)$$

which is determined by the following axioms:

1. For a general complex bundle $E \to X$

   $$c_1^E := c_1^{\det E}$$

   where

   $$\det E = \underbrace{E \wedge \ldots \wedge E}_{\dim_{\mathbb{C}} E} \to X$$

   is an associated $\mathbb{C}$-line bundle.
2. For line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$:

   $$c_1^{\mathcal{L}_1 \otimes \mathcal{L}_2} = c_1^{\mathcal{L}_1} + c_1^{\mathcal{L}_2}$$

The first Chern class [MS74]

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2. For line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$:

$$c_1^{\mathcal{L}_1 \otimes_{\mathbb{C}} \mathcal{L}_2} = c_1^{\mathcal{L}_1} + c_1^{\mathcal{L}_2}$$

and thus (since $\det(E_1 \oplus E_2) = \det(E_1) \otimes \det(E_2)$):

$$c_1^{E_1 \oplus E_2} = c_1^{E_1} + c_1^{E_2}.$$

The first Chern class [MS74]

3. For an *oriented* Riemann surface $u\colon \Sigma \to X$, the value

$$c_1^E[u] \in \mathbb{Z}$$

is equal to the algebraic number of zeros of a generic section in the pull-back $\mathbb{C}$-bundle

$$u^* \det E = \det u^* E \to \Sigma.$$

This is also called the *Chern number* of $u^* \det E \to \Sigma$. Note the dependence on the orientation of Σ as well as the orientation of the fibres of the $\mathbb{C}$-bundle (which we take to be the canonical one)!

The first Chern class [MS74]

Relation to Ricci-curvature [GH94]

When $E = \mathcal{L}$ is a holomorphic line bundle on a complex manifold X the first Chern class with $\mathbb{C}$–coefficients lives in $H^{1,1}(X)$ and can be represented by the Ricci-curvature form

$$\frac{i}{2\pi}\partial\overline{\partial}\log\left(h\|\sigma\|^2\right)$$

where $h\|\cdot\|^2$ is the local expression for a Hermitian metric on $\mathcal{L}$ and σ is a local holomorphic section.

Compare with:

$$\omega_{\mathsf{FS}} = \frac{i}{2}\partial\overline{\partial}\log\rho.$$

Properties of c_1

Useful consequences of the above:

- **Adjunction formula:** Let $\mathcal{N} \to X$ be a $\mathbb{C}$-line bundle. It is immediate that

$$T \operatorname{Tot} \mathcal{N}|_X = TX \oplus \mathcal{N},$$

i.e. $\mathcal{N}$ is the normal bundle of X inside the total space $\operatorname{Tot} \mathcal{N}$ of $\mathcal{N}$. In other words,

$$c_1^{T \operatorname{Tot} \mathcal{N}} = c_1^{TX} + c_1^{\mathcal{N}} \in H^2(X) \cong H^2(\mathcal{N})$$

(using the canonical identification of cohomology groups).

Properties of c_1

Useful consequences of the above:

- **Self-intersection:** For a line bundle $\mathcal{L} \to X$,

$$c_1^{\mathcal{L}} = P.D.([X \cap X'])$$

where $X' \subset \mathcal{N}$ is a generic smooth perturbation, e.g. a smooth section (unlike holomorphic sections, there are always plenty of smooth sections).
The manifold $X \cap X'$ is oriented!

Properties of c_1.

Useful consequences of the above:

- **Formulation in terms of "divisors":** When $\mathcal{L} \to X$ is a holomorphic line bundle on a complex manifold X that admits a *meromorphic* section σ then we have

$$c_1^E = P.D.([\sigma]_0 - [\sigma]_\infty)$$

where $[\sigma]_0 \subset H_{2n-2}(X)$ and $[\sigma]_\infty \in H_{2n-2}(X)$ are the cycles induced by the zeroes and poles of σ (holomorphic subvarieties) counted with multiplicities.

Computation of c_1 for $\mathbb{C}P^n$

Fix the following notation:

- $L \in H_2(\mathbb{C}P^n)$ is the class of a linear embedding $\mathbb{C}P^1 \subset \mathbb{C}P^n$
- $H \in H_{2n-2}(\mathbb{C}P^n)$ is the class a linear embedding $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$, e.g. the hyperplane $\mathbb{C}P^{n-1}_\infty \subset \mathbb{C}P^n$ at infinity.
- We use $T \in H^2(\mathbb{C}P^n)$ to denote $T = P.D.(H)$ i.e. $T(L) = L \bullet H = 1$.

Recall that:

- $H_2(\mathbb{C}P^n) = \mathbb{Z} \cdot L$, $H^2(\mathbb{C}P^n) = \mathbb{Z} \cdot T$, and $H_{2n-2}(\mathbb{C}P^n) = \mathbb{Z} \cdot H$.
- $c_1^{T\mathbb{C}P^1} = 2T \in H^2(\mathbb{C}P^1)$. (Since $\mathbb{C}P^1 \cong S^2$ and $\chi(S^2) = 2$.)

Computation of c_1 for $\mathbb{C}P^1$

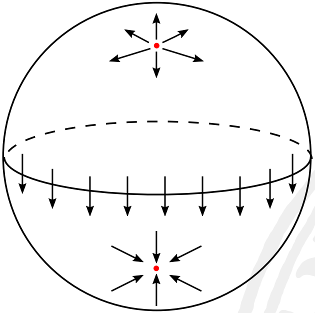

Figure: A vector field in $T\mathbb{C}P^1 = TS^2$ with two elliptic points, each making a contributing of $+1$ to the intersection with the zero-section. This shows that $c_1^{T\mathbb{C}P^1}[\mathbb{C}P^1] = \chi(S^2) = 2$

Computation of c_1 for $\mathbb{C}P^n$

The general case follows from the adjunction formula:

- Recall that

$$\mathbb{C}P^n \setminus \{0\} = \mathcal{O}(1) \to \mathbb{C}P^{n-1}_\infty$$

  and any linear hyperplane $H \subset \mathbb{C}P^n$ disjoint from $0 \in \mathbb{C}^n = \mathbb{C}P^n \setminus \mathbb{C}P^{n-1}_\infty$ is a holomorphic section

$$\sigma \colon \mathbb{C}P^{n-1}_\infty \to \mathcal{O}(1).$$

  with $[\sigma]_\infty = 0$ and $[\sigma_0] = [\mathbb{C}P^{n-2}] \in H_{2(n-1)-2}(\mathbb{C}P^{n-1})$.
- We thus get $c_1^{\mathcal{O}(1)} = P.D.[\mathbb{C}P^{n-2}] = T$.
- Adjunction formula: $c_1^{T\mathbb{C}P^n} = T + c_1^{T\mathbb{C}P^{n-1}_\infty}$.

By induction ($c_1^{T\mathbb{C}P^1} = 2T$) we get

$$c_1^{T\mathbb{C}P^n} = (n+1)T \in H^2(\mathbb{C}P^n)$$

$\mathcal{O}(1)$: A neighbourhood of $\mathbb{C}P^{n-1}_\infty$.

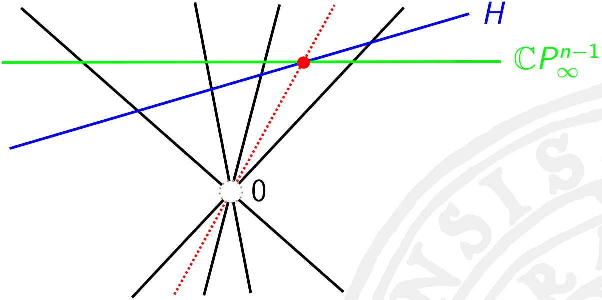

Figure: The dual of the tautological bundle $\mathcal{O}(1)$ with total space $\mathrm{Tot}(\mathcal{O}(1)) = \mathbb{C}P^n \setminus \{0\}$, the zero section is the hyperplane $\mathbb{C}P^{n-1}_\infty$ at infinity, and a hyperplane H which is disjoint from the origin is a holomorphic section. This section vanishes at the intersection $H \cap \mathbb{C}P^{n-1}_\infty$ which is a hyperplane inside $\mathbb{C}P^{n-1}_\infty$ shown as a red dot.

Computation of c_1 for $\mathcal{O}(k)$.

- We have seen that for the line bundle $\mathcal{O}(1) \to \mathbb{C}P^n$ we have

$$c_1^{\mathcal{O}(1)} = T \in H^2(\mathbb{C}P^n)$$

- From the fact that $\mathcal{O}(0) = \mathbb{C} \times \mathbb{C}P^n$ is the trivial line bundle, and hence $c_1^{\mathcal{O}(0)} = 0$, and

$$\mathcal{O}(k_1) \otimes \mathcal{O}(k_2) = \mathcal{O}(k_1 + k_2), \;\; k_i \in \mathbb{Z},$$

we thus get

$$c_1^{\mathcal{O}(k)} = kT \in H^2(\mathbb{C}P^n).$$

Computation of c_1 for $\mathcal{O}(-1)$

In the case of the blowup

$$\mathcal{O}(-1) = \mathrm{Bl}_0\,\mathbb{C}^{n+1} \to \mathbb{C}P^n$$

we thus get that

- $c_1^{\mathcal{O}(-1)} = -T \in H_2(\mathbb{C}P^n)$
- Alternatively: find a meromorphic section $\sigma \colon \mathbb{C}P^n \to \mathcal{O}(-1)$ with a simple pole along $\mathbb{C}P^{n-1}_\infty$. (Linear hyperplanes in

$$\mathrm{Bl}_0\,\mathbb{C}^{n+1} \setminus \mathbb{C}P^n = \mathbb{C}^{n+1} \setminus \{0\}$$

  disjoint from the blow-up locus).
- Writing E for the zero-section of $\mathcal{O}(-1)$ we compute the algebraic intersection number

$$E \bullet E = c_1^{\mathcal{O}(-1)} = -T(L) = -1$$

  in the case of $\mathrm{Bl}_0\,\mathbb{C}^2 \to \mathbb{C}P^1$.

$\mathcal{O}(-1)$: The tautological line bundle.

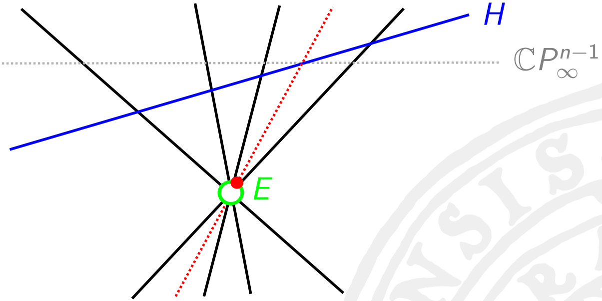

Figure: The tautological bundle $\mathcal{O}(-1)$ with total space $\mathrm{Tot}(\mathcal{O}(-1)) = \mathrm{Bl}_0\,\mathbb{C}^n$, the zero section is the exceptional divisor $E = \mathbb{C}P^{n-1}$, and a hyperplane H which is disjoint from the origin is a meromorphic section. This section has a pole along a hyperplane in the exceptional divisor shown as a red dot.

Today's Application:

Classification of fillings of the standard contact spheres

&

Uniruledness

Standard contact sphere

Recall:

- We have seen two Kähler potentials on $\mathbb{C}^n$ with the standard integrable almost complex structure J_0:

$$\rho(\mathbf{z}) = \log\left(1 + \|\mathbf{z}\|^2\right) \text{ and } \rho_0(\mathbf{z}) = \|\mathbf{z}\|^2.$$

  where $\|\cdot\|$ is the Euclidean norm.
- They give rise to the Kähler forms

$$\omega_{\mathsf{FS}} = -dd^c\frac{\rho}{4} \text{ and } \omega_0 = -dd^c\frac{\rho_0}{4}$$

  equipped with natural primitives $-d^c\frac{\rho}{4}$ and $-d^c\frac{\rho_0}{4}$.
- Both are compatible with J_0, but correspond to *different* Kähler metrics (Fubini–Study and flat metric).

Standard contact sphere

Recall:

- There is a symplectomorphism

$$\left(\mathbb{C}^n, \omega_{\mathrm{FS}} = d\left(-d^c \frac{\rho}{4}\right)\right) \xrightarrow{\cong} \left(B^{2n}, \omega_0 = d\left(-d^c \frac{\rho_0}{4}\right)\right)$$

  which preserves the primitives and which maps

$$S^{2n-1}_r \subset \mathbb{C}^n \text{ to } S^{2n-1}_{\frac{r}{\sqrt{1+r^2}}} \subset B^{2n}.$$

- The Liouville vector field ζ defined by $\iota_\zeta \omega = \lambda$, where λ is the choice of primitive one-form, are outwards pointing and thus give rise to a *contact form*

$$\left(S^{2n-1}, \alpha_0 := -d^c \frac{\rho_0}{4}\bigg|_{TS^{2n-1}}\right)$$

  on the sphere. This is the "round" contact form: $\alpha_0 = \frac{1}{2}\sum_i (x_i dy_i - y_i dx_i)$.

Symplectic fillings

Definition

An odd-dimensional manifold (Y^{2n-1}, α) equipped with $\alpha \in \Omega^1(Y)$ is a *contact manifold with contact form* α if $(Y \times \mathbb{R}_t, d(e^t\alpha))$ is a symplectic manifold (the Liouville vector induced by $e^t\alpha$ is given by $\zeta = \partial_t$).

Definition

Let (X, ω) be a symplectic manifold with boundary together with a choice of primitive $\lambda \in \Omega^1(X)$ (i.e. $d\lambda = \omega$) defined near ∂X, whose corresponding Liouville v.f. $\zeta \in \Gamma(TX)$ points outwards along ∂X. Then $(\partial X, \lambda|_{T\partial X})$ is a contact manifold and we call (X, ω) a *(strong) symplectic filling* of $(\partial X, \lambda|_{T\partial X})$.

Symplectic fillings

Example

The closed $2n$-disc (D^{2n}, ω_0) is thus a symplectic filling of the standard round contact sphere (S^{2n-1}, α_0), with primitive

$$-d^c \frac{\rho_0}{4} = \frac{1}{2} \sum_{i=1}^{n} (x_i dy_i - y_i dx_i)$$

and Liouville vector field

$$\zeta_0 = \frac{1}{2} \sum_{i=1}^{n} (x_i \partial_{x_i} + y_i \partial_{y_i}).$$

Symplectic fillings

The question that we want to study is:

Question

What are the possible symplectic fillings (X, ω) of (S^{2n-1}, α_0), $n > 1$, up to symplectomorphism? Simplifying assumption: $\int_\alpha \omega = 0$ on each $\alpha \in H_2(X)$.

Remark

- By the additional assumption there are no Gromov-limits which contains a "bubble" contained inside X.
- In dimension $\dim X = 4$ i.e. $n = 2$ the answer is: the standard ball. [Gro85]
- In addition, for $n = 2$, one can drop the assumption and thus gets the ball blown up in a number of points.

Symplectic fillings

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Theorem (Eliashberg–Floer–McDuff [McD91b])

Under the above stronger assumptions (X, ω) *is diffeomorphic* D^{2n}.

We will proceed to sketch some important steps of this proof.

The moduli space of spheres

- Using the Liouville flow, one finds a neighbourhood of $S^{2n-1} = \partial X \subset (X^{2n}, \omega)$ which is symplectomorphic to a neighbourhood of $S^{2n-1} = \partial D^{2n} \subset (D^{2n}, \omega_0)$.
- Since
$$(\mathbb{C}P^n \setminus \mathbb{C}P^{n-1}_\infty, \omega_{\mathrm{FS}}) \cong (D^{2n} \setminus S^{2n-1}, \omega_0)$$
we can remove the boundary $X \setminus \partial X$ and add a divisor $\mathbb{C}P^{n-1}_\infty$.
- This produces a closed symplectic manifold
$$\overline{X} := (X \setminus \partial X) \cup \mathbb{C}P^{n-1}_\infty$$
equipped with the symplectic form $\overline{\omega}$.
- Mayer–Vietoris gives
$$H_2(\overline{X}) = H_2(X) \oplus H_2(\mathbb{C}P^{n-1}_\infty) = H_2(X) \oplus L \cdot \mathbb{Z}$$
(Two manifolds glued along a S^{2n-1}, $n \geq 2$)

The moduli space of spheres

Crucial properties:

- Any $\eta \in H_2(\overline{X})$ decomposes as $\eta = \alpha + kL$, where $\alpha \in H_2(X)$ and $k \in \mathbb{Z}$. The simplifying assumption implies
$$\int_{\alpha + kL} \overline{\omega} = \pi \cdot k.$$
- $c_1^{T\overline{X}}(L) = c_1^{T\mathbb{C}P^n}(L) = (n+1)$, since L can be represented by a line in a neighbourhood where $\overline{X}$ coincides with a neighbourhood of $\mathbb{C}P^{n-1}_\infty \subset \mathbb{C}P^n$.
- For a *tame* almost complex structure J on $(\overline{X}, \omega)$ which coincides with J_0 near $\mathbb{C}P^{n-1}_\infty \subset \overline{X}$ there exists plenty of J-holomorphic spheres in class L: e.g. take lines in $\mathbb{C}P^{n-1}_\infty \subset \overline{X}$.

The moduli space of spheres

We define the moduli space of J-holomorphic spheres in class L by

$$\begin{aligned}\mathcal{M}_J(L) &= \\ &= \{u\colon (\mathbb{C}P^1, j) \to (X, J); \overline{\partial}_J u = 0, [u] = L \in H_2(\overline{X})\}/\mathrm{Aut}(\mathbb{C}P^1).\end{aligned}$$

- For generic J equal to J_0 near $\mathbb{C}P^{n-1}_\infty$ the moduli space $\mathcal{M}_J(L)$ is a smooth manifold of dimension

$$\begin{aligned}&\operatorname{index} u - \dim_{\mathbb{R}} \mathrm{Aut}(\mathbb{C}P^1) = \\ &= n\chi(S^2) + 2c_1^{TX}[u] - 2(3 - 3g) \\ &= (n-3)\chi(S^2) + 2c_1^{TX}[u] = 2(n-3) + 2(n+1) = 4n - 4\end{aligned}$$

- Here $\mathrm{Aut}(\mathbb{C}P^1)$ acts without fixed points! (Otherwise: $\mathcal{M}_J(L)$ would be an *orbifold*); the reason is that minimal area pseudoholomorphic curves cannot be branched covers.

Transversality

In order to make coker $D_u\overline{\partial} = 0$ hold for all J-holomorphic spheres (this is necessary to conclude that $\mathcal{M}_J(L)$ is transversely cut out, and hence a smooth manifold), the almost complex structure J must be chosen *generically*.

- In this case all J-holomorphic sphere of minimal energy are necessarily simply covered by a topological argument.
- Transversality can then be achieved by perturbing J within the class of tame almost complex structures.

We will postpone the details of this crucial point to a later lecture.

Gromov's Compactness Theorem

Theorem (Gromov [Gro85])

Assume that $0 < E(u_i) \leq C$ *is uniformly bounded. After passing to a subsequence, we may assume that there exists either:*

1. *A sequence* $\phi_i \in \mathrm{Aut}(\mathbb{C}P^1)$ *of reparametrisations that makes* $\|d(u_i \circ \phi_i)\|$ *uniformly bounded, and the subsequence* $\{u_i \circ \phi_i\}$ *is* C^∞*-convergent to a J-holomorphic sphere* u_∞*.*
2. *A stable nodal pseudoholomorphic sphere* u_∞ *with at least two non-constant components, and reparametrisations* ϕ_i*, such that:*
   - $(\phi_i)^* j$ *is a sequence of complex structures on* $\mathbb{C}P^1$ *which* C^∞_{loc}*-converges to the complex structure* j_∞ *on the nodal sphere;*
   - $u_i \circ \phi_i$ *converges uniformly to* u_∞ *and* C^∞_{loc}*-converges on* $\mathbb{C}P^1 \setminus \Gamma$ *to* u_∞*.*

Stable nodal sphere (a priori limit)

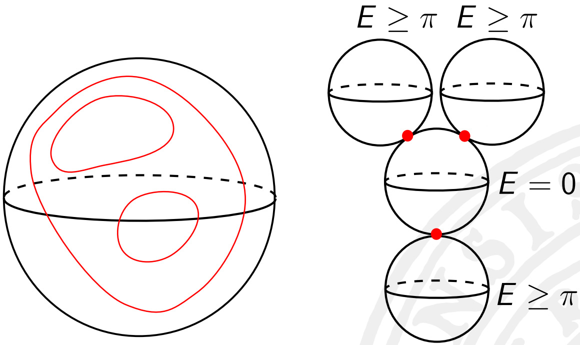

Figure: A *stable* nodal sphere. Since the energies sum to $\int_L \omega_{\mathrm{FS}} = \pi$, which is the minimal positive energy of any class in $H_2(\overline{X})$, there must be precisely one non-constant component in any limit of a sequence of solutions $u_i \in \mathcal{M}_J(L)$.

The moduli space of spheres

We define the moduli space of J-holomorphic spheres in class $L \in H_2(\overline{X})$ to be

$$\mathcal{M}_J(L) = \{u\colon (\mathbb{C}P^1, j) \to (X, J);\ [u] = L \in H_2(\overline{X})\}/\mathrm{Aut}(\mathbb{C}P^1).$$

- Hence $\mathcal{M}_J(L)$ is a smooth manifold of dimension
$$\dim_{\mathbb{R}} \mathcal{M}_J(L) = 4n - 4.$$
- $\mathcal{M}_J(L)$ is compact by Gromov compactness. (There exists no possible stable nodal sphere limits by minimality of energy.)
- There exists a (possibly nontrivial!) $\mathbb{C}P^1$-bundle $\tilde{\mathcal{M}}_J(L) \to \mathcal{M}_J(L)$ whose fibre is the domain that parametrises $u \in \mathcal{M}_J(L)$, and
$$\dim_{\mathbb{R}} \tilde{\mathcal{M}}_J(L) = \dim_{\mathbb{R}} \mathcal{M}_J(L) + \dim_{\mathbb{R}} \mathbb{C}P^1 = (4n-4)+2 = 4n-2,$$

The moduli space of spheres

We have a smooth and compact moduli space

$$\mathbb{C}P^1 \to \tilde{\mathcal{M}}_J(L) \to \mathcal{M}_J(L)$$

of $\dim_{\mathbb{R}} \tilde{\mathcal{M}}_J(L) = 4n - 2$. There is a smooth evaluation map

$$\mathrm{ev} \colon \tilde{\mathcal{M}}_J(L) \to \overline{X}$$

which at $p \in \mathbb{C}P^1$ in the fibre over $u \in \mathcal{M}$ takes the value $u(p) \in \overline{X}$.

- $\mathrm{ev}^{-1}(\mathrm{pt})$ for a generic $\mathrm{pt} \in \overline{X}$ is a submanifold $\mathcal{M}_J(L; \mathrm{pt}) \subset \tilde{\mathcal{M}}_J(L)$ of dimension

$$\dim_{\mathbb{R}} \mathcal{M}_J(L; \mathrm{pt}) = 4n - 2 - \dim_{\mathbb{R}} \overline{X} = 2n - 2.$$

- Pull back the $\mathbb{C}P^1$-bundle to yield a bundle

$$\mathbb{C}P^1 \to \tilde{\mathcal{M}}_J(L; \mathrm{pt}) \to \mathcal{M}_J(L; \mathrm{pt}),$$

$$\dim_{\mathbb{R}} \tilde{\mathcal{M}}_J(L; \mathrm{pt}) = \dim_{\mathbb{R}} \mathcal{M}_J(L; \mathrm{pt}) + \dim_{\mathbb{R}} \mathbb{C}P^1 = 2n$$

Some useuful general nonsense

The evaluation map $\mathrm{ev}\colon \tilde{\mathcal{M}}_J \to \overline{X}$ can be constructed out of general principles:
Write

$$\begin{aligned} G &= \mathrm{Aut}(\mathbb{C}P^1) \\ \mathcal{C} &= \{u\colon (\mathbb{C}P^1, j) \to (X, J);\ [u] = L \in H_2(\overline{X})\}, \\ M &= \mathcal{C}/G = \mathcal{M}_J, \end{aligned}$$

And thus $G \to \mathcal{C} \to M$ is a G-principal bundle. (G acts on points in $\mathcal{C}$ from the right by reparametrisation.)

Some useuful general nonsense

Since $G = \mathrm{Aut}(\mathbb{C}P^1)$ acts naturally on $\mathbb{C}P^1$ from the *left* by

$$\phi \cdot \mathrm{pt} = \phi^{-1}(\mathrm{pt}), \;\; \mathrm{pt} \in \mathbb{C}P^1,$$

we can thus form the induced $\mathbb{C}P^1$-bundle

$$\tilde{\mathcal{M}}_J = \mathcal{C} \times_G \mathbb{C}P^1 = (\mathcal{C} \times \mathbb{C}P^1)/G \to M.$$

The right hand side is the quotient by the diagonal action $g \cdot (u, \mathrm{pt}) = (u \cdot g, g \cdot \mathrm{pt})$.

Some useuful general nonsense

There is also an evaluation map

$$\mathsf{EV}\colon \mathcal{C} \times \mathbb{C}P^1 \to \overline{X},$$
$$(u, \mathrm{pt}) \mapsto u(\mathrm{pt}),$$

which is invariant under the above diagonal G-action.
The evaluation map can then be given as the induced map

$$\mathsf{ev} = [\mathsf{EV}]\colon (\mathcal{C} \times \mathbb{C}P^1)/G \to M$$

on the quotient.

The moduli space of spheres

In the case $\overline{X} = \mathbb{C}P^n$ and $J = J_0$ we get:

- $$\mathcal{M}_{J_0}(L) \cong Gr_2(\mathbb{C}^{n+1})$$
  i.e. the space of complex-linear 2-planes (of $\dim_{\mathbb{C}} = ((n+1)-2)(2))$.
- $$\tilde{\mathcal{M}}_{J_0}(L)$$
  is the "tautological $\mathbb{C}P^1$-bundle" over $Gr_2(\mathbb{C}^{n+1})$.
- $$\mathcal{M}_{J_0}(L; \mathrm{pt}) \cong \mathbb{C}P^{n-1} = Gr_1(\mathbb{C}^n)$$
  i.e. the spaces of lines through some fixed point in $\mathbb{C}P^n$.
- $$\tilde{\mathcal{M}}_{J_0}(L; \mathrm{pt}) \cong \mathrm{Bl}_0(\mathbb{C}^n)$$
  i.e. the tautological $\mathbb{C}P^1$-bundle over $\mathbb{C}P^{n-1}$.

The proof

Theorem (Eliashberg–Floer–McDuff [McD91b])

Under the above assumptions (X, ω) *is diffeomorphic* D^{2n}.

Crucial steps in the proof.

The properties of the smooth map

$$\mathrm{ev}\colon \tilde{\mathcal{M}}_J(L; \mathrm{pt}) \to \overline{X}$$

between equidimensional manifolds will be analysed. □

The proof

Theorem (Eliashberg–Floer–McDuff [McD91b])

Under the above assumptions (X,ω) is diffeomorphic D^{2n}.

Proof (1/2) that ev. map is of degree one.

- Take $J = J_0$ near $\mathbb{C}P^{n-1}_\infty \subset \overline{X}$.
- Take $\mathrm{pt} \in \mathbb{C}P^{n-1}_\infty$ and consider $\tilde{\mathcal{M}}_J(L;\mathrm{pt})$ which is a closed manifold of dimension $2n = \dim_{\mathbb{R}} X$.
- Since $[u] \bullet [\mathbb{C}P^{n-1}_\infty] = 1$ holds when $[u] = L$, and since each intersection of a J-holomorphic curve with a J-holomorphic divisor contributes positively, if $u \in \mathcal{M}_J(L;\mathrm{pt})$ passes through a second point $\mathrm{pt}' \in \mathbb{C}P^{n-1}_\infty$, then u is contained entirely in the divisor. (And is thus a classical linear embedding inside $\mathbb{C}P^{n-1}_\infty$.)

The proof

Theorem (Eliashberg–Floer–McDuff [McD91b])

Under the above assumptions (X, ω) is diffeomorphic D^{2n}.

Proof (2/2) that ev. map is of degree one.

If we compute the degree of

$$\mathrm{ev}\colon \tilde{\mathcal{M}}_J(L; \mathrm{pt}) \to \overline{X}$$

by taking the second point $\mathrm{pt}' \in \mathbb{C}P^{n-1}_\infty$ as well, then by then the same computation as in the classical case $(\mathbb{C}P^n, J_0)$ gives that ev is of degree *one*. □

Since there exists a pseudoholomoprhic line through any two points in $\overline{X}$ we call it *uniruled*.

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