Lie Groups

Karl-Heinz Fieseler

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1 Introduction

The groups one investigates in algebra usually are finite or finitely generated, but for example $\mathbb{R}$ endowed with the addition of real numbers is far from being finitely generated. Or take

$$GL_n(\mathbb{R}) := \{A \in \mathbb{R}^{n,n}; \det(A) \neq 0\},$$

the group of invertible $n \times n$-matrices with real entries, the group law being the multiplication of matrices.

In order to understand even such groups one considers groups with additional structure compatible with the group action.

Definition 1.1. *A topological group is a group G endowed with a Hausdorff topology such that both the group multiplication (group law)*

$$\mu : G \times G \longrightarrow G, (a, b) \mapsto ab,$$

and the "inversion"

$$\iota : G \longrightarrow G, a \mapsto a^{-1}$$

are continuous maps.

Example 1.2.

1. The additive group $G := \mathbb{R}$ endowed with its standard topology.
2. The group
$$G := GL_n(\mathbb{R}) \subset \mathbb{R}^{n,n} \cong \mathbb{R}^{n^2},$$
endowed with the topology as a(n open) subset of $\mathbb{R}^{n^2}$.

Let us comment on the second example: First of all,

$$GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R}^*) \subset \mathbb{R}^{n,n}$$

is an open set of $\mathbb{R}^{n,n}$ as the inverse image of the open set $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ (the punctured real line) with respect to a continuous map, the determinant

$$\det : \mathbb{R}^{n,n} \longrightarrow \mathbb{R}, A = (\alpha_{ij}) \mapsto \det(A) = \sum_{\pi \in \mathbb{S}_n} \mathrm{sign}(\pi) \prod_{i=1}^{n} \alpha_{i,\pi(i)},$$

a polynomial in the entries α_{ij} of A. That matrix multiplication is continuous is immediate, while the continuity of the inversion $A \mapsto A^{-1}$ follows from the following formula

$$A^{-1} = (\gamma_{ij}) \quad \text{with } \gamma_{ij} = (-1)^{i+j} \frac{\det(A_{ji})}{\det(A)} \ ,$$

where $A_{k\ell} \in \mathbb{R}^{n-1,n-1}$ denotes the matrix obtained from $A \in \mathbb{R}^{n,n}$ by deleting the k-th row and the ℓ-th column. With other words, the entries γ_{ij} of A^{-1} are rational functions in the entries of A (with a non-vanishing denominator).

Indeed continuous functions can be quite pathological, while differentiable functions are more easily understood, since locally they can be approximated by linear functions. So it seems natural to look for a notion of "differentiable groups". We give here a very restricted provisional definition:

Definition 1.3. *Assume that $G \subset \mathbb{R}^m$ is both an open subset of $\mathbb{R}^m$ and a group (with a suitable group law). It is called a differentiable group if the group multiplication as well as the inversion are differentiable (i.e. C^∞-)maps.*

Example 1.4.

1. The vector space $\mathbb{R}^m$ endowed with the addition of vectors as group law.
2. $GL_n(\mathbb{R})$ with $m = n^2$.
3. The direct product $G := \mathbb{R}^n \times GL_n(\mathbb{R})$ with $m = n + n^2$.
4. There is an other way to endow $\mathbb{R}^n \times GL_n(\mathbb{R})$ with the structure of a Lie group: Identify a pair $(a, B) \in \mathbb{R}^n \times GL_n(\mathbb{R})$ with the "affine linear map" $\mathbb{R}^n \ni x \mapsto a + Bx \in \mathbb{R}^n$. The composition of two affine linear maps being again affine linear, we obtain the following new group law

$$(a, B)(c, D) := (a + Bc, BD).$$

on $\mathbb{R}^n \times GL_n(\mathbb{R})$.

Now the basic idea in the study of differentiable groups is to replace the commutator map

$$K : G \times G \longrightarrow G, (x, y) \mapsto xyx^{-1}y^{-1}$$

with the "bilinear part" of its Taylor expansion at (e,e) - here $e \in G$ denotes the neutral element of the group G. Let us explain that: Denote $DK(e,e) \in \mathbb{R}^{m,2m}$ the Jacobian matrix of K at (e,e) - the linear part of the Taylor expansion. Then we have for small $\xi, \eta \in \mathbb{R}^m$ the expansion

$$K(e+\xi, e+\eta) = K((e,e)+(\xi,\eta))$$

$$e + DK(e,e)\left(\begin{array}{c} \xi \\ \eta \end{array} \right) + \sum_{1 \le i,j \le m} \frac{\partial K}{\partial x_i \partial y_j}(e,e)\xi_i \eta_j + ...$$

Now the "bilinear term"

$$[\xi, \eta] := \sum_{1 \le i,j \le m} \frac{\partial K}{\partial x_i \partial y_j}(e,e)\xi_i \eta_j$$

defines a bilinear map

$$[..,..] : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^m.$$

It turns out that that map determines the group law near $(e,e) \in G \times G$ completely, so one can replace the local study of differentiable groups with the study of certain bilinear maps $[..,..] : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$.

Let us discuss the example $G = GL_n(\mathbb{R}) \subset \mathbb{R}^{n,n}$ and compute the map $[..,..] : \mathbb{R}^{n,n} \times \mathbb{R}^{n,n} \longrightarrow \mathbb{R}^{n,n}$. For a matrix $A = (\alpha_{ij})$ we define its norm by

$$||A|| := n \max\{|\alpha_{ij}|; 1 \le i,j \le n\}$$

and note that it is even well behaved with respect to products: $||AB|| \le ||A|| \cdot ||B||$. Now denote $E \in GL_n(\mathbb{R})$ the unit matrix (replacing $e \in G$) and take $X, Y \in \mathbb{R}^{n,n}$ of norm < 1 (replacing ξ and η). Then we have $E + X \in GL_n(\mathbb{R})$ with

$$(E+X)^{-1} = E - X + X^2 - X^3 + ...,$$

a convergent series. Consequently

$$K(E+X, E+Y) = (E+X)(E+Y)(E-X+X^2-....)(E-Y+Y^2-...)$$

$$= E + (X+Y-X-Y) + (X^2+Y^2+XY-X^2-XY-YX-Y^2+XY) + ...$$

with the dots representing terms of total degree > 2 in X and Y. Thus the linear term vanishes and

$$[X,Y] = XY - YX$$

is the commutator of the matrices $X, Y \in \mathbb{R}^{n,n}$.

2 Differentiable Manifolds

In Definition 1.3 we require $G \subset \mathbb{R}^m$ (as an open subset) for a differentiable group. That is a very restrictive condition. E.g. it holds for $G := \mathbb{C}^* \subset \mathbb{C} \cong \mathbb{R}^2$, the multiplicative group of non-zero complex numbers, but it does not hold any longer for its closed subgroup

$$\mathbf{S}^1 := \{z \in \mathbb{C}; |z| = 1\},$$

the unit circle. (An open subset of $\mathbb{R}^m$ is never compact.) Nevertheless it locally looks like the real line: There are homeomorphisms

$$\psi_1 : V_1 :=]-\pi, \pi[\longrightarrow U_1 := \mathbf{S}^1 \setminus \{-1\}, \ x \mapsto e^{ix}$$

and

$$\psi_2 : V_2 :=]0, 2\pi[\longrightarrow U_2 := \mathbf{S}^1 \setminus \{1\}, \ x \mapsto e^{ix}.$$

Thus we are led to the notion of a *topological manifold*:

Definition 2.1. An m-dimensional topological manifold M is a Hausdorff topological space admitting an open cover

$$M = \bigcup_{i \in I} U_i$$

with open subset $U_i \subset M$ homeomorphic to open subsets $V_i \subset \mathbb{R}^m$.

Example 2.2.

1. Any open subset of $\mathbb{R}^m$ is an m-dimensional topological manifold.
2. $M := \mathbf{S}^1 = U_1 \cup U_2$ is a one dimensional topological manifold.
3. Denote $||x|| = \sqrt{x_1^2 + ... + x_{n+1}^2}$ the euclidean norm of a vector $x \in \mathbb{R}^{n+1}$. Then the sphere

   $$\mathbf{S}^n := \{x \in \mathbb{R}^{n+1}; ||x|| = 1\}$$

   is an n-dimensional topological manifold: We have

   $$\mathbf{S}^n = U_1 \cup U_2$$

with $U_1 := \mathbf{S}^n \setminus \{-e_{n+1}\}, U_2 := \mathbf{S}^n \setminus \{e_{n+1}\}$, where $U_i \cong \mathbb{R}^n$. For example the maps

$$\sigma_i : U_i \longrightarrow \mathbb{R}^n, x = (x', x_{n+1}) \mapsto \frac{x'}{1-(-1)^i x_{n+1}}, i = 1, 2$$

are homeomorphisms: For $x \in U_i$ the point $(\sigma_i(x), 0)$ is the intersection of the line spanned by x and $-e_{n+1}$ (for $i = 1$) resp. e_{n+1} (for $i = 2$) with the hyperplane $\mathbb{R}^n \times 0$.

Now one could try to study a function $f : M \longrightarrow \mathbb{R}$ on a topological manifold M by considering what one gets by composing f with inverse homeomorphisms $\mathbb{R}^m \supset V \xrightarrow{\psi} U \subset M$ and then apply analysis to the composite $f \circ \psi : V \longrightarrow \mathbb{R}$. But then it will depend on the choice of the homeomorphism ψ, whether $f \circ \psi$ is differentiable or not. One can avoid that difficulty by restricting to a system, "atlas", of "mutually compatible" homeomorphisms, also called "charts":

Definition 2.3. Let M be an m-dimensional topological manifold.

1. A chart on M is a pair (U, φ), where $U \subset M$ is open and $\varphi : U \longrightarrow V$ is a homeomorphism between U and an open subset $V \subset \mathbb{R}^m$. The component functions $\varphi^1, ..., \varphi^m$ then are also called (local) coordinates for M on $U \subset M$.

2. Two charts $(U_i, \varphi_i), i = 1, 2$, on a topological manifold M are called (C^∞-)compatible if either $U_{12} := U_1 \cap U_2$ is empty or the transition map ("coordinate change")

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_{12}) \longrightarrow \varphi_2(U_{12})$$

is a diffeomorphism between the open sets $\varphi_1(U_{12}) \subset V_1 \subset \mathbb{R}^m$ and $\varphi_2(U_{12}) \subset V_2 \subset \mathbb{R}^m$.

3. A differentiable atlas $\mathcal{A}$ on a topological manifold M is a system

$$\mathcal{A} = \{(U_i, \varphi_i); i \in I\}$$

of mutually (C^∞-)compatible charts, such that $M = \bigcup_{i \in I} U_i$.

Example 2.4. 1. On the unit circle $\mathbf{S}^1$ the charts $(U_i, \varphi_i := \psi_i^{-1}), i = 1, 2$ constitute a differentiable atlas: Indeed $\mathbf{S}^1 = U_1 \cup U_2$, and the transition map $\varphi_2 \circ \varphi_1^{-1} :]-\pi, 0[\cup]0, \pi[\longrightarrow]0, \pi[\cup]\pi, 2\pi[$ looks as follows

$$\varphi_2 \circ \varphi_1^{-1}(x) = \begin{cases} x & , \text{ if } x \in]0, \pi[\\ x + 2\pi & , \text{ if } x \in]-\pi, 0[\end{cases} .$$

2. The charts $(U_i, \sigma_i), i = 1, 2$ on $\mathbf{S}^n$, cf. 2.2.3, constitute a differentiable atlas: Again we have $\mathbf{S}^n = U_1 \cup U_2$ and the transition map

$$\sigma_2 \circ \sigma_1^{-1} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\},\ x \mapsto \frac{x}{||x||^2}.$$

3. Let $W \subset \mathbb{R}^n$ be an open subset and

$$F : W \longrightarrow \mathbb{R}^{n-m}$$

a differentiable map, such that for all $a \in M := F^{-1}(0) \subset W$ the Jacobian map

$$DF(a) : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-m}$$

is surjective. For every point $a \in M$ we shall construct a local chart (U_a, φ_a). We may assume that $\frac{\partial F}{\partial(x_{m+1},...,x_n)}(a) \neq 0$. Then the map $\Phi : (x_1, ..., x_m, F_1, ..., F_{n-m}) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ induces, according to the inverse function theorem, a diffeomorphism $\tilde{U} \longrightarrow \tilde{V}$ between an open neighborhood $\tilde{U} \subset W$ of $a \in \mathbb{R}^n$ and an open neighborhood $\tilde{V}$ of $(a_1, ..., a_m, 0) \in \mathbb{R}^n$. As a consequence the map

$$\varphi_a : U_a := \tilde{U} \cap M \longrightarrow V_a := \{y \in \mathbb{R}^m; (y, 0) \in \tilde{V} \subset \mathbb{R}^m \times \mathbb{R}^{n-m}\},$$

$$x \mapsto (x_1, ..., x_m)$$

is a homeomorphism. Then the collection $\mathcal{A} := \{(U_a, \varphi_a); a \in M\}$ constitutes a differentiable atlas on M. Note that all local coordinates are obtained by choosing m suitable restrictions $x_{i_1}|_M, ..., x_{i_m}|_M$ of the coordinate functions $x_1, ..., x_n$, i.e., with the choice of the set $\{i_1, ..., i_m\}$ depending on the point $a \in M$.

The last example shows that a differentiable atlas may depend on a lot of choices and can be unnecessarily big as well. So we need to say when two atlases are "equivalent":

Definition 2.5. *1. Two atlases $\mathcal{A}$ and $\tilde{\mathcal{A}}$ on an m-dimensional topological manifold M are called equivalent if any chart in $\mathcal{A}$ is compatible with any chart in $\tilde{\mathcal{A}}$.*

2. A differentiable structure on a topological manifold is an equivalence class of differentiable atlases.

3. A differentiable manifold M is a topological manifold together with a differentiable structure. We say that a differentiable atlas $\mathcal{A}$ is an atlas for the differentiable manifold M, if $\mathcal{A}$ defines (or belongs to) the differentiable structure of M.

If M is a differentiable manifold, then a "chart (U, φ) on M" means always a chart compatible with all the charts of a (resp. all) atlases defining the differentiable structure of M.

We leave the details of the following remark to the reader:

Remark 2.6. 1. Any open subset $U \subset M$ of a differentiable manifold inherits a natural differentiable structure.

2. The cartesian product $M \times N$ of differentiable manifolds M, N carries a natural differentiable structure.

Definition 2.7. *Let M, N be differentiable manifolds of dimension m, n respectively.*

1. A function $f : M \longrightarrow \mathbb{R}$ is differentiable if the functions $f \circ \varphi^{-1} : V \longrightarrow \mathbb{R}$ are differentiable for all charts $(U, \varphi : U \longrightarrow V) \in \mathcal{A}$ in a differentiable atlas for M. The same definition applies for maps $F : M \longrightarrow \mathbb{R}^n$. We denote

$$C^\infty(M) := \{f : M \longrightarrow \mathbb{R} \text{ differentiable}\}$$

the set of all differentiable functions, indeed a real vector space which is even closed with respect to the multiplication of functions.

2. A continuous map $F : M \longrightarrow N$ is called differentiable if all the maps $\psi \circ (F|_{F^{-1}(W)}) : F^{-1}(W) \longrightarrow \mathbb{R}^n$ are differentiable, where $(W, \psi) \in \mathcal{B}$ is any chart in an atlas $\mathcal{B}$ defining the differentiable structure of N.

3. A diffeomorphism $F : M \longrightarrow N$ between two differentiable manifolds M and N is a bijective differentiable map, such that its inverse $F^{-1} : N \longrightarrow M$ is differentiable as well.

4. *We say that M is diffeomorphic to N and write $M \cong N$ if there is a diffeomorphism $F : M \longrightarrow N$.*

Note that differentiable functions are continuous, and that the definition of differentiability is independent from the choice of the differentiable atlases for M and N.

Remark 2.8. Given a topological manifold M there are a lot of distinct differentiable structures, but the corresponding differentiable manifolds may be diffeomorphic nevertheless: By a smooth type on M we mean the diffeomorphism class of the differentiable manifold defined by some differentiable structure on M.

1. For $\dim M \leq 3$ there is exactly one smooth type.

2. For $\dim M \geq 4$ it may happen that there is no differentiable structure on M at all.

3. A compact topological manifold of dimension at least 5 admits only finitely many smooth types. E.g., the sphere $\mathbf{S}^n$ has the standard smooth type as described above, but there may be other ones: For $n = 5, ..., 20$ we obtain $1, 1, 28, 2, 8, 6, 992, 1, 3, 2, 16256,\ 2, 16, 16, 523264, 24$ smooth types respectively. For $n = 4$ it is not known whether there are exotic smooth types, i.e. smooth types different from the standard smooth type.

4. For $n \neq 4$ there is only the standard smooth type on $\mathbb{R}^n$, while on $\mathbb{R}^4$ there are uncountably many different smooth types; some of them are obtained as follows: One considers an open subset $U \subset \mathbb{R}^4$ with the standard smooth type; if there is a homeomorphism $U \cong \mathbb{R}^4$ one gets an induced differentiable structure on $\mathbb{R}^4$.

Definition 2.9. Let M be an m-dimensional differentiable manifold. A subset $L \subset M$ is called a submanifold of codimension k iff for every point $a \in L$ there is a chart (U, φ), such that $\varphi(U \cap L) = \{x = (x_1, ..., x_m) \in V := \varphi(U); x_{m-k+1} = ... = x_m = 0\}$.

Note that a submanifold $L \subset M$ inherits from M a unique differentiable structure, such that the inclusion $L \hookrightarrow M$ is differentiable.

Remark 2.10. Complex Manifolds: We can literally apply the same strategy as above with $\mathbb{C}$ replacing $\mathbb{R}$. That leads to the notion of a complex structure and complex manifolds. But there are also dramatic changes, since the condition for a map $f : W \longrightarrow \mathbb{C}$ on an open subset $W \subset \mathbb{C}^n$ to have everywhere a (complex) linear approximation, is an extremely strong one. Such functions are called holomorphic, we denote

$$\mathcal{O}(M) := \{f : M \longrightarrow \mathbb{C} \text{ holomorphic}\}$$

the complex vector space of holomorphic functions. A holomorphic function $f \in \mathcal{O}(M)$ has derivatives of any order, in fact it is even analytic, i.e. is locally described by its Taylor series.

Let us, without proof, mention the following difference: Given a compact set $K \subset U$ contained in an open set $U \subset M$ of a differentiable manifold M, there is a function $f \in C^\infty(M)$ with $f|_K \equiv 1$ and $f|_{M\setminus U} \equiv 0$. On the other hand, for a connected complex manifold M the following identity theorem holds: The restriction map

$$\mathcal{O}(M) \longrightarrow \mathcal{O}(W), f \mapsto f|_W$$

is injective for any nonempty open subset $W \subset M$. Indeed if in addition M itself is compact, we have $\mathcal{O}(M) = \mathbb{C}$, i.e. there are only constant holomorphic functions - this is an immediate consequence of the maximum principle for holomorphic functions. On the other hand $\mathbb{C}^m \cong \mathbb{R}^{2m}$ and any holomorphic map $\mathbb{C}^n \supset W \xrightarrow{f} \mathbb{C}^m$ is in particular differentiable as a map $\mathbb{R}^{2n} \supset W \xrightarrow{f} \mathbb{R}^{2m}$. So a complex manifold of dimension m can also be regarded as a differentiable manifold of dimension $2m$, the "underlying differentiable manifold".

3 Lie Groups

Definition 3.1. *A (real) Lie group is a topological group G, which also carries the structure of a differentiable manifold, such that the group multiplication $G \times G \longrightarrow G, (a, b) \mapsto ab$, as well as the inversion $G \longrightarrow G, a \mapsto a^{-1}$ are differentiable maps. If G even carries the structure of a complex manifold and the group operations are holomorphic, we call G a complex Lie group.*

Since real and complex Lie groups often can be treated simultaneously we shall from now on use the letter K in order to denote either $\mathbb{R}$ or $\mathbb{C}$ and use the term K-Lie group in order to refer to real resp. complex Lie groups.

Example 3.2. The general linear group

$$GL_n(K) := \{A \in K^{n,n}; \det(A) \neq 0\}$$

is a K-Lie group. Note that $GL_1(K)$ is nothing but the multiplicative group $K^* := K \setminus \{0\}$ of K.

In the following we shall present a series of closed subgroups $G \subset GL_n(K)$. In order to see that they are even Lie groups we use

Remark 3.3. Let $G \subset GL_n(K)$ be a closed subgroup, such that $G = F^{-1}(0)$ with a map $F : GL_n(K) \longrightarrow K^m$ satisfying $F(AX) = F(X)$ for all $A \in G$. If then $DF(E) : K^{n,n} \longrightarrow K^m$ is onto, the subgroup $G \subset GL_n(K)$ carries a natural differentiable resp. complex structure, and with respect to that structure G is a K-Lie group. According to Example 2.4.3, it suffices to check that

$$DF(A) : K^{n,n} \longrightarrow K^m$$

is onto for all $A \in G$. Denote $\lambda_A : GL_n(K) \longrightarrow GL_n(K), X \mapsto AX$ the left multiplication with A, a diffeomorphism. Since $F = F \circ \lambda_A$, we obtain

$$DF(E) = DF(A) \circ D(\lambda_A)(E) = DF(A) \circ \lambda_A,$$

where we have used the fact that λ_A as a linear map coincides with its own Jacobian - here we denote also λ_A the map $K^{n,n} \longrightarrow K^{n,n}, X \mapsto AX$. But A being invertible, $\lambda_A : K^{n,n} \longrightarrow K^{n,n}$ is an isomorphism of vector spaces, so with $DF(E)$ the map $DF(A)$ is surjective as well. The affine subspace

$$E + \ker DF(E) \subset K^{n,n}$$

is the best approximation of $F^{-1}(0) = G$ at E by an affine subspace, it can naturally be identified with the "tangent space" $T_E G$ of $G = F^{-1}(0)$ at E, to be defined in the next chapter.

Now let us continue with our examples:

Example 3.4. 1. The special linear group

$$SL_n(K) := \{A \in K^{n,n}; \det(A) = 1\}$$

is a closed normal subgroup (as the kernel of the continuous homomorphism $\det : GL_n(K) \longrightarrow K^*$). Take $F(X) = \det(X) - 1$. Since

$$DF(E) = D(\det)(E) = \mathrm{Tr}$$

with the (surjective) trace map $\mathrm{Tr} : K^{n,n} \longrightarrow K, A = (\alpha_{ij}) \mapsto \sum_{i=1}^n \alpha_{ii}$, we can apply Remark 3.3. So $SL_n(K)$ is a K-Lie group.

2. We consider a non degenerate bilinear form $\sigma : K^n \times K^n \longrightarrow K$, write $\sigma(x,y) = x^T S y$ with a matrix $S \in K^{n,n}$. Then a matrix $A \in GL_n(K)$ preserves σ, i.e. $\sigma(Ax, Ay) = \sigma(x,y)$ for all $x, y \in K^n$ iff $A^T S A = S$. Obviously the set of all such "σ-isometries" forms a closed subgroup of $GL_n(K)$. We look at the map

$$F : GL_n(K) \longrightarrow K^{n,n}, X \mapsto X^T S X - S.$$

Then the Jacobian of F at E is

$$DF(E) : K^{n,n} \longrightarrow K^{n,n}, X \mapsto X^T S + SX.$$

Assume now S is either symmetric: $S^T = S$ or antisymmetric: $S^T = -S$. Then $F(GL_n(K)) \subset S_n(K)$ resp. $F(GL_n(K)) \subset A_n(K)$, where $S_n(K)$ denotes the vector space of all symmetric matrices and $A_n(K)$ the vector space of all anti-symmetric matrices. Thus we may replace the target of both F and $DF(E)$ with $S_n(K)$ resp. $A_n(K)$. Since a given matrix $A \in S_n(K)$ resp. $A \in A_n(K)$ is of the form $A = DF(E)(X)$ with $X := \frac{1}{2}(S^{-1}A)$, the Jacobian map of F at E is onto and hence our isometry group even a Lie group. If we take $S = E$ we obtain the K-orthogonal group

$$O_n(K) := \{A \in GL_n(K); A^T A = E\},$$

while for even $n = 2m$ and $S = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ the analogous group

$$Sp_n(K) := \{A \in GL_n(K); A^T S A = S\}$$

is called the K-symplectic group. We have

$$\dim O_n(K) = n^2 - \dim S_n(K) = \frac{1}{2}n(n-1)$$

and

$$\dim Sp_n(K) = n^2 - \dim A_n(K) = \frac{1}{2}n(n+1).$$

We remark that the real orthogonal group $O_n(\mathbb{R})$ is compact, and that $\det(A) = \pm 1$ for $A \in O_n(K)$ as well as for $A \in Sp_n(K)$.

3. Now let us consider the hermitian form $\sigma : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}, \sigma(z,w) = \overline{z}^T w$. The corresponding isometry group is the unitary group

$$U(n) := \{A \in GL_n(\mathbb{C}); \overline{A}^T A = E\},$$

while

$$F : GL_n(\mathbb{C}) \longrightarrow H_n, X \mapsto \overline{X}^T X - E.$$

describes $U(n) = F^{-1}(0)$ as before. Here $H_n \subset \mathbb{C}^{n,n}$ denotes the real subspace of all Hermitian matrices. The Jacobian map

$$DF(E) : \mathbb{C}^{n,n} \longrightarrow H_n, X \mapsto \overline{X}^T + X$$

is again onto. Note that $U(n)$ is not a complex Lie group, since F is not holomorphic!

4. For $G \subset GL_n(K)$ let $SG := G \cap SL_n(K)$. Since matrices $A \in O_n(K)$ have determinant ± 1, $SO_n(K)$ is an open subgroup of $O_n(K)$ of index 2, while $Sp(n)$, as we hopefully shall see later on, is connected, hence not only $\det(A) = \pm 1$, but even $\det(A) = 1$ for all $A \in Sp_n(K)$. The group $SU(n) \subset GL_n(\mathbb{C})$ can be realized as follows: First note that $\det(A) \in \mathbf{S}^1$ for $A \in U(n)$. Let $W := GL_n(\mathbb{C}) \setminus \det^{-1}(\mathbb{R}_{<0})$ and consider the map

$$F : W \longrightarrow H_n \times \mathbb{R},\ X \mapsto (\overline{X}^T X - E, \arg(\det(X)))$$

where, say, $-\pi < \arg(.) < \pi$, with $SU(n) = F^{-1}(0)$ and Jacobian

$$DF(E) : \mathbb{C}^{n,n} \longrightarrow H_n \times \mathbb{R},\ X \mapsto (\overline{X}^T + X, \mathrm{Im}(\mathrm{Tr}(X))),$$

which is onto, since $(A, \lambda) \in H_n \times \mathbb{R}$ has $X = \frac{1}{2}(A + in^{-1}\lambda E)$ as an inverse image.

5. Let $V_0 := \{0\} \subset V_1 \subset ... \subset V_n = K^n$ be an increasing sequence of subspaces (a "flag"). If $V_i = Ke_1 + ... + Ke_i$ the subgroup

$$UT_n(K) := \{A \in GL_n(K); A(V_i) \subset V_i\}$$

is a K-Lie group, consisting of the invertible upper triangular matrices, indeed the underlying differentiable or complex manifold is nothing but $(K^*)^n \times K^{n(n-1)/2}$ and we may argue as in the case of $GL_n(K)$.

There is a canonical homomorphism

$$UT_n(K) \longrightarrow GL(V_n/V_{n-1}) \times ... \times GL(V_2/V_1) \times GL(V_1) \cong (K^*)^n,$$

its kernel $UU_n(K) \subset UT_n$ is K-Lie group, it consists of all upper triangular matrices with diagonal entries equal to 1 (”unipotent” matrices).

All our above Lie groups are closed subgroups of $GL_n(K)$. The below theorem tells us that any such group is indeed a Lie group:

Theorem 3.5. *Let $H \subset G$ be a closed subgroup of the real Lie group G. Then $H \subset G$ is a closed submanifold of G and in particular again a Lie group.*

But since we are far from being able to prove it, we have preferred to give in any particular case a complete argument. – For any topological group G the connected component $G_0 \subset G$ of G containing the neutral element $e \in G$ is a normal subgroup, the group operations being continuous. Since a Lie group G as a topological manifold is locally connected, G_0 is even open, indeed the minimal open subgroup of G, a connected Lie group G being generated by any open neighborhood of $e \in G$.

Remark 3.6. A connected Lie group G is countable at infinity: Take a compact neighborhood $K \subset G$ of the neutral element, w.l.o.g. $K = K^{-1} := \{a^{-1}; a \in K\}$. Then G is the ascending union $G = \bigcup_{n=1}^{\infty} K^n$ with the compact subsets $K^n := \{a_1 \cdot ... \cdot a_n; a_1, ..., a_n \in K\}$: The right hand side is obviously an open subgroup of G, hence equals G, the group G being connected.

4 Vector Fields

On a differentiable manifold M there is no natural notion of a derivative of a function $f \in C^\infty(M)$, since in order to differentiate we need local coordinates. We are now going to define objects with respect to which functions can be differentiated in a point $a \in M$ or globally.

Definition 4.1. *A tangent vector X_a at a point $a \in M$ of a differentiable manifold M is a linear map $X_a : C^\infty(M) \longrightarrow \mathbb{R}$ satisfying the following Leibniz rule:*

$$X_a(fg) = f(a)X_a(g) + X_a(f)g(a).$$

The set of all tangent vectors of M at $a \in M$ forms a vector space T_aM, called the tangent space *of M at $a \in M$.*

Remark 4.2. (1) We have $X_a(\mathbb{R}) = 0$ for every tangent vector $X_a \in T_aM$, since $X_a(1) = X_a(1^2) = X_a(1) + X_a(1)$.

(2) Take a chart $\varphi : U \to V \subset \mathbb{R}^n$ with $a \in U$ and $\varphi(a) = 0$. Then the maps

$$\partial_i^a := \partial_i^{\varphi,a} : f \mapsto \frac{\partial f \circ \varphi^{-1}}{\partial x_i}(0),\ i = 1, ..., n,$$

are tangent vectors at a.

(3) Another, may be more geometric, construction that avoids the choice of charts is the following: To any curve, i.e. differentiable map, $\gamma : \mathcal{I} \to M$ defined on an open interval $\mathcal{I} \subset \mathbb{R}$ with $\gamma(t_0) = a$ for some $t_0 \in \mathcal{I}$ we can associate the tangent vector $\dot{\gamma}(t_0) \in T_aM$ defined by

$$\dot{\gamma}(t_0) : f \mapsto (f \circ \gamma)'(t_0)\ .$$

The vector $\dot{\gamma}(t_0)$ is called the tangent vector of the curve $\gamma : \mathcal{I} \to M$ at $t_0 \in \mathcal{I}$. In particular the tangent space $T_a\mathbb{R}^n$ is naturally isomorphic to $\mathbb{R}^n$ itself: associate to $x \in \mathbb{R}^n$ the tangent vector $\dot{\gamma}_x(0)$ with the curve $\gamma_x(t) := a + tx$. The adjective "natural" means here that it only depends on $\mathbb{R}^n$ as vector space, not on the choice of a particular base (e.g. the standard base) of $\mathbb{R}^n$.

Theorem 4.3. *Using the notation of Remark 4.2.2 we have*

$$T_aM = \bigoplus_{i=1}^{n} \mathbb{R} \cdot \partial_i^a,$$

i.e. the tangent vectors $\partial_i^a := \partial_i^{\varphi,a}$ form a basis of the tangent space T_aM.

For the proof we need

Lemma 4.4. *1. If $f \in C^\infty(M)$ vanishes in a neighborhood of $a \in M$, then $X_a(f) = 0$ for all tangent vectors $X_a \in T_aM$.*

2. Let $U \subset M$ be open. Denote $\varrho : C^\infty(M) \longrightarrow C^\infty(U), f \mapsto f|_U$ the restriction from M to U. Then the map

$$T_aU \longrightarrow T_aM, X_a \mapsto X_a \circ \varrho,$$

is an isomorphism.

Proof. 1.) If f vanishes near a, take a function $g \in C^\infty(M)$ with $g = 1$ near a and $fg = 0$. Then $0 = X_a(fg) = g(a)X_a(f) + f(a)X_a(g) = X_a(f)$.
2.) Injectivity: Assume $X_a \circ \varrho = 0$. Take any function $f \in C^\infty(U)$. Choose $\tilde{f} \in C^\infty(M)$ with $\tilde{f} = f$ near a. Then, according to the first part, we have $X_a(f) = X_a(\tilde{f}|_U) = 0$. Now the function $f \in C^\infty(U)$ being arbitrary, we obtain $X_a = 0$.
Surjectivity: For $Y_a \in T_aM$ define $X_a \in T_aU$ by its value on $f \in C^\infty(U)$ as follows

$$X_a(f) := Y_a(\tilde{f}),$$

where again $\tilde{f} \in C^\infty(M)$ with $\tilde{f} = f$ near a. Then $X_a(f)$ is well defined as a consequence of the first part and obviously $X_a \circ \varrho = Y_a$. □

Proof of 4.3. As a consequence of 4.4 we may assume, with the notation of Rem.4.2.2, $M = U = V \subset \mathbb{R}^n$ and show that the tangent vectors $\partial_i^0 \in T_0V$ with $\partial_i^0(f) := \frac{\partial f}{\partial x_i}(0)$ form a basis of T_0V. Since $\partial_i^0(x_j) = \delta_{ij}$ they are linearly independent. On the other hand, for any $X_0 \in T_0V$ we have

$$X_0 = \sum_{i=1}^n X_0(x_i)\partial_i^0.$$

Take $f \in C^\infty(V)$. After, may be, a shrinking of V we may assume $f = f(a) + \sum_{i=1}^n x_i f_i$ with $f_i \in C^\infty(V)$ and then obtain $X_a(f) = \sum_{i=1}^n X_a(x_i)f_i(a) = \sum_{i=1}^n X_a(x_i)\partial_i^a(f)$. □

Tangent vectors on a complex manifold (Only for readers feeling at home in complex analysis!): For a complex manifold M we have to modify the definition of a tangent vector, since $\mathcal{O}(M)$ may be too small. Instead of working with $\mathcal{O}(M)$ we have to introduce the concept of a germ of a function near $a \in M$. Let us do that simultaneously for differentiable and complex manifolds: We consider pairs (U, f), where $f : U \longrightarrow K$ is any K-valued function defined on an open neighborhood U of $a \in M$. A *germ* of a function is an equivalence class of such pairs, where two pairs (U, f) and (V, g) are equivalent if there is an open neighborhood $W \subset U \cap V$ of a with $f|_W = g|_W$. Given $f : U \longrightarrow K$ the corresponding germ is denoted f_a. We leave it to the reader to define the sum and product of germs. Then we obtain the vector spaces C_a^∞ and $\mathcal{O}_a$ of germs of differentiable resp. holomorphic functions near $a \in M$. It follows from the above remark 4.2.2, that any tangent vector $X_a : C^\infty(M) \longrightarrow \mathbb{R}$ factorizes uniquely through a linear map $C_a^\infty \longrightarrow \mathbb{R}$ satisfying the Leibniz rule, and on the other hand, every such map yields a tangent vector by composition with the natural (surjective!) map

$$C^\infty(M) \longrightarrow C_a^\infty, f \mapsto f_a,$$

associating to a function $f \in C^\infty(M)$ its germ at $a \in M$. Since for a complex manifold the corresponding map

$$\mathcal{O}(M) \longrightarrow \mathcal{O}_a, f \mapsto f_a,$$

is never surjective for $\dim M > 0$, we have to define a complex tangent vector at $a \in M$ as a $\mathbb{C}$-linear map $Z_a : \mathcal{O}_a \longrightarrow \mathbb{C}$ satisfying the Leibniz rule. Again we obtain a basis $\partial_1^a, ..., \partial_n^a$ of T_aM given by complex differentiation with respect to local complex coordinates. Furthermore, given a holomorphic "curve", i.e. a holomorphic map $\gamma : G \longrightarrow M$ defined on an open subset $G \subset \mathbb{C}$ we may define its tangent vector $\dot{\gamma}(z_0)$ for any $z_0 \in G$ literally as in the real case.

Given a complex manifold M, denote $M_\mathbb{R}$ the underlying differentiable manifold. Then there is a natural identification of real tangent vectors at a, i.e. tangent vectors of the differentiable manifold $M_\mathbb{R}$, and complex tangent vectors: More precisely, for $a \in M$, there is a natural isomorphism of real vector spaces (Only the target is even a complex vector space!)

$$T_a(M_\mathbb{R}) \longrightarrow T_aM, X_a \mapsto Z_a := X_a^\mathbb{C}|_{\mathcal{O}_a},$$

where

$$X_a^\mathbb{C} := X_a + iX_a : C_a^{\infty,\mathbb{C}} = C_a^\infty \oplus iC_a^\infty \longrightarrow \mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$$

with the vector space $C_a^{\infty,\mathbb{C}} = C_a^\infty \oplus iC_a^\infty$ of germs of complex valued differentiable functions near $a \in M$. To be less mysterious and more explicit: Given complex local coordinates $z_1 = x_1 + iy_1, ..., z_n = x_n + iy_n$ near a, the above map acts as follows

$$\frac{\partial^a}{\partial x_k} \mapsto \frac{\partial^a}{\partial z_k}, \ \frac{\partial^a}{\partial y_k} \mapsto i\frac{\partial^a}{\partial z_k},$$

since the above derivations coincide on $\mathcal{O}_a \subset C_a^{\infty,\mathbb{C}}$.

Differentiable maps induce linear maps between tangent spaces:

Definition 4.5. Given a differentiable map $F : M \to N$ between the differentiable manifolds M and N, there is an induced homomorphism of tangent spaces:

$$F_* := T_a F : T_a M \to T_{F(a)} N$$

defined by

$$F_*(X_a) : C^\infty(N) \to \mathbb{R}, f \mapsto X_a(f \circ F) \ .$$

It is called the *tangent map* of F at $a \in M$.

Obviously we have for a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = a$ that

$$F_*(\dot{\gamma}(0)) = \dot{\delta}(0), \text{ where } \delta := F \circ \gamma \ .$$

For explicit computations we note that, if $F = (F_1, ..., F_m) : U \to W$ is a differentiable map between the open sets $U \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$, and $b = F(a)$ for $a \in U$, then with respect to the bases $\partial_1^a, ..., \partial_n^a$ of $T_a U$ and $\partial_1^b, ..., \partial_m^b$ of $T_b W$ the linear map $T_a F$ has the matrix:

$$DF(a) = \left(\frac{\partial F_i}{\partial x_j}(a) \right)_{1 \le i \le m, 1 \le j \le n} \in \mathbb{R}^{m,n} \ ,$$

the Jacobi matrix of F at $a \in U$.

Furthermore it is immediate from the definition, that the tangent map behaves functorially, i.e. if $F : M_1 \to M_2$ and $G : M_2 \to M_3$ are differentiable maps, then $G \circ F : M_1 \to M_3$ is again differentiable and the chain rule

$$T_a(G \circ F) = T_{F(a)} G \circ T_a F$$

holds.

All the tangent vectors at points in a differentiable n-manifold M form a differentiable n^2-manifold:

Definition 4.6. Let M be a differentiable n-manifold. The *tangent bundle* TM is, as a set, the disjoint union

$$TM := \bigcup_{a \in M} T_a M$$

of all tangent spaces at points $a \in M$. Denote $\pi : TM \to M$ the map, which associates to a tangent vector $X_a \in T_aM$ its "base point" $a \in M$. Now given a chart $\varphi : U \xrightarrow{\cong} V \subset \mathbb{R}^n$ on M, we consider the bijective map (*trivialization*)

$$T\varphi := (\varphi \circ \pi, \varphi_*) : \pi^{-1}(U) \to V \times \mathbb{R}^n,$$

where we use the natural isomorphism $T_a\mathbb{R}^n \cong \mathbb{R}^n$ as explained above and $\varphi_*|_{T_aM} = T_a\varphi$. We endow TM with a topology: a set $W \subset TM$ is open if $T\varphi(W \cap U) \subset \mathbb{R}^n \times \mathbb{R}^n$ is open for all charts (U, φ) in an atlas $\mathcal{A}$ for M. Finally, the charts $(\pi^{-1}(U), T\varphi)$ with $(U, \varphi) \in \mathcal{A}$ define an atlas on TM.

In order to see that this idea works we need to understand the coordinate changes for two charts $(\pi^{-1}(U), T\varphi)$ and $(\pi^{-1}(\tilde{U}), T\tilde{\varphi})$ of TM: We may assume $\tilde{U} = U$ and then obtain with $F := \tilde{\varphi} \circ \varphi^{-1} : V \longrightarrow \tilde{V}$ the following coordinate change on the tangent level:

$$T\tilde{\varphi} \circ (T\varphi)^{-1} : V \times \mathbb{R}^n \longrightarrow \tilde{V} \times \mathbb{R}^n, (x, y) \mapsto (F(x), DF(x)y).$$

Now we can generalize Definition 4.5: given a differentiable map $F : M \to N$ the pointwise tangent maps $T_aF : T_aM \to T_{F(a)}N$ combine to a differentiable map $TF : TM \to TN$, i.e.

$$TF|_{T_aM} := T_aF : T_aM \to T_{F(a)}N.$$

Indeed, the map TF fits into a commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{TF} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{F} & N \end{array},$$

i.e. $\pi_N \circ TF = F \circ \pi_M$ holds with the projections $\pi_M : TM \longrightarrow M$ and $\pi_N : TN \to N$ of the respective tangent bundles.

Remark 4.7. For a complex manifold M the above construction works as well, leading to a complex manifold TM, its *holomorphic tangent bundle*. On the other hand, we already have seen that for a complex manifold M, there is a natural isomorphism

$$T_a(M_{\mathbb{R}}) \cong T_aM$$

for every point $a \in M$. Since that isomorphism depends "differentiably" on the base point $a \in M$, there is a natural identification

$$T(M_{\mathbb{R}}) = TM$$

of differentiable manifolds.

Definition 4.8. A (holomorphic) *vector field* X on an open subset $U \subset M$ of a differentiable (complex) manifold M is a differentiable (holomorphic) section of the projection $\pi : TM \to M$, i.e., a differentiable (holomorphic) map

$$X : U \to TM$$

satisfying $\pi \circ X = \mathrm{id}_U$, with other words, $X(a) \in T_aM$ for all $a \in U$. In that case we also write $X_a := X(a)$. We denote $\Theta(U)$ the set of all (holomorphic) vector fields on $U \subset M$.

Remark 4.9. (1) The set $\Theta(U)$ carries, with the argument-wise algebraic operations, in a natural way the structure of a real vector space. In fact the scalar multiplication

$$\mathbb{R} \times \Theta(U) \to \Theta(U)$$

can be extended to a multiplication by functions:

$$C^\infty(U) \times \Theta(U) \to \Theta(U), (f, X) \mapsto fX,$$

where

$$(fX)_a := f(a)X_a \ .$$

That follows immediately from the fact that the maps $T_a\varphi : T_aM \longrightarrow \mathbb{R}^n$ are linear isomorphisms.

(2) Let (U, φ) be a chart. Then

$$\partial_i := \partial_i^\varphi : U \longrightarrow TM, a \mapsto \partial_i^a \quad \text{for } i = 1, ..., n,$$

with

$$\partial_i^a(f) := \frac{\partial f \circ \varphi^{-1}}{\partial x_i}(\varphi(a))$$

are vector fields on U, the "coordinate vector fields" associated to the local chart (or local coordinates) $x_i = \varphi_i(a), i = 1, ..., n$. Since $T_aM = \bigoplus_{i=1}^n \mathbb{R}\partial_i^a$, any section $X : U \longrightarrow TM$ of $\pi : TM \longrightarrow M$ can be written

$$X = \sum_{i=1}^{n} g_i \partial_i,$$

with functions $g_i : U \longrightarrow \mathbb{R}$, and we have

$$X \in \Theta(U) \Longleftrightarrow g_1, ..., g_n \in C^\infty(U).$$

(3) Note that on an arbitrary differentiable manifold M it is in general not possible to find vector fields $X_1, ..., X_n \in \Theta(M)$, such that $(X_1)_a, ..., (X_n)_a$ is a *frame* at a, i.e., a basis of T_aM, for all $a \in M$. If such vector fields exist, the manifold M is called *parallelizable.* So the open subsets U, where a local chart $\varphi : U \longrightarrow V$ is defined, are always parallelizable. As we shall see in the next section the underlying manifold of a Lie group is always parallelizable.

(4) The vector fields on M can be identified with derivations $D : C^\infty(M) \to C^\infty(M)$, i.e. linear maps satisfying the Leibniz rule $D(fg) = D(f)g + fD(g)$ for all $f, g \in C^\infty(M)$. Given a vector field $X \in \Theta(M)$ the corresponding derivation $X : C^\infty(M) \to C^\infty(M), f \mapsto X(f)$ is defined by $(X(f))(a) := X_a(f)$. In fact, every derivation $D : C^\infty(M) \to C^\infty(M)$ is obtained from a vector field: Take $X \in \Theta(M)$ with

$$X_a : C^\infty(M) \to \mathbb{R}, f \mapsto D(f)(a) \ .$$

(5) For an open subset $U \subset M$ the tangent bundle TU is identified, in a natural way, with the open subset $\pi^{-1}(U) \subset TM$.

(6) Let $F : M \to N$ be a differentiable map. Given a vector field $X \in \Theta(M)$, we can consider $TF \circ X : M \to TN$, but that map does not in general factor through N, e.g. if F is not injective. But it does if $F : M \to N$ is a diffeomorphism: then we may define a map

$$F_* : \Theta(M) \to \Theta(N), X \mapsto F_*(X) := TF \circ X \circ F^{-1} \in \Theta(N),$$

the push forward of vector fields with respect to a diffeomorphism.

The vector space $\Theta(M)$ carries a further algebraic structure: though the compositions XY and YX of two derivations $X, Y : C^\infty(M) \to C^\infty(M)$ are no longer derivations, their commutator is:

$$\begin{array}{rcl} (XY - YX)fg & = & XY(fg) - YX(fg) \\ & = & X(fYg + gYf) - Y(fXg + gXf) \\ & = & fXYg + (Xf)Yg + gXYf + (Xg)Yf \\ & - & fYXg - (Yf)(Xg) - gYXf - (Yg)(Xf) \\ & = & fXYg - fYXg + gXYf - gYXf \\ & = & f(XY - YX)g + g(XY - YX)f. \end{array}$$

Definition 4.10. The *Lie bracket* $[X, Y] \in \Theta(M)$ of two vector fields $X, Y \in \Theta(M)$ is the commutator of the derivations $X, Y : C^\infty(M) \to C^\infty(M)$, i.e.

$$[X, Y] := XY - YX,$$

or, in other words, the vector field $[X, Y]$ satisfying

$$[X, Y]_a(f) := X_a(Y(f)) - Y_a(X(f))$$

for all differentiable functions $f \in C^\infty(M)$ at every point $a \in M$.

Note that the tangent vector $[X, Y]_a$ is not a function of the values $X_a, Y_a \in T_aM$ only, since the local behavior of the vector fields X, Y near $a \in M$ also enters in the computation rule. If $x_1, ..., x_n$ are local coordinates on $U \subset M$, and $X, Y \in \Theta(U)$ have representations

$$X = \sum_{i=1}^{n} f_i \, \partial_i, \; Y = \sum_{i=1}^{n} g_i \, \partial_i,$$

then

$$[X, Y] = \sum_{i=1}^{n} (X(g_i) - Y(f_i)) \, \partial_i \; .$$

So, in particular, $[\partial_i, \partial_j] = 0$ for coordinate vector fields. On the other hand we mention:

Theorem 4.11. (Frobenius Theorem) *Let $X_1, ..., X_n \in \Theta(M)$ be pairwise commuting vector fields, i.e. $[X_i, X_j] = 0$ for $1 \le i, j \le n$. Then every* *point $a \in M$, such that $(X_1)_a, ..., (X_n)_a$ is a frame at a (i.e. a basis of the tangent space T_aM) admits a neighborhood $U \subset M$ with local coordinates $x_1, ..., x_n \in C^\infty(U)$ such that*

$$X_i|_U = \partial_i.$$

The proof relies on the fact that given the flows (μ_t) and $(\tilde{\mu}_t)$ of commuting vector fields $X, \tilde{X} \in \Theta(M)$, one has $\mu_s \circ \tilde{\mu}_t = \tilde{\mu}_t \circ \mu_s$ for all sufficiently small $s, t \in \mathbb{R}$.

The Lie bracket defines on the vector space $\Theta(M)$ the structure of a *Lie algebra*, a notion which plays a key rôle in the investigation of Lie groups:

Definition 4.12. A *Lie algebra* $\mathfrak{g}$ over K is a K-vector space endowed with an antisymmetric (or alternating) bilinear map $[..,..] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, i.e.

$$[X, Y] = -[Y, X], \ \forall \ X, Y \in \mathfrak{g}, \text{ in particular } [X, X] = 0,$$

such that the *Jacobi identity* holds:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

We shall need later on

Proposition 4.13. *Let $F : M \longrightarrow N$ be a diffeomorphism resp. a biholomorphic map. Then the push forward of vector fields $F_* : \Theta(M) \longrightarrow \Theta(N)$ is an isomorphism of Lie algebras.*

Proof. Regarding $X \in \Theta(M)$ as a derivation $X : C^\infty(M) \longrightarrow C^\infty(M)$ the derivation $F_*(X) : C^\infty(N) \longrightarrow C^\infty(N)$ satisfies

$$F_*(X)g = X(g \circ F) \circ F^{-1}.$$

Hence

$$F_*([X, Y])g = XY(g \circ F) \circ F^{-1} - YX(g \circ F) \circ F^{-1},$$

while

$$F_*(X)F_*(Y)g = F_*(X)(Y(g \circ F) \circ F^{-1}) = XY(g \circ F) \circ F^{-1}$$

and in the same way

$$F_*(Y)F_*(X)g = YX(g \circ F) \circ F^{-1}.$$

This implies our claim. □

Definition 4.14. Let $X \in \Theta(M)$ be a vector field. A smooth curve $\gamma : \mathcal{I} \to M$ resp. a holomorphic curve $\gamma : G \to M$ defined on an open interval $\mathcal{I} \subset \mathbb{R}$ resp. on an open domain $G \subset \mathbb{C}$ is called an *integral curve* of the vector field X, if $\dot{\gamma}(t) = X_{\gamma(t)}$ for all $t \in \mathcal{I}$ resp. $t \in G$.

Remark 4.15. The basic theorem in the theory of ordinary differential equations says that, given a vector field $X \in \Theta(M)$ on a differentiable manifold M and a point $a \in M$, there is an integral curve $\gamma : \mathcal{I} \to M$ defined on an open interval $\mathcal{I} \ni 0$ such that $\gamma(0) = a$, and if $\tilde{\gamma} : \tilde{\mathcal{I}} \to M$ is a second such curve, then γ and $\tilde{\gamma}$ coincide on the intersection $\mathcal{I} \cap \tilde{\mathcal{I}}$. If M is compact, we can always assume $\mathcal{I} = \mathbb{R}$. Moreover, there is a differentiable map $\mu : U \to M$ defined on an open neighborhood $U \subset M \times \mathbb{R}$ of $M \times \{0\} \hookrightarrow M \times \mathbb{R}$, such that $U \cap (\{x\} \times \mathbb{R})$ is an interval for all $x \in M$ and $t \to \mu(x, t)$ an integral curve of X satisfying $\mu(x, 0) = x$. If, moreover, M is compact, this map is defined on all of $M \times \mathbb{R}$. The *flow* of the vector field X then is the family $(\mu_t)_{t \in \mathbb{R}}$ of the differentiable maps

$$\mu_t : M \to M, x \mapsto \mu(x, t).$$

In fact, we have

$$\mu_0 = \mathrm{id}_M, \mu_{s+t} = \mu_s \circ \mu_t,$$

since integral curves are uniquely determined by their initial values and $t \mapsto \gamma(s + t)$ is an integral curve with the value $\gamma(s)$ at $t = 0$. Since $\mu_0 = \mathrm{id}_M$, it follows that each map $\mu_t : M \to M$ is a diffeomorphism with inverse μ_{-t}.

5 The Lie Algebra of a Lie Group

For a Lie group G there are distinguished vector fields: Given an element $a \in G$ we denote $\lambda_a : G \longrightarrow G, x \mapsto ax$, resp. $\varrho_a : G \longrightarrow G, x \mapsto xa$, the left resp. right multiplication ("left resp. right translation") with $a \in G$. They define diffeomorphisms (resp. biholomorphic maps) $G \longrightarrow G$ and induce Lie algebra isomorphisms

$$(\lambda_a)_*, (\varrho_a)_* : \Theta(G) \longrightarrow \Theta(G).$$

Definition 5.1. *A vector field $X \in \Theta(G)$ on a Lie group G is called left resp. right invariant if $(\lambda_a)_*(X) = X$ resp. $(\varrho_a)_*(X) = X$ for all $g \in G$. We denote $\mathrm{Lie}(G) \subset \Theta(G)$ or simply $\mathfrak{g} := \mathrm{Lie}(G)$ the subspace of all left invariant vector fields.*

In the following we usually consider only left invariant vector fields, but everything holds - mutatis mutandis - for right invariant vector fields as well.

Remark 5.2. 1. The subspace $\mathfrak{g} \subset \Theta(G)$ is a Lie subalgebra, i.e. closed with respect to the Lie bracket:

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}.$$

This follows immediately from the fact that the $(\lambda_a)_* : \Theta(G) \longrightarrow \Theta(G)$ are Lie group homomorphisms.

2. A vector field $X \in \Theta(G)$ is left invariant iff $X_a = T_e(\lambda_a)(X_e)$ for all $a \in G$. As a consequence the evaluation map

$$\mathfrak{g} \longrightarrow T_eG, X \mapsto X_e$$

is an isomorphism of vector spaces. Hence $\mathfrak{g}$ is a finite dimensional vector space with $\dim \mathfrak{g} = \dim G$ (where $\dim G$ denotes the dimension of G as differentiable or complex manifold). Indeed, usually one identifies $\mathfrak{g}$ with T_eG via the above isomorphism.

Example 5.3. Let us consider $G = GL_n(K)$ and denote $\Xi := (\xi_{ij})$ a variable matrix in $GL_n(K)$. Furthermore we need the following matrix

$$\Delta := (\partial_{ij}),$$

the entries of which are the coordinate vector fields with respect to the coordinates ξ_{ij}. To simplify notation we introduce a scalar product

$$A \cdot B := \sum_{i,j} A_{ij} B_{ij}$$

for matrices $A = (A_{ij}), B = (B_{ij})$. Since $T_E\lambda_C = \lambda_C$, the left invariant vector field X with $X_E = A \cdot \Delta^E = \sum_{i,j} A_{ij}\partial_{ij}^E$ looks as follows

$$X = (\Xi A) \cdot \Delta \ = \sum_{i,j} (\Xi A)_{ij} \partial_{ij}.$$

Let us now compute the Lie bracket of vector fields $X = (\Xi A) \cdot \Delta, Y = (\Xi B) \cdot \Delta$. We already know that

$$[X, Y] = (X(\Xi B) - Y(\Xi A)) \cdot \Delta,$$

where X and Y apply to each entry of the matrices ΞB and ΞA separately. Now

$$X(\Xi B) = X(\Xi)B$$

and

$$\partial_{ij}\Xi = E(i,j)$$

with the matrix $E(i,j)$ such that $E(i,j)_{k\ell} = \delta_{ki}\delta_{\ell j}$. Thus, using

$$X(\Xi) = ((\Xi A)\cdot\Delta)(\Xi) = \sum_{i,j}(\Xi A)_{ij}\partial_{ij}(\Xi) = \sum_{i,j}(\Xi A)_{ij}E(i,j) = \Xi A,$$

we obtain

$$X(\Xi B) = \Xi AB.$$

Since by symmetry

$$Y(\Xi A) = \Xi BA,$$

we finally arrive at

$$[X,Y] = (\Xi\,[A,B])\cdot\Delta$$

with the commutator

$$[A,B] = AB - BA$$

of the matrices $A, B \in K^{n,n}$.

Definition 5.4. *A homomorphism $\varphi : G \longrightarrow H$ between Lie groups G, H is a differentiable resp. holomorphic group homomorphism. We denote* $\mathrm{Hom}(G,H)$ *the set of all such homomorphisms.*

Remark 5.5. The kernel and image of a homomorphism $\varphi : G \longrightarrow H$ of Lie groups:

1. The kernel $\ker(\varphi) \subset G$ is clearly a closed subgroup. Indeed it is easily seen to be also a submanifold: The tangent map $T_a\varphi : T_aG \longrightarrow T_{\varphi(a)}H$ has for all $a \in G$ the same rank, and as a consequence of that, there is for every $a \in G$ an open neighborhood $U \subset G$ of a and $W \subset H$ of $\varphi(a)$, such that with respect to suitable local coordinates $x_1, ..., x_n$ on U and $y_1, ..., y_m$ on W the map φ takes the form $(x_1, ..., x_n) \mapsto (x_1, ..., x_k, 0, ...0)$ with some $k \leq n$.

2. The image $\varphi(G) \subset H$ need neither be a closed subgroup of H nor a topological manifold with respect to the relative topology as a subspace of H. As an example consider the injective homomorphism $\varphi : \mathbb{R} \longrightarrow \mathbf{S}^1 \times \mathbf{S}^1, t \mapsto (e^{it}, e^{i\sqrt{2}t})$ – the real line wound up around a life belt. But if we refine the relative topology then it is: Define a base for the topology of $\varphi(G)$ to consist of the sets $\varphi(U)$ with $U \subset G$ as above. The differentiable structure now is inherited from H, the sets $\varphi(U)$ being submanifolds of H. So there is a unique Lie group structure on $\varphi(G)$! On the other hand, if $\varphi(G) \subset H$ is closed and G connected, then both Lie group structures coincide, that one as a closed subgroup of H and that as an image of a homomorphism of Lie groups. In order to see that one uses the fact that G is countable at infinity and that an open subset of $\mathbb{R}^m$ never is the countable union of closed submanifolds of dimension $< m$ (use e.g. measure theory - a compact part of such a submanifold is a Lebesgue zero set in $\mathbb{R}^m$.).

Proposition 5.6. *Any homomorphism of Lie groups $\varphi : G \longrightarrow H$ induces a homomorphism of the corresponding Lie algebras $d\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$.*

Proof. The homomorphism $d\varphi$ is defined as $\mathfrak{g} \cong T_eG \xrightarrow{T_e\varphi} T_eH \cong \mathfrak{h}$, and we have to show that $d\varphi$ preserves the Lie bracket. We may assume that both G and H are connected and then consider separately $G \longrightarrow \varphi(G)$ (resp. the case of a surjective homomorphism) and the inclusion $\varphi(G) \longrightarrow H$ resp. an injective homomorphism.

The first case is easy: Since $\varphi : G \longrightarrow H$ is onto, the pull back $\varphi^* : C^\infty(H) \longrightarrow C^\infty(G)$ is injective. Now $\mathfrak{g}$ can be regarded as the set of all derivations $D : C^\infty(G) \longrightarrow C^\infty(G)$ commuting with left translation, i.e. $D(f \circ \lambda_a) = (Df) \circ \lambda_a$ and $d\varphi$ is nothing but the restriction $D \mapsto D|_{C^\infty(H)}$. The claim follows, since the Lie bracket is nothing but the commutator of two derivations. But note that a general derivation does not satisfy $D(C^\infty(H)) \subset C^\infty(H)$.

The case of an injective homomorphism (resp. an inclusion) $\varphi : G \longrightarrow H$: Given $X \in \mathfrak{g}$ the vector field $\hat{X} := d\varphi(X) \in \mathfrak{h}$ is the unique left invariant extension of X, i.e. $\hat{X}|_G = X$. Now the claim follows from the equality

$$[\hat{X}, \hat{Y}]|_G = [X, Y],$$

the left hand side being a left invariant vector field. It is easily checked using local coordinates $x_1, ..., x_n$ with $G \subset H$ being defined by $x_{k+1} = ... = x_n = 0$.

If we write

$$\hat{X} = \sum_{i=1}^{n} f_i \partial_i, \ \hat{Y} = \sum_{i=1}^{n} g_i \partial_i,$$

then $g_i|_G = 0 = f_i|_G$ for $i \geq k+1$. Hence

$$[\hat{X}, \hat{Y}] = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} f_j \partial_j g_i - g_j \partial_j f_i \right) \partial_i$$

and

$$[\hat{X}, \hat{Y}]|_G = \sum_{i=1}^{k} \left(\sum_{j=1}^{k} f_j \partial_j g_i - g_j \partial_j f_i \right) \partial_i = [X, Y],$$

since we have $g_j|_G = 0 = f_j|_G$ for $j \geq k+1$ and $\partial_j g_i = 0 = \partial_j f_i$ for $j \leq k, i \geq k+1$. □

Example 5.7. Let us discuss the Lie algebras $\mathfrak{g} = \mathrm{Lie}(G)$ for the Lie groups of Example 3.3. There $G \subset GL_n(K)$ is a closed subgroup of $GL_n(K)$. Then $\mathfrak{g} \subset \mathfrak{gl}_n(K) = K^{n,n}$ is a subalgebra, the kernel of the map $DF(E) : \mathfrak{gl}_n(K) \longrightarrow K^m$. We obtain:

1. $\mathfrak{sl}_n(K) = \{A \in K^{n,n}; \mathrm{Tr}(A) = 0\}$.

2. If $G \subset GL_n(K)$ is the isometry group of a the bilinear form $\sigma(x, y) = x^T S y$, then $\mathfrak{g} = \{A \in K^{n,n}; A^T S + SA = 0\}$, in particular

   $$\mathfrak{so}_n(K) = \{A \in K^{n,n}; A^T + A = 0\}$$

   consists of the skew symmetric matrices.

3. $\mathfrak{u}_n = \{A \in \mathbb{C}^{n,n}; \overline{A}^T = -A\}$.

4. $\mathfrak{su}_n = \{A \in \mathbb{C}^{n,n}; \overline{A}^T = -A, \mathrm{Tr}(A) = 0\}$.

5. $\mathfrak{ut}_n(K) := \mathrm{Lie}(UT_n(K))$ consists of all upper triangular matrices and

6. $\mathfrak{uu}_n(K) := \mathrm{Lie}(UU_n(K))$ of all upper triangular matrices with only zeros on the diagonal.

6 Homogeneous Spaces

Let $H \subset G$ be a closed subgroup of the Lie group G. We want to explain how to make the left coset space $B := G/H$ a differentiable manifold. Denote $\pi : G \longrightarrow B, g \mapsto gH$, the quotient projection. We endow B with the π-quotient topology. Then π becomes an open map, since $\pi^{-1}(\pi(V)) = \bigcup_{h\in H} Vh$.

Now let us show that B is Hausdorff: Consider two different points $x = aH, y = bH$. Equivalently $(a, b) \in G \times G \setminus L$ with the closed set $L = \psi^{-1}(H)$, where $\psi : G \times G \longrightarrow G, (\xi, \eta) \mapsto \xi^{-1}\eta$, is continuous. Hence $L \subset G \times G$ is closed, and there is a neighbourhood $U \times V \subset G \times G \setminus L$ of (a, b) with open $U, V \subset G$. Then $\pi(U)$ and $\pi(V)$ are disjoint open neighbourhoods of the points $x, y \in G$.

Indeed $L \subset G \times G$ is even a submanifold as the inverse image of a submanifold with respect to a submersion: A differentiable map $f : M \longrightarrow N$ is called a submersion, if all its tangent maps $T_af : T_aM \longrightarrow T_{f(a)}N$ are onto (:=surjective). Since

$$g\psi(\xi, \eta)\tilde{g} = \psi(\xi g^{-1}, \eta\tilde{g}),$$

and $(\xi, \eta) \mapsto (\xi g^{-1}, \eta\tilde{g})$ is a diffeomorphism $G \times G \longrightarrow G \times G$ as well as $\zeta \mapsto g\zeta\tilde{g}$ a diffeomorphism $G \longrightarrow G$, it suffices to check that

$$T_{(e,e)}\psi : T_{(e,e)}(G \times G) \cong T_eG \oplus T_eG \longrightarrow T_eG$$

is onto; indeed $T_{(e,e)}\psi(X_e, Y_e) = Y_e - X_e$. This follows from the fact that the group law $\mu : G \times G \longrightarrow G, (\xi, \eta) \mapsto \xi\eta$ has differential $T_{(e,e)}\mu(X_e, Y_e) = X_e + Y_e$ as a consequence of $\mu(., e) = \mathrm{id}_G = \mu(e, .)$. It follows immediately that the inversion $\iota : G \longrightarrow G, \xi \mapsto \xi^{-1}$ has differential $T_e\iota(X_e) = -X_e$.

Our next aim is to construct on a suitable neighbourhood U of $x_0 := eH$ a continuous section $\sigma : U \longrightarrow G$ of the quotient projection π, i.e., such that $\pi \circ \sigma := \mathrm{id}_U$. As candidate for $\sigma(U)$ we choose a slice $S \subset G$ to the submanifold H at e, i.e. $S \subset G$ is a submanifold, $e \in S$ and $T_eG = T_eS \oplus T_eH$. If G has dimension n and H codimension $\ell := \dim G - \dim H$, then we may take $S \cong B^\ell$ with an open ball $B^\ell \subset K^\ell$. The main point now is to prove that after a shrinking of S the image $U := \pi(S)$ is open in B and the map

$$\tau : S \times H \longrightarrow \pi^{-1}(U), (s, h) \mapsto sh$$

diffeomorphic resp. biholomorphic. First of all, the map σ is diffeomorphic (biholomorphic) near (e, e). If $\sigma : V_1 \times V_2 \longrightarrow W \ni e$ has that property, replace S with V_1; then we have $\pi^{-1}(U) = \bigcup_{h\in H} Wh$, so $U := \pi(S)$ is open and our map $\tau : S \times H \longrightarrow \pi^{-1}(U)$ a surjective locally diffeomorphic resp. biholomorphic map (since $\pi(s, h) = \pi(s, e)h$). It remains to show that it is injective. That is true if $\pi|_S$ is injective resp.

$$(S \times S) \cap L = \Delta := \{(s, s); s \in S\}.$$

If we can prove that $(S \times S) \cap L$ is an ℓ-dimensional submanifold, we are done, since $\Delta \subset (S \times S) \cap L$ is as well, hence not only closed, but also open in $(S \times S) \cap L$, i.e. it is the connected component of $(S \times S) \cap L$ containing (e, e). Thus a shrinking of S leads to the desired equality $(S \times S) \cap L = \Delta$.

In order to see that $(S \times S) \cap L$ is, after some shrinking, an ℓ-dimensional manifold, we note that

$$T_{(e,e)}(G \times G) = T_{(e,e)}(S \times S) + T_{(e,e)}L,$$

with the RHS being nothing but

$$T_eS \times T_eS + (T_eH \times \{0\} + \Delta_{T_eG \times T_eG})$$

$$= T_eG \times T_eS + \Delta_{T_eG \times T_eG} = T_eG \times T_eG,$$

and apply

Proposition 6.1. *Let M be a differentiable manifold and $a \in L \cap N$ a point in the intersection of two submanifolds $L, N \subset M$. If $T_aM = T_aL + T_aN$ the intersection $L \cap N$ is near a a submanifold of dimension* $\dim N + \dim L - \dim M$.

Proof. Let $n = \dim N, \ell = \dim L, m = \dim M$. We may assume

$$N = F^{-1}(0)$$

with a submersion $F : U \longrightarrow \mathbb{R}^{m-n}$ defined on a neighbourhood U of $a \in M$. Since $T_aN = \ker(T_aF)$ our assumption $T_aM = T_aL + T_aN$ implies that even $T_a(F|_{(L\cap U)})$ is onto. Hence after a shrinking of U the restriction $F|_{L\cap U}$ is a submersion as well, in particular $L \cap N \cap U = (F|_{U\cap L})^{-1}(0)$ is a submanifold of dimension $\ell - (m - n)$. $\square$

In order to write down explicitly a differentiable atlas for $B = G/H$ we use that B carries a natural (left) G-action:

$$G \times B \longrightarrow B, (g, x = aH) \mapsto gx := (ga)H.$$

Now, for $a \in G$ take $U_a := aU \subset B$, as local coordinates on U_a consider the map
$$\varphi_a : U_a \longrightarrow B^\ell, x \mapsto F(\sigma(a^{-1}x)),$$
where $F : S \longrightarrow B^\ell$ is a fixed diffeomorphism (biholomorphic map) with a ball $B^\ell \subset K^\ell$..

Corollary 6.2. *Let $H \subset G$ be a closed normal K-Lie subgroup of the K-Lie group G. Then G/H is again a K-Lie group with the quotient map $G \longrightarrow G/H$ being a homomorphism of K-Lie groups.*

The map $\pi : G \longrightarrow B$ has a remarkable geometric structure: It is an H-principal bundle:

Definition 6.3. *Let H be a Lie group. An H-*principal bundle *consists of*

1. *a differentiable/holomorphic right action*
$$E \times H \longrightarrow E$$
on a differentiable/complex manifold E, also called the total space *of the bundle*

2. *and the* bundle projection
$$\pi : E \longrightarrow B,$$
a surjective differentiable/holomorphic H-invariant map to a differentiable or complex manifold B – the base (space) *of the bundle – , such that every point $b \in B$ admits an open neighbourhood $U \subset B$, over which there is an H-equivariant trivialization of the bundle projection*
$$\tau : U \times H \longrightarrow \pi^{-1}(U),$$
i.e. a diffeomorphism/biholomorphic map satisfying
$$\pi \circ \tau = \mathrm{pr}_U : U \times H \longrightarrow U$$
and
$$\tau(x, h\tilde{h}) = \tau(x, h)\tilde{h}$$
for all $x \in U, h, \tilde{h} \in H$, i.e. τ transforms the right action of H on $U \times H$ by right translation on the second factor into the right action of H on E.

Remark 6.4. Given an H-principal bundle $\pi : E \longrightarrow B$ there is an open cover $B = \bigcup_{i\in I} U_i$ together with equivariant *trivializations*

$$\tau_i : U_i \times H \longrightarrow \pi^{-1}(U_i).$$

The bundle $\pi : E \longrightarrow B$ then can be reconstructed from that cover and the transition functions

$$f_{ij} : U_{ij} := U_i \cap U_j \longrightarrow H,$$

defined by the condition that the composition

$$U_{ij} \times H \xrightarrow{\tau_i} \pi^{-1}(U_{ij}) \xrightarrow{\tau_j^{-1}} U_{ij} \times H$$

satisfies

$$(x, e) \mapsto (x, f_{ij}(x)).$$

Since $\tau_j^{-1} \circ \tau_i$ is H-equivariant, it then takes the form

$$\tau_j^{-1} \circ \tau_i : (x, h) \mapsto (x, f_{ij}(x)h).$$

Note that the bundle $E \longrightarrow B$ can (up to a bundle isomorphism) be reconstructed from the cover $(U_i)_{i\in I}$ and the functions f_{ij}.

Indeed, whenever there is an open cover $B = \bigcup_{i\in I} U_i$ together with functions $f_{ij} : U_{ij} \longrightarrow H$ for all $(i, j) \in I^2$ satisfying the *cocycle condition*:

$$f_{ik} = f_{jk} f_{ij}$$

on $U_{ijk} = U_i \cap U_j \cap U_k$ for all $(i, j, k) \in I^3$, we may construct an associated H-principal bundle $E \longrightarrow B$ as follows:

$$E = \left(\bigcup_{i\in I} U_i \times H \right) / \sim \quad ,$$

where the union is regarded as a disjoint union and the equivalence relation identifies as follows:

$$U_i \times H \supset U_{ij} \times H \ni (x, h) \sim (x, f_{ij}(x)h) \in U_{ij} \times H \subset U_j \times H.$$

Example 6.5. Our left coset map $\pi : E := G \longrightarrow B := G/H$ is an H-principal bundle: First of all

$$G/H = \bigcup_{a \in G} U_a$$

with $U_a := aU$. As next define the section $\sigma_a : U_a \longrightarrow E = G$ of the bundle projection $\pi : E \longrightarrow B$ by $\sigma_a(x) := a\sigma(a^{-1}x)$, and finally

$$\tau_a : U_a \times H \longrightarrow \pi^{-1}(U_a), (x, h) \mapsto \sigma_a(x)h.$$

The corresponding function $f_{a,b} : U_{a,b} = U_a \cap U_b \longrightarrow H$ then turns out to be

$$f_{a,b}(x) = \sigma_b(x)^{-1}\sigma_a(x).$$

7 The Exponential Map

In order to study the geometry of a K-Lie group one investigates its "one parameter subgroups", i.e. the homomorphisms $\gamma : K \longrightarrow G$ from the additive group K to G.

Theorem 7.1. *The map*

$$\mathrm{Hom}(K, G) \longrightarrow T_eG, \gamma \mapsto \dot{\gamma}(0)$$

from the set $\mathrm{Hom}(K, G)$ *of all one parameter subgroups of* G *to the tangent space* T_eG *of* G *at* e *is a bijection. Indeed, any* $\gamma \in \mathrm{Hom}(K, G)$ *is an integral curve of the vector field* $X \in \mathfrak{g}$ *with* $X_e = \dot{\gamma}(0)$.

Before we give the proof we discuss the example $G = GL_n(K)$.

Example 7.2. For the general linear group $GL_n(K)$ with Lie algebra $\mathfrak{gl}_n(K) = K^{n,n}$ the left invariant vector field taking the value $A \in K^{n,n}$ at E is, according to Example 5.3 given by $\Xi \mapsto (\Xi A) \cdot \Delta$, hence the differential equation for the corresponding one parameter subgroup is $\dot{\gamma} = \gamma A$ with the solution $\gamma(t) = e^{tA}$, where

$$e^{tA} := \sum_{\nu=0}^{\infty} \frac{t^n A^n}{n!}.$$

Proof. For $\gamma \in \mathrm{Hom}(K, G)$ and $s \in K$ we have $\gamma(s+t) = \gamma(s)\gamma(t)$, hence

$$\dot\gamma(s) = (\lambda_{\gamma(s)})_*(\dot\gamma(0)) = (\lambda_{\gamma(s)})_*(X_e) = X_{\gamma(s)},$$

i.e. γ is an integral curve of X. Furthermore the injectivity is now also obvious, since an integral curve γ of a vector field X is completely determined by its initial value $\gamma(0)$. To get surjectivity, let us first consider the real case. Given $X_e \in T_eG$, we know that there is an integral curve $\gamma : I \longrightarrow G$ of the vector field $X \in \mathfrak{g}$, where $I \subset \mathbb{R}$ is some open interval containing 0. Let $I \subset \mathbb{R}$ be a maximal open interval such that γ can be defined on I. If $a \in \partial I$, we can easily extend γ by defining $\gamma(t)$ as $\gamma(\frac{a}{2})\gamma(t - \frac{a}{2})$ near a. So necessarily $\partial I = \emptyset$ resp. $I = \mathbb{R}$. Now argue as above in order to see that a globally defined integral curve of a left invariant vector field is a one parameter subgroup. In the complex case, we consider $Z_e \in T_eG$ as a real tangent vector and denote $\gamma_\vartheta : \mathbb{R} \longrightarrow G$ the one parameter subgroup with $\dot\gamma_\vartheta = e^{i\vartheta} Z_e$. Then $re^{i\vartheta} \mapsto \gamma_\vartheta(r)$ defines an integral "curve" $\gamma : \mathbb{C} \longrightarrow G$ of $Z \in \mathfrak{g}$. □

Corollary 7.3. *The connected real Lie groups of dimension one are* $\mathbb{R}$ *and* $\mathbf{S}^1$*, the connected complex Lie groups of dimension one are* $\mathbb{C}, \mathbb{C}^*$ *and the tori* $\mathbb{C}/\Lambda$ *with a lattice* $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ *(where* $\omega_1, \omega_2 \in \mathbb{C}$ *are linearly independent over* $\mathbb{R}$*.)*

Proof. Take any nonconstant one parameter subgroup $\gamma : K \longrightarrow G$. Since G is connected and $\gamma(K)$ contains a neighborhood of $e \in G$ we obtain $G = \gamma(K)$. The kernel $\ker\gamma \subset K$ contains 0 as isolated point, hence is a discrete subgroup and $G \cong K/\ker(\gamma)$.

For $\mathbb{R}$ such subgroups are either trivial or of the form $\mathbb{Z}\omega$ with some $\omega \in \mathbb{R} \setminus \{0\}$, hence $G \cong \mathbb{R}$ or $G \cong \mathbb{R}/\mathbb{Z}\omega \cong \mathbf{S}^1$. For $\mathbb{C}$ there is a further possibility for $\ker\gamma$, namely $\ker\gamma = \Lambda$, a lattice. So either $G \cong \mathbb{C}$ or $G \cong \mathbb{C}^*$ or $G \cong \mathbb{C}/\Lambda$. □

Note that in the real case $\mathbb{R} \cong \mathbb{R}_{>0}$ via the exponential function, while $\mathbb{C}^* \cong \mathbb{R}_{>0} \times \mathbf{S}^1$ resp. $\mathbb{C}/\Lambda \cong \mathbf{S}^1 \times \mathbf{S}^1$ as real Lie groups. On the other hand $\mathbb{C}/\Lambda \cong \mathbb{C}/\tilde\Lambda$ as complex Lie groups (or even as complex manifolds) iff $\tilde\Lambda = \alpha\Lambda$ with a nonzero complex number $\alpha \in \mathbb{C}^*$. In particular the complex tori constitute a continuous family of pairwise non isomorphic Lie groups or complex manifolds!

In the next step we collect all one parameter subgroups in a family: For a left invariant vector field $X \in \mathfrak{g}$ denote $\gamma_X \in \mathrm{Hom}(K, G)$ the one parameter subgroup with $\dot{\gamma}_X(0) = X_e$.

Definition 7.4. *The* exponential map

$$\exp := \exp_G : T_eG \longrightarrow G$$

is defined as

$$\exp(X_e) := \gamma_X(1),$$

with the left invariant vector field $X \in \mathfrak{g}$ taking the value $X_e \in T_eG$ at $e \in G$.

Example 7.5. Let $G = GL_n(K)$. For $A \in K^{n,n} \cong \mathfrak{gl}_n(K)$ we have $\gamma_A(t) = e^{tA}$ and thus $\exp(A) = e^A$.

Since the solution of the equation for the integral curve of a vector field does not only depend differentiably on the prescribed initial value, but even varies differentiably with the vector field itself, we obtain that $\exp : T_eG \longrightarrow G$ is differentiable. We compute its differential at the origin:

Proposition 7.6. *The Jacobian of the exponential map*

$$T_0(\exp) : T_eG \cong T_0((T_eG)) \longrightarrow T_eG$$

at the origin is the identity on T_eG. In particular it induces a diffeomorphism $\exp|_U : U \longrightarrow V$ from an open neighbourhood of $0 \in T_eG$ onto an open neighbourhood V of $e \in G$.

Proof. The curve $t \mapsto tX_e$ has tangent vector X_e at the origin, thus $T_0(\exp)(X_e)$ is the tangent vector of $t \mapsto \exp(tX_e) = \gamma_{tX}(1) = \gamma_X(t)$. But $\dot{\gamma}_X(0) = X_e$ by definition. $\square$

In order to avoid cumbersome notation, we will from now on use capital letters X, Y in order to denote both left invariant vector fields in $\mathfrak{g} \subset \Theta(G)$ as well as tangent vectors at $e \in G$, associated one to the other via the evaluation isomorphism $\mathfrak{g} \xrightarrow{\cong} T_eG$. With that convention we have

$$\exp((s+t)X) = \gamma_X(s+t) = \gamma_X(s)\gamma_X(t) = \exp(sX)\exp(tX)$$

and thus $\exp(X+Y) = \exp(X)\exp(Y)$ if X and Y are linearly dependent. Otherwise, as a consequence of the fact that exp is a diffeomorphism near $0 \in T_eG$ we may write

$$\exp(X)\exp(Y) = \exp(C(X,Y))$$

with a differentiable (holomorphic) function $C : U \times U \longrightarrow \mathfrak{g}$ and an open neighbourhood $U \subset T_eG \cong \mathfrak{g}$ of 0. The interesting fact about the function $C : U \times U \longrightarrow \mathfrak{g}$ is that it can be written as a convergent series

$$C(X,Y) = \sum_{n=1}^{\infty} C_n(X,Y),$$

where

$$C_n : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

is a homogeneous Lie bracket polynomial of degree n with rational coefficients independent of the Lie group G and the Lie algebra $\mathfrak{g}$. That is, $C_n(X,Y)$ is a rational linear combination of terms

$$[[...[Z_1, Z_2]..., Z_{n-1}], Z_n], Z_i \in \{X, Y\}$$

with coefficients only depending on the function $\{1,...,n\} \longrightarrow \{X,Y\}, i \mapsto Z_i$ (and not on the Lie algebra $\mathfrak{g}$.)

Thus the Lie algebra determines completely the group law in a neighbourhood of $e \in G$: Using the local inverse $\log := (\exp|_U)^{-1}$ as a local coordinate, it is given by $C : U \times U \longrightarrow \mathfrak{g}$. In order to obtain that result one derives a differential equation for the $\mathfrak{g}$-valued function

$$F(t) := C(tX, tY)$$

defined on a neighbourhood of $0 \in K$. For $Z \in \mathfrak{g}$ denote $\mathrm{ad}(Z) : \mathfrak{g} \longrightarrow \mathfrak{g}$ the linear map $X \mapsto [Z, X]$ (in fact a derivation as a consequence of the Jacobi identity). Then

$$\dot{F}(t) = f(\mathrm{ad}(F(t)))(X+Y) + \frac{1}{2}[X - Y, F(t)],$$

where f is a convergent power series in t with rational coefficients, indeed

$$f(t) = \frac{t}{1-e^{-t}} - \frac{t}{2}.$$

Together with the initial condition $F(0) = 0 \in \mathfrak{g}$ we obtain then a recursive formula for the homogeneous Lie bracket polynomials $C_n(X, Y)$. We mention here only

$$C_1(X, Y) = X + Y, \ C_2(X.Y) = \frac{1}{2}[X, Y]$$

and

$$C_3(X, Y) = \frac{1}{12}([[X, Y], Y] - [[X, Y], X])$$

as well as

$$C_4(X, Y) = -\frac{1}{48}([Y, [X, [X, Y]]] + [X, [Y, [X, Y]]]).$$

8 Subgroups and Subalgebras

In analogy to one parameter subgroups we define connected subgroups of a Lie group G:

Definition 8.1. *A connected (K-Lie) subgroup of a K-Lie group G is a pair (H, ι) with a connected Lie group H together with an injective (K-Lie group) homomorphism $\iota : H \longrightarrow G$. Two such subgroups $(H, \iota), (\tilde{H}, \tilde{\iota})$ are equivalent if there is a Lie group isomorphism $\psi : H \longrightarrow \tilde{H}$ with $\iota = \tilde{\iota} \circ \psi$.*

We remark that two connected subgroups are equivalent iff their image groups coincide. The image $\iota(H)$ is closed in G iff it is a submanifold (and then $H \cong \iota(H)$ even as Lie groups). The implication "$\Longrightarrow$" is a nontrivial statement we can not prove here. On the other hand, a submanifold $\iota(H)$ is locally closed and thus open in its closure $\overline{\iota(H)}$, a connected topological group. But then $\iota(H)$ is an open subgroup of the connected topological group $\overline{\iota(H)}$ and hence coincides with it.

Since $\iota_* : \mathfrak{h} \longrightarrow \mathfrak{g}$ is a Lie algebra homomorphism we can associate to any Lie subgroup a Lie subalgebra of $\mathfrak{g} := \mathrm{Lie}(G)$, namely $\iota_*(\mathfrak{h}) \cong \mathfrak{h}$. We want to prove that every subalgebra $\mathfrak{h} \subset \mathfrak{g}$ arises in that way. In order to get the corresponding Lie subgroup one could look at $\exp(\mathfrak{h})$. But that set need not be even a group nor can we take its closure in G. Instead one has to realize the corresponding subgroup as a "maximal integral submanifold of an involutive subbundle $E \subset TG$". Let us start explaining these new notions:

Definition 8.2. *A subbundle $E \subset TM$ (of rank k) of the tangent bundle TM of a differentiable/complex manifold M is a union $E := \bigcup_{x \in M} E_x$, where*

$E_x \subset T_xM$ is a k-dimensional vector subspace of T_xM for every $x \in M$, such that every point $a \in M$ has an open neighbourhood $U \subset M$ together with vector fields $X_1, ..., X_k \in \Theta(U)$, such that the tangent vectors $X_{1,x}, ..., X_{k,x} \in T_xM$ constitute a basis of E_x for all $x \in U$. We denote

$$\Theta_E(M) := \{X \in \Theta(M); X_a \in E_a \ \forall \ a \in M\}$$

the vector subspace of all vector fields taking values in $E \subset TM$.

Note that any vector field $X \in \Theta(M)$ without zeros defines a subbundle $E \subset TM$, namely $E_a := KX_a$ for $a \in M$. Integral curves then generalize to integral submanifolds:

Definition 8.3. *An integral manifold of a subbundle $E \subset TM$ is an injective immersion $\iota : N \longrightarrow M$ from a connected differentiable (complex) manifold N into M, such that $T_a\iota(T_aN) = E_a$ for all $a \in M$. Two integral submanifolds $\iota : N \longrightarrow M$ and $\tilde{\iota} : \tilde{N} \longrightarrow M$ are called equivalent if there is a diffeomorphism (biholomorphic map) $\psi : N \longrightarrow \tilde{N}$ with $\iota = \tilde{\iota} \circ \psi$. We say that two integral submanifolds agree at $a \in M$ if $a = \iota(c) = \tilde{\iota}(\tilde{c})$ and $\iota|_U$ is equivalent to $\tilde{\iota}|_{\tilde{U}}$ for open neighbourhoods $U, \tilde{U}$ of $c, \tilde{c}$.*

Note that in the above situation every point $c \in N$ admits an open neighbourhood $U \ni c$, such that $\iota(U) \subset M$ is a submanifold of M and $\iota|_U : U \longrightarrow \iota(U)$ a diffeomorphism.

Example 8.4. Let $M = K^n$ and $E \subset T(K^n)$ be the subbundle spanned at each point by the values of the first k coordinate vector fields $\partial_1, ..., \partial_k \in \Theta(K^n)$. Then, up to equivalence, the integral submanifolds of E are the maps

$$U \longrightarrow K^n, (x_1, ..., x_k) \mapsto (x_1, ..., x_k, a_{k+1}, ..., a_n)$$

with a connected open subset $U \subset K^k$.

Definition 8.5. *A subbundle $E \subset TM$ is called* involutive *(or an* involutive system*) if for all open subsets $U \subset M$ we have*

$$X, Y \in \Theta_E(U) \Longrightarrow [X, Y] \in \Theta_E(U).$$

Example 8.6. 1. Take $X = \partial_1, Y = \cos(x)\partial_2 + \sin(x)\partial_3 \in \Theta(K^3)$ and $E_a := KX_a + KY_a$. Then $E = \bigcup_{a \in K^3} E_a \subset T(K^3)$ is a subbundle, but not an involutive one, since $[X, Y] = -\sin(x)\partial_2 + \cos(x)\partial_3 \notin \Theta_E(K^3)$.

2. A subbundle admitting at every point $a \in M$ an integral submanifold $\iota_a : N_a \longrightarrow M$, i.e. with $a \in \iota_a(N_a)$, is involutive.

Theorem 8.7. *Let $E \subset TM$ be an involutive subbundle. Then every point $a \in M$ admits an open neighbourhood with local coordinates $x_1, ..., x_n$, such that, with the corresponding coordinate vector fields $\partial_1, ..., \partial_n \in \Theta(U)$, one has $E_b = K\partial_{1,b} + ... + K\partial_{k,b}$ for all $b \in U$.*

Corollary 8.8. *An involutive subbundle $E \subset TM$ admits integral manifolds at every point $a \in M$, and any two such integral submanifolds agree near a.*

Proof. We do induction on the rank k of the subbundle $E \subset TM$.
The case $k = 1$: Take a submanifold $S \ni a$ of M of dimension $n - 1$ with $X_a \notin T_aS$, where $X \in \Theta_E(U)$. Then the inverse of the flow

$$\mu : S \times I \longrightarrow M$$

of the vector field $X \in \Theta(U)$ (here $I \subset K$ denotes a small open interval or disc containing $0 \in K$) provides local coordinates with $X = \partial_1$. Here the flow means the map

$$\mu(x, t) := \gamma_x(t),$$

where $\gamma_x : I \longrightarrow M$ denotes the integral curve of X with initial value $\gamma_x(0) = x$.
The case $k > 1$: Denote $X_1, ..., X_k \in \Theta_E(U)$ spanning vector fields on a neighbourhood U of $a \in M$. From the case $k = 1$ we know that there are local coordinates $y_1, ..., y_n$ near $a \in M$, such that $X_1 = \partial_1^y$. We shall now replace the remaining spanning vector fields $X_2, ..., X_k \in \Theta_E(U)$ with vector fields $Y_2, ..., Y_k \in \Theta_E(U)$ tangent to any submanifold $\{t\} \times K^{n-1}$ and satisfying

$$[X_1, Y_i] = 0 \ , \ i = 2, ..., k.$$

Indeed in that case we have

$$Y_i = \sum_{j=2}^{n} f_{ij} \partial_j^y$$

with coefficient functions

$$f_{ij} = f_{ij}(y_2, ..., y_n)$$

not depending on y_1. Denote $E_0 \subset TV$, where $V \subset K^{n-1}$ is an open neighbourhood of $0 \in K^{n-1}$, the subbundle generated by the vector fields Y_i. Indeed, it is involutive, since E is. Now E_0 having rank $k-1$ we find coordinates $x_2, ..., x_n$ on V (may be after a shrinking) such that $\partial_2, ..., \partial_k$ span E_0. Finally $y_1, x_2, ..., x_n$ are the local coordinates on $K^n = K \times K^{n-1}$ we are looking for.
The construction of the vector fields $Y_2, ..., Y_k \in \Theta_E(U)$: We may assume that the vector fields $X_2, ..., X_k \in \Theta_E(U)$ are tangent to $\{t\} \times K^{n-1}$ for all $t \in K$ near the origin, i.e. that

$$X_j = \sum_{\ell=2}^{n} h_{j\ell} \partial_\ell^y$$

holds for $j \geq 2$: Replace X_j with $X_j - h_{j1}X_1 = X_j - h_{j1}\partial_1^y$. Now we consider a vector field

$$Y = \sum_{j=2}^{k} \varphi_j X_j \in \Theta_E(U),$$

and study how to choose the coefficient functions $\varphi_2, ..., \varphi_k$ in order to have

$$0 = [X_1, Y] = \sum_{j=2}^{k} \partial_1^y(\varphi_j) X_j + \varphi_j [X_1, X_j].$$

Since E is involutive, we have

$$[X_1, X_j] = \sum_{\ell=2}^{n} \partial_1^y(h_{j\ell}) \cdot \partial_\ell^y \in \Theta_E(U)$$

and thus

$$[X_1, X_j] = \sum_{\ell=2}^{k} g_{j\ell} \cdot X_\ell,$$

where the vector field X_1 does not show up in the expansion of $[X_1, X_j] \in \Theta_E(U)$ as a linear combination of $X_1, ..., X_k$. Hence

$$0 = \sum_{j=2}^{k} \left(\partial_1^y(\varphi_j) X_j + \varphi_j \sum_{\ell=2}^{k} g_{j\ell} \cdot X_\ell \right).$$

or equivalently

$$\partial_1^y(\varphi_j) = \sum_{\ell=2}^{k} g_{\ell j}\varphi_\ell,\ j = 2, ..., k.$$

This is an ODE with respect to the variable y_1, keeping fixed $y_2, ..., y_n$. Thus we may prescribe the functions $\varphi_j(0, y_2, ..., y_n), j = 2, ..., k$, and obtain uniquely determined functions $y \mapsto \varphi_j(y, y_2, ..., y_n)$ satisfying the above system of differential equations. Now taking $\varphi_j(0, y_2, ..., y_n) \equiv \delta_{ij}$ for $i = 2, ..., k$, we obtain the vector fields $Y_i, i = 2, ..., k$. $\square$

As a consequence we can easily compare two integral submanifolds $\iota : N \longrightarrow M$ and $\tilde{\iota} : \tilde{N} \longrightarrow M$ of an involutive subbundle $E \subset TM$, namely: There are open subsets $U \subset N, \tilde{U} \subset \tilde{N}$ with $\iota(U) = \iota(N) \cap \tilde{\iota}(\tilde{N}) = \tilde{\iota}(\tilde{U})$, such that $\iota|_U$ and $\tilde{\iota}|_{\tilde{U}}$ are equivalent. In particular we can apply Zorn's lemma and see that through every point $a \in M$ there is a unique maximal integral submanifold (i.e. unique up to equivalence).

Proposition 8.9. *Let G be a Lie group. The map*

$$(H, \iota) \mapsto \iota_*(\mathfrak{h}) \subset \mathfrak{g}$$

defines a bijection between the set of connected Lie subgroups of G (up to equivalence) and the Lie subalgebras of $\mathfrak{g} = \mathrm{Lie}(G)$.

Proof. Given a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ we define an involutive subbundle $E \subset TG$ by $E_a := T_e\lambda_a(\mathfrak{h}_e)$ (with $\mathfrak{h}_e := \{X_e; X \in \mathfrak{h}\}$). Indeed, if $X_1, ..., X_k \in \mathfrak{h}$ is a basis, then $E_a = KX_{1,a} \oplus ... \oplus KX_{k,a}$. In particular $\mathfrak{h} \subset \mathfrak{g}$ being a subalgebra, we see that $E \subset TM$ is involutive. Now take $\iota : H \longrightarrow G$ as a maximal integral submanifold of E at $e \in G$. To simplify notation we treat ι as an inclusion. We then have $aH = H$ for all $a \in H$: Since E is λ_a-invariant, $aH \ni a$ is a maximal integral submanifold, but $H \ni a$ is as well, hence $aH = H$. Since on the other hand $e \in H$, this immediately gives that $H \subset G$ is a subgroup (in the algebraic sense). We leave it to the reader to check, that H becomes a Lie group with this group law (note once again that in general the topology and differentiable structure is only locally that one induced by G). On the other hand every connected K-Lie subgroup (H, ι) clearly is an integral submanifold of the left invariant (involutive) subbundle $E \subset TM$ with $E_e = T_e\iota(T_eH)$. It follows the injectivity of our map. $\square$

Let us now investigate what it means that $\iota(H) \subset G$ is a normal subgroup of G. In order to obtain a differential interpretation of that property we consider the adjoint representation:

Definition 8.10. *A* representation *of a (K-)Lie group G is a Lie group homomorphism $G \longrightarrow GL(V)$ from G into the general linear group $GL(V)$ of a finite dimensional K-vector space V. The adjoint representation of a Lie group G with Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ is the homomorphism $\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g}), a \mapsto T_e(\kappa_a)$, where $\kappa_a : G \longrightarrow G, g \mapsto aga^{-1}$ is conjugation with a.*

Proposition 8.11. *The differential*

$$\mathrm{ad} := T_e\mathrm{Ad} : \mathfrak{g} = \mathrm{Lie}(G) \longrightarrow \mathfrak{gl}(\mathfrak{g}) = \mathrm{Lie}(GL(\mathfrak{g}))$$

of the adjoint representation

$$\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g}), a \mapsto T_e(\kappa_a)$$

satisfies

$$\mathrm{ad}(X) = [X, ..]$$

Proof. We start with the equality

$$\mathrm{Ad}(\exp(X)) = e^{\mathrm{ad}(X)}$$

of endomorphisms of $\mathfrak{g}$. Evaluating it at some $Y \in \mathfrak{g}$ gives

$$\mathrm{Ad}(\exp(X))(Y) = e^{\mathrm{ad}(X)}Y$$

and then apply both sides to a function $f \in C^\infty(G)$

$$Y(f \circ \kappa_{\exp(X)}) = (T_e\kappa_{\exp(X)}(Y))f = (e^{\mathrm{ad}(X)}Y)f,$$

i.e.,

$$Yf(\exp(X) \cdot .. \cdot \exp(-X)) = (e^{\mathrm{ad}(X)}Y)f.$$

Now replace X with $sX, s \in K$, in order to get

$$Yf(\exp(sX) \cdot .. \cdot \exp(-sX)) = (e^{s \cdot \mathrm{ad}(X)}Y)f.$$

Differentiation at $s = 0$ yields:

$$\frac{d}{ds}(Yf(\exp(sX) \cdot .. \cdot \exp(-sX)))_{s=0} = (\mathrm{ad}(X)Y)f$$

resp.

$$\frac{\partial^2}{\partial s \partial t}(f(\exp(sX)\exp(tY)\exp(-sX)))_{s=0,t=0} = (\mathrm{ad}(X)Y)f.$$

The right hand side can be rewritten with the chain rule as

$$(\frac{\partial^2}{\partial s \partial t}f(\exp(sX)\exp(tY)) + \frac{\partial^2}{\partial t \partial s}f(\exp(tY)\exp(-sX)))_{s=t=0}.$$

Since for a left invariant vector field $Y \in \mathfrak{g}$ one has

$$\frac{d}{dt}f(a\exp(tY))_{t=0} = Y(f \circ \lambda_a)(e) = ((Yf) \circ \lambda_a)(e) = (Yf)(a)$$

we obtain (with $a = \exp(sX)$ resp. $a = \exp(tY)$) finally

$$(XYf)(e) + (Y(-X)f)(e) = (XYf)(e) - (YXf)(e) = ([X,Y]f)(e)$$

resp. $[X,Y] = \mathrm{ad}(X)(Y)$. □

In order to simplify notation we shall from now on identify a connected Lie subgroup (H, ι) with its image $\iota(H)$ and its Lie algebra $\mathfrak{h}$ with the subalgebra $\iota_*(\mathfrak{h}) \subset \mathfrak{g}$.

Corollary 8.12. *A connected Lie subgroup $H \subset G$ of a connected Lie group G is a normal subgroup if and only if its Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, i.e. $X \in \mathfrak{g}, Y \in \mathfrak{h} \Longrightarrow [X,Y] \in \mathfrak{h}$.*

Proof. The subgroup H is normal iff $\kappa_a(H) = H$ for all $a \in G$. Obviously that implies $T_e\kappa_a(\mathfrak{h}) = \mathfrak{h}$ for all $a \in G$, in particular for $a = \exp(X)$ with $X \in \mathfrak{g}$. So $\mathfrak{h} \ni T_e(\kappa_{\exp(X)})(Y) = \mathrm{Ad}(\exp(X))(Y) = e^{\mathrm{ad}(X)}Y$ gives once again, after replacing X with $sX, s \in K$ and differentiation with respect to s at $s = 0$ that

$$\mathfrak{h} \ni \frac{d}{ds}(e^{s\cdot\mathrm{ad}(X)}Y)_{s=0} = \mathrm{ad}(X)(Y) = [X,Y],$$

the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ being closed in $\mathfrak{g}$. (Every finite dimensional K-vector space carries a natural topology obtained from a linear isomorphism $V \cong K^n$ and not depending on the choice of that isomorphism.)

On the other hand let us assume that the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is even an ideal. We show that the normalizer

$$N_G(H) := \{a \in G; \kappa_a(H) = H\}$$

contains an open neighbourhood $U \subset G$ of the neutral element $e \in G$. The group G being connected, it follows $N_G(H) = G$. Since $\exp(\mathfrak{g})$ contains such a neighbourhood, it suffices to consider $a = \exp(X)$ with $X \in \mathfrak{g}$. But $\mathrm{ad}(X)(Y) \in \mathfrak{h}$ for $Y \in \mathfrak{h}$ implies

$$T_e\kappa_{\exp(X)}Y = \mathrm{Ad}(\exp(X))(Y) = e^{\mathrm{ad}(X)}Y \in \mathfrak{h}.$$

Finally if we can prove that $\kappa_{\exp(X)}$ maps an open neighbourhood of $e \in H$ into H, we are done, since H as connected group is generated by any open neighbourhood of e. But any element in such a neighbourhood can again be taken in the form $\exp(Y)$ and then

$$\kappa_{\exp(X)}(\exp(Y)) = \exp(T_e\kappa_{\exp(X)}Y) \in \exp(\mathfrak{h}) \subset H.$$

$\square$

9 Lie Algebras of Dimension ≤ 3

For Lie algebras there is as well the notion of a derivation:

Definition 9.1. *Let $\mathfrak{g}$ be a Lie algebra. A linear map $D : \mathfrak{g} \longrightarrow \mathfrak{g}$ is called a derivation, if it satisfies the Leibniz rule with respect to the Lie bracket:*

$$D[X, Y] = [DX, Y] + [X, DY], \ \forall\ X, Y \in \mathfrak{g}.$$

We denote $\mathrm{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ *the subalgebra of* $\mathfrak{gl}(\mathfrak{g})$ *consisting of all derivations of* $\mathfrak{g}$.

Remark 9.2.

1. We have $\mathrm{id}_{\mathfrak{g}} \in \mathrm{Der}(\mathfrak{g})$ iff $\mathfrak{g}$ is abelian. Indeed in that case $\mathrm{Der}(\mathfrak{g}) = \mathfrak{gl}(\mathfrak{g})$.

2. Given any element $Z \in \mathfrak{g}$, the map $\mathrm{ad}(Z) : \mathfrak{g} \longrightarrow \mathfrak{g}, X \mapsto [Z, X]$ is a derivation: In fact the Leibniz rule is nothing but the Jacobi identity for Z, X, Y.

3. The Jacobi identity may even be reformulated in a more sophisticated way by saying that the map $\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}), Z \mapsto \mathrm{ad}(Z)$, is a homomorphism of Lie algebras, the *adjoint representation* of $\mathfrak{g}$. (A representation of a Lie algebra is any Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, where V is a K-vector space.)

4. If $\lambda, \mu \in K$ are eigenvalues of a derivation $D \in \mathrm{Der}(\mathfrak{g})$, then either so is $\lambda + \mu$ or $[X, Y] = 0$ for any two eigenvectors $X, Y \in \mathfrak{g}$ of D corresponding to λ resp. μ.

5. If $D \in \mathrm{Der}(\mathfrak{g})$ and $\dim \mathfrak{g} < \infty$, then $e^D : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a Lie algebra automorphism, i.e., a linear isomorphism with $e^D([X, Y]) = [e^D X, e^D Y]$ for all $X, Y \in \mathfrak{g}$.

6. If $D \in \mathrm{Der}(\mathfrak{h})$, we can form a new Lie algebra $\mathfrak{h}_D$. As vector space it is the cartesian product (or direct sum)

$$\mathfrak{h}_D := \mathfrak{h} \times K,$$

where $Z := (0, 1) \in \mathfrak{h}_D$ satisfies $[(0, 1), (X, 0)] = DX$. Here, the fact that D is a derivation gives the Jacobi identity on $\mathfrak{h}_D$. Note that $\mathfrak{h}_D \cong \mathfrak{h}_{\tilde{D}}$, if $\tilde{D}$ is similar to λD for some $\lambda \in K^*$. Furthermore

$$[\mathfrak{h}_D, \mathfrak{h}_D] = ([\mathfrak{h}, \mathfrak{h}] + D\mathfrak{h}) \times \{0\}.$$

Remark 9.3. If H is a Lie group with Lie algebra $\mathfrak{h}$, then a Lie group with Lie algebra $\mathfrak{h}_D$ should be a semidirect product, e.g.

$$H \times_\sigma K,$$

where $\sigma : K \longrightarrow \mathrm{Aut}(H)$ is a homomorphism, i.e. $\sigma_{t+s} = \sigma_t \circ \sigma_s$. Here we need that

$$\sigma_t : H \longrightarrow H$$

satisfies

$$T_e(\sigma_t) = e^{tD} : \mathfrak{h} \longrightarrow \mathfrak{h}, \ \forall\ t \in K.$$

In general it is not true, that the automorphism e^{tD} is induced by some (indeed unique, if it exists) $\sigma_t : H \longrightarrow H$. But such a "lifting" exists always, if H is simply connected. For example let us consider $\mathfrak{h} = K, D = \mathrm{id}$. For $K = \mathbb{R}$ we could take $H = \mathbb{R}_{>0} \subset \mathbb{R}^*$. Then $\sigma_t(x) = x^t$ is the lifting we are looking for. But if $H = \mathbb{C}^*$ and $D = \mathrm{id}_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}$, there is some $\sigma_t : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ only for $t \in \mathbb{Z}$.

Om the other hand, if we remember that the Lie algebra of K^* is K as well, we may try

$$H \times_\sigma K^*$$

and succeed with (now we need $\sigma_{ts} = \sigma_t \circ \sigma_s$)

$$\sigma_t : K \longrightarrow K, x \mapsto tx.$$

The group we obtain is isomorphic to $\begin{pmatrix} K^* & K \\ 0 & 1 \end{pmatrix}$, or as well to the group of affine linear transformations of K.

For two subspaces $U, V \subset \mathfrak{g}$ we denote

$$[U, V] := \mathrm{span}\{[X, Y]; X \in U, Y \in V\}.$$

Now the *commutator* $[\mathfrak{g}, \mathfrak{g}]$ of a Lie algebra $\mathfrak{g}$ is not only a subalgebra, but even an ideal, and any isomorphism between two Lie algebras transforms the respective commutators one to the other: It is a "characteristic ideal". Indeed if $X_1, ..., X_n$ is a basis of the K-vector space $\mathfrak{g}$, then $[\mathfrak{g}, \mathfrak{g}]$ is spanned by the Lie brackets $[X_i, X_j], i < j$.

Let us now come to our classification:

1. $\dim \mathfrak{g} = 1$: A one dimensional Lie algebra is obviously abelian.

2. $\dim \mathfrak{g} = 2$: Any two dimensional non-abelian Lie algebra $\mathfrak{g}$ is of the form $\mathfrak{g} = \mathfrak{h}_{\mathrm{id}}$ with a one dimensional (abelian) Lie algebra $\mathfrak{h}$: Necessarily $\dim[\mathfrak{g}, \mathfrak{g}] = 1$, say $[\mathfrak{g}, \mathfrak{g}] = KZ$. If we take any $X \notin [\mathfrak{g}, \mathfrak{g}]$, then $[X, Z] = \lambda Z$ with $\lambda \in K^*$, and after a rescaling of X we may assume $[X, Z] = Z$. We need later on the following

   Lemma 9.4. *For the two dimensional non-abelian Lie algebra $\mathfrak{g}$ one has*
   $$\mathrm{Der}(\mathfrak{g}) = \{D \in \mathfrak{gl}(\mathfrak{g}); D\mathfrak{g} \subset [\mathfrak{g}, \mathfrak{g}]\}.$$

   Proof. It is easily checked, that a linear map $D : \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying $DX = \lambda Z, DZ = \mu Z$ is a derivation.

   Now let $D : \mathfrak{g} \longrightarrow \mathfrak{g}$ be a derivation. Since $D[\mathfrak{g}, \mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}]$, we have $DZ = \lambda Z$. For $\lambda = 0$ we get

   $$0 = D[X, Z] = [DX, Z] + [X, DZ] = [DX, Z]$$

   and thus $DX \in KZ = [\mathfrak{g}, \mathfrak{g}]$ as desired. Now assume $\lambda \neq 0$. If D is diagonalizable we may assume that $DX = \mu X$ with a $\mu \in K$. Since

$[X, Z] = Z$, we conclude $\mu + \lambda = \lambda$, i.e. $\mu = 0$ and thus $D\mathfrak{g} \subset KZ = [\mathfrak{g}, \mathfrak{g}]$. If D is not diagonalizable, we have $DX - \lambda X \in K^* Z$. But then $Z = [X, Z]$ satisfies

$$D[X, Z] = [DX, Z] + [X, DZ] = [\lambda X + \beta Z, Z] + [X, \lambda Z] = 2\lambda[X, Z].$$

So $\lambda = 2\lambda$, i.e. $\lambda = 0$, a contradiction. $\square$

3. $\dim \mathfrak{g} = 3$:

   (a) $\dim[\mathfrak{g}, \mathfrak{g}] = 0$: Then $\mathfrak{g}$ is abelian.

   (b) $\dim[\mathfrak{g}, \mathfrak{g}] = 1$, say $[\mathfrak{g}, \mathfrak{g}] = KZ$:

      i. If $[Z, \mathfrak{g}] = \{0\}$ (or equivalently $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] = \{0\}$), we have

      $$\mathfrak{g} \cong (K^2)_D,$$

      where K^2 is the two dimensional abelian Lie algebra and $D : K^2 \longrightarrow K^2$ a nontrivial nilpotent derivation (linear map). Indeed, we find a basis X, Y, Z with $[X, Y] = Z$. Then $\mathfrak{a} := KX + KZ$ is an abelian ideal and we find $\mathfrak{g} = \mathfrak{a}_D$ with the nilpotent derivation $D : \mathfrak{a} \longrightarrow \mathfrak{a}, X \mapsto -Z \mapsto 0$.

      ii. Otherwise $[Z, \mathfrak{g}] = KZ$ (or equivalently $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}]$). Then we have

      $$\mathfrak{g} \cong \mathfrak{h} \oplus K \ (\cong \mathfrak{h}_0),$$

      with the two dimensional non-abelian Lie algebra $\mathfrak{h}$. We choose $X \in \mathfrak{g}$ with $[X, Z] = Z$. Choose then $Y \notin KX + KZ$ with $[Y, Z] = 0$. That is possible, since $[X, \mathfrak{g}] = KZ$ in any case. Furthermore we can assume $[X, Y] = 0$; if that was not the case right from the beginning, we may replace Y with $Y - \lambda Z$, where $[X, Y] = \lambda Z$ with $\lambda \in K^*$. Hence $\mathfrak{g} = \mathfrak{h} \oplus KY$ is the direct sum of the two dimensional non-abelian Lie algebra $\mathfrak{h}$ and a one dimensional Lie algebra. We could also write $\mathfrak{g} \cong \mathfrak{h}_0$, where $0 \in \mathrm{Der}(\mathfrak{h})$ denotes the trivial derivation.

   (c) $\dim[\mathfrak{g}, \mathfrak{g}] = 2$: Then we have

   $$\mathfrak{g} = (K^2)_D,$$

   where K^2 denotes the two dimensional abelian Lie algebra and $D \in GL_2(K)$ is any automorphism of the vector space K^2. To see that, take any $X \notin \mathfrak{a} := [\mathfrak{g}, \mathfrak{g}]$ and $D := \mathrm{ad}(X)|_{\mathfrak{a}}$. By Lemma 9.4, if $\mathfrak{a}$ was not abelian, we had $D\mathfrak{a} \subset [\mathfrak{a}, \mathfrak{a}]$ and hence

$$\mathfrak{a} = [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{a}, \mathfrak{a}] + D\mathfrak{a} = [\mathfrak{a}, \mathfrak{a}],$$

a contradiction to our assumption $\dim \mathfrak{a} = 2$. So $\mathfrak{a}$ is abelian and D an isomorphism because of $\mathfrak{a} = [\mathfrak{g}, \mathfrak{g}] = D\mathfrak{a}$. So $\mathfrak{g} = \mathfrak{a}_D$ with a two dimensional abelian Lie algebra $\mathfrak{a}$ and any linear automorphism $D \in \mathfrak{gl}(\mathfrak{a}) = \mathrm{Der}(\mathfrak{a})$.

4. $\dim[\mathfrak{g}, \mathfrak{g}] = 3$: Given a basis X_1, X_2, X_3 of $\mathfrak{g}$, the Lie brackets $[X_1, X_2]$, $[X_1, X_3], [X_2, X_3]$ form a basis as well because of $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

   First we hunt for some element $X \in \mathfrak{g} \setminus \{0\}$, such that the linear map

$$\mathrm{ad}(X) = [X, ..] : \mathfrak{g} \longrightarrow \mathfrak{g}$$

   is easily understood. Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, its image $[X, \mathfrak{g}]$ is a two dimensional subspace and $\ker(\mathrm{ad}(X)) = KX$. We now are looking for an $\mathrm{ad}(X)$-invariant subspace, complementary to KX. Let us try with $[X, \mathfrak{g}] = \mathrm{ad}(X)(\mathfrak{g})$: Indeed if $X \notin [X, \mathfrak{g}]$, we have

$$\mathfrak{g} = KX \oplus [X, \mathfrak{g}]$$

   as vector spaces. So we first show that the assumption

$$\forall\, X \in \mathfrak{g} : \ X \in [X, \mathfrak{g}]$$

   leads to a contradiction: Given any $X \in \mathfrak{g} \setminus \{0\}$, choose $Y \in \mathfrak{g}$ with $X = [X, Y]$ and then $Z \in \mathfrak{g}$ with $Y = [Y, Z]$. We claim that the elements X, Y, Z are linearly independent. Apply $D := \mathrm{ad}(Y)$ to a relation

$$\alpha X + \beta Y + \gamma Z = 0$$

   and obtain

$$-\alpha X + \gamma Y = 0.$$

   Since X, Y are linearly independent, that implies $\alpha = 0 = \gamma$, whence $\beta = 0$ as well. Hence we can write

$$[X, Z] = \alpha X + \beta Y + \gamma Z$$

and thus

$$D[X,Z] = \alpha[Y,X] + \gamma[Y,Z],$$

while on the other hand

$$D[X,Z] = [DX,Z] + [X,DZ] = -[X,Z] + [X,Y].$$

Since the elements $[X,Y],[Y,Z],[X,Z]$ are linearly independent, that is not possible.

Now let us return to the map $D := \mathrm{ad}(X)$, where $X \notin [X,\mathfrak{g}]$. It induces a linear automorphism of $[X,\mathfrak{g}]$. If it is diagonalizable there are two linearly independent eigenvectors $Y,Z \in [X,\mathfrak{g}]$, say $DY = \lambda Y, DZ = \mu Z$. Since $[Y,Z] \neq 0$, we find $\lambda + \mu \in \{\lambda,\mu,0\}$ – the eigenvalues of $\mathrm{ad}(X)$ – resp. $\lambda + \mu = 0$ because of $\lambda,\mu \in K^*$. After a rescaling of X we can assume $\lambda = 2, \mu = -2$. Finally $0 \neq [Y,Z] \in \ker(\mathrm{ad}(X)) = KX$, such that we after a rescaling of Y, say, obtain $[Y,Z] = X$. Indeed, the linear map

$$\mathfrak{g} \xrightarrow{\cong} \mathfrak{sl}_2(K)$$

with

$$X \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Y \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Z \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

turns out to be an isomorphism of Lie algebras.

Now let us exclude the possibility that D is not diagonalizable with one eigenvalue $\lambda \neq 0$. Then we find an eigenvector $Z \in [X,\mathfrak{g}]$, i.e. $DZ = \lambda Z$, and a vector $Y \in [X,\mathfrak{g}]$ with $DY - \lambda Y = Z$, and thus

$$D[Y,Z] = [DY,Z] + [Y,DZ] = [\lambda Y + Z, Z] + [Y,\lambda Z] = 2\lambda[Y,Z].$$

Since $[Y,Z] \neq 0$ that implies $2\lambda \in \{0,\lambda\}$, but that is impossible.

This finishes our discussion for $K = \mathbb{C}$. For $K = \mathbb{R}$ we have to deal with the case that $D = \mathrm{ad}(X)|_{[X,\mathfrak{g}]}$ has no real eigenvalus. We pass to the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$. The complexification $D_{\mathbb{C}} := D + iD$ has two eigenvalues $\lambda, -\lambda$. Since they are zeros of a real polynomial, we have $-\lambda = \overline{\lambda}$, i.e. λ is purely imaginary. After a (real) rescaling of X we may thus assume $\lambda = i$ and thus $D^2 = -\mathrm{id}_{[X,\mathfrak{g}]}$. Now for any $Y \in [X,\mathfrak{g}]$ the elements $Y, Z := DY$ form a basis of $[X,\mathfrak{g}]$, and obviously $DZ = -Y$. Hence $D[Y,Z] = [DY,Z] + [Y,DZ] = 0$ and thus $\ker(D) = KX$ tells us that $[Y,Z] = \lambda X$ with a $\lambda \in \mathbb{R}^*$. By dividing

Y, Z with $\sqrt{|\lambda|}$ we finally get $[Y, Z] = \pm X$. If $[Y, Z] = X$ we obtain the vector product multiplication table of the vectors in an orthonormal basis of $\mathbb{R}^3$. More explicitly there is an isomorphism

$$\mathfrak{g} \xrightarrow{\cong} \mathfrak{so}_3(\mathbb{R})$$

with

$$X \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, Y \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, Z \mapsto \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using complex matrices we have even an isomorphism

$$\mathfrak{g} \xrightarrow{\cong} \mathfrak{su}_2 \subset \mathfrak{sl}_2(\mathbb{C})$$

with

$$X \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, Y \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Z \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Note that $\mathfrak{su}_2 \cong \mathfrak{so}_3(\mathbb{R})$ is not isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ since in the former algebra no derivation $\mathrm{ad}(X), X \in \mathfrak{g} \setminus \{0\}$ is diagonalizable.

If on the other hand $[Y, Z] = -X$ the derivation $\mathrm{ad}(Y)$ is diagonalizable with the eigenvalues $1, -1$ and corresponding eigenvectors $X-Z, X+Z$. So $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{R})$ in that case.

10 The Universal Covering Group

A Lie group homomorphism $\varphi : G \longrightarrow H$ induces a Lie algebra homomorphism $\varphi_* : \mathfrak{g} \longrightarrow \mathfrak{h}$. In this section we ask when for given Lie groups G, H and a Lie algebra homomorphism $\psi : \mathfrak{g} \longrightarrow \mathfrak{h}$ we can find a Lie group homomorphism $\varphi : G \longrightarrow H$ inducing ψ, i.e. such that $\psi = \varphi_*$. The strategy is as follows: If $\varphi : G \longrightarrow H$ is a Lie group homomorphism, then its graph

$$\Gamma_\varphi := \{(g, \varphi(g)); g \in G\} \subset G \times H$$

is a Lie subgroup of $G \times H$ with Lie algebra

$$\mathrm{Lie}(\Gamma_\varphi) = \{(X, \varphi_*(X)); X \in \mathfrak{g}\}.$$

So given $\psi : \mathfrak{g} \longrightarrow \mathfrak{h}$ we look at the connected Lie subgroup $\Gamma \subset G \times H$ with

$$\mathrm{Lie}(\Gamma) = \{(X, \psi(X)); X \in \mathfrak{g}\}.$$

The inclusion followed by the projection onto the first factor

$$\pi : \Gamma \hookrightarrow G \times H \stackrel{pr_G}{\longrightarrow} G$$

has obviously bijective differential

$$\pi_* : \mathrm{Lie}(\Gamma) \longrightarrow \mathfrak{g}.$$

Since both groups, G and Γ are connected, it is a surjective homomorphism with discrete kernel. And if it is even an isomorphism, we can take $\varphi := pr_H \circ \pi^{-1}$. Indeed, there are groups, where π necessarily is an isomorphism.

In order to understand that phenomenon we need an excursion to topology. The basic notion is that of of a *covering*:

Definition 10.1. *A continuous map $\pi : X \longrightarrow Y$ between topological spaces X and Y is called a* covering *iff every point $b \in Y$ admits an open neighbourhood $V \subset Y$, such that its inverse image is the disjoint union*

$$\pi^{-1}(V) = \bigcup_{i \in I} U_i$$

of open subsets $U_i \subset X$ with $\pi|_{U_i} : U_i \longrightarrow V$ being a homeomorphism for every $i \in I$.

Example 10.2. A surjective Lie group homomorphism $\varphi : G \longrightarrow H$ with a connected Lie group G and discrete kernel $D \subset G$ (e.g. if $\varphi_* : \mathfrak{g} \longrightarrow \mathfrak{h}$ is an isomorphism) is a covering: Choose an open neighbourhood $U \subset G$ of $e \in G$ with $U \cdot U^{-1} \subset G \setminus D^*$, where $D^* := D \setminus \{e\}$. Then for $V := \pi(U)$ we have

$$\pi^{-1}(V) = \bigcup_{a \in D} aU,$$

a disjoint union.

Now, given a covering $\pi : X \longrightarrow Y$ we want to study spaces Z, such that any continuous map $\varphi : Z \longrightarrow Y$ admits a lift $\hat{\varphi} : Z \longrightarrow X$, i.e. such that $\varphi = \pi \circ \hat{\varphi}$. For $Z = [0, 1]$, the unit interval in $\mathbb{R}$, we have:

Proposition 10.3. *Let $\pi : X \longrightarrow Y$ be a covering of the locally path connected space Y. Then, given a path $\gamma : [0,1] \longrightarrow Y$ and a point $x_0 \in X, \pi(x_0) = \gamma(0)$, there is a unique path $\hat{\gamma} : [0,1] \longrightarrow X$ with $\hat{\gamma}(0) = x_0$.*

Proof. Write $[0,1] = I_1 \cup ... \cup I_n$ with $I_k = [\frac{k-1}{n}, \frac{k}{n}]$. For sufficiently big $n \in \mathbb{N}$ every piece $\gamma(I_k) \subset \gamma(I)$ is contained in an open path connected set $V = V_k \subset Y$, such that $\pi^{-1}(V)$ is a disjoint union as in Def. 10.1. Now assume we have found a lift

$$\hat{\gamma}_k : \big[0, k/n\big] \longrightarrow X,$$

of $\gamma_k := \gamma|_{[0,\frac{k}{n}]}$. Choose $U \subset \pi^{-1}(V_{k+1})$ with $\pi|_U : U \longrightarrow V_{k+1}$ being homeomorphic and $\hat{\gamma}_k(\frac{k}{n}) \in U$. Now define $\hat{\gamma}_{k+1}$ by

$$\hat{\gamma}_{k+1}|_{[0,k/n]} := \hat{\gamma}_k \ , \quad \hat{\gamma}_{k+1}|_{I_{k+1}} := (\pi|_U)^{-1}|_{I_{k+1}}.$$

$\square$

The idea now is to fix a base point z_0 in a (path connected) topological space Z and to define for a given continuous map $\varphi : Z \longrightarrow Y$ a lift as follows: Choose a point $x_0 \in X$ above $y_0 := \varphi(z_0)$ (i.e. $\pi(x_0) = y_0$). Now, given a point $z \in Z$, take a path $\beta_z : [0,1] \longrightarrow Z$ with $\beta_z(0) = z_0, \beta_z(1) = z$. Denote $\hat{\gamma}_z : [0,1] \longrightarrow X$ the lift of $\gamma_z := \varphi \circ \beta_z$ with $\hat{\gamma}_z(0) = x_0$ and take $\hat{\varphi}(z) := \hat{\gamma}_z(1)$. It remains to show that under suitable assumptions on Z different choices of $\beta_z : [0.1] \longrightarrow Z$ give the same value $\hat{\varphi}(z)$.

For that we need the notion of *homotopic paths*:

Definition 10.4. *Two paths $\alpha, \beta : [0,1] \longrightarrow Y$ with same start and end point are called homotopic: "$\alpha \sim \beta$", if there is a* homotopy *from α to β, i.e., a continuous map $F : [0,1] \times [0,1] \longrightarrow Y$ with the following properties:*

$$F(0,t) = \alpha(0) = \beta(0), F(1,t) = \alpha(1) = \beta(1)$$

and $F_t(s) := F(s,t)$ satisfies

$$F_0 = \alpha, \ F_1 = \beta.$$

Remark 10.5. 1. To be homotopic is an equivalence relation on the set of paths from a given point $x \in Y$ to another given point $y \in Y$. We denote $[\gamma]$ the equivalence class (homotopy class) of the path γ. If $\tau : [0,1] \longrightarrow [0,1]$ is a continuous map with $\tau(0) = 0, \tau(1) = 1$ (a "reparametrization"), then $\gamma \circ \tau \sim \gamma$.

2. Given paths $\alpha, \beta : [0,1] \longrightarrow Y$, such that $\beta(0) = \alpha(1)$, we define the concatenation $\alpha\beta : [0,1] \longrightarrow Y$ by

$$(\alpha\beta)(s) = \begin{cases} \alpha(2s) & , \quad \text{if } 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1) & , \quad \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

3. If $\alpha \sim \tilde{\alpha}, \beta \sim \tilde{\beta}$ and the end point of α is the starting point of β, then $\alpha\beta \sim \tilde{\alpha}\tilde{\beta}$, in particular we can concatenate homotopy classes. Note that in general $\alpha(\beta\gamma) \neq (\alpha\beta)\gamma$, but that $\alpha(\beta\gamma) \sim (\alpha\beta)\gamma$, i.e. on the level of homotopy classes concatenation becomes associative.

4. We can not only compose paths, but there is also the notion of an inverse path: Given $\alpha : [0,1] \longrightarrow Y$, we denote $\alpha^{-1} : [0,1] \longrightarrow Y$ the path $\alpha^{-1}(s) := \alpha(1-s)$. Note that $\alpha^{-1}\alpha \sim \alpha(0) \sim \alpha\alpha^{-1}$.

Proposition 10.6. *Let $\pi : X \longrightarrow Y$ be a covering and $\alpha, \beta : [0,1] \longrightarrow Y$ homotopic paths. Then a homotopy $F : [0,1]^2 \longrightarrow Y$ between α and β can be lifted to a homotopy $\hat{F} : [0,1]^2 \longrightarrow X$ between any two lifts $\hat{\alpha}, \hat{\beta} : [0,1] \longrightarrow X$ of α, β with the same starting point. In particular they have the same end point as well: $\hat{\alpha}(1) = \hat{\beta}(1)$.*

Proof. Fix $n \in \mathbb{N}$. We consider the subdivision of the unit square

$$[0,1]^2 = \bigcup_{1 \leq i,j \leq n} Q_{ij}$$

with

$$Q_{ij} := \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right], \quad 1 \leq i,j \leq n.$$

For sufficiently big $n \in \mathbb{N}$ every $F(Q_{ij}) \subset Y$ is contained in an open connected set $V = V_{ij} \subset Y$, such that $\pi^{-1}(V)$ is a disjoint union as in Def. 10.1. Now assume we have found a lift

$$\hat{F}_{ij} : B_{ij} := \bigcup_{(k,\ell) \prec (i,j)} Q_{k\ell} \longrightarrow X,$$

of $F|_{B_{ij}}$, where $\prec$ is the lexicographic order on $\{1, ..., n\}^2$. Since $B_{ij} \cap Q_{ij}$ is connected, we have $F_{ij}(B_{ij} \cap Q_{ij}) \subset U$ for one of the subset $U \subset \pi^{-1}(V)$ with $\pi|_U : U \longrightarrow V$ being homeomorphic. Hence we may extend $\hat{F}_{ij}$ to $B_{ij} \cup Q_{ij}$ defining it on Q_{ij} as $(\pi|_U)^{-1}$. □

Definition 10.7. *A path connected topological space Z is called* simply connected *if it is connected and any closed path $\gamma : [0,1] \longrightarrow Z$ is nullhomotopic, i.e. homotopic to the constant path $\equiv \gamma(0) = \gamma(1)$.*

Remark 10.8. 1. It suffices to check the above condition for one base point as prescribed starting and end point for the closed path γ.

2. In a simply connected topological space Z any two paths with the same start and the same end points are homotopic.

Example 10.9. 1. Obviously K^n is simply connected.

2. A path connected space $X = U \cup V$, which is the union of two open simply connected subsets $U, V \subset X$ with a path connected intersection $U \cap V$, is simply connected. In particular the spheres $\mathbf{S}^n, n \geq 2$, are simply connected. – To see this take a base point $x_0 \in U \cap V$ and consider a closed path $\gamma : [0,1] \longrightarrow X$. Then for sufficiently big $n \in \mathbb{N}$ every interval $I_k := [\frac{k-1}{n}, \frac{k}{n}]$ satisfies $\gamma(I_k) \subset U$ or $\gamma(I_k) \subset V$. Choose a path α_k from x_0 to $\gamma(\frac{k}{n})$ within U resp. V if $\gamma(\frac{k}{n}) \in U$ resp. $\gamma(\frac{k}{n}) \in V$. That is possible, since $U \cap V$ is connected. Then $\gamma \sim \beta_1 ... \beta_n := (..(\beta_1\beta_2)....\beta_n)$ with $\beta_1 := \gamma_1\alpha_1^{-1}, \beta_k := \alpha_{k-1}\gamma_k\alpha_k^{-1}, 2 \leq k < n$ and $\beta_n := \alpha_{n-1}\gamma_n$. Since both U and V are simply connected and $\beta_k([0,1]) \subset U$ or $\beta_k([0,1]) \subset V$, we get $\beta_k \sim x_0, 1 \leq k \leq n$, and thus $\gamma \sim x_0$.

As a consequence of Proposition 10.6 we obtain:

Proposition 10.10. *Let $\pi : X \longrightarrow Y$ be a covering and $\varphi : Z \longrightarrow Y$ be a continuous map from the simply connected topological space Z to Y. Then given points $z_0 \in Z, x_0 \in X$ with $\varphi(z_0) = \pi(x_0)$, there is a unique lifting $\hat{\varphi} : Z \longrightarrow X$ of φ with $\hat{\varphi}(z_0) = x_0$.*

Corollary 10.11. *A covering $\pi : X \longrightarrow Y$ from a connected space X to a simply connected space Y is a homeomorphism. In particular for a simply connected Lie group G the mapping*

$$\mathrm{Hom}(G, H) \longrightarrow \mathrm{Hom}(\mathfrak{g}, \mathfrak{h}), \varphi \mapsto \varphi_*$$

is bijective for any Lie group H.

Proof. The identity $\mathrm{id} := \mathrm{id}_Y : Y \longrightarrow Y$ admits a lifting $\widehat{\mathrm{id}}$. Its image in X is non-empty and both open and closed, hence equals X. □

Definition 10.12. *A covering $\pi : \hat{X} \longrightarrow X$ is called the universal covering of X, if $\hat{X}$ is simply connected.*

We call a topological space locally simply connected if every point has an open simply connected neighbourhood.

Theorem 10.13. *Every connected and locally simply connected topological space X admits a covering $\pi : \hat{X} \longrightarrow X$ with a simply connected $\hat{X}$, called* the universal covering of X.

Proof. The construction is motivated by the following observation: Assume there is a universal covering $\pi : \hat{X} \longrightarrow X$ and choose base points $x_0 \in X$ as well as $\hat{x}_0 \in \hat{X}$ above $x_0 \in X$, i.e. $\pi(\hat{x}_0) = x_0$. Then the points in the fiber $\pi^{-1}(x)$ of a point $x \in X$ are in one-to-one correspondence with the homotopy classes of paths in X from the base point $x_0 \in X$ to x: Given a point $\hat{x} \in \pi^{-1}(x)$, choose a path $\hat{\gamma}$ from $\hat{x}_0$ to $\hat{x}$. Then associate to $\hat{x}$ the homotopy class $[\pi \circ \hat{\gamma}]$; indeed that provides a well defined map, $\hat{X}$ being simply connected. On the other hand, given a homotopy class $[\gamma]$ of a path $\gamma : [0,1] \longrightarrow X$ from x_0 to x, denote $\hat{\gamma} : [0,1] \longrightarrow \hat{X}$ the unique lift of γ with $\hat{\gamma}(0) = \hat{x}_0$. The point $\hat{x} \in \pi^{-1}(x)$, we associate to $[\gamma]$ then is $\hat{x} := \hat{\gamma}(1)$, "welldefinedness" being guaranteed by Proposition 10.6.

So in order to obtain the universal covering we may apply the following strategy: We choose a base point $x_0 \in X$ and define $\hat{X}$ to be the set of all homotopy classes of paths with the base point x_0 as start point. The map $\pi : \hat{X} \longrightarrow X$ then is defined as $\pi([\gamma]) := \gamma(1)$. The topology on $\hat{X}$ is defined as follows: Given a point $\hat{x} := [\gamma]$ with $x := \pi(\hat{x})$ and a simply connected neighbourhood U of x, we set

$$U(\hat{x}) := \{[\gamma\delta]; \delta : [0,1] \longrightarrow U, \delta(0) = x\} \ , \ \hat{x} = [\gamma].$$

Then the sets $U(\hat{x})$ with an open neighbourhood $U \subset X$ of $x \in X$ constitute a basis for the topology of $\hat{X}$. We claim, that for simply connected U we have $U(a) \cap U(b) = \emptyset$ for $a, b \in \pi^{-1}(x), a \neq b$. Let $a = [\alpha], b = [\beta]$. Assume that $[\alpha\delta] = [\beta\delta']$ with paths $\delta, \delta' : [0,1] \longrightarrow U$. Take a path $\gamma : [0.1] \longrightarrow U$ from $\delta(1) = \delta'(1)$ to $\alpha(1) = \beta(1)$. Now, U being simply connected, we have $\delta\gamma \sim 0 \sim \delta'\gamma$ and thus

$$\alpha\delta \sim \beta\delta' \Longrightarrow \alpha\delta\gamma \sim \beta\delta'\gamma \Longrightarrow \alpha \sim \beta,$$

a contradiction. It follows easily that $\hat{X}$ is Hausdorff and $\pi : \hat{X} \longrightarrow X$ a covering.

Finally we show that $\hat{X}$ is simply connected: Let $\hat{\alpha} : [0,1] \longrightarrow \hat{X}$ be a closed path with start end point $\hat{x}_0 := [x_0]$ (where x_0 is regarded as the constant path). Consider the path $\alpha := \pi \circ \hat{\alpha}$. We have $\hat{\alpha}(1) = [\alpha]$, since $\hat{\alpha}$ is a lift of α with starting point $\hat{x}_0$ as well as the path $t \mapsto [\alpha_t]$ with the path $\alpha_t : [0,1] \longrightarrow \hat{X}, s \mapsto \alpha(ts)$. So because of the unique lifting property we obtain $\hat{\alpha}(1) = [\alpha_1] = [\alpha]$. But $\hat{\alpha}$ was a closed path, i.e. $[\alpha] = \hat{\alpha}(1) = \hat{\alpha}(0) = x_0$, i.e. $\alpha \sim x_0$. By Proposition 10.6 we obtain $\hat{\alpha} \sim \hat{x}_0$. □

Now let us consider the situation where $X = G$ is the topological space underlying a connected Lie group.

Proposition 10.14. *The universal covering $\hat{G}$ of a connected Lie group G carries, after the choice of a neutral element $\hat{e} \in \pi^{-1}(e)$, a unique group structure, such that $\pi : \hat{G} \longrightarrow G$ becomes a group homomorphism.*

Proof. For every point $\hat{a} \in \hat{G}$ we define the map $\lambda_{\hat{a}} : \hat{G} \longrightarrow \hat{G}$ as the unique lift of the map $\lambda_a \circ \pi : \hat{G} \longrightarrow G, a := \pi(\hat{a})$, satisfying $\lambda_{\hat{a}}(\hat{e}) = \hat{a}$. We leave it to the reader to check that the resulting map

$$\hat{G} \times \hat{G} \longrightarrow \hat{G}, (\hat{a}, x) \mapsto \lambda_{\hat{a}}(x)$$

defines a group law on $\hat{G}$, such that $\pi : \hat{G} \longrightarrow G$ becomes a group homomorphism. □

Remark 10.15. Fundamental Group: The construction of the universal covering $\pi : \hat{X} \longrightarrow X$ can be used to associate to any connected and locally simply connected space a group, namely the set

$$\mathrm{Deck}(X) := \{f : \hat{X} \longrightarrow \hat{X} \text{ homeomorphism}; \pi \circ f = \pi\}$$

of all π-fiber preserving homeomorphisms of $\hat{X}$ ("deck transformations") with the composition of maps as group law. From our above reasoning it follows that, given a base point $x_0 \in X$, the restriction $\mathrm{Deck}(X) \longrightarrow \mathbb{S}(\pi^{-1}(x_0)), f \mapsto f|_{\pi^{-1}(x_0)}$ is injective. On the other hand, given points $a, b \in \pi^{-1}(x_0)$ there is exactly one $f \in \mathrm{Deck}(X)$ with $f(a) = b$. If one wants to avoid the universal covering $\pi : \hat{X} \longrightarrow X$ in the definition of $\mathrm{Deck}(X)$, one can construct an isomorphic group as follows: Take again a base point $x_0 \in X$ and define the *fundamental group* of X as the set

$$\pi_1(X, x_0) := \{[\gamma];\ \gamma \text{ path in } X, \gamma(0) = x_0 = \gamma(1)\}$$

of homotopy classes of closed paths in X with start and end point x_0, the group law being the concatenation of paths representing homotopy classes:

$$[\alpha][\beta] := [\alpha\beta].$$

Then there is a natural isomorphism

$$\pi_1(X, x_0) \cong \mathrm{Deck}(X)$$

as follows: Given $[\gamma]$ take any lifting $\hat{\gamma}$ of γ, then the unique $f \in \mathrm{Deck}(X)$ with $f(\hat{\gamma}(1)) = \hat{\gamma}(0)$ is the image of $[\gamma]$.

Furthermore note that a path α from $x_0 \in X$ to $x_1 \in X$ induces an isomorphism

$$\pi_1(X.x_0) \xrightarrow{\cong} \pi_1(X, x_1), [\gamma] \mapsto [\alpha^{-1}\gamma\alpha].$$

If G is a connected Lie group, we have

$$\pi_1(G, e) \cong \mathrm{Deck}(G) \cong K := \ker(\pi : \hat{G} \longrightarrow G),$$

where we associate to $a \in K$ the left translation $\lambda_a : \hat{G} \longrightarrow \hat{G}$. Now a discrete normal subgroup of a Lie group G is central, i.e. contained in its center $Z(G)$, and hence abelian. So we have seen:

Corollary 10.16. *The fundamental group $\pi_1(G, e)$ of a connected Lie group is abelian.*

In order to compute the fundamental group of a connected Lie group G, one often considers a connected closed Lie subgroup $H \subset G$, usually realized as the stabilizer $H = G_x$ of a point $x \in M$, when given a differentiable G-action $G \times M \longrightarrow M$ on some manifold M. Then $E := G \longrightarrow B := G/H \cong Gx$ is an H-principal bundle, and in that situation we can apply the following theorem:

Theorem 10.17. *Let H be a connected Lie group and $p : E \longrightarrow B$ be an H-principal bundle over the path connected base B. Choose base points $b_0 \in B$ and $x_0 \in \pi^{-1}(b_0) \subset E$. Then there is an exact sequence*

$$\pi_1(H, e) \xrightarrow{\iota_*} \pi_1(E, x_0) \xrightarrow{p_*} \pi_1(B, b_0) \longrightarrow 1,$$

where $\iota : H \longrightarrow E, h \mapsto x_0 h$ identifies (H, e) with $(p^{-1}(b_0), x_0)$, and ι_*, p_* denote the induced homomorphisms on the level of the fundamental groups. Recall that exactness means:

$$\ker(p_*) = \mathrm{im}(\iota_*),\ \mathrm{im}(p_*) = \pi_1(B, b_0).$$

Proof. Since the map $p : E \longrightarrow B$ is locally trivial (i.e. B can be covered with open subsets U, such that $U \times H \cong p^{-1}(U) \xrightarrow{p} U \subset B$ is the projection onto U), every path $\gamma : I \longrightarrow B$ admits a lifting to a path $\hat{\gamma} : I \longrightarrow E$. This implies the surjectivity of p_*. Furthermore obviously $p_* \circ \iota_* \equiv [b_0] \in \pi_1(B, b_0)$.

It remains to show the inclusion $\ker(p_*) \subset \operatorname{im}(\iota_*)$: So let $\gamma : I \longrightarrow E$ be a closed path with start and end point x_0, such that $p \circ \gamma$ is nullhomotopic in B. We have to find a homotopy between γ and a suitable closed path in $p^{-1}(b_0)$. Consider a homotopy $F : I^2 \longrightarrow B$ with $F_0 = p \circ \gamma, F_1 \equiv b_0$. As in the proof of Proposition 10.6 we are looking for a lifting $\hat{F} : I^2 \longrightarrow E$ of F with $\hat{F}_0 = \gamma, \hat{F}_t(0) = x_0 = \hat{F}_t(1)$ for all $t \in I$. Then $\hat{F}_1 : I \longrightarrow E$ is the desired path in the fiber over b_0 homotopic to γ.

Subdivide again I^2 into small squares Q_{ij}, such that $Q_{ij} \subset U_{ij}$ for some open set U_{ij} with $p^{-1}(U_{ij}) \cong U_{ij} \times H$. Define

$$B_{ij} := B \cup \bigcup_{(k,\ell) \prec (i,j)} Q_{k\ell}$$

with $B := I \times \{0\} \cup \partial I \times I$. On $B_{00} = B$ the lift is given by $\hat{F}_0 = \gamma, \hat{F}_t(0) = x_0 = \hat{F}_t(1)$. Now assume we have found a lift $\hat{F}_{ij} : B_{ij} \longrightarrow E$ of $F|_{B_{ij}}$. Take an open subset $U \supset Q_{ij}$ with $p^{-1}(U) \cong U \times H$. Denote

$$q : p^{-1}(U) \cong U \times H \longrightarrow H$$

the projection onto H. Now the intersection $B_{ij} \cap Q_{ij}$ is connected and never the entire boundary ∂Q_{ij} of the tiny square Q_{ij}, but consists of two or three of its sides. So we can first run through the sides contained in $B_{ij} \cap Q_{ij}$ and then through the remainig ones, defining there an extension of $q \circ \hat{F}_{ij}$, such that one simply returns to the starting point of $q \circ \hat{F}_{ij}|_{B_{ij} \cap Q_{ij}}$ by inverting the parametrization. The resulting closed path $\partial I^2 \longrightarrow H$ is nullhomotopic in H, hence it can be extended to a continuous map $f : Q_{ij} \longrightarrow H$. Now extend $\hat{F}_{ij}$ to B_{ij} by defining it on the square Q_{ij} as

$$Q_{ij} \longrightarrow U \times H \cong p^{-1}(U), (s,t) \mapsto (F(s,t), f(s,t)).$$

□

11 Structure Theory for Lie Algebras

The last section gives a brief survey without proofs over some important results on the structure of real and complex Lie algebras. We start with the following "embedding theorem":

Theorem 11.1 (Theorem of Ado). *Any finite dimensional K-Lie algebra $\mathfrak{g}$ is isomorphic to a subalgebra $\mathfrak{h} \subset \mathfrak{gl}_n(K)$ for some $n \in \mathbb{N}$.*

As a consequence we see that

1. any Lie algebra $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathrm{Lie}(G)$ of a Lie group G, and

2. any connected Lie group G is "locally isomorphic" to a linear group H ($:=$ a connected subgroup $H \subset GL_n(K)$ for some $n \in \mathbb{N}$), i.e., their simply connected covering groups are isomorphic: $\hat{G} \cong \hat{H}$. Indeed, that is the best we can say, since a factor group G/K with a discrete normal (indeed central) subgroup $K \subset G$ or the universal covering group $\hat{G}$ of a linear group G need not be linear again!

In Remark 9.2.6 we have discussed a procedure how to construct a new Lie algebra $\mathfrak{g}_D$ starting from a Lie algebra $\mathfrak{g}$ and a derivation $D \in \mathrm{Der}(\mathfrak{g})$. The class of all *solvable Lie algebras*, see Def.11.3, turns out to consist exactly of those Lie algebras which can be obtained from the abelian Lie algebra $\mathfrak{g} = K$ by an iterated application of that step.

Remark 11.2. Let $\mathfrak{g} = \mathrm{Lie}(G)$ with a simply connected Lie group G and $D \in \mathrm{Der}(\mathfrak{g})$. For $t \in K$ denote $\sigma_t : G \longrightarrow G$ the automorphism with $(\sigma_t)_* = e^{tD} : \mathfrak{g} \longrightarrow \mathfrak{g}$. Then $\sigma : K \longrightarrow \mathrm{Aut}(G), t \mapsto \sigma_t$, is a group homomorphism, and the semidirect product $G \times_\sigma K$ is the simply connected Lie group with Lie algebra isomorphic to $\mathfrak{g}_D$. Remember that the underlying manifold is just the cartesian product $G \times K$, while the group law looks as follows:

$$(g,t)(g',t') = (g\sigma_t(g'), t+t').$$

In particular we see that the manifold underlying a simply connected solvable Lie group is isomorphic to K^n for some $n \in \mathbb{N}$.

Back to Lie algebras! Note first that, given an ideal $\mathfrak{a} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ we can endow the factor vector space with a natural Lie bracket

$$[X+\mathfrak{a}, Y+\mathfrak{a}] := [X,Y] + \mathfrak{a},$$

the resulting Lie algebra being called the factor algebra $\mathfrak{g}/\mathfrak{a}$ of $\mathfrak{g}$ mod(ulo) the ideal $\mathfrak{a}$. Furthermore that $\mathfrak{g} = \mathfrak{a}_D$ if $\dim \mathfrak{g} = \dim \mathfrak{a} + 1$ and $D = \mathrm{ad}(X)$ for some $X \in \mathfrak{g} \setminus \mathfrak{a}$.

Definition 11.3. *The Lie algebra* $\mathfrak{g}$ *is called* solvable *if there is a finite sequence of subalgebras*

$$\mathfrak{g}_0 = 0 \subset \mathfrak{g}_1 \subset ... \subset \mathfrak{g}_r = \mathfrak{g},$$

such that $\mathfrak{g}_i \subset \mathfrak{g}_{i+1}$ *is an ideal of* $\mathfrak{g}_{i+1}$ *for* $i < r$ *with abelian factor algebra* $\mathfrak{g}_{i+1}/\mathfrak{g}_i$ *(or equivalently* $[\mathfrak{g}_{i+1}, \mathfrak{g}_{i+1}] \subset \mathfrak{g}_i$*).*

Example 11.4.

1. The Lie algebra $\mathfrak{ut}_n(K) := \mathrm{Lie}(UT_n(K))$ is solvable. Indeed take

   $$\mathfrak{g}_i := \{A \in \mathfrak{ut}_n(K); A(V_k) \subset V_{k-(n-i)}\ k = 0, ..., n\},$$

   where we set $V_k = K^k \times \{0\} \subset K^n$ and $V_k = 0$ for $k \leq 0$. For $A, B \in \mathfrak{g}_n$ we have $[A, B] \in \mathfrak{g}_{n-1}$, since $\mathfrak{gl}(V_k/V_{k-1})$ is abelian, while for $A, B \in \mathfrak{g}_i, i < n$, we already have $AB, BA \in \mathfrak{g}_{i-1}$.

2. If $\mathfrak{g}$ is solvable, then any subspace $\mathfrak{h} \subset \mathfrak{g}$ with $\mathfrak{g}_i \subset \mathfrak{h} \subset \mathfrak{g}_{i+1}$ is a subalgebra and even an ideal in $\mathfrak{g}_{i+1}$. Hence we may refine a given strictly increasing sequence as in Def. 11.3 in such a way that finally $r = \dim \mathfrak{g}$ and $\dim \mathfrak{g}_{i+1} = \dim \mathfrak{g}_i + 1$. In particular we see, that a solvable algebra can be constructed by a repeated application of the $\mathfrak{g}_D$-construction for a Lie algebra $\mathfrak{g}$ together with a derivation $D \in \mathrm{Der}(\mathfrak{g})$.

3. Denote $C(\mathfrak{g}) := [\mathfrak{g}, \mathfrak{g}]$ the commutator subalgebra of $\mathfrak{g}$. A Lie algebra is solvable iff the decreasing sequence of successive commutator subalgebras $C^i(\mathfrak{g})$, i.e.

   $$C^0(\mathfrak{g}) := \mathfrak{g}, C^{i+1}(\mathfrak{g}) := C(C^i(\mathfrak{g}))$$

   terminates at the trivial subalgebra.

4. Let $\mathfrak{a} \subset \mathfrak{g}$ be an ideal. If $\mathfrak{a}$ is solvable as well as $\mathfrak{g}/\mathfrak{a}$, so is $\mathfrak{g}$. In particular, if $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are solvable ideals, so is $\mathfrak{a} + \mathfrak{b}$. Hence there is a unique maximal solvable ideal in a Lie algebra $\mathfrak{g}$:

Definition 11.5. *The maximal solvable ideal* $\mathfrak{r} \subset \mathfrak{g}$ *is called the* radical *of the Lie algebra* $\mathfrak{g}$.

Example 11.6. For $\mathfrak{g} = \mathfrak{gl}_n(K)$ we find $\mathfrak{r} = KE$.

Obviously the factor algebra $\mathfrak{g}/\mathfrak{r}$ has trivial radical. The next theorem tells us that there is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ projecting isomorphically onto the factor algebra $\mathfrak{g}/\mathfrak{r}$. It is clear that there is a subspace $U \subset \mathfrak{g}$, such that $\mathfrak{g} = \mathfrak{r} \oplus U$ and thus $U \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{r}$ is an isomorphism of vector spaces. The point is, that we can even find a complementary subspace, which is closed with respect to the Lie bracket, with other words a complementary Lie subalgebra.

Theorem 11.7. *For any Lie algebra $\mathfrak{g}$ there is a subalgebra $\mathfrak{h}$ (a "Levi subalgebra") complementary to the radical $\mathfrak{r} \subset \mathfrak{g}$, i.e. such that $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{h}$ as vector spaces (but in general not as Lie algebras!).*

Example 11.8. For $\mathfrak{g} = \mathfrak{gl}_n(K)$ the subalgebra $\mathfrak{h} := \mathfrak{sl}_n(K)$ is a Levi subalgebra. Note that in this case

$$\mathfrak{gl}_n(K) = \mathfrak{r} \oplus \mathfrak{sl}_n(K)$$

even as Lie algebras.

Remark 11.9. Given a Levi-Malcev-decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{h}$ of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ of the simply connected Lie group G, the group G is the semidirect product of the simply connected Lie groups R, H of $\mathfrak{r}$ and $\mathfrak{h}$. Indeed

$$G \cong R \times_\sigma H$$

with the group homomorphism $\sigma : H \longrightarrow \mathrm{Aut}(\mathfrak{r}) \cong \mathrm{Aut}(R), h \mapsto \mathrm{Ad}(h)$.

But note that $GL_n(K)$ in general is not isomorphic to $K^*E \times SL_n(K)$, instead one has to factor out the finite subgroup

$$K^*E \cap SL_n(K) \cong C_n(K)$$

with the group of n-th roots of unity $C_n(K) := \{a \in K; a^n = 1\}$. So

$$GL_n(K) \cong (K^*E \times SL_n(K))/C_n(K)E.$$

For the further study of Lie algebras it turns out to be useful to investigate in general homomorphisms $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ for a K-vector space V (also called *representations* of $\mathfrak{g}$ in V). Slightly changing the point of view (but not the objects under consideration!) we arrive at the notion of a $\mathfrak{g}$-*module*:

Definition 11.10. *Let $\mathfrak{g}$ be a Lie algebra.*

1. *A (finite dimensional)* $\mathfrak{g}$-module V *is a (finite dimensional)* K*-vector space* V *together with a bilinear map*

$$\mathfrak{g} \times V \longrightarrow V, (X, v) \mapsto Xv,$$

such that $[X, Y]v = X(Yv) - Y(Xv)$.

2. *A* $\mathfrak{g}$*-module is called*

   (a) irreducible *or* simple *if* V *and* $\{0\}$ *are the only* $\mathfrak{g}$*-submodules.*

   (b) semisimple *if any* $\mathfrak{g}$*-submodule* $U \subset V$ *admits a complementary* $\mathfrak{g}$*-submodule* $W \subset V$*, i.e. such that* $V = U \oplus W$.

3. *A Lie algebra* $\mathfrak{g}$ *is called semisimple, if all its finite dimensional* $\mathfrak{g}$*-modules are semisimple.*

So $\mathfrak{g}$-modules are in one-to-one correspondence with K-vector spaces V together with a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. In particular, regarding K as abelian Lie algebra, a K-module is not just a K-vector space, but rather a pair (V, f) with a K-vector space V and an endomorphism $f \in \mathfrak{gl}(V)$: The Lie algebra homomorphisms $K \longrightarrow \mathfrak{gl}(V)$ are uniquely determined by their value $f \in GL(V)$ at $1 \in K$, and that value can be prescribed arbitrarily.

For solvable Lie algebras we have:

Theorem 11.11 (Theorem of Lie)**.** *Any (finite dimensional) module* V *over a solvable complex Lie algebra* $\mathfrak{g}$ *admits a one dimensional submodule* $L \subset V$*. In particular an irreducible* $\mathfrak{g}$*-module over a solvable complex Lie algebra* $\mathfrak{g}$ *has dimension* $\dim V = 1$.

Corollary 11.12.

1. *Any (finite dimensional) module* V *over a solvable complex Lie algebra* $\mathfrak{g}$ *admits an invariant flag*

$$0 = V_0 \subset V_1 \subset ... \subset V_n = V$$

of submodules $V_i \subset V$ *of dimension* i.

2. *A solvable complex Lie algebra is isomorphic to a subalgebra of* $\mathfrak{ut}_n(\mathbb{C})$ *for some* $n \in \mathbb{N}$.

Before we discuss semisimple algebras and modules let us study an important subclass of all solvable algebras:

Definition 11.13. *A Lie algebra* $\mathfrak{g}$ *is called nilpotent if the following decreasing sequence of subalgebras* $N^i(\mathfrak{g})$ *terminates at* $\{0\}$*:*

$$N^0(\mathfrak{g}) := \mathfrak{g}, N^{i+1}(\mathfrak{g}) = [\mathfrak{g}, N^i(\mathfrak{g})].$$

Remark 11.14. We have $N(\mathfrak{g}) = C(\mathfrak{g})$ and $C^i(\mathfrak{g}) \subset N^i(\mathfrak{g})$ for $i > 1$, hence a nilpotent algebra is solvable.

Example 11.15.

1. The Lie algebra $\mathfrak{uu}_n(K)$ consisting of all upper triangular matrices with zeros on the diagonal is nilpotent.
2. The Lie algebra $\mathfrak{g} = KX + KZ$ with $[X, Z] = Z$ is solvable, but not nilpotent: We have $N^i(\mathfrak{g}) = KZ$ for all $i \in \mathbb{N}_{>0}$.
3. Abelian Lie algebras are nilpotent.
4. Subalgebras and factor algebras of nilpotent Lie algebras are nilpotent.

In a nilpotent Lie algebra $\mathfrak{g}$ the Baker-Campbell-Hausdorff series

$$C(X,Y) = \sum_{i=1}^{\infty} C_i(X,Y) = (X+Y) + \frac{1}{2}[X,Y] + C_3(X,Y) + ...$$

is finite and hence defines a polynomial map

$$C : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}.$$

Indeed, it provides $\mathfrak{g}$ with a group law: Consider the exponential map $\exp : \mathfrak{g} \longrightarrow G$ for the simply connected Lie group with $\mathrm{Lie}(G) \cong \mathfrak{g}$. Then, exp being diffeomorphic near $0 \in \mathfrak{g}$, the conditions for a group law are satisfied near the origin and hence everywhere by the identity theorem for polynomial maps (A map $V \times V \longrightarrow K$ for a K-vector space V is called polynomial if it is a K-linear combination of products of linear forms in one of both variables. A map $V \times V \longrightarrow V$ is polynomial, if the composition with any linear functional $V \longrightarrow K$ is polynomial.). Indeed the exponential map $\exp : \mathfrak{g} \longrightarrow G$ turns out to be an isomorphism of Lie groups. So we can replace G with $(\mathfrak{g}, C(.,.))$, the expression for C in terms of Lie monomials being independent from the nilpotent Lie algebra $\mathfrak{g}$. Note that the n-th power of $X \in \mathfrak{g}$ for $n \in \mathbb{Z}$ with respect to $C(.,.)$ is nX.

Example 11.16. The exponential map $\exp : \mathfrak{uu}_n(K) \longrightarrow UU_n(K)$ is polynomial:

$$\exp(X) = \sum_{i=0}^{n-1} \frac{X^i}{i!}$$

and has the inverse

$$\log : UU_n(K) \longrightarrow \mathfrak{uu}_n(K),\ \log(Y) = \sum_{i=1}^{n-1} (-1)^{i+1} \frac{(Y-E)^i}{i}.$$

In order to understand all Lie groups with nilpotent Lie algebra we have to consider factor groups of $(\mathfrak{g}, C(.,.))$ mod central discrete subgroups $D \subset (\mathfrak{g}, C(.,.))$. We claim that the center of the Lie group $(\mathfrak{g}, C(.,.))$ is the center $\ker(\mathrm{ad})$ of the Lie algebra $\mathfrak{g}$: Since $C(X,0) = 0 = C(0,Y)$, any of the Lie bracket monomials in $C(X,Y)$ contains both X and Y as factor, hence $C(Z,X) = Z + X = X + Z = C(X,Z)$ for $Z \in \ker(\mathrm{ad})$ and any $X \in \mathfrak{g}$. On the other hand, for a central element Z of the Lie group $(\mathfrak{g}, C(.,.))$ all its integral powers nZ are central as well; hence $C(nZ, mX) = C(mX.nZ)$ for all $n, m \in \mathbb{Z}$. Both expressions being polynomials in $n, m \in \mathbb{Z}$, comparison of the bilinear term yields $[Z,X] = [X,Z]$ resp. $[Z,X] = 0$. So central discrete subgroups are exactly the lattices in the subspace $\ker(\mathrm{ad}) \subset \mathfrak{g}$. Note furthermore that the connected Lie subgroups of $(\mathfrak{g}, C(.,.))$ are exactly the subalgebras $\mathfrak{h} \subset \mathfrak{g}$ (the exponential map being the identity on $\mathfrak{g}$ and subspaces being maximal connected submanifolds).

Theorem 11.17.

1. *A Lie algebra $\mathfrak{g}$ is solvable iff its commutator algebra $C(\mathfrak{g})$ is nilpotent.*
2. *A Lie algebra is nilpotent iff $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent for every $X \in \mathfrak{g}$.*
3. *Any nilpotent Lie algebra is isomorphic to a subalgebra of $\mathfrak{uu}_n(K)$ for some $n \in \mathbb{N}$.*

Example 11.18. A non-linear Lie group: Consider $G = (\mathfrak{g}, C(.,.))/D$ with $D = \mathbb{Z} \cdot [X,Y]$ with a central element $[X,Y]$. Assume $\varphi : G/D \xrightarrow{\cong} H \subset GL_n(K)$ is an isomorphism of Lie groups. Then, $\mathfrak{g}$ being solvable we may assume $\mathfrak{h} \subset \mathfrak{ut}_n(K)$, see Theorem 11.11 resp. $H \subset UT_n(K)$, see Cor. 11.12. Consider the element $Z := 2^{-1}[X,Y]$. Since $C(\mathfrak{ut}_n(K)) \subset \mathfrak{uu}_n(K)$ we find $\varphi_*(Z) \in \mathfrak{uu}_n(K)$, hence $\varphi(Z) = \exp(\varphi_*(Z)) \in UU_n(K)$ (note that G is here identified with its Lie algebra), but $UU_n(K)$ contains no non-trivial elements of finite order!

Now let us consider semisimple algebras and modules:

Corollary 11.19. *A semisimple $\mathfrak{g}$-module V is the direct sum of irreducible $\mathfrak{g}$-modules*

$$V = \bigoplus_{i=1}^{s} V_i,$$

the factors $V_1, ..., V_s$ being unique up to isomorphy and order.

But note that for $v \in V$ the subspace $\mathfrak{g}v := \{Xv; X \in \mathfrak{g}\}$ in general is not a ($\mathfrak{g}$-)submodule, so the irreducible factors are not necessarily factor algebras of $\mathfrak{g}$ (e.g. $X(Yv)$ need not belong to $\mathfrak{g}v$). On the other hand $V := \mathfrak{g}$ is a $\mathfrak{g}$-module with the Lie bracket as "scalar multiplication" (corresponding to the adjoint representation $\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}), X \mapsto \mathrm{ad}(X)$). Then the irreducible factors are ideals of $\mathfrak{g}$. Calling a Lie algebra $\mathfrak{g}$ simple if it is semisimple and admits no nontrivial ideals we obtain:

Theorem 11.20. *A semisimple Lie algebra $\mathfrak{g}$ is the direct sum of simple Lie algebras*

$$\mathfrak{g} = \bigoplus_{i=1}^{s} \mathfrak{g}_i,$$

the factors $\mathfrak{g}_1, ..., \mathfrak{g}_s$ being unique up to isomorphy and order.

Ideals of semisimple algebras are direct factors and thus semisimple as well. As a consequence, no nontrivial solvable algebra is semisimple, since otherwise we would find that the one-dimensional Lie algebra K is semisimple. Furthermore, a semisimple algebra has trivial radical. Indeed the reverse implication holds as well:

Theorem 11.21 (Theorem of Weyl)**.** *A Lie algebra $\mathfrak{g}$ with trivial radical is semisimple.*

Example 11.22.

1. A real Lie algebra $\mathfrak{g}$, such that the simply connected Lie group G with $\mathrm{Lie}(G) \cong \mathfrak{g}$ is compact, is semisimple: The Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ defining a $\mathfrak{g}$-module is induced from a Lie group homomorphism $\varphi : G \longrightarrow GL(V)$. According to the next point there is an inner product $\sigma : V \times V \longrightarrow K$, such that all maps

$\varphi(g) : V \longrightarrow V, v \mapsto gv := \varphi(g)v$ are σ-isometries. Now, given a $\mathfrak{g}$-submodule $U \subset V$, the σ-orthogonal complement $W := U^\perp$ is both G-invariant and a complementary $\mathfrak{g}$-module. So $\mathfrak{so}_n(\mathbb{R})$ and $\mathfrak{su}_n$ are semisimple for $n > 1$.

We remark, that any real semisimple algebra is of the type described above.

2. Invariant inner products: On any Lie group G there is a left-invariant σ-finite measure μ on the Borel subsets of G, the "Haar measure", unique up to a scalar factor. For compact G, the Haar measure is even right invariant and $\mu(G) < \infty$. Then we may take any inner product $\tau : V \times V \longrightarrow K$ in order to get by means of averaging over G the desired G-invariant inner product:

$$\sigma(v, w) := \int_G \tau(gv, gw) d\mu(g).$$

3. Complex semisimple Lie algebras: The first point applies only to real Lie algebras, since there are no compact simply connected Lie groups except the trivial group. (The only connected compact complex Lie groups are the tori $G = \mathbb{C}^m/\Lambda$ with a lattice $\Lambda \cong \mathbb{Z}^{2m}$ of maximal rank.) But we can weaken our assumption: It is sufficient that $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ with a real Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ belonging to a simply connected compact real Lie group K (not to be confused with the base field). Given now a $\mathfrak{g}$-submodule $U \subset V$, its orthogonal complement with respect to a K-invariant inner product on V is a K-invariant complex vector subspace, hence also $\mathfrak{k}$ and $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$-invariant. As example take

$$\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) = \mathfrak{su}_n \oplus i\mathfrak{su}_n.$$

or

$$\mathfrak{g} = \mathfrak{so}_n(\mathbb{C}) = \mathfrak{so}_n(\mathbb{R}) \oplus i\mathfrak{so}_n(\mathbb{R}).$$

Again, any complex semisimple Lie algebra is obtained in that way.

4. Haar measure (Only for those knowing differential forms): Let G be a real Lie group of dimension n. As with vector fields, any n-form $\omega_e \in \bigwedge^n T_e^*G$ extends uniquely to a left invariant n-form $\omega \in \Omega^n(G)$. Then $\mu(A) := \int_A \omega$ (using the orientation of G defined by ω itself) provides a Haar measure on the Borel subsets $A \subset G$.

In the remaining part of this section we explain the classification of complex simple Lie algebras.

Definition 11.23. *A subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *of a Lie algebra* $\mathfrak{g}$ *is called a* Cartan subalgebra *(CSA), if it is nilpotent and satisfies*

$$[X, \mathfrak{h}] \subset \mathfrak{h} \Longrightarrow X \in \mathfrak{h}, \ \forall \ X \in \mathfrak{g}.$$

Example 11.24. For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ the subalgebra $\mathfrak{h} := \mathfrak{sd}_n(\mathbb{C})$ consisting of all diagonal matrices in $\mathfrak{sl}_n(\mathbb{C})$ is a CSA.

Proposition 11.25. *In a complex Lie algebra* $\mathfrak{g}$ *any two CSA* $\mathfrak{h}, \mathfrak{h}' \subset \mathfrak{g}$ *are conjugate under an automorphism* $f = e^{\mathrm{ad}(X)}$ *for some* $X \in \mathfrak{g}$*, i.e.* $\mathfrak{h}' = f(\mathfrak{h})$*.*

For a semisimple algebra we have

Proposition 11.26. *Let* $\mathfrak{g}$ *be a complex semisimple algebra. Then a subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *is a CSA if and only if the following conditions are satisfied:*

1. *The subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *is a maximal abelian subalgebra.*
2. *All homomorphisms* $\mathrm{ad}(H) : \mathfrak{g} \longrightarrow \mathfrak{g}$ *are diagonalizable.*

Since $\mathfrak{h}$ is abelian, the endomorphisms $\mathrm{ad}(H)$ for $H \in \mathfrak{h}$ commute one with the other, hence can be diagonalized simultaneously, and satisfy $\mathrm{ad}(H)|_{\mathfrak{h}} = 0$. So defining $\mathfrak{g}_\alpha \subset \mathfrak{g}$ for a linear form $\alpha : \mathfrak{h} \longrightarrow \mathbb{C}$ by

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}; \mathrm{ad}(H)(X) = \alpha(H)X, \ \forall \ H \in \mathfrak{h}\}$$

we can write

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

with the following finite subset $\Delta \subset \mathfrak{h}^*$:

$$\Delta := \{\alpha \in \mathfrak{h}^* \setminus \{0\}; \mathfrak{g}_\alpha \neq \{0\}\}.$$

The case $\alpha = 0$ does not occur, $\mathfrak{h}$ being a maximal abelian subalgebra.

Example 11.27. For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}), \mathfrak{h} = \mathfrak{sd}_n(\mathbb{C})$(:= the diagonal matrices with trace 0) we have

$$\Delta = \{\alpha_{ij} := \beta_i - \beta_j; 1 \leq i, j \leq n, i \neq j\}$$

where

$$\beta_i : \mathfrak{sd}_n(\mathbb{C}) \longrightarrow \mathbb{C}, D = (z_k \delta_{k\ell}) \mapsto z_i$$

with

$$\mathfrak{g}_{ij} := \mathfrak{g}_{\alpha_{ij}} = \mathbb{C}E_{ij}$$

with the matrix $E_{ij} := (\varepsilon_{k\ell} := \delta_{ki}\delta_{\ell j})$.

Since $\mathrm{ad}(H) \in \mathrm{Der}(\mathfrak{g})$ we find

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta},$$

indeed

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}.$$

Furthermore one can show

$$\dim \mathfrak{g}_\alpha = 1, \ \forall \ \alpha \in \Delta$$

and that

$$\Delta \cap \mathbb{C}\alpha = \{\pm\alpha\}, \ \forall \ \alpha \in \Delta.$$

The set Δ spans a real subspace

$$\mathfrak{h}_{\mathbb{R}}^* := \sum_{\alpha \in \Delta} \mathbb{R}\alpha \subset \mathfrak{h}^*$$

with $\mathfrak{h}^* = \mathfrak{h}_{\mathbb{R}}^* \oplus i\mathfrak{h}_{\mathbb{R}}^*$. For a more detailed description of Δ we need a natural inner product on $\mathfrak{h}_{\mathbb{R}}^*$.

First of all on a Lie algebra $\mathfrak{g}$ one defines:

Definition 11.28. *Let $\mathfrak{g}$ be a Lie algebra. The* Cartan-Killing form *is the following bilinear symmetric form*

$$\langle ., . \rangle : \mathfrak{g} \times \mathfrak{g} \longrightarrow K, (X, Y) \mapsto \langle X, Y \rangle := \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y)).$$

Let us mention:

Theorem 11.29. *Let $\mathfrak{g}$ be a Lie algebra with Cartan-Killing form $\langle .,. \rangle$: $\mathfrak{g} \times \mathfrak{g} \longrightarrow K$. Then $\mathfrak{g}$ is*

1. *solvable iff $\langle \mathfrak{g}, C(\mathfrak{g}) \rangle = \{0\}$ and*
2. *semisimple iff its Cartan-Killing form is nondegenerate.*

Moreover in the latter case its restriction to $\mathfrak{h} \times \mathfrak{h}$ with a CSA $\mathfrak{h} \subset \mathfrak{g}$ is nondegenerate as well, and its dual form, also denoted

$$\langle .,. \rangle : \mathfrak{h}^* \times \mathfrak{h}^* \longrightarrow \mathbb{C},$$

is real valued on $\mathfrak{h}^*_{\mathbb{R}} \times \mathfrak{h}^*_{\mathbb{R}}$ and even positive definite.

Remark 11.30. Define $H_\alpha \in \mathfrak{h}$ by $\langle H_\alpha, H \rangle = \alpha(H)$. Then

$$[X, Y] = \langle X, Y \rangle H_\alpha \neq 0$$

for $X \in \mathfrak{g}_\alpha \setminus \{0\}, Y \in \mathfrak{g}_{-\alpha} \setminus \{0\}$.

In particular we see that

$$\mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \cong \mathfrak{sl}_2(\mathbb{C}).$$

The classification of complex semisimple Lie algebras now depends on a better understanding of the set $\Delta \subset \mathfrak{h}^*_{\mathbb{R}}$. Indeed it has a remarkable symmetry property: It forms a *root system*:

Definition 11.31. *A finite subset $\Delta \subset E \setminus \{0\}$ of a finite dimensional euclidean vector space E with inner product $\langle .,. \rangle$ is called a root system if the following conditions are satisfied:*

1. *$E = \operatorname{span}(\Delta)$*
2. *$\Delta \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in \Delta$.*
3. *For any root (i.e. element) $\alpha \in \Delta$ the reflection*

$$s_\alpha : E \longrightarrow E, v \mapsto v - \frac{2\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

on the hyperplane $\alpha^\perp \subset E$ leaves Δ invariant:

$$s_\alpha(\Delta) = \Delta, \ \forall \ \alpha \in \Delta.$$

4. For all $\beta, \alpha \in \Delta$ we have

$$\chi(\beta, \alpha) := \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}.$$

The fourth condition is a very strong condition on the possible angles between the roots: Denote $\vartheta \in [0, \pi)$ the angle between the non proportional roots α, β. Then

$$\chi(\beta, \alpha)\chi(\alpha, \beta) = 4\cos^2(\vartheta) \in \mathbb{Z},$$

hence $\cos(\vartheta) \in \{0, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{2}\}$ resp. $\vartheta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$.

Furthermore $\chi(\beta, \alpha) = \pm 1, \pm 2, \pm 3$, if α, β are neither proportional nor orthogonal. A more detailed analysis of that situation shows that in case $||\beta|| \geq ||\alpha||$ we have

$$||\beta||^2 = 4\cos^2(\vartheta) \cdot ||\alpha||^2 = \chi(\beta, \alpha)\chi(\alpha, \beta)||\alpha||^2.$$

Indeed, that means nothing but $\beta = \pm(\alpha - s_\beta(\alpha))$, the sign depending on whether $\vartheta < \frac{\pi}{2}$ or $\vartheta > \frac{\pi}{2}$.

The proof that the above conditions hold for $\Delta \subset \mathfrak{h}^*$ relies on the knowledge of the irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules. Indeed, given nonproportional α, β, the subspace

$$\bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$$

is a module over the subalgebra

$$\mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \cong \mathfrak{sl}_2(\mathbb{C}).$$

Now let us concentrate on abstract root systems:

Definition 11.32. *Let $\Delta \subset E$ be a root system, and demote $O(E)$ the group of linear isometries of the euclidean space E. The Weyl group $W(\Delta) \subset O(E)$ is defined as the subgroup of $O(E)$ generated by the reflections $s_\alpha, \alpha \in \Delta$.*

The symmetries of a root system Δ given by the action of the Weyl group $W(\Delta)$ make it possible to compress the information contained in it in a *basis* B:

Definition 11.33. *A subset $B \subset \Delta$ is called a* basis *of the root system $\Delta \subset E$ if B is a basis of the vector space E and every $\beta \in \Delta$ is an integral linear combination*

$$\beta = \sum_{\alpha \in B} k_\alpha \cdot \alpha,$$

where the coefficients $k_\alpha \in \mathbb{Z}$ satisfy either $k_\alpha \geq 0$ for all $\alpha \in B$ or $k_\alpha \leq 0$ for all $\alpha \in B$.

A basis of a root system $\Delta \subset E$ gives rise to a decomposition

$$\Delta = \Delta_B + (-\Delta_B),$$

with $\Delta_B := \Delta \cap (\sum_{\alpha \in B} \mathbb{N}_{\geq 0}\alpha)$. On the other hand, starting with certain decompositions we get all the bases of a root system Δ:

Proposition 11.34. *Let $\Delta \subset E$ be a root system and $H \subset E$ a hyperplane with $H \cap \Delta = \emptyset$. Given a connected component E_0 of $E \backslash H$ the indecomposable elements in $\Delta \cap E_0$, i.e. those which can not be written as a sum $\beta_1 + \beta_2$ with $\beta_1, \beta_2 \in \Delta \cap E_0$, constitute a basis of the root system Δ. Indeed, any basis of Δ is obtained in that way. In particular any root can be realized as an element of a suitable basis $B \subset \Delta$. Furthermore the angle between two base vectors is obtuse, i.e. $\in [\frac{\pi}{2}, \pi)$.*

Proposition 11.35. *Let $\Delta \subset E$ be a root system.*

1. *The action of the Weyl group $W(\Delta)$ on the set of bases of Δ is simply transitive.*

2. *Given a basis B, the reflections $\sigma_\alpha, \alpha \in B$, generate $W(\Delta)$.*

As a consequence we see that $|W(\Delta)| < \infty$. Furthermore that given a basis B we can recover the Weyl group $W(\Delta)$ as well as $\Delta = W(\Delta)B$.

Let us explain a little bit more in detail the different bases a root system Δ admits: Writing a hyperplane as $H = P_\gamma := \gamma^\perp$ for $\gamma \in E$ we see that it is a separating hyperplane for Δ, i.e., $P_\gamma \cap \Delta = \emptyset$, iff $\gamma \notin E \setminus \bigcup_{\alpha \in \Delta} P_\alpha$.

Definition 11.36. *Let $\Delta \subset E$ be a root system. The connected components of $E \setminus \bigcup_{\alpha \in \Delta} P_\alpha$ are called* Weyl chambers. *An element $\gamma \in E$ is called regular if $\gamma \notin E \setminus \bigcup_{\alpha \in \Delta} P_\alpha$. For such an element γ denote $Ch(\gamma)$ the Weyl chamber containing γ.*

Proposition 11.37. *For a regular element γ denote $B_\gamma \subset \Delta$ the unique basis of Δ with $\langle \gamma, B_\gamma \rangle > 0$. Then $B_\delta = B_\gamma$ for all $\delta \in Ch(\gamma)$ and $Ch(\gamma) \mapsto B_\gamma$ is a bijection between the set of Weyl chambers of Δ and the set of bases of Δ.*

Now the information contained in a basis $B \subset \Delta$ can be encoded in a so called *Dynkin diagram*, a graph whose vertices are the base roots $\alpha \in B$. Two vertices α, β are connected by $\chi(\beta, \alpha)\chi(\alpha, \beta) = 4\cos^2(\vartheta)$ edges, i.e. by one edge, if the angle $\vartheta \in [0, \pi)$ equals $\frac{2\pi}{3}$, by two edges, if $\vartheta = \frac{3\pi}{4}$ and by three edges if $\vartheta = \frac{5\pi}{6}$. In the last two cases the two or three edges are even oriented, the arrow pointing from the longer root to the smaller one. Note that the diagram does not depend on the choice of the basis B.

Let us come back to Lie algebras: A Lie algebra can be reconstructed - up to isomorphy - from its root system, and a root system from one of its bases resp. - again up to isomorphy - from its Dynkin diagram. First of all, it is connected if and only if the corresponding algebra is simple, and there is a complete classification of the connected Dynkin diagrams. Here it is, the index counting the number of vertices:

1. $A_\ell, \ell \geq 1$: A linear string with ℓ vertices and only simple edges.

2. $B_\ell, \ell \geq 2$: A linear string with ℓ vertices with $\ell - 2$ simple edges and one double edge at one of its ends, the arrow pointing to the end point (for $\ell \geq 3$).

3. $C_\ell, \ell \geq 3$: A linear string with ℓ vertices with $\ell - 2$ simple edges and one double edge at one of its ends, the arrow pointing to the inner point. Note that $C_2 = B_2$.

4. $D_\ell, \ell \geq 4$: A string with $\ell - 2$ vertices, with the two remaining vertices connected to the same end point of the string. All edges are simple. Note that $D_3 = A_3$.

5. $E_\ell, \ell = 6, 7, 8$: A string with $\ell - 1$ vertices, the ℓ-th vertex being connected to the third vertex of the string. All edges are simple.

6. F_4: A linear string with 4 vertices, the outer edges being simple, the inner one being a double edge (the orientation is not important, since both choices give isomorphic Dynkin diagrams).

7. G_2: Two vertices connected by three edges.

Indeed, for every of the above Dynkin diagrams, there is a complex simple Lie algebra realizing it.

Now let us look at complex semisimple Lie groups, i.e. complex connected Lie groups G with semisimple Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$.

First of all the adjoint representation $\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g})$ is a covering of the subgroup $\mathrm{Ad}(G) \subset \mathrm{Aut}(\mathfrak{g}) \subset GL(\mathfrak{g})$, since its differential $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ is injective for a semisimple algebra $\mathfrak{g}$, its center being trivial. So there is not only a maximal Lie group - the simply connected one - but also a minimal Lie group with Lie algebra $\mathfrak{g}$, since $\mathrm{Ad}(G)$ only depends on $\mathfrak{g}$: It is the connected Lie subgroup of $\mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$. Indeed, for a semisimple algebra we have $\mathrm{ad}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$ and hence $\mathrm{Ad}(G) = \mathrm{Aut}^0(\mathfrak{g})$, the component of the identity of $\mathrm{Aut}(\mathfrak{g})$.

If G is simply connected, any other connected Lie group with Lie algebra $\mathfrak{g}$ is of the form G/D with a discrete subgroup $D \subset Z(G)$. Indeed the center $Z(G) \subset G$ is finite, it can even be read off from the root system Δ associated to a CSA $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}(G)$: If we denote $\Gamma_0 \subset E := \mathfrak{h}^*_{\mathbb{R}}$ the lattice generated by Δ (in fact $\Gamma_0 = \bigoplus_{\alpha \in B} \mathbb{Z}\alpha$ with any basis $B \subset \Delta$) and $\Gamma := \{\gamma \in E; \chi(\gamma, \Delta) \subset \mathbb{Z}\}$, then, obviously $\Gamma_0 \subset \Gamma$ and (less obviously)

$$Z(G) \cong \Gamma/\Gamma_0.$$

The connected Lie subgroup $H \subset G$ is a maximal (complexified) torus $(\mathbb{C}^*)^\ell$, a closed subgroup of G containing the center $Z(G)$.

Furthermore semisimple Lie groups can be realized as algebraic (in particular closed) subgroups of $GL(V)$ for some complex vector space V; and even better, any homomorphism $G \longrightarrow GL(W)$ for an arbitrary vector space W is algebraic. Indeed this follows from the knowledge of all irreducible $\mathfrak{g}$-modules W resp. all irreducible representations $\mathfrak{g} \longrightarrow \mathfrak{gl}(W)$.

Here is the table of all simply connected complex simple Lie groups:

Dynkin diagram	simply connected Lie group	Dimension	Center
$A_\ell, \ell \geq 1$	$SL_{\ell+1}(\mathbb{C})$	$\ell(\ell+2)$	$\mathbb{Z}_{\ell+1}$
$B_\ell, \ell \geq 2$	$Spin_{2\ell+1}(\mathbb{C})$	$\ell(2\ell+1)$	$\mathbb{Z}_2$
$C_\ell, \ell \geq 3$	$Sp_{2\ell}(\mathbb{C})$	$\ell(2\ell+1)$	$\mathbb{Z}_2$
$D_\ell, \ell \geq 4$, even	$Spin_{2\ell}(\mathbb{C})$	$\ell(2\ell-1)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$D_\ell, \ell \geq 5$, odd	$Spin_{2\ell}(\mathbb{C})$	$\ell(2\ell-1)$	$\mathbb{Z}_4$
E_6	—	78	$\mathbb{Z}_3$
E_7	—	133	$\mathbb{Z}_2$
E_8	—	248	0
F_4	—	52	0
G_2	$\mathrm{Aut}(\mathbb{O}(\mathbb{C}))$	14	0

Here $Spin_n(\mathbb{C})$ denotes the universal covering group of $SO_n(\mathbb{C})$, and $\mathbb{O}(\mathbb{C})$ is the complexified algebra of Cayley numbers (octonians). The remaining exceptional group do not have an immediate geometric realization.

Note that an arbitrary semisimple complex Lie group is of the form G/D, where G is a finite product of copies of the above Lie groups and $D \subset Z(G)$, with $Z(G)$ being the direct product of the centers of the simple factors.

Complex semisimple and real compact Lie groups: Finally let us comment on the relation between complex semisimple groups and real compact Lie groups. Let us start with a semisimple Lie algebra $\mathfrak{g}$ and look for a "real compact form" $\mathfrak{k} \subset \mathfrak{g}$, i.e., a real subalgebra $\mathfrak{k} \cong \mathrm{Lie}(K)$ for some real compact Lie group K satisfying $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. We start with a CSA $\mathfrak{h} \subset \mathfrak{g}$ and write

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

noting that, with respect to the Cartan Killing form $\langle .,. \rangle$ we have $\mathfrak{h} \perp \mathfrak{g}_\alpha$ for all $\alpha \in \Delta$ as well as $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ for $\alpha + \beta \neq 0$. We shall represent $\mathfrak{k} = Fix(\tau)$ as fix algebra of a conjugation

$$\tau : \mathfrak{g} \longrightarrow \mathfrak{g},$$

i.e. an involutive automorphism of $\mathfrak{g}$ as real Lie algebra ($\tau^2 = \mathrm{id}_\mathfrak{g}$) satisfying $\tau(iX) = -iX$. Setting

$$\mathfrak{g}_0 = \mathfrak{h}_\mathbb{R} \oplus \bigoplus_{\alpha \in \Delta} \mathbb{R} Z_\alpha \subset \mathfrak{g}$$

with $\mathfrak{h}_\mathbb{R} := \mathrm{span}(H_\alpha; \alpha \in \Delta)$ (see Remark 11.30) we have $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ and a conjugation $\sigma := \mathrm{id}_{\mathfrak{g}_0} \oplus -\mathrm{id}_{i\mathfrak{g}_0}$.

We want to take $\tau = \varphi \circ \sigma$ with a complex automorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{g}$: In order to define φ, we choose elements $Z_\alpha \in \mathfrak{g}_\alpha, Z_{-\alpha} \in \mathfrak{g}_{-\alpha}$ with $\langle Z_\alpha, Z_{-\alpha} \rangle = -1$. Then the linear map $\varphi : \mathfrak{g} \longrightarrow \mathfrak{g}$ with $\varphi|_{\mathfrak{h}} = -\mathrm{id}_{\mathfrak{h}}$ and $\varphi(Z_\alpha) = Z_{-\alpha}$ is the desired Lie algebra automorphism.

Example 11.38. For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}), \mathfrak{h} = \mathfrak{so}_n(\mathbb{C})$ we have $\sigma(A) = \overline{A}$ and $\varphi(A) = -A^T$.

Since $X = iH + \sum c_\alpha Z_\alpha \in \mathfrak{k}$ satisfies $H \in \mathfrak{h}_\mathbb{R}$ and $\overline{c}_\alpha = c_{-\alpha}$ we find

$$\langle X, X \rangle = -\langle H, H \rangle - 2 \sum_{\alpha \in \Delta} |c_\alpha|^2 < 0,$$

so the negative Cartan-Killing form $-\langle ., . \rangle$ is positive definite on $\mathfrak{k}$. So we have obtained an inner product on $\mathfrak{k}$ invariant under any automorphism of $\mathfrak{k}$ (an automorphism of $\mathfrak{k}$ extends to an automorphism of $\mathfrak{g}$). Hence the closed subgroup $\mathrm{Aut}(\mathfrak{k}) \subset O(\mathfrak{k}, -\langle ., . \rangle)$ is compact. On the other hand one can prove that $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{k})) = \mathrm{Der}(\mathfrak{k}) = \mathrm{ad}(\mathfrak{k})$, whence $\mathfrak{k} = \mathrm{Lie}(K)$ with $K := \mathrm{Aut}(\mathfrak{k})$.

Theorem 11.39. *Given a complex semisimple algebraic group G there is a compact real Lie subgroup $K \subset G$ (unique up to conjugacy) such that $\mathfrak{k} := \mathrm{Lie}(K)$ is a compact real form of $\mathfrak{g} := \mathrm{Lie}(G)$, and the map*

$$K \times i\mathfrak{k} \longrightarrow G, (x, X) \mapsto x \exp(X)$$

is a diffeomorphism.

Trevlig Sommar!