Holomorphic Curve Theories in Symplectic Geometry

Lecture IV

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Goal of lecture

Today:

- Lagrangians from Hamiltonian torus actions.
- Local model of Lagrangians.
- The Fredholm index and Maslov index of discs.
- Gromov's Compactness for pseudoholomorphic discs.

Next time:

- "Uniruledness" of Lagrangians in $(\mathbb{C}P^n, \omega_{\mathsf{FS}}, J)$ for any tame J.

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Plan

1. Goal of lecture
2. Lagrangian submanifolds
3. Local model of Lagrangians
4. Pseudoholomorphic discs
5. Deformation theory for discs
6. References

Take-home message

The moduli spaces of pseudoholomorphic curves with boundary behave like real algebraic 1-dim varieties. I.e. they are more complicated.

- The nodal Gromov limits of pseudoholomorphic closed curves form a moduli space of (expected) codimension *two*.
- The nodal Gromov limits of pseudoholomorphic curves with boundary form a moduli space of (expected) codimension *one*.

Example: Real conics

We consider the family $C_t = \{z_1 z_2 = t\} \subset \mathbb{C}P^2$ where $t \in \mathbb{R}$ is a real parameter, and thus the Lagrangian $\mathbb{R}P^2 \subset \mathbb{C}P^2$ divides C_t into two discs with boundary on the Lagrangian.

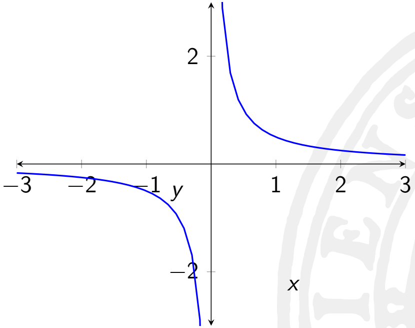

Figure: The bdy. in $\mathbb{R}P^2$ of the disc components from $C_{0.5} \setminus \mathbb{R}P^2$.

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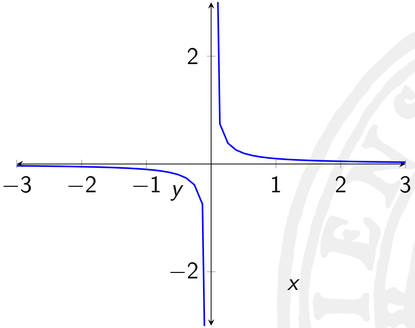

Figure: The bdy. in $\mathbb{R}P^2$ of the disc components from $C_{0.1} \setminus \mathbb{R}P^2$.

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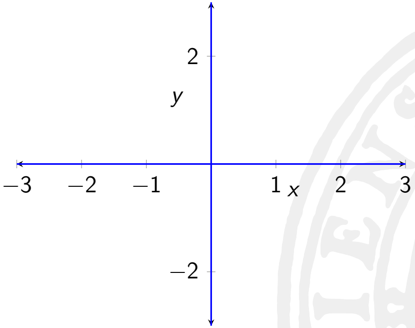

Figure: The bdy. in $\mathbb{R}P^2$ of the disc components from $C_0 \setminus \mathbb{R}P^2$.

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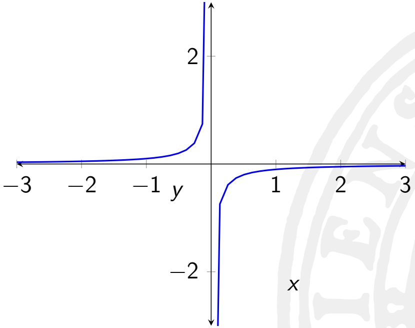

Figure: The bdy. in $\mathbb{R}P^2$ of the disc components from $C_{-0.1} \setminus \mathbb{R}P^2$.

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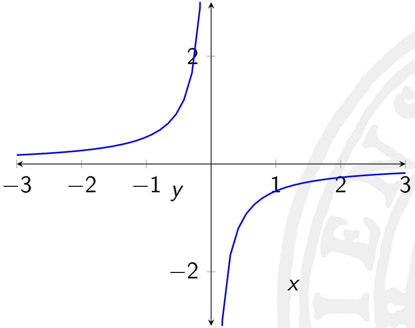

Figure: The bdy. in $\mathbb{R}P^2$ of the disc components from $C_{-0.5} \setminus \mathbb{R}P^2$.

Example: Real conics

- When passing through $t = 1$ to $t = -1$ we must pass the nodal configuration at $t = 0$ if we want to pass through real conics.
- In the closed case, we could avoid $t = 0$ by utilizing the freedom to choose $t \in \mathbb{C}^*$ as a complex variable. (These solutions, however, are *not* doubled pseudoholomorphic discs boundary on $\mathbb{R}P^n$.)

Lagrangian submanifolds

Definition

- A half-dimensional manifold $L^n \subset (X^{2n}, \omega)$ of a symplectic manifold is called *Lagrangian* if the pullback of the symplectic form vanishes, i.e. $\omega|_{TL} \equiv 0$.
- A half-dimensional manifold $L^n \subset (X^{2n}, d\lambda)$ of an exact symplectic manifold with primitive λ of the symplectic form is called an *exact Lagrangian submanifold* if the pullback of the primitive is exact, i.e. $\lambda|_{TL} = dg$ is exact ($g\colon L \to \mathbb{R}$ a smooth function on L).

Prime examples: real parts

Fixed-point loci of anti-symplectic involutions

$$I\colon (X^{2n},\omega) \to (X^{2n},-\omega)$$

which are of maximal dimension (i.e. $\dim_{\mathbb{R}} = n$).

Example

- Products of projective spaces (the n-torus when all $n_i = 1$)

$$\mathbb{R}P^{n_1} \times \ldots \times \mathbb{R}P^{n_k} \subset (\mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_k}, \omega_{\mathrm{FS}} \oplus \ldots \oplus \omega_{\mathrm{FS}})$$

- The 2-sphere

$$\{(z,\overline{z})\} \subset (\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{\mathrm{FS}} \oplus \omega_{\mathrm{FS}})$$

Prime examples: orbits in toric systems

Another important example comes from *integrable systems*. Assume that we are given

- a symplectic $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ Lie-group action on (X^{2n}, ω).
- which moreover is a *Hamiltonian group action*, i.e. there is a map

$$T_e\mathbb{T}^n \to C^\infty(X, \mathbb{R}),$$
$$V \mapsto H_V,$$

which is a morphism of Lie algebras that satisfies

$$e^{tV} = \phi^t_{H_V} \colon (X, \omega) \to (X, \omega).$$

(The action of $e^{tV} \in \mathbb{T}^n$ is given by the Hamiltonian time-t map $\phi^t_{H_V}$ generated by $H_V \colon X \to \mathbb{R}$.)

The Poisson bracket

Definition

The *Poisson bracket* of two autonomous Hamiltonians $H, G \in C^\infty(X, \mathbb{R})$ on a symplectic manifold (X, ω) is the autonomous Hamiltonian

$$\{H, G\} := \omega(X_H, X_G)$$

where $X_H := \dot{\phi}_H^t$ and $X_G := \dot{\phi}_G^t$ are the infinitesimal generators.

Proposition (See [MS98])

$$\dot{\phi}^t_{\{H,G\}} = [X_H, X_G].$$

Prime examples: orbits in toric systems

Proposition

Any regular n-dimensional orbit $\mathbb{T}^n \cdot \mathrm{pt} \subset X$ *of a Hamiltonian* $\mathbb{T}^n$*-action is a Lagrangian n-torus.*

Proof.

- $\mathbb{T}^n$ is an abelian Lie group, so its Lie algebra $(T_e\mathbb{T}^n, [\cdot,\cdot])$ is abelian. (The Lie bracket vanishes.)
- For $U, V \in T_e\mathbb{T}^n$ we have

$$\omega(U, V) = \{H_U, H_V\} = H_{[U,V]} = H_0 = 0.$$

(So any two tangent vectors in the orbit have vanishing symplectic pairing.)

□

A non-example

The symplectic 2-torus

$$(\mathbb{T}^2 = \mathbb{C}/(2\pi\mathbb{Z} \oplus i2\pi\mathbb{Z}), \omega_0)$$

has an obvious symplectic $\mathbb{T}^2$-action by translation.
However, the vector-fields ∂_x or ∂_y are *not* Hamiltonian on the torus:
Reason: The two-forms

$$-\iota_{\partial_x}\omega_0 = -dy \text{ and } -\iota_{\partial_y}\omega_0 = dx$$

are *not* exact on the torus. (Even though they are exact on $(\mathbb{C}, \omega_0)$.)

The toric action on $\mathbb{C}P^n$

The Fubini–Study form

$$\omega_{\mathsf{FS}} = \frac{i}{2}\partial\overline{\partial}\log\left(1 + \|\mathbf{z}\|^2\right)$$

is preserved under the action by

$$U(1)^n \subset U(n) \subset \mathrm{Aut}(\mathbb{C}P^n, \omega_{\mathsf{FS}}) = \mathbb{P}SU(n+1)$$

where $U(1)^n = \mathbb{T}^n$.

The toric action on $\mathbb{C}P^n$

Two general results:

- Since $\mathbb{C}P^n$ is simply connected $\pi_1(\mathbb{C}P^n) = 0$, it follows that $H^1(\mathbb{C}P^n, \mathbb{R}) = 0$ and hence
  any symplectic action is Hamiltonian.
- When there is a compatible almost complex structure $J \in \mathcal{J}^{comp}(X, \omega)$, the gradient of the Hamiltonian satisfies

$$\omega(\cdot, J\nabla_g H) = g_{\omega,J}(\cdot, \nabla_g H) = dH(\cdot)$$

  and hence $\boxed{J\nabla_g H = X_H = \dot{\phi}^t_H.}$ (Recall that $g_{\omega,J}(\cdot,\cdot) = \omega(\cdot, J\cdot)$.)

The toric action on $\mathbb{C}P^n$

We now describe the map

$$\boldsymbol{\mu}\colon \mathbb{C}P^n \to \mathbb{R}^n$$

whose i:th value is $\mu_i := H_{\partial_{\theta_i}}$, where

$$T_e U(1)^n = \mathbb{R}^n = \langle \partial_{\theta_1}, \ldots, \partial_{\theta_n} \rangle$$

I.e. μ_i is the Hamiltonian that generates the S^1-action

$$\{e\}^{i-1} \times U(1) \times \{e\}^{n-i} \subset U(1)^n.$$

Lemma

The $U(1)^n$-action on $(\mathbb{C}P^n, \omega_{\mathrm{FS}})$ is generated by the Hamiltonian

$$\partial_{\theta_i} \mapsto \mu_i := H_{\partial_{\theta_i}} = \frac{\|z_i\|^2}{2(1 + \|\mathbf{z}\|^2)}$$

The toric action on $\mathbb{C}P^n$

Proof (1/2).

- Use the symplectomorphism

$$(\mathbb{C}^n = \mathbb{C}P^n \setminus \mathbb{C}P^{n-1}_\infty, \omega_{\mathrm{FS}}) \xrightarrow{\cong} (B^{2n}, \omega_0),$$
$$(r_i, \theta_i) \mapsto \left(\frac{r_i}{\sqrt{1 + \|\mathbf{r}\|^2}}, \theta_i \right)$$

which obviously pulls back the primitive $-d^c\frac{\rho_0}{4} = \sum_i \frac{r_i^2}{2} d\theta_i$ of ω_0 to the primitive $-d^c\frac{\rho}{4} = \sum_i \frac{r_i^2}{2(1+\|\mathbf{r}\|^2)} d\theta_i$ of ω_{FS}.

The toric action on $\mathbb{C}P^n$

Proof (2/2).

- The induced action of $U(1)^n$ on B^{2n} again is the standard $U(1)^n$ action, and hence

$$\partial_{\theta_i} \mapsto \mu_i = H_{\partial_{\theta_i}} = \|z_i\|^2/2 = r_i^2/2.$$

Indeed the gradient of $r^2/2$ in the Euclidean metric $\omega_0(\cdot, J_0 \cdot)$ is $r_i \partial_{r_i}$ and

$$J_0 \cdot r_i \partial_{r_i} = \partial_{\theta_i}.$$

□

The momentum map

Definition

The map

$$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n) \colon X \to \mathbb{R}^n, \ \ \mu_i = H_{\partial_{\theta_i}},$$

is called the *momentum map* of the $\mathbb{T}^n$-action.

- The splitting $\boldsymbol{\mu} = (\mu_i)$ depends on a choice of basis of $T_e\mathbb{T}^n$
- Since

$$dH(\dot{\phi}_H^t) = dH(X_H) = \omega(X_H, X_H) = 0$$

it follows that H is constant along Hamiltonian orbits of ϕ_H^t.

The momentum map

More generally, since $\mathbb{T}^n$ is abelian and $T_e\mathbb{T}^n \ni V \mapsto H_V \in C^\infty(X,\omega)$ is a morphism of Lie algebras, we get

$$0 = \{\mu_i, \mu_j\} = \omega(X_{\mu_i}, X_{\mu_j}) = d\mu_j(X_{\mu_i})$$

which means that the momentum map $\boldsymbol{\mu}$ is invariant under the $\mathbb{T}^n$-action. This, in turn, implies that:

Lemma

Preimages of regular values of the momentum map are $\mathbb{T}^n$-orbits and thus they are Lagrangian n-tori.

The momentum map on $\mathbb{C}P^1$

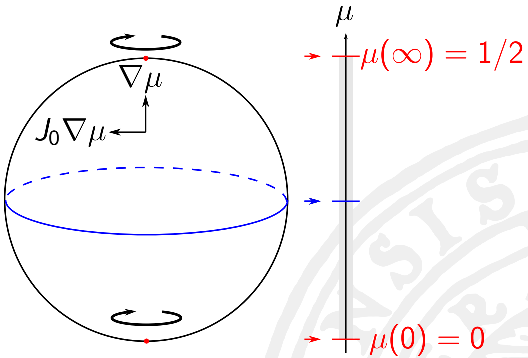

Figure: The image of the momentum map μ of the $U(1)$ action of $\mathbb{C}P^1$. The Hamiltonian action is a rotation of the sphere which fixes the south pole 0 and north pole ∞, turning clockwise when seen from above the north pole. (Here $\mu = \frac{r^2}{2(1+r^2)}$ in affine polar coordinates.)

The momentum map on $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2$

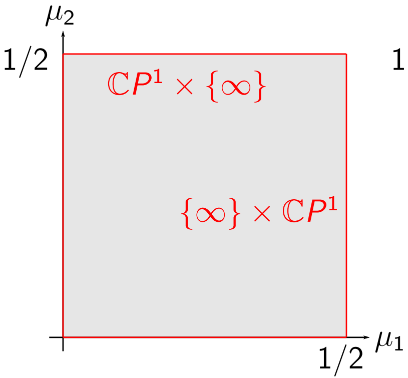

Figure: The image of the momentum map of the $U(1)^2$ action on $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2$ for the standard basis of $T_eU(1)$. In polar Darboux coordinates, the momentum map is given by $\boldsymbol{\mu} = \frac{\sum_i r_i^2}{2}$. (Recall that $(\mathbb{C}P^n \setminus \mathbb{C}P^{n-1}_\infty, \omega_{\mathrm{FS}}) = (B^{2n}, \omega_0)$)

Local model of Lagrangians

We will now spend some time on understanding the local model of a Lagrangian submanifold. We refer to [MS98] for a good introduction to the techniques on which these results are based.

Weinstein's Lagrangian neighbourhood theorem

- Recall that the Darboux theorem implies that a symplectic manifold locally is symplectomorphic to the standard symplectic vector space.
- In other words, there is no interesting local features of a symplectic manifold.
- Similarly, Lagrangians have no interesting local features, since they admit a standard neighbourhood by a classical result [Wei71] due to Weinstein.
- We proceed to describe the local model of a neighbourhood of a Lagrangian $L \subset (X, \omega)$: its cotangent bundle $(T^*L, d\theta_L)$ (more precisely: a neighbourhood of the zero section).

Local model

The cotangent bundle T^*M of any smooth n-dimensional manifold M has a tautological one-form $\theta \in \Omega^1(T^*M)$ defined in the following manner. Let $\pi\colon T^*M \to M$ be the canonical bundle projection. Let $\alpha \in T^*M$ be a point. The point α is itself a one-form on some tangent space $T_{\mathrm{pt}}M$ of M. We can now define:

$$\theta_\alpha\colon T_\alpha(T^*M) \to \mathbb{R},$$
$$\theta_\alpha = \pi^*\alpha$$

Local model

The cotangent bundle T^*M of any smooth n-dimensional manifold M has a tautological one-form $\theta \in \Omega^1(T^*M)$ which in local coordinates $q_i \in M$ takes the form

$$\theta = \sum_i p_i dq_i$$

where p_i are the *canonical conjugate momenta* corresponding to the coordinates q_i. In other words,

$$(q_i, p_i) \mapsto \sum_i p_i dq_i \in T^*M$$

is a local diffeomorphism. Note that the exterior differential

$$d\theta = \sum_i dp_i \wedge dq_i$$

is the linear symplectic form in local coordinates!

Local model

For any smooth n-dimensional manifold M we thus have an exact symplectic $2n$-dimensional manifold $(T^*M, d\theta_M)$ with a canonical choice of primitive θ_M.

- The Liouville vector field is of the form $\zeta_M = \sum_i p_i \partial_{p_i}$ in local coordinates.
- Since the graph of a form $\alpha \in \Omega^1(M)$ is a section $\alpha \colon M \to T^*M$ which satisfies $\alpha^* \theta_M = \alpha$, any graph of a closed (resp. exact) one-form on M is a Lagrangian (resp. exact Lagrangian) section of T^*M. This is why θ_M is called the *tautological* form!

Local model

For any smooth n-dimensional manifold M we thus have an exact symplectic $2n$-dimensional manifold $(T^*M, d\theta_M)$ with a canonical choice of primitive θ_M.

Another aspect of the tautological nature of θ_M:

- Any diffeomorphism $\phi\colon M \to M$ induces a pull-back which itself is a diffeomorphism

$$\Phi := \phi^*\colon T^*M \to T^*M,$$
$$\alpha \mapsto \phi^*\alpha.$$

Lemma

$$\Phi^*\theta_M = \theta_M$$

Local model

Proof.

Recall that at the point $\alpha \in T^*M$ we have

$$\theta_\alpha = \pi^*\alpha$$

by construction. We now compute

$$\begin{aligned}(\Phi^*\theta)_\alpha = & \\ = \; & \Phi^*(\pi^*(\phi^{-1})^*\alpha) \\ = \; & \pi^*(\phi^*((\phi^{-1})^*\alpha)) \\ = \; & \pi^*\alpha = \theta_\alpha.\end{aligned}$$

where we have used that $\pi \circ \Phi = \phi$. □

Local model

Theorem (Weinstein [Wei71], also see [MS98])

For any Lagrangian embedding $L \subset (X, \omega)$ *there exists a neighbourhood (called Weinstein neighbourhood)* $U \supset L$ *and a symplectomorphism*

$$\Phi\colon (U, \omega) \hookrightarrow (T^*L, d\theta_L)$$

such that $\Phi|_L$ *is the canonical inclusion of the zero-section in the cotangent bundle.*

Local deformations

Weinstien's neighbourhood theorem can be used to classify the space of C^∞-small deformations of a Lagrangian submanifold $L \subset (X, \omega)$ *up to Hamiltonian isotopy*.

- The C^∞-small deformations of L are C^∞-small Lagrangian sections sections Γ_α, $\alpha \in \Omega^1(L)$, inside the Weinstein neighbourhood T^*L of L; i.e. $d\alpha = 0$.
- Two such sections Γ_α and $\Gamma_{\alpha'}$ are Hamiltonian isotopic inside the neighbourhood if and only if $\alpha - \alpha' = df$.

Proof.

The Hamiltonian $f \circ \pi \colon T^*L \to \mathbb{R}$ induces a Hamiltonian isotopy which takes

$$\phi^1_{f\circ\pi}(\Gamma_{\alpha'}) = \Gamma_\alpha.$$

□

Local deformations

Weinstien's neighbourhood theorem can be used to classify the space of C^∞-small deformations of a Lagrangian submanifold $L \subset (X, \omega)$ *up to Hamiltonian isotopy*. In other words:

Theorem

The space of C^∞-small Lagrangian perturbations of $L \subset (X, \omega)$ up to Hamiltonian isotopy can be naturally identified with a neighbourhood of the origin $0 \in H^1(L, \mathbb{R})$.

Remark

The *global* classification question, or even the C^0-close classification question, is typically beyond current technology (except for a handful of cases in low dimension).

Example of local model: T^*S^1

The case of the torus $\mathbb{T}^n = (S^1)^n$ is of particular importance:

$$(T^*\mathbb{T}^n, d\theta_M) = \left(\mathbb{C}^n / i2\pi\mathbb{Z}^n, d\sum_i p_i d\theta_i\right),$$

where

$$[y_i] = \theta_i \in S^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

is an angular coordinate and

$$p_i = x_i \in \mathbb{R}$$

is the corresponding "conjugate momentum."

The momentum map on $T^*\mathbb{T}^n$

The cotangent bundle

$$(\mathbb{C}/i2\pi\mathbb{Z} = T^*S^1, d(p, d\theta)), \;\; p = x \in \mathbb{R}, \theta = [y] \in S^1$$

has a Hamiltonian S^1-action which can be seen in either of the following ways:

- the S^1-action on $M = S^1$ by diffemorphism lifted to $T^*M = T^*S^1$,
- a rotation in the "imaginary direction" of $\mathbb{C}/i2\pi\mathbb{Z}$, or
- the Hamiltonian $x = p \colon \mathbb{C}/i2\pi\mathbb{Z} \to \mathbb{R}$.

The momentum map on $T^*\mathbb{T}^n$

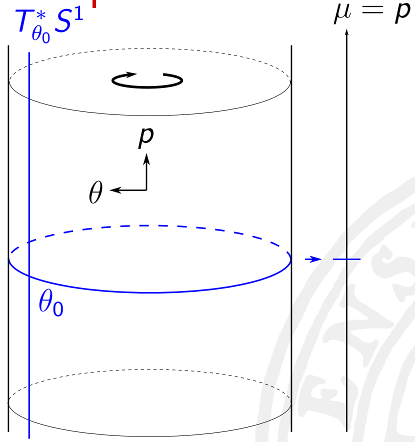

Figure: The image of the momentum map p on T^*S^1. The S^1-action is a rotation of the angular coordinate in the positive direction. The cotangent fibre $T^*_{\theta_0}S^1$ is a Lagrangian section of the momentum map.

The momentum map on $T^*\mathbb{T}^n$

Since

$$(T^*\mathbb{T}^n, d\theta_{\mathbb{T}^n}) = ((T^*S^1)^n, d\theta_{\mathbb{T}^1} \oplus \ldots \oplus d\theta_{\mathbb{T}^1})$$

there is an Hamiltonian $\mathbb{T}^n$-action on $T^*\mathbb{T}^n$ with momentum map

$$\begin{gathered} \boldsymbol{\mu}_{T^*\mathbb{T}^n} \colon T^*\mathbb{T}^n \to \mathbb{R}^n, \\ (p_i = x_i, \theta_i = [y_i]) \mapsto p_i = x_i. \end{gathered}$$

In this case of regular fibres of momentum maps, the local model can be easily identified with the above model, while
preserving the momentum map.

Note the existence of the Lagrangian sections $T^*_{\theta_0}\mathbb{T}^n$ of the momentum map (the cotangent fibres).

Local model of the momentum map

Theorem (Arnol'd–Liouville [Arn89])

Any regular torus fibre $\mu^{-1}(\mathrm{pt}) \subset (X, \omega)$ *of a momentum map on a symplectic manifold* (X, ω) *with a Hamiltonian* $\mathbb{T}^n$*-action has a neighbourhood which is symplectomorphic to a neighbourhood of* $\mu^{-1}_{T^*\mathbb{T}^n}(\mathrm{pt})$, *where the symplectomorphism* ϕ *moreover satisfies* $\mu = \mu_{T^*\mathbb{T}^n} \circ \phi$.

Sketch of proof.

- The momentum map $\boldsymbol{\mu}$ defines n local coordinates on X;
- Find a local Lagr. section of $\boldsymbol{\mu}$, together with the $\mathbb{T}^n$-action we get angular coordinates on the fibres.
- In these coordinates ω becomes $d\theta_{\mathbb{T}^n}$; The obtained local coordinates on (X, ω) is called *action-angle coordinates*.

The moduli space of psh. discs

We have seen that closed pseudoholomorphic curves contain important information about the symplectic manifold. Similarly, pseudoholomorphic curves with boundary on a Lagrangian submanifold contain important information about the Lagrangian submanifolds. For any $A \in H_2(X, L)$ or $A \in \pi_2(X, L)$ we are interested in the moduli space

$$\begin{aligned}\mathcal{M}_J(A) = \\ = \ \{u \colon (D^2, j) \to (X, J); \overline{\partial}_J u = 0, [u] = A\}/\mathrm{Aut}(D^2).\end{aligned}$$

for $J \in \mathcal{J}^{tame}(X, \omega)$.

The moduli space of psh. discs

- The *Uniformisation Theorem* implies that there is a unique closed and simply connected Riemann surface: The Riemann sphere $(\mathbb{C}P^1, j)$.
- In addition, there is a unique simply connected Riemann surface with boundary: The disc (D^2, j).
- $\mathrm{Aut}(\mathbb{C}P^1, j) = \mathbb{P}\mathrm{GL}_{\mathbb{C}}(2)$ is the $\dim_{\mathbb{R}} = 6$ non-compact group of Möbius transformations.
- Since $(D^2, j) \subset (\mathbb{C}P^1, j)$ embeds in $\mathbb{C}P^1$ with boundary $\partial D^2 = \mathbb{R}P^1$. Thus $\mathrm{Aut}(D^2, j) \subset \mathrm{Aut}(\mathbb{C}P^1, j)$ is the $\dim_{\mathbb{R}} = 3$ non-compact group of "real" Möbius transformations.

The Fredholm index

- The linearised problem $D_u\overline{\partial}_J$ is again Fredholm (for a suiable functional-analytic setup).
- The Fredholm index is given by

$$\text{index}(u) = \dim_{\mathbb{R}} \ker D_u\overline{\partial}_J - \dim_{\mathbb{R}} \operatorname{coker} D_u\overline{\partial}_J = n \cdot \chi(D^2) + \mu_L[A]$$

where $\mu_L \colon H_2(X, L) \to \mathbb{Z}$ is the *Maslov class* (see below).

- The expected (virtual) dimension of the moduli-space is

$$v\dim(u) = \text{index}(u) - \dim_{\mathbb{R}} \mathrm{Aut}(D^2) = \text{index}(u) - 3 = n - 3 + \mu_L[A].$$

The Fredholm index

- The boundary-value problem should be thought of as the *real-part* of the complex moduli space.
- This is literally true when L is the fixed-point locus of an anti-symplectic and anti-holomorphic involution

$$I\colon (X,\omega,J) \to (X,-\omega,-J).$$

- Namely: A J-holomorphic disc u with boundary on the fixed-point locus of L can be *doubled* to a pseudoholomorphic sphere $u^{dbl}\colon \mathbb{C}P^1 \to X$ which satisfies
  - $u^{dbl}(z) = u(z)$ for $z \in D^2 \subset \mathbb{C}P^1$, and
  - $I \circ u^{dbl}(z) = u^{dbl}(\overline{z})$
- However, not all deformations of u^{dbl} remain conjugation-invariant: only the *real deformations*.

The Fredholm index

- The real part of a complex space is half-dimensional,so the index of a disc u on the fixed point locus should be half of the index of the index of the doubled sphere u^{dbl},i.e.

$$\text{index}(u) = \frac{1}{2}\,\text{index}(u^{dbl}) = \frac{1}{2}(n\chi(\mathbb{C}P^1) + 2c_1^{TX}[u^{dbl}]).$$

- This yields the sought formula if we set $\boxed{\mu_L[u] = c_1^{TX}[u^{dbl}]}$ where v is the double. This is indeed the case!

Discs with boundary on $\mathbb{R}P^n$

- A real line $\mathbb{R}P^1 \subset \mathbb{R}P^n \subset \mathbb{C}P^n$ can be realised as a complex line

$$[x_1 : x_2] \to x_1 \cdot P_1 + x_2 \cdot P_2 \in \mathbb{C}P^n, \;\; P_1 \neq P_2 \in \mathbb{R}P^n,$$

  (its complexification) intersected with the Lagrangian $\mathbb{R}P^n \subset \mathbb{C}P^n$.
- The complex line $\mathbb{C}P^1 \subset \mathbb{C}P^n$ lives in a moduli space of virtual dimension

$$v\dim = n2 + 2(n+1) - 6.$$

  (Recall: $c_1^{T\mathbb{C}P^n}(L) = n+1$ where L is the line class.)
- The discs given by either side of $\mathbb{C}P^1 \setminus \mathbb{R}P^n$ lives in a moduli space of virtual dimension

$$v\dim = n + (n+1) - 3 = 2n - 2.$$

Real lines

There are real J_0-holomorphic lines $\mathbb{C}P^1 \to \mathbb{C}P^n$ through every pair of points on $\mathbb{R}P^n$. Each real line splits into *two* J_0-holomorphic discs with coinciding boundaries, equal to a *real* line $\mathbb{R}P^1 \subset \mathbb{R}P^n$.

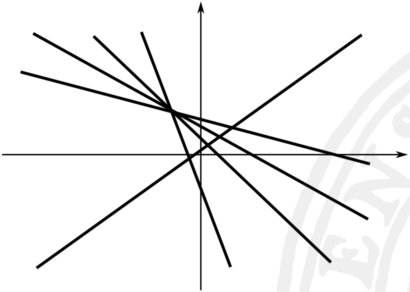

Figure: Real lines $\mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \mathbb{C}P^2$. Each is a boundary of two J_0-hol. disc in a moduli space of virtual dimension is $2n-2=2$. The two discs join to form a complex line.

The Maslov class

Assume that we are given a continuous map of Riemann surface $u\colon (\Sigma, \partial\Sigma) \to (X, L)$ with boundary on a Lagrangian $L \subset (X, \omega)$.

1. Double Σ to a closed Riemann-surface $\Sigma^{dbl} = \Sigma \cup \overline{\Sigma}$ with an anti-holomorphic (orientation-reversing suffices) involution
$$I\colon \Sigma^{dbl} \to \Sigma^{dbl}$$
whose fixed point locus is exactly $\partial\Sigma \subset \Sigma^{dbl}$.
2. Extend the complex vector bundle $u^*TX \to \Sigma$ to $E \to \Sigma^{dbl}$ so that I lifts to an anti-complex involution of E which fixes the real sub-bundle
$$u^*TL \subset u^*TX|_{\partial\Sigma} = E|_{\partial\Sigma} \to \partial\Sigma.$$
3. Define the *Maslov index* $\mu_L[u] := c_1^E$ of $[u]$ as the first Chern number of $E \to \Sigma^{dbl}$.

The Maslov class

Assume that we are given a continuous map of Riemann surface with boundary $u\colon (\Sigma, \partial\Sigma) \to (X, L)$. An equivalent definition of $\mu_L[u]$ is the following:

1. $u^*TX \to \Sigma$ is a $\mathbb{C}$-vector bundle (trivial since Σ is a surface with non-empty boundary).
2. $\det_{\mathbb{C}} u^*TX = (u^*TX)^{\wedge_{\mathbb{C}} \dim_{\mathbb{C}} TX} \to \Sigma$ is a (trivial) $\mathbb{C}$-bundle.
3. $\det_{\mathbb{R}} u^*TX \to \partial\Sigma$ is a $\mathbb{R}$-bundle bundle.
4. If the latter bundle is *trivial* (the same as *orientable* since $\dim_{\mathbb{R}}(\partial\Sigma) = 1$), then $\mu_L[u] = 2\,\mathrm{wind}(\sigma)$ where σ is a non-zero section of $\det_{\mathbb{R}} u^*TX \to \partial\Sigma$. (The winding number is computed in the given trivialisation.)
5. The latter bundle is not always trivial. An *odd* Maslov nr. is equivalent to TL being *non*-orientable along $\partial\Sigma$.

The Maslov class relative a trivialisation

Remark

- The above definition of the Maslov index only uses the trivialisation of $\det_{\mathbb{C}} u^* TX|_{\partial \Sigma}$.
- Given a *choice* of trivialisation of $\det_{\mathbb{C}} TX$, one can define a Maslov class

$$\mu_L \colon H_1(L) \to \mathbb{Z}$$

which coincides with the previous definition on the image of the connecting homomorphism

$$\delta \colon H_2(X, L) \to H_1(L).$$

Example: Product tori

Example

- An embedded curve $\gamma \subset \mathbb{C}P^1$ bounds two *embedded* holomorphic discs: u and v. The Maslov index is invariant up to homotopy, and since $\mathbb{R}P^1$ is the real part we compute

$$\mu_\gamma(u) = \mu_\gamma(v) = c_1^{T\mathbb{C}P^1}[\mathbb{C}P^1] = \chi(\mathbb{C}P^1) = 2.$$

- Similarly, $c_1^{T(\mathbb{C}P^1)^n}[\{0\}^{i-1} \times \mathbb{C}P^1 \times \{0\}^{n-i}] = 2$ implies

$$\mu_{(\mathbb{R}P^1)^n}[\{0\}^{i-1} \times u \times \{0\}^{n-i}] = 2$$

as well as

$$\mu_{(\mathbb{R}P^1)^n}[\{0\}^{i-1} \times v \times \{0\}^{n-i}] = 2.$$

A basis of holomorphic discs

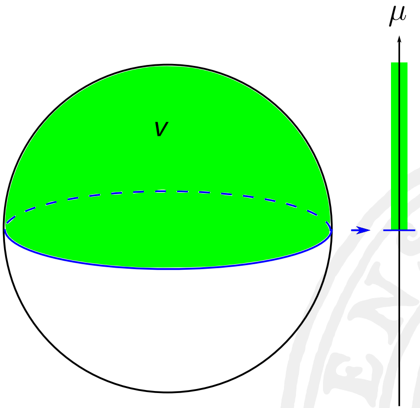

Figure: A pseudoholomorphic disc v inside $\mathbb{C}P^1$ with Maslov index two with boundary on a fiber of the momentum map μ.

A basis of holomorphic discs

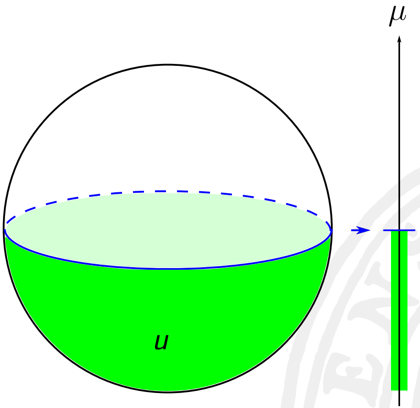

Figure: A pseudoholomorphic disc u inside $\mathbb{C}P^1$ with Maslov index two with boundary on a fiber of the momentum map μ.

A basis of holomorphic discs

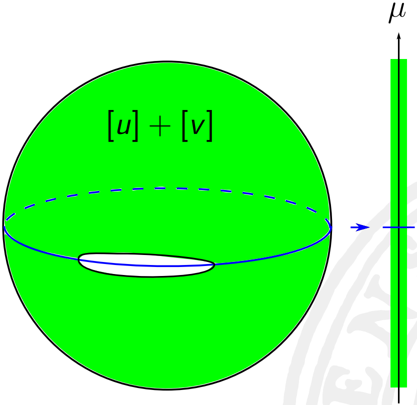

Figure: A pseudoholomorphic disc inside $\mathbb{C}P^1$ of Maslov index four with boundary on a fiber of the momentum map μ, which lives in the homology class $[u]+[v] \in H_2(\mathbb{C}P^1, \mu^{-1}\mathrm{pt})$.

Example: Product tori

Hence: For product tori

$$L = \gamma_1 \times \ldots \times \gamma_n \subset (\mathbb{C}P^1)^n = X$$

(e.g. the fibres of the standard momentum map
$\boldsymbol{\mu}\colon (\mathbb{C}P^1)^n \to [0, 1/2]^n$)
there is a basis of $H_2(X, L) = \mathbb{Z}^{2n}$ which can be represented by embedded J_0-holomorphic discs u_i, v_i, $i = 1, \ldots, n$, and for which

$$\mu_L[u_i] = \mu_L[v_i] = 2.$$

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