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Holomorphic Curve Theories in Symplectic Geometry

Lecture VII

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Goal of lecture

Today:

- More about Floer homology in different geometric settings.
- Moduli spaces of discs with boundary punctures and operations in Floer homology.
- Application: Continuation Elements.

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Take-home Message

If a closed Lagrangian can be displaced by a Hamiltonian isotopy, then it admits a non-constant pseudoholomorphic disc.

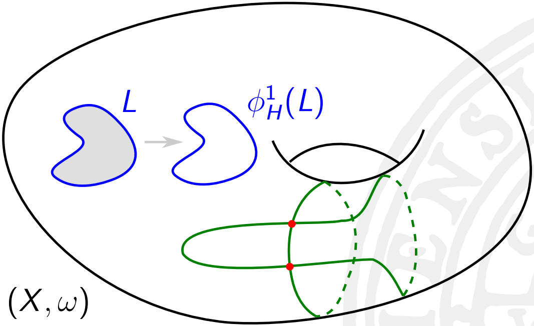

Figure: The blue Lagrangian L is displaceable by a Hamiltonian isotopy and bounds a holomorphic disc; the green Lagrangian which is homologically essential is not Hamiltonian displaceable.

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Plan

Discs with boundary punctures

Later today we will consider a general moduli space of smooth pseudoholomorphic maps from the disc with

- a finite number of boundary punctures in some arbitrary position,
- a boundary condition in certain Lagrangian submanifolds on each boundary arc between the punctures, and
- punctures mapping to transverse intersection points of these Lagrangians.

In particular, such a disc has a finite symplectic area.

Discs with boundary punctures

Last lecture we only say examples when the disc has two boundary punctures; A disc with two boundary points removed is biholomorphic to the strip (we will see this later in today's lecture).

Recall:

- The differential counts strips (discs with two boundary punctures) of index one;
- The continuation map counts trips (discs with two boundary punctures) of index zero.

Discs with boundary punctures

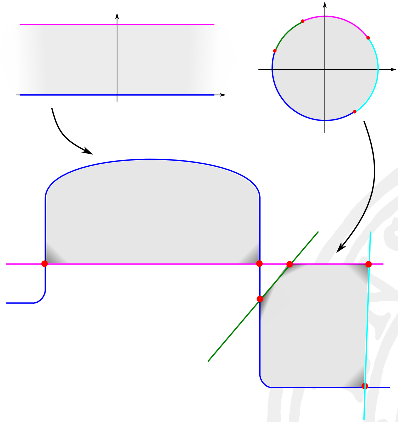


Figure: Pseudoholomorphic discs with two and four boundary punctures. Recall that a strip is biholomorphic to a disc with two boundary punctures removed.

Discs with boundary punctures

It is important to spell out the precise definition of the index in this case:

By the *index* we mean the expected dimension of the solution space where:

- we consider the moduli space of *maps* (i.e. not the quotient "maps modulo reparametrisation"),
- if there are $d+1$ boundary punctures, then we fix positions of the first three (or any other choice) and allow only the position of the remaining $d-2$ punctures to vary. (In the case $d \leq 2$ this is means that the position of all punctures are fixed.)

Discs with boundary punctures

Recall: $\mathrm{Aut}(D^2)$ is three-dimensional and each element is uniquely determined by its value on three distinct boundary points.

Remark

Assume that we have three or more boundary punctures. Then

- Reparametrisations do not contribute to the index, since the identity is the unique automorphism which fixes the three boundary punctures which are required to remain fixed in the definition of the index; and hence
- The moduli space of discs as above with *three or more* boundary punctures, whose positions all are allowed to vary freely, has the same expected dimension as the above index after taking the quotient by reparametrisation.

The two-dimensional case

- When $\dim_{\mathbb{R}} X = 2n = 2$ (i.e. $n = 1$ and the Lagrangian is one-dimensional) the index, as well as the transversality question (i.e. whether the cokernel vanishes or not), is easy to determine in view of the below proposition.

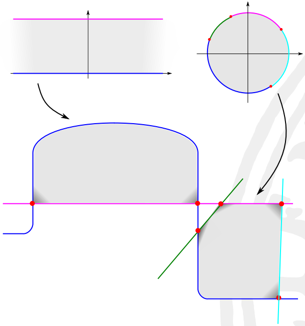


Figure: In the two-dimensional case this is an honest figure.

Strips and disc in dimension two

Proposition (Folklore)

Consider the moduli space of discs with boundary on pairwise transverse one-dim. $L_0, L_1, L_2, \ldots$*. Any holomorphic strip u (resp. disc with more than three punctures) has index one (resp. zero) if and only if*

- *It is an immersion everywhere away from the punctures (in particular, there are no branched points in the interior, nor along the boundary away from the punctures); and*
- *Near a boundary puncture* p_i*, the interior of the disc is an embedding into the complement of* $\cup_i L_i$ *(in other words, it covers a single quadrant near the corr. intersection).*

In these cases, the solution is moreover transversely cut out.

The two-dimensional case

- Conversely, it is easy to see that any immersion of the above form is the image of a holomorphic disc; Roughly: pull back the almost complex structure on the target space to the domain.
- **In conclusion:** In dimension $\dim_{\mathbb{R}} X = 2$ the pseudoholomorphic maps of the above form can be found purely by combinatorial means.

The two-dimensional case

Remark

Assume that $p \in L_0 \cap L_1$ is a transverse intersection of two n-dimensional Lagrangians.

- A disc with $2k$ boundary punctures which maps constantly into p, with an alternating condition on $L_0, L_1, L_0, L_1, \ldots$, has index equal to $n + 2k - nk - 3$ (the expected dimension after the quotient by reparam.)
- However, we know these solutions explicitly, and the moduli space is clearly equal to $\dim \mathcal{R}_{2k-1} = 2k - 3$ when $k \geq 2$ (regardless of the dimension of n).

This is the reason why we require the Lagrangians $L_0, L_1, L_2, \ldots$ to be *pairwise* transverse.

The two-dimensional case

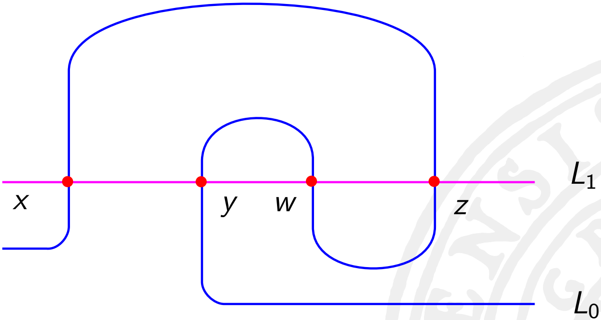
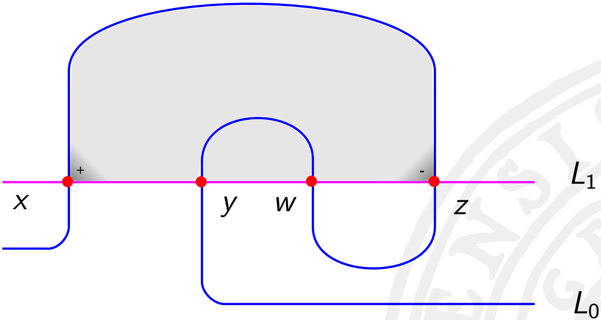


Figure: The Floer complex $CF(L_0, L_1)$: $\partial(x) = y + z$, $\partial(y) = w$, $\partial(z) = w$

The two-dimensional case



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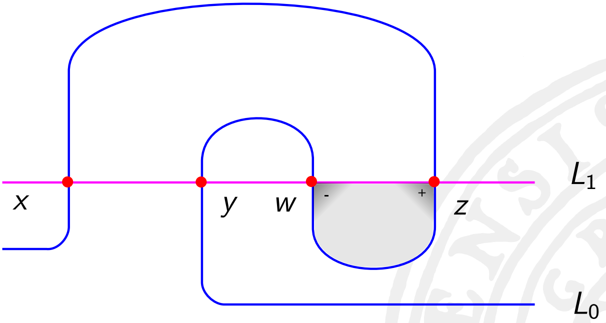

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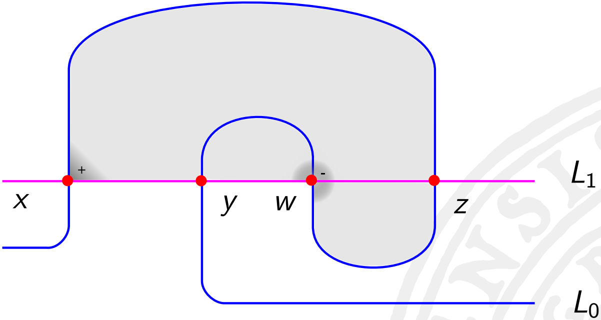


Figure: A strip with “input” at x and “output” at w. It is not rigid, since the corner at w covers more than one quadrant; instead, it lives in a one-dimensional moduli space.

The two-dimensional case

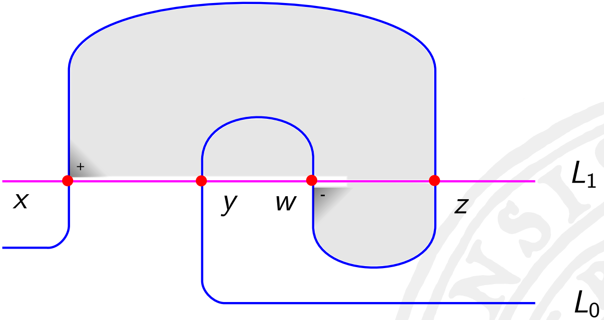
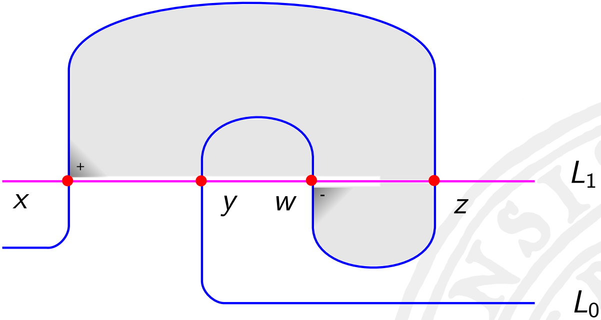


Figure: A strip with “input” at x and “output” at w which comes in a one-parameter family. We start to approach a nodal strip which contributes to $\partial^2(x) = \mathbf{w} - w = 0$. There is a branched point at the boundary (the strip is not immersed there).

The two-dimensional case



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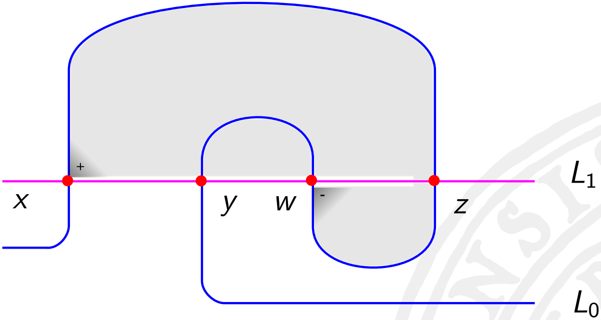

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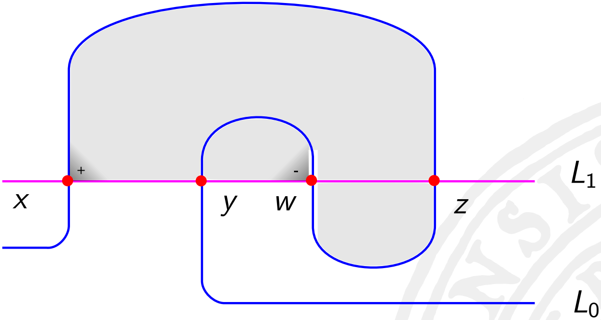

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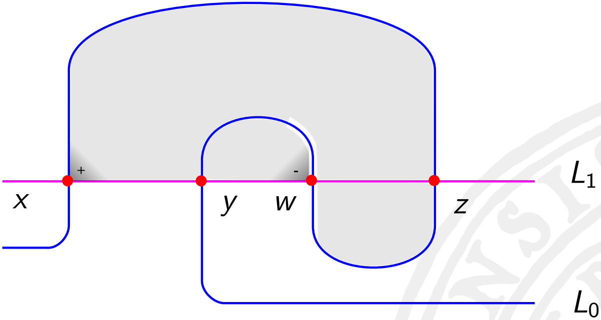

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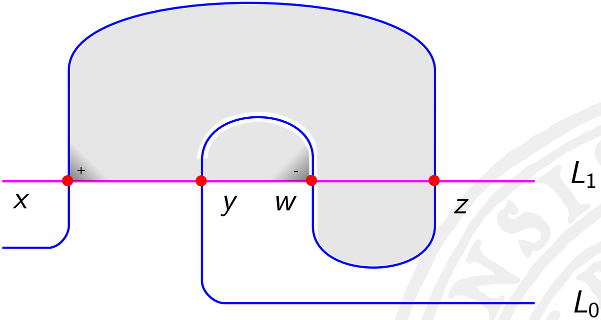

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Floer homology in the non-exact case

The setting of exact Lagrangians is very limited; e.g. the ambient manifold cannot be closed.
The two main reasons why Floer homology in the non-exact case is badly behaved:

1. Divergent sums when counting strips (or other moduli spaces); This happens when e.g. a moduli space $\mathcal{M}(x, y)$ for fixed x, y (differential, continuation, homotopy, etc.) does not satisfy an a priori area bound; **Solution:** Coefficients in the Novikov field.
2. Bubbling of pseudoholomorphic discs with boundary on L; this is sometimes an actual obstruction to defining Floer homology.

Floer homology in the non-exact case

These two other settings are important cases where Floer homology works well:

- Weakly exact Lagrangians:

$$\int_\alpha \omega = 0 \text{ for all } \alpha \in \pi_2(X, L)$$

- Monotone Lagrangians; e.g. the Clifford torus from [Lecture 5].

Floer homology in the weakly exact case

Unlike in the exact case, the *weakly* exact condition does not imply that the symplectic energy of strips in $\mathcal{M}(x, y)$ only depends on x,y. The difference of two strips in $\mathcal{M}(x, y)$, thought of as continuous chains, can be considered as a continuous *annulus* with boundary on $L_0 \cup L_1$:

Two strips in $\mathcal{M}(x, y)$ have a symplectic area which differ by the symplectic area of some continuous annulus with boundary on $L_0 \cup L_1$.

In other words

In the weakly exact case, Novikov coefficients must be used unless all annuli as above have vanishing symplectic area.

Floer homology in the weakly exact case

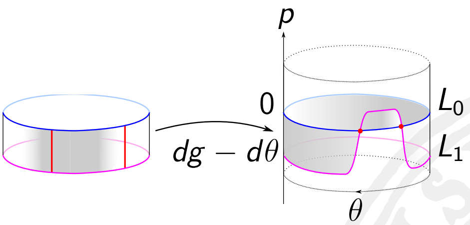


Figure: The difference of two strips can be considered to be a continuous *annulus* with boundary on the pair L_0, L_1. For the Lagrangians $L_0 \cup L_1 \subset T^*S^1$ in the figure, these annuli are of area $k2\pi$, $k \in \mathbb{Z}$. The area of a strip in $\mathcal{M}(x, y)$ is determined by x, y only modulo 2π.

Weakly exact example

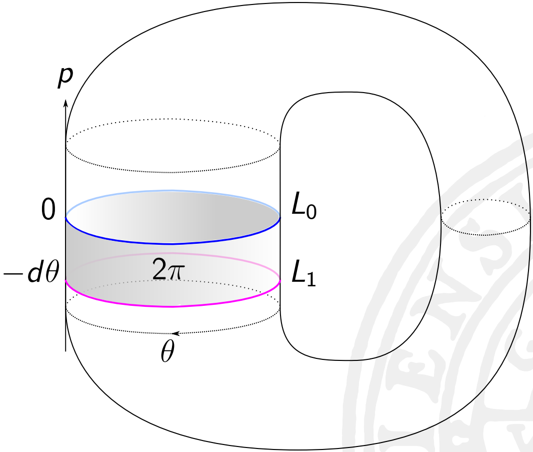


Figure: Two disjoint embedded weakly exact Lagranginans in $\mathbb{T}^2$ bound many annuli of nonzero symplectic area; the one in the figure has area 2π.

Weakly exact example

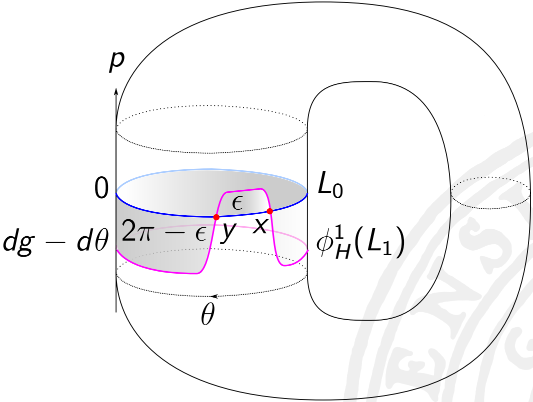


Figure: After a Hamiltonian isotopy the two disjoint weakly exact Lagrangians intersect in two points. There are precisely two rigid Floer strips, one of area $2\pi - \epsilon$ and one of area ϵ, both contribute to the coefficient $\langle \partial(x), y \rangle$.

Weakly exact example

In the above example:

- Clearly
$$CF(L_0, L_1) = \{0\}$$
since the Lagrangians are disjoint there are no generators.
- However,
$$CF(L_0, \phi^1_H(L_1)) = \mathbb{Z}_2 x \oplus \mathbb{Z}_2 y$$
with trivial differential when $\mathbb{Z}_2$-coefficients are used (there are two rigid Floer strips that contribute to $\partial(x)$).
- This contradicts invariance under Hamiltonian isotopy! (First complex is acyclic, second is not!)

Weakly exact example

With Novikov coefficients

$$\Lambda^{\mathbb{Z}_2} := \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i}, \;\; a_i \in \mathbb{Z}_2, \lambda_i \in \mathbb{R}, \lim_{i \to +\infty} \lambda_i = +\infty \right\}.$$

we get the correct answer: The complex

$$CF(L_0, \phi_H^1(L_1)) = \Lambda^{\mathbb{Z}_2} x \oplus \Lambda^{\mathbb{Z}_2} y$$

with differential

$$\partial(x) = (T^{2\pi - \epsilon} + T^{\epsilon}) y$$

is *acyclic*.

Remark

$$(T^{2\pi-\epsilon} + T^{\epsilon})^{-1} = T^{-\epsilon} \sum_{i=0}^{\infty} T^{i(2\pi - 2\epsilon)} \in \Lambda^{\mathbb{Z}_2}$$

Exercise

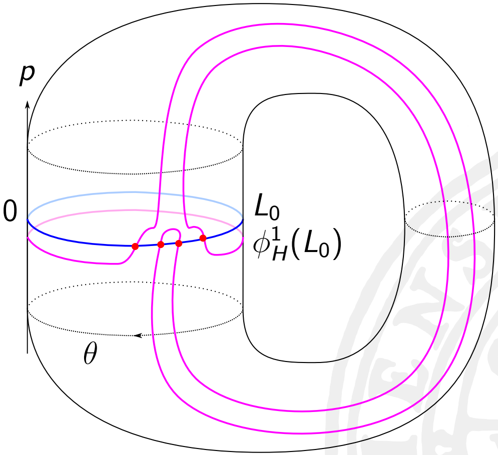


Figure: Compute $CF(L_0, \phi^1_H(L_0))$ for the above pair of Hamiltonian isotopic curves on the two-dimensional symplectic torus.

Floer homology in the monotone case

Definition

We say that a Lagrangian submanifold $L \subset (X, \omega)$ is *monotone* if there exists a constant $c \geq 0$ so that

$$\int_{[u]} \omega = c\mu^L[u]$$

is satisfied.

Floer homology in the monotone case

In the case of monotone Lagrangians whose minimal positive Maslov index is $N \geq 3$:

- $CF(L, \phi_H^1(L))$ is a well-defined complex with coefficients in $\mathbb{Z}_2$ (non-orientable case) or $\mathbb{Q}$ (orientable case); this complex has a $\mathbb{Z}_N$-grading and differential of degree $-1 \in \mathbb{Z}_N$.
- Monotonicity does not imply that annuli have vanishing symplectic area, but when L_0 and L_1 are Hamiltonian isotopic, the Maslov index of such an annulus is proportional to the area. Hence, the strips in $\mathcal{M}(x, y)$ of the same index also have the same area.

Floer homology in the monotone case

In the case of monotone Lagrangians whose minimal positive Maslov index is $N \geq 3$:

- Nevertheless, the complex $CF(L, \phi^1_H(L))$ might be acyclic! (The unit in degree zero is possibly killed by something in degree $1 \in \mathbb{Z}_N$.)
- For two monotone Lagrangians which are not Hamiltonian isotopic, $CF(L_0, L_1)$ can *in general* only be defined with Novikov coefficients (even when their minimal Maslov numbers coincide).

Floer homology in the monotone case

Example

- The Lagrangian sphere

$$\{(z, \bar{z})\} \subset (\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{\mathrm{FS}} \oplus \omega_{\mathrm{FS}})$$

  has minimal positive Maslov index

$$N = 2 \cdot c_1^{T\mathbb{C}P^1}[\mathbb{C}P^1] = 2 \cdot 2 = 4.$$

- The standard Lagrangian projective plane

$$\mathbb{R}P^n \subset (\mathbb{C}P^n, \omega_{\mathrm{FS}})$$

  has minimal positive Maslov index $N = n + 2$.

Floer homology in the monotone case

In the case of monotone Lagrangians whose minimal positive Maslov index is $N = 2$:

- $CF(L, \phi^1_H(L))$ is a complex with coefficients in $\mathbb{Z}_2$ (non-orientable case) or $\mathbb{Q}$ (orientable case); the complex has a $\mathbb{Z}_2$-grading.
- There is a possibility of bubbling, but a non-trivial argument due to Oh shows that $\partial^2 = 0$ still holds in this case; see [FOOO09, Section 3.6].
- $CF(L_0, L_1)$ is not necessarily a complex, even with Novikov coefficients, due to the possibility of bubbles.

Floer homology in the monotone case

Example

The Clifford tori

- $$(\mathbb{R}P^1)^n = \boldsymbol{\mu}^{-1}(1/4, \ldots, 1/4) \subset (\mathbb{C}P^1)^n$$

  as well as

- $$\boldsymbol{\mu}^{-1}(1/(2(n+1)), \ldots, 1/(2(n+1))) \subset (\mathbb{C}P^n, \omega_{\mathsf{FS}})$$

have minimal positive Maslov index $N = 2$.

The Floer complex: A monotone example

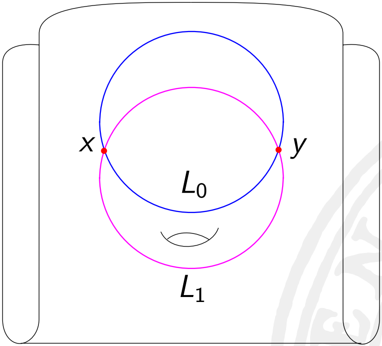


Figure: A monotone Lagrangian L_0 of minimal Maslov index 2 and a weakly exact & monotone Lagrangian L_1 inside the punctured torus.

The Floer complex: A monotone example

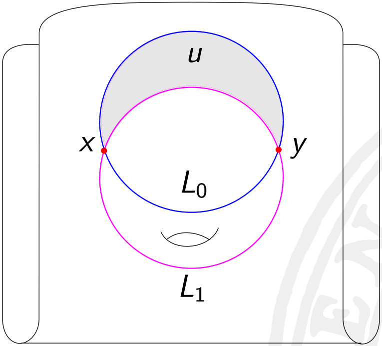


Figure: The single strip u which contributes to $\partial(x) = y$

The Floer complex: A monotone example

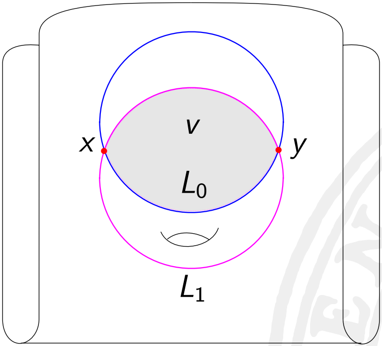


Figure: The single strip v which contributes to $\partial(y) = x$

The Floer complex: A monotone example

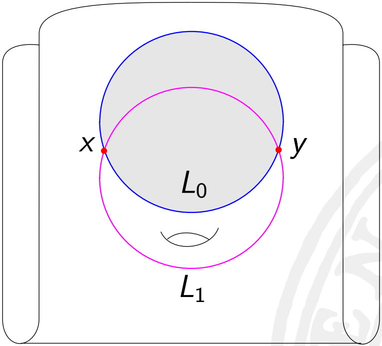


Figure: $\partial^2(x) = x \neq 0$ implies the existence of bubbles: This is a pseudoholomorphic disc of Maslov index two with boundary on L_0.

The Floer complex: A monotone example

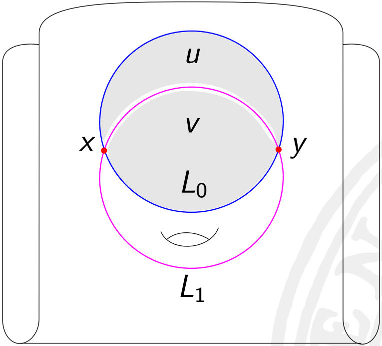
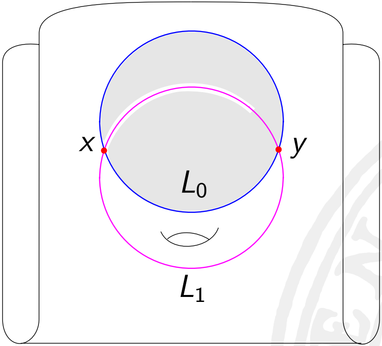


Figure: The two strips u and v that contribute to $\partial^2(x) = x$ can be glued to form a strip which lives in a one-dimensional moduli space.

The Floer complex: A monotone example



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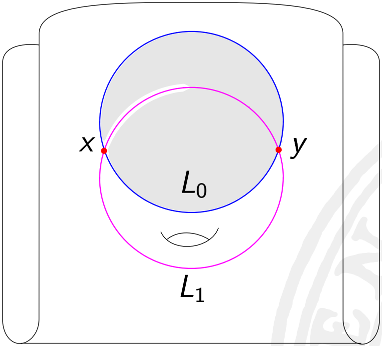

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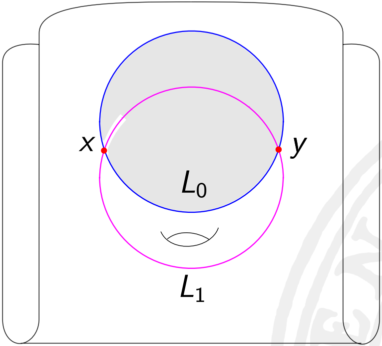

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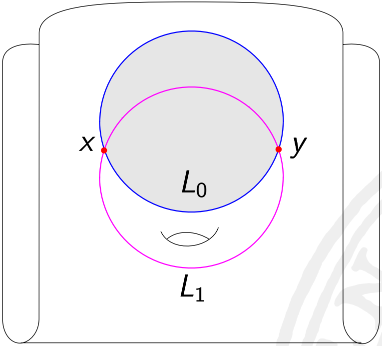


Figure: One boundary point of the moduli space is a constant strip attached to a non-constant disc of Maslov index two. This boundary point of the moduli space is *not* a configurations that contributes to $\partial^2(x)$; this explains why $\partial^2(x) \neq 0$.

Moduli space of punctured discs

Today we will need to consider a general moduli space which combines the moduli spaces that we got acquainted in the previous lecture, i.e.:

- The moduli space of Floer strips with boundary on $L_0 \cup L_1$;
- The moduli space of continuation strips with the *moving boundary condition* $L_0 \cup \phi_H^{-s}(L_1)$ (s parametrises a point in the domain!)
- The moduli space $\mathcal{R}_d$ of $d+1$ boundary punctures on D^2 up to reparam.

Strips and discs

First, recall that the strip is biholomorphic to the disc:

$$\{x+iy;\ \ y\in[0,1]\}\xrightarrow{\cong}\mathbb{H}\setminus\{0\}=\{x+iy;\ \ y\geq 0\}\setminus\{0\},$$
$$z\mapsto e^{\pi z}$$

is a biholomorphism to the upper half-plane, which in turn is mapped by

$$\mathbb{H}\setminus\{0\}\xrightarrow{\cong}D^2\setminus\{-1,1\}$$
$$z\mapsto\frac{z-i}{z+i}$$

biholomorphically to the disc. The composition is

$$\psi\colon\{x+iy;\ \ y\in[0,1]\}\xrightarrow{\cong}D^2\setminus\{-1,1\}$$
$$z\mapsto\frac{e^{\pi z}-i}{e^{\pi z}+i}$$

Strips and discs

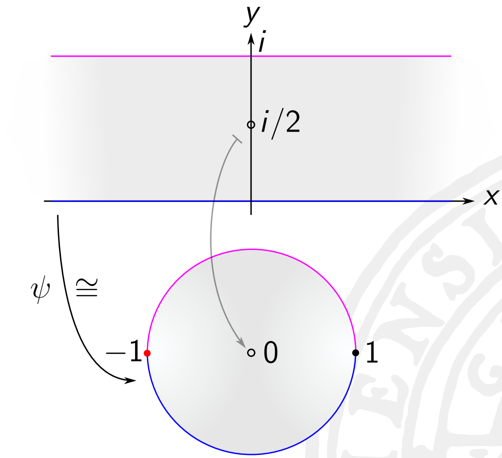


Figure: A biholomorphism from the strip $\{\Im\mathfrak{m} z \in [0,1]\}$ to D^2. Uniquely determined by its values at the points $-\infty, +\infty, i/2$!

Strips and discs

The inverse ψ^{-1} is given the composition

$$D^2 \setminus \{-1, 1\} \xrightarrow{\cong} \mathbb{H} \setminus \{0\},$$
$$z \mapsto \frac{z+1}{z-1}$$

composed with

$$\mathbb{H} \setminus \{0\} \xrightarrow{\cong} \{x + iy; \;\; y \in [0,1]\},$$
$$z \mapsto \frac{1}{\pi} \log z$$

i.e.

$$z \mapsto \psi^{-1}(z) = \frac{\pi^{-1} \log z + 1}{\pi^{-1} \log z - 1}$$

Strips and discs

In other words, instead of considering maps

$$u(s+it) \in (X,\omega),\;\; u(s+ij) \in L_j, j \in \{0,1\},$$
$$du(\partial_t) = J_t du(\partial_s)$$

we can consder maps

$$u(s+it) \in (X,\omega),\;\; u(e^{i(-1)^{j+1}\theta}) \in L_j, j \in \{0,1\},$$
$$du(\partial_t) = J_{\mathfrak{Im}\psi^{-1}(z)} du(\partial_s)$$

Strips and discs

Conversely, near every point $p = e^{i\theta} \in \partial D^2$ we can use the above biholomorphism ψ precomposed with a suitable rotation to obtain the coordinates

$$s + it \mapsto e^{i\theta}\psi(s + it)$$

where $s + it \in \{t \in [0,1]\} \subset \mathbb{C}^2$ are coordinates on the strip, and the point p thus corresponds to the limit as $s \to +\infty$.

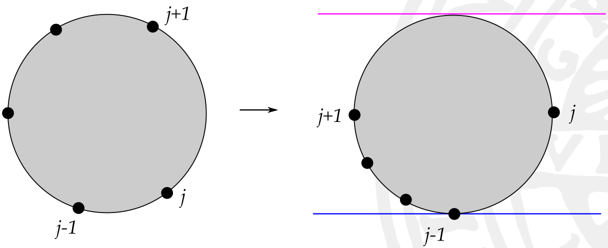
- We say that these coordinates are choices of *strip-like ends at* $p \in \partial D^2$.
- There are ambiguities in the choice of a striplike end:
  - Translation of the coordinate s in the domain.
  - Choice of point $p' \in \partial D^2 \setminus \{p\}$ to which the coordinates are asymptotic as $s \to -\infty$.

Strips and discs

The following immediate result is useful for specifying strip-like ends:

Lemma

Three consecutive boundary punctures p_{j-1}, p_j, p_{j+1} gives rise to a unique strip-like end on the disc for the puncture p_j: we can send e.g. p_{j-1} to the origin in the strip, p_j to $s = +\infty$, and p_{j+1} to $s = -\infty$.



Moduli space of punctured discs

The general moduli space of punctured discs will be denoted by $\mathcal{M}_{\mathbf{J}}(X, \mathbf{L})$ and depends on *several* choices of data.
Roughly speaking it consists of pseudoholomorphic discs *modulo reparametrisation* where:

- The boundary has at least three boundary punctures,
- Varying boundary conditions at a collection **L** of *families* of Lagrangians induced by Hamilonian isotopies,
- Asymptotics to intersection points at *some* of its boundary punctures.

In order to pin-point the above conditions, one must work on the level of choices of *representatives* of $\mathcal{R}_d$, but remembering that reparametrisation must *preserve the conditions*. (So that we can pass to a quotient.)

General moduli space

The moduli space consists of *equivalence classes* (up to reparametrisation) of

- Smooth maps
$$u\colon D^2 \to (X,\omega)$$
together with a configuration $\{p_0,\ldots,p_d\} \subset \partial D^2$ of $d+1 \geq 3$ disjoint boundary punctures.
- We choose an almost complex structure $J_{z,r}$ on X which depends on both the point in the domain $z \in D^2$ and on the configuration r of the boundary punctures, and we require that
$$du \circ j = J_{z,r} \circ du$$
as well as the invariance $J_{\phi(z),\phi(r)} = J_{z,r}$ with respect to reparam.
- Since $d+1 \geq 3$ there are canonically identified strip-like ends near each boundary puncture p_j; we choose $J_{z,r}$ to be t-dependent near the punctures with respect to these coord.

General moduli space

We also choose $d+1$ Hamiltonian isotopies of Lagrangian submanifolds

$$\phi^s_{H_{(0),r}}(L_0), \phi^s_{H_{(1),r}}(L_1), \phi^s_{H_{(2),r}}(L_2), \ldots, \phi^s_{H_{(d),r}}(L_d)$$

where

- $H_{(i),r}$ is a Hamiltonian which depends on $r \in \mathcal{R}_d$; we allow both non-autonomous Hamiltonians, but also constant ones,
- $\phi^s_{H_{(i),r}}(L_i)$ is constant outside of $s \in [0,1]$,
- The Lagrangian boundary conditions $\phi^\alpha_{H_{(i),r}}(L_i)$ for different $\alpha = 0, 1$ and i is a collection of Lagrangians in which pairs either *coincide* or *intersects transversely*.

Strips and discs

We then specify the boundary condition by the requirement that:

- On the j:th boundary arc $e^{i\theta}$ at the angle $\theta \in [\arg p_j, \arg p_{j+1}]$ we have the *varying boundary condition*

$$u(e^{i\theta}) \in \phi_{H_{(j),r}}^{\frac{\theta - \arg p_j}{\theta - \arg p_{j+1}}}(L_j).$$

  This is constructed by careful choices of representatives of $r \in \mathcal{R}_d$, and then extended to be invariant under reparam.

- The map u is asymptotic to some intersection $L_j \cap L_{j+1}$ at p_j if the boundary condition is discontinuous there.

Strips and discs

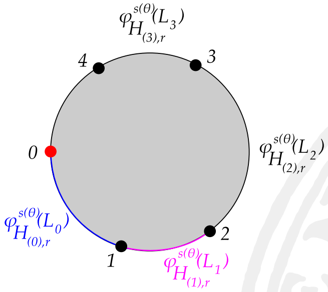


Figure: A disc in the interior of the moduli space

Gromov's compactness theorem

Theorem (Seidel [Sei08])

For a suitable smooth extension of the boundary value conditions over the compactifications $\overline{\mathcal{R}}_d$, the components of the aforementioned moduli space that satisfies a fixed bound on the symplectic energy $\int_u \omega$ is compact in the Gromov sense, where a nodal limit has components that are either:

- *Discs with at least three boundary punctures including nodes (i.e. solutions in the interior of a moduli space of the same type), or*
- *Non-constant discs with two boundary punctures with locally constant boundary condition (e.g. "Floer strips"), or*
- *Non-constant disc/sphere bubbles with a constant boundary conditions and a single node.*

Gromov's compactness theorem

Limits to a nodal disc can occur in two different ways, depending on the underlying limit $r_\infty \in \overline{\mathcal{R}}_d$ of the configuration of boundary punctures:

- If $r_\infty \in \mathcal{R}_d$ is contained in the interior. In this case a disc or sphere bubbled off just as in the previous version of Gromov compactness: due to a blow-up of the gradient. (The disc bubble can also carry with it a single boundary puncture, but only in the case when the boundary condition does not switch Lagrangians there.)
- If $r_\infty \in \mathcal{R}_d$ is contained in the interior, it is also possible that a "Floer strip" has broken off; this is a disc bubble that (except for the node) also captures precisely one of the boundary punctures in the limit where the boundary condition makes a jump;

Gromov's compactness theorem

Limits to a nodal disc can occur in two different ways, depending on the underlying limit $r_\infty \in \overline{\mathcal{R}}_d$ of the configuration of boundary punctures:

- If $r_\infty \in \overline{\mathcal{R}}_d \setminus \mathcal{R}_d$ is contained in the boundary. In this case a blow-up of the gradient may or may not have happend, but due to the nature of the compactification of $\mathcal{R}_d$, we get a nodal limit nevertheless.

Strips and discs

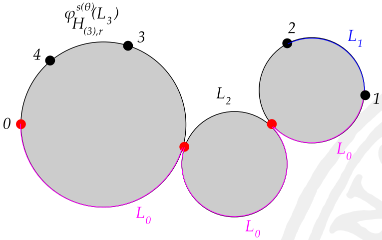


Figure: A limit configuration in which the component in the middle is a non-constant Floer strip. This configuration lives in the boundary of $\overline{\mathcal{R}}_d$.

Strips and discs

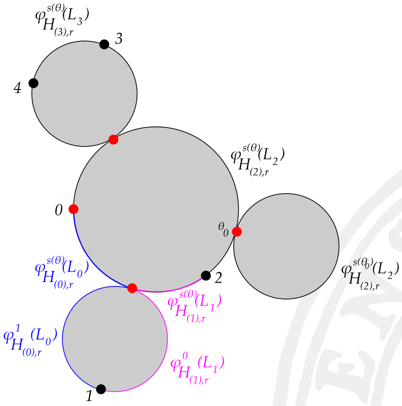


Figure: A limit configuration: on the right there is a disc bubble, and on the bottom there is a breaking of a "Floer strip." This configuration lives in the boundary of $\overline{\mathcal{R}}_d$.

Strips and discs

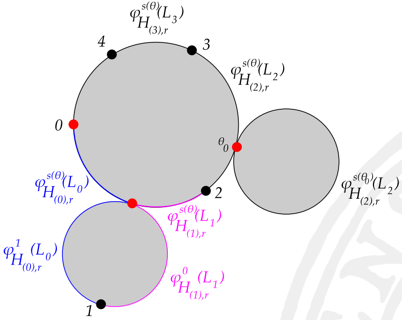


Figure: This nodal configuration has boundary punctures $r \in \mathcal{R}_d$ in the interior; one component is a Floer strip, one component is a disc bubble, and one component has $d+1$ boundary punctures.

Strips and discs

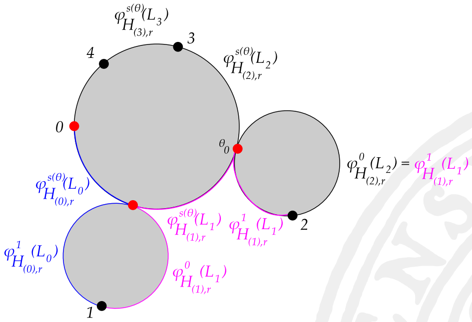


Figure: This nodal configuration has boundary punctures $r \in \mathcal{R}_d$ in the interior; two components are Floer strips and one component has $d+1$ boundary punctures.

The index

From now on we will mainly work in the *exact setting*. Hence, there are no disc/sphere bubbles. In this case:

Lemma

The index of a smoothed nodal disc in $\overline{\mathcal{R}}_{d+1}$ can be obtained by:

- *Summing the indices of the components involved.*
- *Adding the codimension of the stratum of $\overline{\mathcal{R}}_{d+1}$ in which the nodal disc lives.*

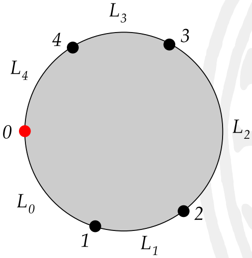
I.e. if nodal disc lives in the interior $\mathcal{R}_{d+1}$, then you simply sum the indices of all components; if the the nodal disc lives in a generic boundary point, then you have to sum indices and add one; etc.

Operations

The count of transversely cut out rigid solutions in the above moduli spaces – i.e. *index zero and transversely cut out* – with the appropriate boundary conditions and asymptotics, gives rise to operations

$$\mu_d \colon CF(L_{d-1}, L_d) \otimes CF(L_{d-2}, L_{d-1}) \otimes \ldots \otimes CF(L_1, L_2) \to CF(L_0, L_d)$$

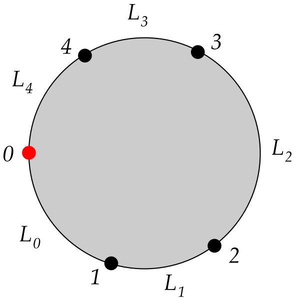
with $d \geq 2$. Next lecture we will see that these operations satisfy the A_∞-relations.

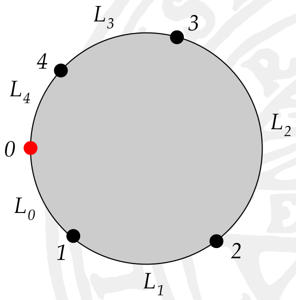


Operations

Remark

The punctured discs that contribute to μ^d are rigid; there might be several different conformal structures $r_i \in \mathcal{R}_d$ which admit solutions that contribute to the value of $\mu^d(x_1, \ldots, x_d)$.





Invariance of the Floer complex

Theorem

Floer [Flo88] When $L_i \subset (X, d\lambda)$, $i = 0, 1$, are closed exact Lagrangian submanifolds and J_t is cylindrical outside of a compact subset then

1. *The boundary ∂ of $CF(L_0, L_1)$ is well-defined;*
2. $\partial^2(x) = 0$;
3. *A compactly supported Hamiltonian isotopy ϕ_H^t of $(X, d\lambda)$, and choice of two-parameter family of almost complex structures $J_{s,t}$, induces a chain map*

$$\Phi_{H,J_{s,t}} \colon CF(L_0, L_1; J_{-1,t}) \to CF(L_0, \phi_H^1(L_1); J_{1,t})$$

which induces isomorphism in homology.

Proof: Invariance of the Floer complex

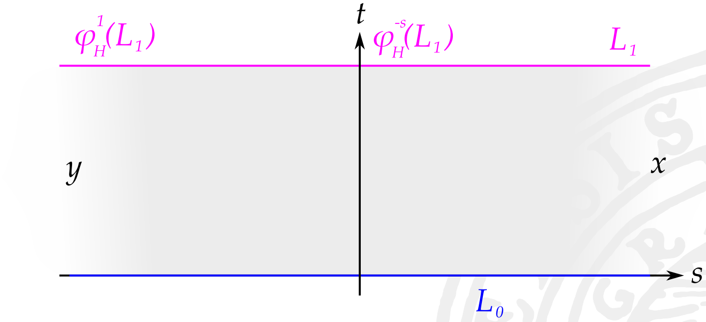


Figure: **Recall:** The continuation map $\Phi_{H,J_{s,t}}$ counts strips of index zero with a moving boundary condition.

Proof: Invariance of the Floer complex

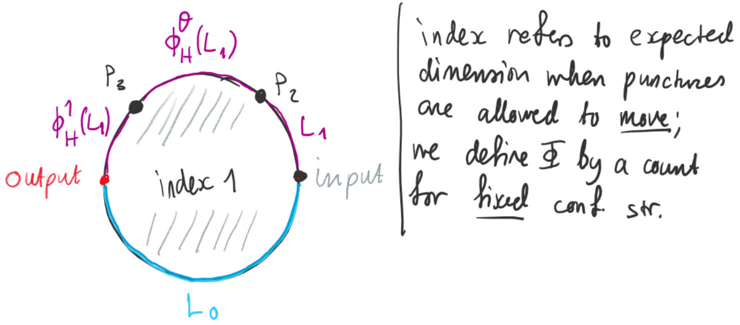


Figure: We reformulate this in our new moduli space: count index *one* discs with *four* boundary punctures, but for a *fixed* conformal structure $r \in \mathcal{R}_3$. (Index one now implies that the solution is rigid!)

Proof: Invariance of the Floer complex

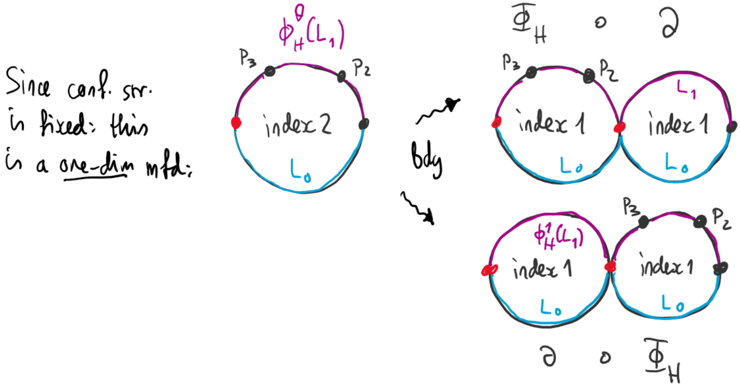


Figure: The chain map property follows from counting such configurations of index *two*; we again keep $r \in \mathcal{R}_3$ fixed, so we obtain a one-dimensional space.

Proof: Invariance of the Floer complex

To show that the continuation map is invertible on homology level, we consider the inverse Hamiltonian isotopy

$$\phi_G^t = (\phi_H^t)^{-1}.$$

A direct computation shows that

Lemma

The Hamiltonian G can be taken to be the smooth function determined by

$$G_t \circ \phi_G^t = -H_t.$$

Proof: Invariance of the Floer complex

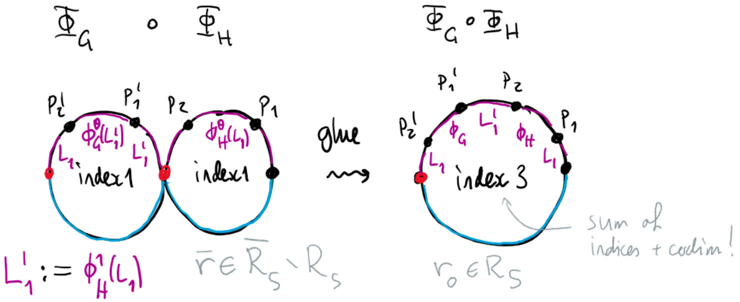


Figure: The pairs of index one strips that define the counts $\Phi_G \circ \Phi_H$ can be glued to form index three strips, again counted with some fixed conformal structure $r_0 \in \mathcal{R}_d$ near the boundary.

Proof: Invariance of the Floer complex

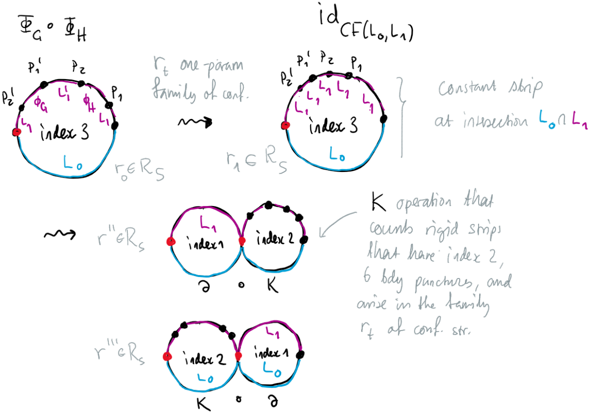


Figure: Interpolate to the constant boundary condition on $L_0 \cup L_1$ in a one-parameter family r_t of conf. structures.

Proof: Invariance of the Floer complex

We finally get the sought chain homotopy relation

$$\Phi_G \circ \Phi_H = \mathrm{Id}_{CF(L_0,L_1)} + K \circ \partial - \partial \circ K$$

where the latter two terms come from the fact that the strips might break when varying the conformal structure in the one-parameter family r_t. □

Continuation element

By a similar argument, the continuation map is chain homotopic to the operation

$$m^2(c,\cdot)\colon CF(L_0,L_1)\to CF(L_0,\phi^1(L_1))$$

obtained by inserting the *continuation element* $c\in CF(L_0,L_1)$ into μ^2, where c is given by counting rigid discs with a "single jump" (see next slide). This will be explored further next lecture.

Continuation element

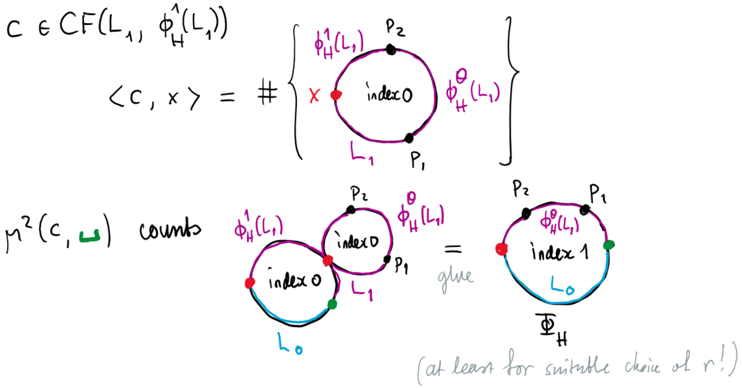


Figure: The continuation element and its insertion into the μ^2-product give the continuation map.

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