Holomorphic Curve Theories in Symplectic Geometry

Lecture III

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Goal of lecture

Today:

- "Uniruledness" of $(\mathbb{C}P^n, \omega_{\mathrm{FS}}, J)$ for any tame J.
  - Positivity of intersection.
  - Cobordisms of moduli spaces.
- Continuation of proof: Restriction of the topology of "symplectic fillings" of the round contact sphere (S^{2n-1}, α_0) (Gromov [Gro85], Eliashberg–Floer–McDuff [McD91b]).
- Definition of Lagrangian submanifolds.

Take-home message

There are J_0-holomorphic lines $\mathbb{C}P^1 \to \mathbb{C}P^n$ through every pair of points, this property remains for all tame J.

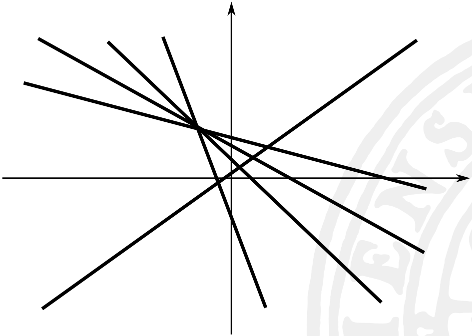

Figure: Lines inside $\mathbb{C}P^2$.

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Plan

Uniruledness

We show that $\mathbb{C}P^n$ is *uniruled* for any tame almost complex structure $J \in \mathcal{J}^{tame}(\mathbb{C}P^n, \omega_{\mathsf{FS}})$. First we need to recall the definitions from last time. Let $L \in H_2(\mathbb{C}P^n)$:

$$\begin{aligned}\mathcal{M}_J(L) &= \\ &= \{u\colon (\mathbb{C}P^1, j) \to (\mathbb{C}P^n, J); \overline{\partial}_J u = 0, [u] = L\}/\mathrm{Aut}(\mathbb{C}P^1).\end{aligned}$$

which is a smooth manifold of dim $2n + 2(n+1) - 6 = 4n - 4$; and

$$\mathbb{C}P^1 \to \tilde{\mathcal{M}}_J(L) \to \mathcal{M}_J(L)$$

the associated $\mathbb{C}P^1$-bundle with evaluation map

$$\mathsf{ev}\colon \tilde{\mathcal{M}}_J(L) \to \mathbb{C}P^n.$$

We also need the $2n$-dim. moduli space $\tilde{\mathcal{M}}_J(L; \mathrm{pt})$ with fibration:

$$\mathbb{C}P^1 \to \tilde{\mathcal{M}}_J(L; \mathrm{pt}) \to \mathcal{M}_J(L; \mathrm{pt}) = \mathsf{ev}^{-1}(\mathrm{pt}).$$

Uniruledness

By *uniruledness* we mean that:

Theorem (Gromov [Gro85])

The evaluation map

$$\tilde{\mathcal{M}}_J(L;\mathrm{pt}) \to \mathbb{C}P^n$$

is of degree <u>one</u> for any generic

$$J \in \mathcal{J}^{tame}(\mathbb{C}P^n, \omega_{\mathsf{FS}})$$

and arbitrary $\mathrm{pt} \in \mathbb{C}P^n$*.*

The proof relies on:

- The property is true for $J_0 = J$. We know *all* solutions in the linear case (uses positivity of intersection).
- A cobordism argument for the moduli space.

Uniruledness for $J = J_0$

For the standard integrable complex structure $J = J_0$ on $\mathbb{C}P^n$ something even stronger is true:

Proposition

There exists a unique holomorphic curve of degree one (i.e. homologous to $L \in H_2(\mathbb{C}P^n) = \mathbb{Z} \cdot L$) that passes through two given points $P_1 \neq P_2 \in \mathbb{C}P^n$, up to reparametrisation. This is the complex line

$$\mathbb{C}P^1 \to \mathbb{C}P^n,$$
$$[x_1 : x_2] \mapsto x_1 \cdot P_1 + x_2 \cdot P_2.$$

The proof relies heavily on *positivity of intersection* between complex curves ($\dim_{\mathbb{C}} = 1$) and complex hypersurfaces ($\dim_{\mathbb{C}} = n - 1$).

Positivity of intersection

Proposition (Positivity of intersection, Bézout)

Consider a connected holomorphic curve $u \colon \Sigma \to X$ *and a holomorphic hypersurface* $D \subset X$, *i.e.* $\dim_{\mathbb{C}} = \dim_{\mathbb{C}} X - 1$, *such that* u *is not contained inside* D. *Then:*

- u *and* D *intersect in a discrete subset;*
- *each geometric intersection point gives a positive contribution to the algebraic intersection number* $[u] \bullet [D] \geq 0$*; and*
- *if an intersection point moreover is not a transverse intersection (e.g. a tangency or an intersection of* D *and a singular point of* u*), then that geometric point contributes at least* $+2$.

Sketch of proof.

Non-contant holomorphic maps have a *positive* local degree. □

Uniruledness for $J = J_0$

Proposition

There exists a unique holomorphic curve of degree one (i.e. homologous to $L \in H_2(\mathbb{C}P^n) = \mathbb{Z} \cdot L$) that passes through two given points $P_1 \neq P_2 \in \mathbb{C}P^n$, up to reparametrisation. This is the complex line

$$\mathbb{C}P^1 \to \mathbb{C}P^n,$$
$$[x_1 : x_2] \mapsto x_1 \cdot P_1 + x_2 \cdot P_2.$$

Proof.

If a curve $u\colon (\Sigma, j) \to (\mathbb{C}P^n, J_0)$ in class $[u] = L$ is not of the above form, then we can find a linear hyperplane $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$ which is tangent to the curve at some point, but which does not contain it.

Uniruledness for standard J_0.

Proposition

There exists a unique holomorphic curve of degree one (i.e. homologous to $L \in H_2(\mathbb{C}P^n) = \mathbb{Z} \cdot L$) that passes through two given points $P_1 \neq P_2 \in \mathbb{C}P^n$, up to reparametrisation. This is the complex line

$$\mathbb{C}P^1 \to \mathbb{C}P^n,$$
$$[x_1 : x_2] \mapsto x_1 \cdot P_1 + x_2 \cdot P_2.$$

Proof.

Positivity of intersection of the curve and the hyperplane implies that $H \bullet [u] \geq 2$ (each geometric intersection contributes positively, and a tangency contributes at least $+2$). This contradicts $H \bullet L = 1$. □

Uniruledness

By *uniruledness* we mean that:

Theorem (Gromov [Gro85])

The evaluation map

$$\tilde{\mathcal{M}}_J(L; \mathrm{pt}) \to \mathbb{C}P^n$$

is of degree one for any generic

$$J \in \mathcal{J}^{tame}(\mathbb{C}P^n, \omega_{\mathsf{FS}})$$

- We have now established the property for $J_0 = J$.
- Now we outline the cobordism argument.

Cobordisms of moduli spaces

In general, any k-parameter family

$$\mathbf{J}\colon I^k \to \mathcal{J}^{\mathrm{tame}}(X,\omega)$$

of almost complex structures with

$$J_{\mathbf{s}} := \mathbf{J}(\mathbf{s}),\ \ \mathbf{s} \in I^k,$$

gives rise to a moduli space $\mathcal{M}_{\mathbf{J}} \to I^k$ where the fibres over $\mathbf{s} \in I^k$ is the moduli space $\mathcal{M}_{J_{\mathbf{s}}}$ of $J_{\mathbf{s}}$-holomorphic curves in X

Gromov compactness holds in the setting when u_i is a sequence of J_i-holomorphic curves where J_i are tame almost complex structures on (X,ω) which C^∞-converge to some tame J_∞ as $i \to \infty$.

Cobordisms of moduli spaces

Gromov compactness with parameter.

We can construct a tame almost complex structure of the form $\mathbf{J} \oplus J_Y$ on $(X \times Y, \omega_X \oplus \omega_Y)$ for a suitable symplectic manifold (Y, ω_Y, J_Y) equipped with a tame almost complex structure, which contains a smooth embedding $I^k \hookrightarrow Y$. I.e.

- $\mathbf{J} \oplus J_Y = J_{\mathbf{s}} \otimes J_Y$ over

$$X \times \{\mathbf{s}\} \subset X \times I^k \hookrightarrow X \times Y$$

  which is both a *symplectic* and an *almost complex* submanifold.
- $\mathcal{M}_{J_{\mathbf{s}}}$ consists of those curves contained entirely inside

$$X \times \{\mathbf{s}\} \subset X \times I^k \hookrightarrow X \times Y.$$

Cobordisms of moduli spaces

Remark

The choice of Y here is irrelevant, we can take e.g.

$$(Y, \omega, J_Y) = (\mathbb{C}P^N, \omega_{\mathsf{FS}}, J_0).$$

Cobordisms of moduli spaces

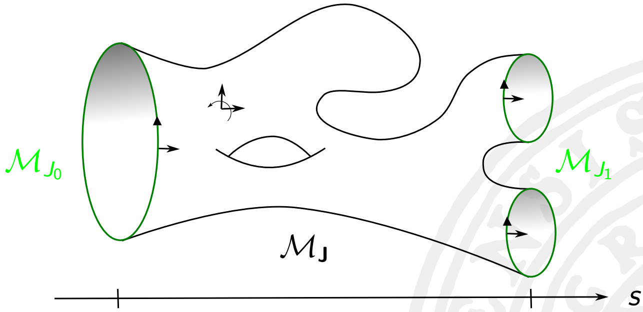

Figure: A cobordism $W = \mathcal{M}_{\mathbf{J}}$ of moduli spaces *from* $X_- = \mathcal{M}_{J_0}$ *to* $X_+ = \mathcal{M}_{J_1}$.

Cobordisms of moduli spaces

Recall the following basic fact in differential topology about degrees and cobordisms:

Lemma

Let $F\colon W^{n+1} \to M^n$ be a smooth map between compact oriented manifolds, where $\partial M = \emptyset$. Choose a decomposition of ∂W into compact manifolds $X_\pm$, i.e. $\partial W^{n+1} = X_+ \sqcup X_-$. If we orient X_+ (resp. X_-) along (resp. against) the boundary orientation, then we have

$$\deg(F|_{X_+}) = \deg(F|_{X_-}).$$

- In the above case we call W^{n+1} a smooth compact cobordism from X_- to X_+.

Cobordisms of moduli spaces

Recall the following basic fact in differential topology about degrees and cobordisms:

Lemma

Let $F\colon W^{n+1} \to M^n$ be a smooth map between compact oriented manifolds, where $\partial M = \emptyset$. Choose a decomposition of ∂W into manifolds $X_\pm$, i.e. $\partial W^{n+1} = X_+ \sqcup X_-$. If we orient X_+ (resp. X_-) along (resp. against) the boundary orientation, then we have

$$\deg(F|_{X_+}) = \deg(F|_{X_-}).$$

- The pieces $X_\pm$ of the boundary of W^{n+1} may themselves be disconnected manifolds. In this case, $\deg(F|_{X_\pm}) = 1$ does not imply that that there exists a single component for which the degree is one.

Uniruledness for arbitrary tame J

Theorem (Gromov [Gro85])

The evaluation map $\tilde{\mathcal{M}}_J(L; \mathrm{pt}) \to \mathbb{C}P^n$ is of degree one for any generic tame J.

Proof.

- Recall Gromov's lemma that $J^{tame}(\mathbb{C}P^n, \omega_{\mathsf{FS}})$ is contractible. In particular we can find a one-parameter family $\mathbf{J}$ which connects the standard J_0 to $J_1 = J$.
- Consider the moduli space $\tilde{\mathcal{M}}_{\mathbf{J}}(L; \mathrm{pt})$, which is a cobordism from $\tilde{\mathcal{M}}_{J_0}(L; \mathrm{pt})$ to $\tilde{\mathcal{M}}_{J_1}(L; \mathrm{pt})$. It admits a compactification by Gromov's compactness thm. (cobordism version).

Uniruledness for arbitrary tame J

Theorem (Gromov [Gro85])

The evaluation map $\tilde{\mathcal{M}}_J(L;\mathrm{pt}) \to \mathbb{C}P^n$ is of degree <u>one</u> for any generic tame J.

Proof.

- Since L is the class of smallest positive symplectic area in $(\mathbb{C}P^n, \omega_{\mathrm{FS}})$, $\int_L \omega_{\mathrm{FS}} = \pi$, Gromov's compactness theorem implies the above moduli space already is *compact*.
- A transversality argument shows that $\tilde{\mathcal{M}}_{\mathbf{J}}(L)$ is a compact manifold of dimension $2n+1$ with smooth boundary when the path $\mathbf{J}$ is generic. (We gloss over this point.)

Uniruledness for arbitrary tame J

Theorem (Gromov [Gro85])

The evaluation map $\tilde{\mathcal{M}}_J(L;\mathrm{pt}) \to \mathbb{C}P^n$ is of degree one for any generic tame J.

Proof.

- The moduli space $\tilde{\mathcal{M}}_{\mathbf{J}}(L;\mathrm{pt})$ is a cobordism from $\tilde{\mathcal{M}}_{J_0}(L;\mathrm{pt})$ to $\tilde{\mathcal{M}}_J(L;\mathrm{pt})$, and the evaluation map extends to the entire cobordism: there is a fibration

$$\mathbb{C}P^1 \to \tilde{\mathcal{M}}_{\mathbf{J}}(L;\mathrm{pt}) \to \mathcal{M}_{\mathbf{J}}(L;\mathrm{pt}).$$

Uniruledness for arbitrary tame J

Theorem (Gromov [Gro85])

The evaluation map $\tilde{\mathcal{M}}_J(L; \mathrm{pt}) \to \mathbb{C}P^n$ is of degree one for any generic tame J.

Proof.

- Since the evaluation map

$$\tilde{\mathcal{M}}_{\mathbf{J}}(L; \mathrm{pt}) \to \mathbb{C}P^n$$

restricts to the evaluation map

$$\tilde{\mathcal{M}}_{J_0}(L; \mathrm{pt}) \to \mathbb{C}P^n,$$

which is of degree one (the classical holomorphic case), the differential topological lemma shows the claim.

Classification of fillings

The question that we want to study is:

Question

What are the possible symplectic fillings (X, ω) of (S^{2n-1}, α_0), $n > 1$, up to symplectomorphism? Simplifying assumption: $\int_\alpha \omega = 0$ on each $\alpha \in H_2(X)$.

Theorem (Eliashberg–Floer–McDuff [McD91b])

Under the above assumptions (X, ω) *is diffeomorphic* D^{2n}.

For the proof, the properties of the smooth map

$$\mathrm{ev}\colon \tilde{\mathcal{M}}_J(L; \mathrm{pt}) \to \overline{X}$$

between equidimensional manifolds will be analysed.

The proof

Recall that

$$\overline{X} = (X \setminus \partial X) \sqcup \mathbb{C}P^{n-1}_\infty$$

with the induced symplectic form $\overline{\omega}$.
In particular:

- $\overline{X}$ is a closed symplectic manifold with a "divisor at infinity" $\mathbb{C}P^{n-1}_\infty$ whose neighbourhood is symplectomorphic to a neighbourhood of

$$\mathbb{C}P^{n-1}_\infty \subset (\mathbb{C}P^n, \omega_{\mathsf{FS}}).$$

- For an almost complex structure J which is equal to J_0 near this divisor, we know all pseudoholomorphic lines near the divisor. (They are the same as those in $\mathbb{C}P^n$).

Lines near $\mathbb{C}P^{n-1}_\infty$.

A neighbourhood of $\mathbb{C}P^{n-1}_\infty \subset \overline{X}$ is biholomorphic to a neighbourhood of $\mathbb{C}P^{n-1}_\infty \subset (\mathbb{C}P^n, J_0)$.

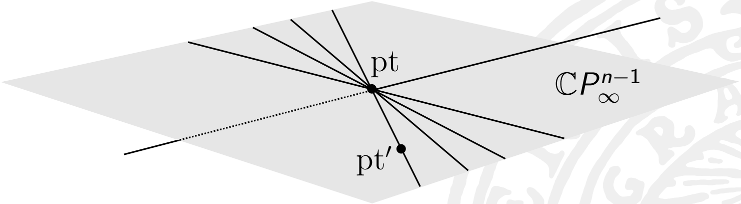

Figure: Lines near $\mathbb{C}P^{n-1}_\infty \subset \overline{X}$ are standard holomorphic lines.

The proof

Theorem (Eliashberg–Floer–McDuff [McD91b])

Under the above assumptions (X, ω) is diffeomorphic D^{2n}.

Proof that evaluation map is of degree one.

- Take $J = J_0$ near $\mathbb{C}P^{n-1}_\infty \subset \overline{X}$.
- Take $\mathrm{pt} \in \mathbb{C}P^{n-1}_\infty$ and consider $\tilde{\mathcal{M}}_J(L; \mathrm{pt})$ which is a closed manifold of dimension $2n = \dim_{\mathbb{R}} X$.
- Since $[u] \bullet [\mathbb{C}P^{n-1}_\infty] = 1$ holds when $[u] = L$, and since each intersection of a J-holomorphic curve with a J-holomorphic divisor contributes positively, if $u \in \mathcal{M}_J(L; \mathrm{pt})$ passes through a second point $\mathrm{pt}' \in \mathbb{C}P^{n-1}_\infty$, then u is contained entirely in the divisor. (And is thus a classical linear embedding inside $\mathbb{C}P^{n-1}_\infty$.)

□

The proof

Theorem (Eliashberg–Floer–McDuff [McD91b])

Under the above assumptions (X, ω) is diffeomorphic D^{2n}.

Proof that evaluation map is of degree one.

If we compute the degree of

$$\mathrm{ev}\colon \tilde{\mathcal{M}}_J(L; \mathrm{pt}) \to \overline{X}$$

by taking the second point $\mathrm{pt}' \in \mathbb{C}P^{n-1}_\infty$ as well, then the *same* classical argument as in the case of $(\mathbb{C}P^n, J_0)$ gives that ev is of degree *one*. □

Lines near $\mathbb{C}P^{n-1}_\infty$.

A neighbourhood of $\mathbb{C}P^{n-1}_\infty \subset \overline{X}$ is biholomorphic to a neighbourhood of $\mathbb{C}P^{n-1}_\infty \subset (\mathbb{C}P^n, J_0)$.

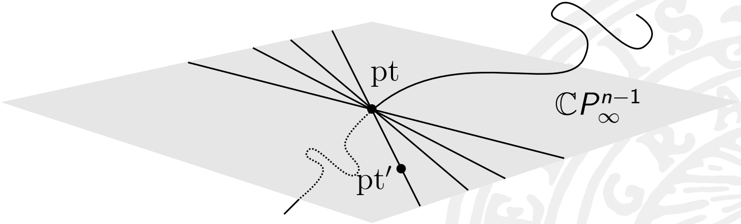

Figure: Lines can also enter the interior of X.

Lines near $\mathbb{C}P^{n-1}_\infty$.

A neighbourhood of $\mathbb{C}P^{n-1}_\infty \subset \overline{X}$ is biholomorphic to a neighbourhood of $\mathbb{C}P^{n-1}_\infty \subset (\mathbb{C}P^n, J_0)$.

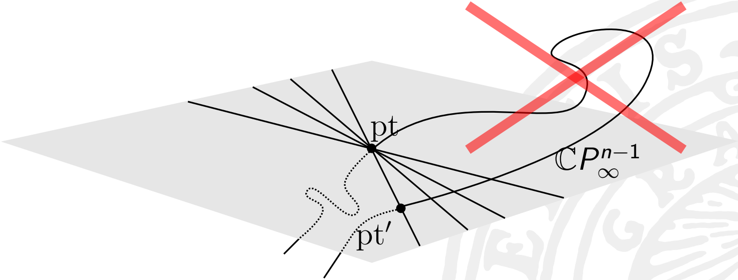

Figure: Lines can *not* enter the interior of X and then touch the divisor at a second point pt'.

The proof

Theorem (Eliashberg–Floer–McDuff [McD91b])

Under the above assumptions (X, ω) is diffeomorphic D^{2n}.

Proof that $\pi_1(\overline{X}) = 0$.

- Pass to the universal cover $(\tilde{X}, \tilde{J}, \tilde{\omega}) \to (\overline{X}, J, \omega)$. Note that $\mathbb{C}P^{n-1}_\infty$ lifts to a number $|\pi(\overline{X})|$ of disjoint divisors.
- Pseudoholomorphic spheres admit lifts that touch precisely one of the divisors.
- The lift of the evaluation is again of "degree one", which contradicts positivity of intersection with the other divisors.
- Use Seifert–van Kampen to deduce $\pi_1(X) = 0$.

□

The proof

Theorem (Eliashberg–Floer–McDuff [McD91b])

Under the above assumptions (X, ω) *is diffeomorphic* D^{2n}.

Proof.

See Ghiggini–Niederkrüger's recent work [GN20] for the rest of the proof in dimension ≥ 6.
Punchline: X is a simply connected $\mathbb{Z}$-homology ball. Use Smale's *h*-cobordism theorem to produce the diffeomorphism.
The *h*-cobordism theorem applies only in dimension ≥ 6! □

The proof in dimension four

In dimension four, however, something much stronger is true:

Theorem (Gromov [Gro85])

When $2n = 4$ any symplectic filling (X^4, ω) of S^3 is <u>*symplectomorphic*</u> *to (D^4, ω_0), after a finite number of symplectic blow-downs.*

First we will need to use some facts about pseudoholomorphic curves in dimension four.

In dimension four

When $\dim X = 2n = 4$ (i.e. $n = 2$) positivity of intersection holds between pseudoholomorphic curves. More precisely:

Proposition (McDuff [McD91a])

Consider two connected pseudoholomorphic curves u and v in a four-dimensional almost complex manifold that are not branched covers of some common underlying curve. Then

- *u and v intersect in a discrete subset;*
- *each geometric intersection point gives a positive contribution to the algebraic intersection number $[u] \bullet [v] \geq 0$; and*
- *if an intersection point between the two curves moreover is not a transverse intersection (e.g. a tangency of u and v), then that geometric intersection contributes at least $+2$.*

The proof in dimension four

A second intermediate result which holds for symplectic manifolds (X^4, ω) of dimension $2n = 4$:

Lemma

An embedded pseudoholomorphic sphere of self-intersection $[u] \bullet [u] = k$ has Fredholm index

$$\text{index}(u) = n\chi(\mathbb{C}P^1) + 2c_1^{TX}[u] = 4 + 2(2 + k) = 8 + 2k.$$

In particular, the expected (virtual) dimension of the moduli space of the curve is

$$\mathrm{vdim}(u) := \text{index}(u) - \dim_{\mathbb{R}} \mathrm{Aut}(\mathbb{C}P^1) = 2 + 2k$$

(after taking quotient by reparam.)

The proof in dimension four

Proof.

In order to compute $c_1^{TX}[u]$ we use $c_1^{T\mathbb{C}P^1} = \chi(\mathbb{C}P^1) = 2$ and the adjunction formula.
(The normal bundle of the sphere is a $\mathbb{C}$-bundle of Chern number k by the assumption $[u] \bullet [u] = k$.) □

Negative self-int. spheres

A second intermediate result which holds for symplectic manifolds (X^4, ω) of dimension $2n = 4$:

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In particular, the expected (virtual) dimension of the moduli space of the curve is

$$\mathrm{vdim}(u) := \text{index}(u) - \dim_{\mathbb{R}} \mathrm{Aut}(\mathbb{C}P^1) = 2 + 2k.$$

(This is the actual $\dim_{\mathbb{R}}$ *if the transversality is achieved, i.e. when* coker $D_u\overline{\partial}_J = 0$.*)*

Negative self-int. spheres

Since embedded curves can be made transversely cut out for generic J, after a generic choice of tame almost complex structure J on (X^4, ω) one can conclude that:

Lemma

There exists no embedded pseudoholomorphic spheres of self-intersection strictly less than -1 for generic tame J, and the embedded spheres of self-intersection number -1 which satisfy some fixed bound on their energy form a 0-dimensional compact manifold; i.e. they form a finite set of points.

The proof in dimension four

Proof.

The fact that the manifold compact (and thus a finite nr. of points) follows by Gromov's compactness, but one needs to make sure that there exists no nodal spheres that are potential limits. □

We have used the sub-additivity of expected dimension which holds in dimension four.

$$([u] + [v])^2 = [u]^2 + 2[u] \bullet [v] + [v]^2$$

$$\mathrm{vdim}([u + v]) = \mathrm{vdim}([u]) + \mathrm{vdim}([v]) + 2$$

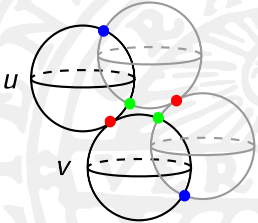

Figure: The self-intersection of a nodal sphere

The proof in dimension four

- A solution which lives in a moduli space of expected negative dimension cannot exist if transversality is achieved. (There are no manifolds of negative dimension!)
- Sub-additivity of the expected dimension comes from the fact that a node $z_1 z_2 = 0$ can be smoothed by deforming the right-hand side with the complex one-dimensional (real two-dimensional) parameter $\epsilon \in \mathbb{C}$.

Sub-additivity of the expected dimension

The sub-additivity of expected dimension which holds in dimension four:

Lemma

The (expected) dimension of moduli space of the psh. spheres u_i for $i \gg 0$ and the (expected) dimensions d_k of the moduli spaces that contain the components u_∞^k in the nodal limit of u_i satisfy the relation

$$d = \sum_k d_k + 2N$$

where $N > 0$ is the number of nodes of the limit.

The proof in dimension four

Theorem (Gromov [Gro85])

When $2n = 4$ any symplectic filling (X^4, ω) of S^3 is symplectomorphic to (D^4, ω_0) after a finite number of symplectic blow-downs.

Proof that moduli space is cpct. after blow-down.

- The 2-dimensional manifold $\mathcal{M}_J(L; \mathrm{pt})$ need not be compact. However..
- By above a nodal limit must consist of one embedded sphere of self-intersection -1 (expected dimension 0) disjoint from $\mathbb{C}P^1_\infty$ and one embedded sphere of self-intersection 0 (expected dimension 2) which passes through pt.

□

A nodal line.

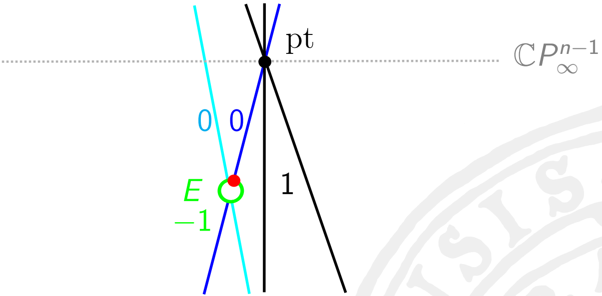

Figure: The numbers denote the self-intersection indices of the different lines. E is the exceptional divisor (line) of a blow-up. The red point is the unique intersection between the line of self intersection 0 and E. The lines of self-intersection number one (shown in black) converge to the nodal line consisting of a line of self-intersection 0 (blue) and the exceptional line (green).

The proof in dimension four

Theorem (Gromov [Gro85])

When $2n = 4$ any symplectic filling (X, ω) of S^3 is symplectomorphic to (D^4, ω_0) after a finite number of symplectic blow-downs.

Proof that moduli space is cpct. after blow-down.

- To preclude a non-compact $\mathcal{M}_J(L)$ it thus suffices to blow down all exceptional spheres of self-intersection -1 of symplectic area less than $\int_L \omega_{\mathrm{FS}} = \pi$.
- There number such spheres is finite by the previous lemma!
- We also need the fact that: a sphere of self-int. -1 has a nbhd. which is symplectomorphic to a nbhd. of the exceptional divisor $E \subset Bl_{D^4_{\sqrt{\lambda}}}\mathbb{C}^2$ for some $0 < \lambda < 1$.

□

The proof in dimension four

Theorem (Gromov [Gro85])

When $2n = 4$ any symplectic filling (X, ω) of S^3 is symplectomorphic to (D^4, ω_0) after a finite number of symplectic blow-downs.

Idea of proof of diffeomorphism.

In this case positivity of intersection implies that $\mathcal{M}_J(L; \mathrm{pt}) \cong \mathbb{C}P^1$ and that

$$\mathrm{ev}\colon \tilde{\mathcal{M}}_J(L; \mathrm{pt}) \to \overline{X}$$

is *foliation* (i.e. coordinate system) of $\overline{X}$ by pseudoholomorphic lines $\mathbb{C}$ away from $\mathrm{pt} \in \overline{X}$.

Reason: There is a unique line with each tangency at pt. (Just as for standard J_0.) □

Exact symplectic manifolds

Definition

A symplectic manifold $(X^{2n}, d\lambda)$ with a choice of primitive λ for the symplectic form is said to be *exact*.

- Stoke's theorem together with $\int_X d\lambda^{\wedge n} > 0$ implies that closed symplectic manifolds are never exact.
- Recall that λ induces the Liouville vector field ζ via $\iota_\zeta \omega = \lambda$.
- A Kähler potential σ for a symplectic Kähler form $\omega = i\partial\overline{\partial}\sigma$ on a complex manifold (X, J) induces the primitive $\lambda = -d^c\sigma/2$.
- For example: $\omega_0 = d\lambda_0$ where

$$\lambda_0 = -d^c\|\mathbf{z}\|^2/4 = \frac{1}{2}\sum_i (x_i dy_i - y_i dx_i),$$

$$\zeta_0 = \frac{1}{2}\sum_i (x_i \partial_{x_i} + y_i \partial_{y_i}).$$

Lagrangian submanifolds

Definition

- A half-dimensional manifold $L^n \subset (X^{2n}, \omega)$ of a symplectic manifold is called *Lagrangian* if the pullback of the symplectic form vanishes, i.e. $\omega|_{TL} \equiv 0$.
- A half-dimensional manifold $L^n \subset (X^{2n}, d\lambda)$ of an exact symplectic manifold with primitive λ of the symplectic form is called an *exact Lagrangian submanifold* if the pullback of the primitive is exact, i.e. $\lambda|_{TL} = dg$ is exact ($g \colon L \to \mathbb{R}$ a smooth function on L).

Lagrangian submanifolds

Weinstein's creed

"Everything is a Lagrangian submanifold"

- Never the less: Existence of Lagrangians is a difficult problem! When do they exist? We have only partial answers.
- What is meant is that many constructions in symplectic topology can be translated into statements about Lagrangian submanifolds. Main example on next page.

Lagrangian submanifolds

Graphs of symplectomorphisms

The graph

$$\Gamma_\phi = \{(x,y) \in X_1 \times X_2;\ \ y = \phi(x)\} \subset (X_1 \times X_2, \omega_1 \oplus -\omega_2).$$

is Lagrangian if and only if

$$\phi\colon (X_1, \omega_1) \xrightarrow{\cong} (X_2, \omega_2)$$

is a symplectomorphism. Note the sign $-\omega_2$!

Proof.

The inlcusion $\mathrm{Id} \times \phi\colon X_1 \to X_1 \times X_2$ pulls back $\omega_1 \oplus -\omega_2$ to $\omega_1 - \omega_1 = 0$. □

Lagrangian submanifolds

More constructions of *closed* Lagrangians in *closed* symplectic manifolds:

- Curves $\gamma_i \subset (X_i^2, \omega_i)$ of dimension $\dim_{\mathbb{R}} \gamma_i = 1$ in a symplectic surface, and their products

$$\gamma_1 \times \ldots \times \gamma_n \subset (X_1 \times \ldots \times X_n, \omega_1 \oplus \ldots \oplus \omega_1).$$

- Fixed-loci of anti-symplectic involutions

$$I \colon (X, \omega) \xrightarrow{\cong} (X, -\omega), \;\; I^2 = \mathrm{Id}_X$$

  (whenever they are half-dimensional).
- Leaves of integrable systems (Hamiltonain torus actions; more details next lecture).

The sphere in $\mathbb{C}P^1 \times \mathbb{C}P^1$

Consider the diagonal sphere

$$\Delta \subset (\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{\mathsf{FS}} \oplus -\omega_{\mathsf{FS}})$$

which is a Lagrangian sphere. Complex conjugation $z \to \overline{z}$ in an affine chart induces an anti-symplectic involution

$$I \colon (\mathbb{C}P^1, \omega_{\mathsf{FS}}) \to (\mathbb{C}P^1, -\omega_{\mathsf{FS}}).$$

The sphere in $\mathbb{C}P^1 \times \mathbb{C}P^1$

Recall that $\omega_{\mathrm{FS}} = \frac{i}{2}\partial\overline{\partial}\rho$ becomes

$$\begin{aligned} I^*\omega_{\mathrm{FS}} &= \\ &= \frac{i}{2} I^*(\partial\overline{\partial}\rho) = \frac{i}{2}\overline{\partial} I^* \overline{\partial}\rho = \\ &= \frac{i}{2}\overline{\partial}\partial\overline{\rho} = \frac{i}{2}\overline{\partial}\partial\rho = \\ &= -\frac{i}{2}\partial\overline{\partial}\rho = -\omega_{\mathrm{FS}} \end{aligned}$$

under complex conjugation. (ρ is a *real* function!) Consequently

$$S = \{(z, \overline{z}) \in \mathbb{C}P^1 \times \mathbb{C}P^1\} \subset (\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{\mathrm{FS}} \oplus \omega_{\mathrm{FS}})$$

is a Lagrangian sphere.

The sphere in $\mathbb{C}P^1 \times \mathbb{C}P^1$

Alternatively the Lagrangian sphere

$$S = \{(z, \overline{z}) \in \mathbb{C}P^1 \times \mathbb{C}P^1\} \subset (\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{\mathsf{FS}} \oplus \omega_{\mathsf{FS}})$$

is the fixed-point locus of the anti-symplectic involution

$$I \colon (\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{\mathsf{FS}} \oplus \omega_{\mathsf{FS}}) \to (\mathbb{C}P^1 \times \mathbb{C}P^1, -\omega_{\mathsf{FS}} \oplus -\omega_{\mathsf{FS}}),$$
$$(z_1, z_2) \mapsto (\overline{z}_2, \overline{z}_1).$$

$\mathbb{R}P^n$ inside $\mathbb{C}P^n$

Complex conjugation in $\mathbb{C}P^n$ is an anti-symplectic involution with fixed-point locus

$$\mathbb{R}P^n \subset (\mathbb{C}P^n, \omega_{\mathsf{FS}}).$$

In the particular case $n = 1$ we get the equator $S^1 = \mathbb{R}P^1 \subset \mathbb{C}P^1$.

Remark

The Lagrangian $\mathbb{R}P^1 \subset \mathbb{C}P^1$ divides $\mathbb{C}P^1$ into two hemispheres which by symmetry each bound a symplectic area equal to $\pi/2$. This will be important next lecture.

$\mathbb{R}P^1$ inside $\mathbb{C}P^1$

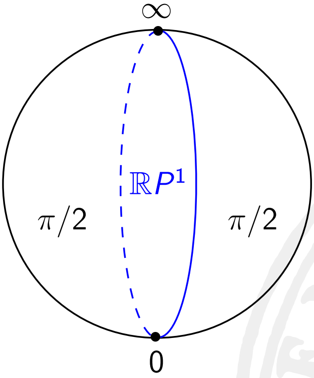

Figure: The Lagrangian $\mathbb{R}P^1 \subset \mathbb{C}P^1$. The south pole is the origin $0 \in \mathbb{C}$ in the affine chart, and the north pole is ∞.

Tori inside $(\mathbb{C}P^1)^n$

Example

The n-fold product

$$(\mathbb{R}P^1)^n \subset ((\mathbb{C}P^1)^n, \omega_{\mathsf{FS}} \oplus \ldots \oplus \omega_{\mathsf{FS}})$$

is a Lagrangian n-dimensional torus which is the fixed-point locus of an anti-symplectic involution.
This torus is sometimes called the *Clifford torus*.

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