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Holomorphic Curve Theories in Symplectic Geometry

Lecture VIII

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Goal of lecture

Today:

- Displaceability implies existence of pseudoholomorphic discs (Or contrapositive: No holomorphic discs implies non-vanishing Floer homology in the closed case.)
- The A_∞-structure in Floer homology and the Fuakay category for closed exact Lagrangians.

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Take-home Message

The operations defined by counting pseudoholomorphic curves of a certain type inherit algebraic relations from the geometry of the moduli space.

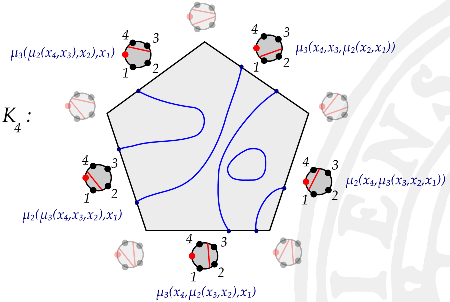

Figure: The A_∞-relations arise by summing the boundary points of the one-dimensional moduli spaces of discs with punctures (the blue curves).

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Plan

1. Goal of lecture
2. Displaceability implies bubbling
3. A_∞-operations
4. References

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Section 2

Displaceability implies bubbling

Bubbles from displaceability

Gromov proved in his original paper [Gro85] that:

Theorem (Gromov [Gro85], Hofer, Oh)

For any closed Lagrangian $L \subset (X, \omega)$ which can be displaced by a Hamiltonian isotopy, and $J \in \mathcal{J}^{tame}(X, \omega)$ which is well-behaved outside of a compact subset (e.g. cylindrical), there exists a non-constant J-holomorphic disc

$$u\colon (D^2, j_0) \to (X, J)$$

with $u(\partial D^2) \subset L$ (or a non-constant J-holomorphic sphere).

Corollary

There exists no closed exact Lagrangians inside $(\mathbb{C}^n, d\lambda_0 = \omega_0)$.

Invariance of the Floer complex

The original proof was not referring to Floer homology. We will prove a (stronger version) of the contrapositive statement to the one on the previous slide (formulated in terms of Floer homology).
Roughly:

If there are no pseudoholomorphic discs with boundary on L (and no psh. sphere) then Floer homology is non-trivial and invariant, and therefore L is not Hamiltonian displaceable.

Invariance of the Floer complex

More precisely, we begin with:

Theorem (Floer)

If J is a tame almost complex structure for which L admits no pseudoholomorphic discs and X admits no non-constant J-holomorphic spheres, then $CF(L, \phi_H^1(L))$ is well-defined and invariant under the choice of Hamiltonian $\phi_H^t(L)$ if the paths of almost complex structures on the strips are of the form

- *$J_{s,t} = (D\phi_H^{t\rho(s)})_* J$ (fully domain dependent) in the case where the boundary condition on the upper boundary arc $\{t = 1\}$ of the strip is taken in $\phi_H^{\rho(s)}(L)$; (for the Floer strips $\rho(s) \equiv 1$, i.e. no s-dependence.)*
- *In particular $J_{s,0} = J$ along the entire lower boundary arc $\{t = 0\}$ of the strip.*

Invariance of the Floer complex

- We need Novikov coefficients for well-definedness and invariance in the above setting, unless we e.g. assume exactness.
- Recall the strategy of the proofs of well-definedness and invariance: study one-dimensional moduli spaces, and show that the operations

  $$\partial^2,\ \ \partial \circ \Phi_H - \Phi_H \circ \partial,\ \ \text{etc.}$$

  count *boundary points* of some moduli space; hence they all vanish as sought. (The previous lecture this was proved in the exact case.)
- Under the above assumptions, our choice of almost complex structure $J_{s,t}$ on the strips prevents bubbles of pseudoholomorphic discs from forming (see the version of Gromov compactness from the previous lecture). The well-def. and invariance thus follows as in the exact case.

Invariance of the Floer complex

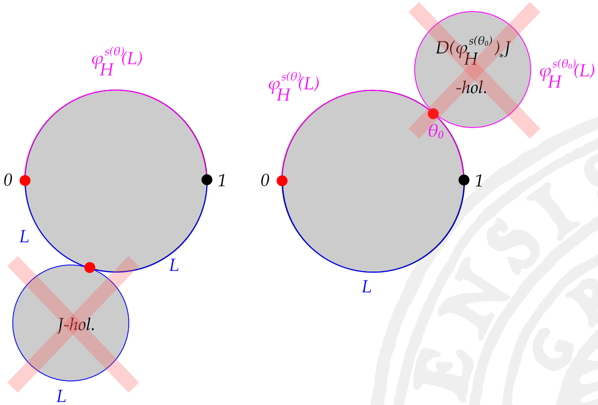


Figure: The symplectomorphism ϕ_H^1 gives a bijection between J-holomorphic discs with boundary on L (which do not exist by assumption) and $D(\phi_H^1)_* J$-holomorphic discs with boundary on $\phi_H^1(L)$ (and consequently there are no such discs either).

Invariance of the Floer complex

The existence of psh. discs for *any* tame almost complex structure in the case when L admits a Hamiltonian displacement follows immediately from the following result that we now prove:

Theorem

When L is closed and admits a tame almost complex structure J for which there are no non-constant pseudoholomorphic discs with boundary on L, and no non-constant pseudoholomorphic spheres, then the continuation element $c_{L,H} \in CF(L, \phi^1_H(L))$ is a cycle which is non-trivial in homology $HF(L, \phi^1_H(L))$.)

Remark

The proof of the existence of psh. discs does actually not need the Floer complex, it suffices to consider the argument on the level of moduli spaces.

Invariance of the Floer complex

In fact, something stronger is true (c.f. [Lecture 6])

Theorem (Floer, [FOOO09a] generalising Floer [Flo88])

If H is sufficiently C^∞-small so that $\phi^1_H(L)$ is the section of the exact one-form dg in a Weinstein neighbourhood of L for a Morse function $g\colon L \to \mathbb{R}$, then

- *$CF(L, \phi^1_H(L))$ is the Morse homology complex of L for the Morse function $-g$ for a suitable choice of a.c.s., and*
- *the continuation element $c_{L,H}$ is the fundamental class (maximum class) of this Morse complex; (With our conventions it lives in degree zero, and should be considered as the unit in Morse cohomology.)*

Invariance of the Floer complex

Remark

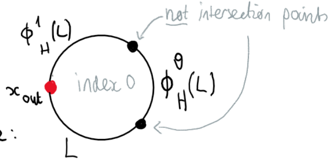
- The Morse homology complex is never acyclic. (Why?)
- Similar considerations show that $c_{L,H}$ is non-trivial in homology.

Proof that $c_{H,L}$ is nontrivial

Proof (that $c_{H,L} \in CF(L, \phi^1_H(L))$ is a nontrivial cycle)

$c_{H,L} := \sum \#\text{sol} \cdot x_{out}$

where solutions are:



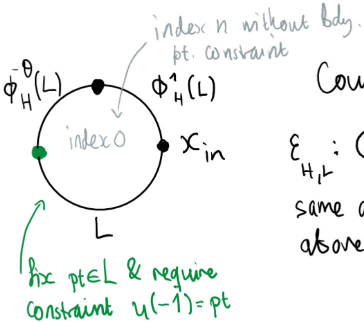
Cycle

count gives $\partial(c_{H,L}) = 0$

since we count boundary pts

Proof that $c_{H,L}$ is nontrivial

<u>Not a boundary</u>: consider the "counit/augmentation" def. by



Count defines a map

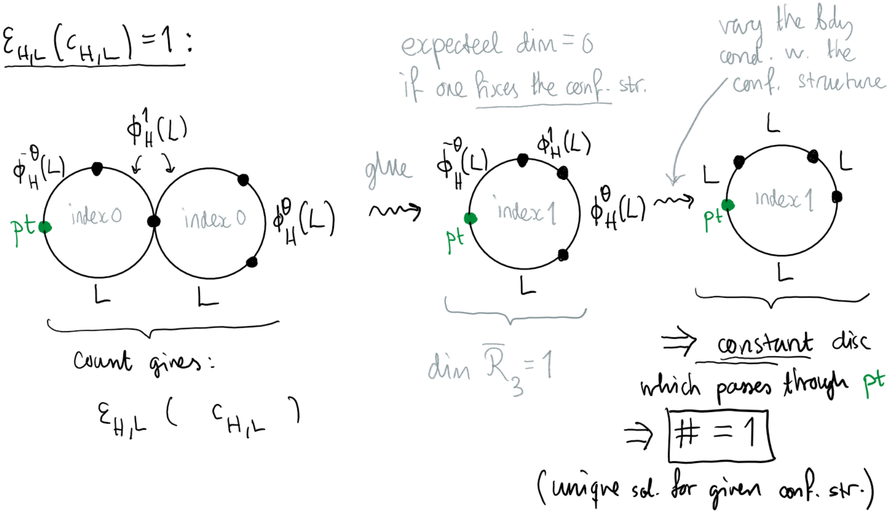
$$\varepsilon_{H,L} : CF(L, \phi_H^1(L)) \longrightarrow \mathbb{F}$$

same argument as above gives

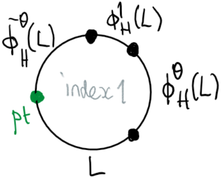
$$\boxed{\varepsilon_{H,L} \circ \partial = 0}$$

we proceed to show $\varepsilon_{H,L}(c_{H,L}) = 1$, which implies that $c_{H,L}$ is <u>not</u> a boundary

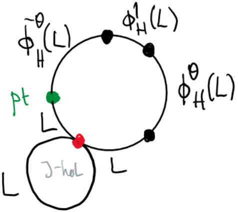
Proof that $c_{H,L}$ is nontrivial

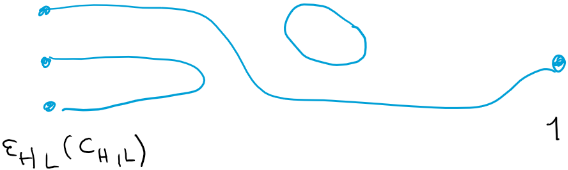


Proof that $c_{H,L}$ is nontrivial



Recall that the assumption precludes bubbles such as





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Section 3

A_∞-operations

From now on: All Lagrangians are assumed to be *exact*.

A_∞-operations

Recall:

- The boundary $\partial\colon CF(L_0, L_1) \to CF(L_0, L_1)$ counts Floer strips of index one which admit a natural $\mathbb{R}$-action; thus they are rigid after quotient by reparam.
- The continuation map

$$\Phi_{H,J_s}\colon CF(L_0, L_1) \to CF(L_0, \phi^1_H(L_1))$$

counts continuation strips of index zero.

Similarly one defines maps

$$\mu_d\colon CF(L_{d-1}, L_d) \otimes CF(L_{d-2}, L_{d-1}) \otimes \ldots \otimes CF(L_1, L_2) \to CF(L_0, L_d).$$

by counts of moduli spaces of discs of *index zero* with $d+1$ punctures; the one at -1 is the output, while the remaining d punctures are inputs.

A_∞-operations

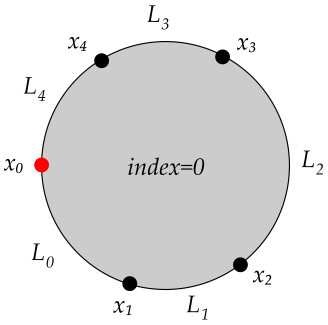


Figure: Rigid disc with punctures asymptotic to intersection points that contributes to the count $\langle \mu_4(x_4, x_3, x_2, x_1), x_0 \rangle$. Recall that: The conformal structure (i.e. position of the boundary punctures) is *not fixed*, while we identify discs which differ by reparametrisation.)

A_∞-operations

The above operations are defined for counts of index zero discs with $d + 1 \geq 3$ punctures. We also define a version of the differential with a twisted sign:

$$\mu_1 \colon CF(L_0, L_1) \to CF(L_0, L_1)$$
$$x_1 \mapsto (-1)^{|x_1|}\partial(x_1),$$

which (for the same reason as δ) is defined by counts of strips of index one.
(The sign depends on the degree of the generators, we will say some more words about this below.)

A_∞-operations

Proposition

The above maps $\{\mu_d\}$, $d = 1, 2, 3, \ldots$, *satisfy the* A_∞*-relations*

$$\sum_{\substack{d_1+d_2=d+1\\0\leq k\leq d_1}} (-1)^{\maltese^k} \mu_{d_1}(x_d, \ldots, x_{k+d_2+1}, \mu_{d_2}(x_{k+d_2}, \ldots, x_{k+1}), x_k, \ldots, x_1)$$

for the sign

$$\maltese^k = k + \sum_{i=1}^{k} |x_i|.$$

There is one relation for each $d = 1, 2, 3, \ldots$, and we proceed to spell out the first three of them explicitly.

A_∞-operations

Unfortunately, neither *signs* of discs nor *gradings* of the generators of $CF(L_0, L_1)$ will be explained at this point. About we grading we simply state the following:

- The generators have a grading which is induced by the Maslov class of a certain capping operator.
- Each basis element $x \in CF(L_0, L_1)$ is an intersection point; it can of course naturally be identified with a basis element $x^\vee \in CF(L_1, L_0)$ as well; the degrees satisfies the relation

$$|x| = \dim L_0 - |x^\vee|.$$

- The index of the disc with input punctures $x_1, \ldots, x_d$ and output x_0 is equal to

$$|x_1| + \ldots + |x_d| - |x_0| + d - 2$$

A_∞-operations

In other words:

- The operations μ_d are of degree $d-2$, i.e. it take an element $x_d \otimes \ldots \otimes x_1$ of homogeneous degree i to a sum of elements of degree $i-d+2$;
- In particular $\mu_1 = \partial$ decreases the degree by one, μ_2 preserves the grading (of the tensor product), and μ_3 increases the degree (of the tensor product) by one.
- The continuation element $c_{H,L}$ lives in degree *zero*. (We should take $d=2$ here.)

A_∞-relations

The A_∞-relations are proven by considering the corresponding moduli spaces of pseudoholomorphc discs with punctures mapping to intersection points, but of index (i.e. dimension) *one higher* than the index of those solutions whose counts define the corresponding operation.

Example

Recall the proof that $\mu_1^2 = 0$: considering discs of index 2 (i.e. a one-dimensional moduli space after quotient by reparam.) and use the fact that the index of a nodal strip is the sum of the indices of components.

A_∞-relation for $d = 1$:

$\mu_1^2 = 0$

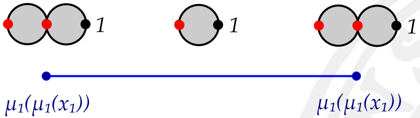


Figure: The fact that the boundary points of a one-dimensional moduli space is even gives the relation $\langle \mu_1^2(x_1), x_0 \rangle = 0$ for any two fixed generators x_0, x_1. The boundary of this moduli space consists of two strips of index one, while the boundary consists of two strips of index two.

A_∞-relation for $d = 2$:

We proceed to show in pictures the one-dimensional moduli spaces which give rise to the first A_∞-operations.

A_∞-relations

$$\mu_1(\mu_2(x_2, x_1)) = \mu_2(x_2, \mu_1(x_1)) + (-1)^{1+|x_1|}\mu_2(\mu_1(x_2), x_1)$$

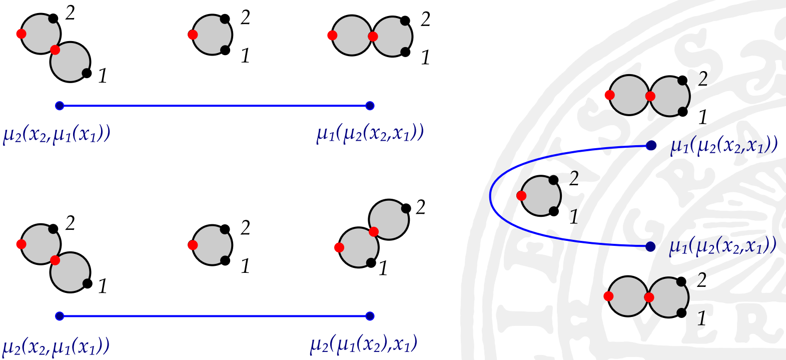


Figure: The associahedron $\mathcal{R}_2 = K_2$ is just a point, so only breaking of strips can occur. The index is additive.

A_∞-relations for $d = 3$:

$$\mu_1(\mu_3(x_3, x_2, x_1)) = A(x_3, x_2, x_1) + B(x_3, x_2, x_1)$$

where

$$A(x_1, x_2, x_3) = \mu_2(x_3, \mu_2(x_2, x_3)) + (-1)^{1+|x_1|}\mu_2(\mu_2(x_3, x_2), x_1)$$

is a signed version of the *associator* (counts "stable" broken strips) while

$$B(x_1, x_2, x_3) = \mu_3(x_3, x_2, \mu_1(x_1)) \\ + \ (-1)^{1+|x_1|}\mu_3(x_3, \mu_1(x_2), x_1) + (-1)^{2+|x_1|+|x_2|}\mu_3(\mu_1(x_3), x_2, x_1)$$

counts "unstable" broken strips.

A_∞-relation for $d = 3$:

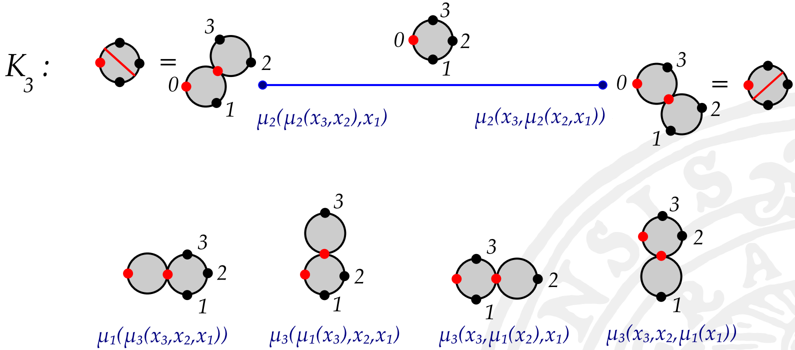


Figure: The one dimensional moduli space shown on the top has only stable breakings (for these the index is sub-additive, since the boundary of the moduli space lies in the boundary of the space of conformal structures $\overline{\mathcal{R}}_3 = K_3$). The other nodal configurations are unstable, and the index is additive.

A_∞-relation for $d = 4$:

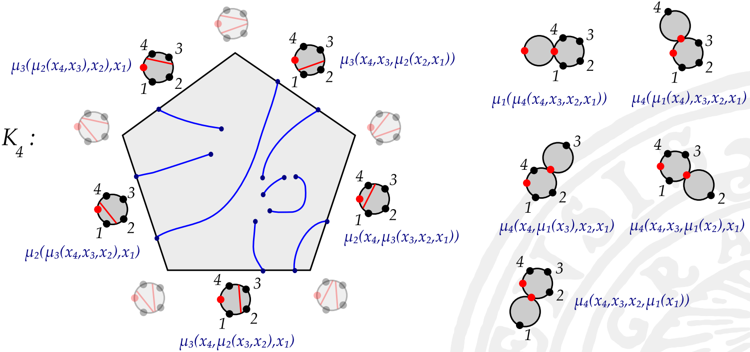


Figure: One-dimensional moduli spaces with five boundary punctures. The unstable breakings happen in the interior of $\overline{\mathcal{R}}_4 = K_4$ (index is additive), while the stable breakings happen in the boundary (index is subadditive).

A_∞-relations

Observe that:

- $\partial(x_1) = (-1)^{|x_1|}\mu_1(x_1)$ is a boundary operator;
- $x_2 \cdot x_1 := (-1)^{|x_1|}\mu_2(x_2, x_1)$ is a product which satisfies the graded Leibniz rule

$$\partial(x_2 \cdot x_1) = \partial(x_2) \cdot x_1 + (-1)^{|x_2|} x_2 \cdot \partial(x_1)$$

  with respect to ∂.

- μ_3 induces a null-homotopy of the associator

$$x_3 \cdot (x_2 \cdot x_1) - (x_3 \cdot x_2) \cdot x_1.$$

A_∞-relations

In other words: On the homology level the above product is

- *well-defined* (by the Leibniz rule), and
- *associative* (by the μ_3-relation).

A_∞-category

The so-called *Fukaya category* is a unital A_∞-category of closed Lagrangians was constructed in [FOOO09a],[FOOO09b] by Fukaya–Ohta–Ono–Oh, and in the exact case by Seidel [Sei08]. Roughly it consists of

- **Objects:** exact closed Lagrangians (equipped with additional data);
- **Morphisms:** elements in the Floer complexes $Hom(L_0, L_1) = CF(L_0, L_1)$.
- **Composition:** defined by the product.

Remark

- Composition is not associative on the chain level, the higher operations are also a part of the data of this category;
- We have not yet defined the *endomorphisms* of this category.

A_∞-category

The endomorphisms cannot be defined as $CF(L, L)$, since $L \cap L = L$ is not transverse.

Solution [Sei08]:

- Equip each L with the additional data of a push-off $\phi^1_H(L)$ for a C^∞-small H, and define

$$CF(L, L) := CF(L, \phi^1_H(L)).$$

- Define the operations

$$\mu_d \colon CF(L, L) \otimes \ldots \otimes CF(L, L) \to CF(L, L)$$

(and so on) by *suitably perturbing* the boundary conditions;

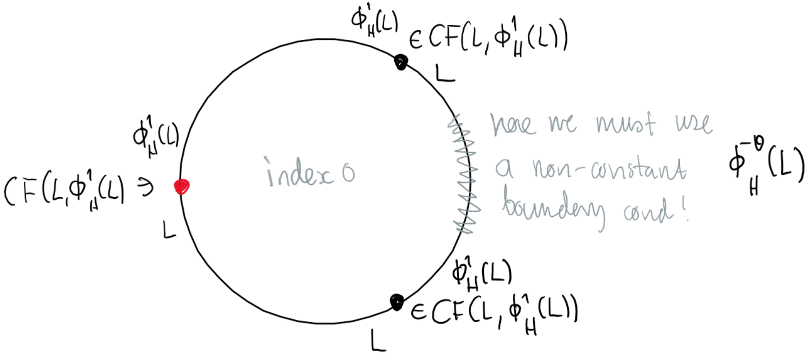
- **Unit:** Homology level unit is the continuation element

$$e_L := c_{H,L} \in CF(L, L).$$

Perturbations for the endomorphisms

Ex Definition of $\mu_2 : CF(L,L) \otimes CF(L,L) \to CF(L,L)$

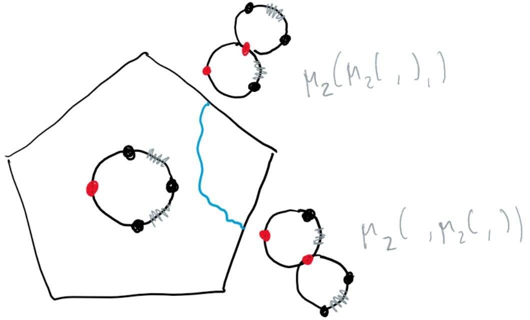
where $CF(L,L) := CF(L, \phi^1_H(L))$



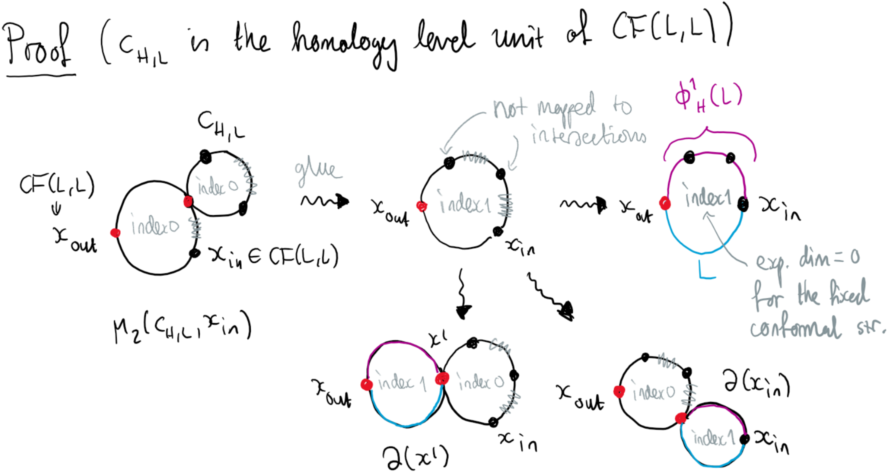
Perturbations for the endomorphisms

Problem To get the A_∞-relations, one must coherently extend the perturbation of the bdy cond to all $\overline{R}_d$. inductively

$\overline{R}_3 = K_3$



$c_{H,L}$ is the homology unit



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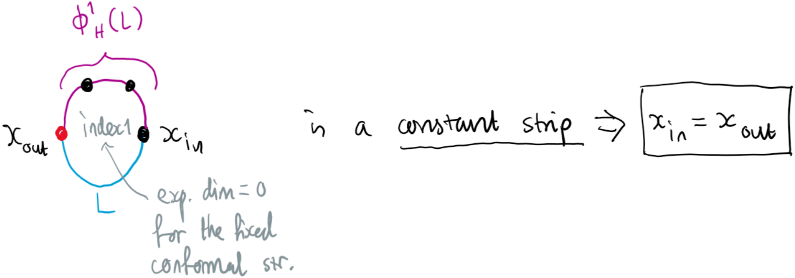

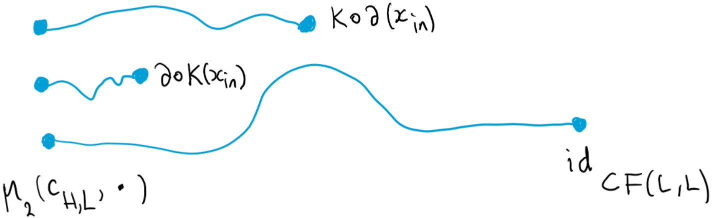

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References

M. Abouzaid and P. Seidel.
An open string analogue of Viterbo functoriality.
Geom. Topol., 14(2):627–718, 2010.

A. Floer.
Morse theory for Lagrangian intersections.
J. Differential Geom., 28(3):513–547, 1988.

K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono.
Lagrangian intersection Floer theory: anomaly and obstruction. Part I, volume 46 of *AMS/IP Studies in Advanced Mathematics*.
American Mathematical Society, Providence, RI, 2009.

K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono.
Lagrangian intersection Floer theory: anomaly and obstruction. Part II, volume 46 of *AMS/IP Studies in Advanced Mathematics*.
American Mathematical Society, Providence, RI, 2009.

M. Gromov.
Pseudoholomorphic curves in symplectic manifolds.
Invent. Math., 82(2):307–347, 1985.

P. Seidel.
Fukaya categories and Picard-Lefschetz theory.
Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.