Algebraic Geometry

Karl-Heinz Fieseler and Ludger Kaup

Uppsala 2012

Contents

1 Introduction: Plane Curves

In order to give an idea what algebraic geometry is about we discuss in the introductory section plane curves in a not too rigorous manner:

Definition 1.1. Let k be a field. A plane curve C over k is a subset $C \subset k^2$, which can be written

$$C = V(f) := \{(x, y) \in k^2; f(x, y) = 0\}$$

with a polynomial $f \in k[X, Y] \setminus k$.

Example 1.2.

1. For $f = X^2 + Y^2 - 1$ and $k = \mathbb{R}$ we find $V(f) \subset \mathbb{R}^2$ is the unit circle.
2. For $f = X^2 + Y^2$ and $k = \mathbb{R}$ we find $V(f) = \{0\}$.

Only the unit circle seems to be an "honest plane curve". In order to avoid points to be plane curves, we shall only consider an algebraically closed field k. Nevertheless it turns out to be very useful to look at pictures in $\mathbb{R}^2$, namely given $f \in \mathbb{R}[X, Y] \subset \mathbb{C}[X, Y]$ one has $V(f) \subset \mathbb{C}^2$ and draws its "real trace" $V(f) \cap \mathbb{R}^2$. Only one has to understand, which features are due to the fact that $\mathbb{R}$ is not algebraically closed and which ones really reflect the complex situation.

Remark 1.3. The polynomial f can always be assumed to be square free.

A natural aim when studying a plane curve C is to try to parametrize it, i.e. to write

$$C = \{\varphi(t); t \in U \subset k\},$$

where $\varphi = (g, h) : U \longrightarrow C \subset k^2$.

In the next theorem we classify plane curves $C = V(f)$ resulting from a polynomial of degree ≤ 2. Here the sign "$\cong$" means, that there is a "reasonable" bijective map between the left and the right hand side.

Theorem 1.4. *Let k be algebraically closed. Then we have $V(f) \cong k$ for $\deg f = 1$. For $\deg(f) = 2$ we have the following possibilities*

1. *$V(f) \cong k$,*
2. *$V(f) \cong k^* := k \setminus \{0\}$,*

3. $V(f) \cong (k \times \{0\}) \cup (k \times \{1\}) \subset k^2$ *is the union of two parallel lines,*

4. $V(f) \cong (k \times \{0\}) \cup (\{0\} \times k) \subset k^2$ *is the coordinate cross.*

Proof. For $\deg f = 1$, the curve $V(f) \subset k^2$ is a line. For a reducible polynomial $f = gh$ of degree 2, we are in one of the cases 1, 3, 4, where the first case corresponds to $h \in k^* g$. Let us now assume that f is irreducible. We may assume that $0 \in V(f)$ and write

$$f = f_2 + f_1$$

with homogeneous polynomials of degree 2 resp. 1. Since k is algebraically closed we can factorize

$$f_2 = gh$$

with homogeneous polynomials $g, h \in k[X, Y]$ of degree 1. If g, h only differ by a nonzero constant factor, we may assume - after a linear change of coordinates - that $f_2 = X^2$, otherwise $f_2 = XY$. In the first case we have

$$f = X^2 + aX + bY$$

with $b \neq 0$. We may solve for Y, and the map

$$t \mapsto (t, -b^{-1}(t^2 + at)),$$

is an isomorphism $k \longrightarrow V(f)$. In the remaining case we have

$$f = XY + aX + bY$$

with $ab \neq 0$. Then we have

$$V(f) \cap (\{-b\} \times k) = \emptyset$$

and

$$t \mapsto \left(t - b, -a + \frac{ab}{t}\right)$$

provides an isomorphism $k^* \longrightarrow V(f)$. $\square$

Then we might conjecture

1. Every plane curve $C \subset k^2$ is the finite union

$$C = C_1 \cup ... \cup C_r$$

   of "irreducible" curves C_i, i.e. of curves which do not admit a further decomposition.

2. For every irreducible curve C there is a bijective map

$$k \supset U \longrightarrow C, t \mapsto (g(t), h(t)),$$

   where $U \subset k$ has finite complement and $g, h : U \longrightarrow k$ are rational functions (quotients of polynomials) with poles outside U.

The first statement is true: If $f_1, ..., f_r$ are the irreducible divisors of f, we have $C_i = V(f_i)$.

The second one is practically always wrong. Let us discuss an example: Take

$$f = Y^2 - X^2(X + a).$$

For $k = \mathbb{C}$ and $a \in \mathbb{R}_{>0}$ the curve $V(f) \cap \mathbb{R}^2 \subset \mathbb{C}^2$ is a noose. Given $t \in k$ we consider the line $L_t := k(1, t)$, our parametrization $\varphi : k \longrightarrow V(f)$ associates to $t \in k$ the unique intersection point $\neq (0, 0)$ in $L_t \cap V(f)$. Solving

$$0 = f(\lambda(1, t)) = \lambda^2 t^2 - \lambda^2(\lambda + a)$$

we find

$$\varphi(t) = (t^2 - a, t(t^2 - a)).$$

Note that for $a \neq 0$ and $\mathrm{char}(k) \neq 2$ this is unfortunately not an injective parametrization: The self intersection point $(0, 0)$ has the inverse images $\pm b$ where $b^2 = a$. On the other hand, for $a = 0$ the above parametrization becomes $\varphi(t) = (t^2, t^3)$, it is bijective, but not an isomorphism, since its inverse can not be defined by the restriction to $V(f) \subset k^2$ of a polynomial function on k^2. In that case

$$V(f) = \{(x, y); y^2 = x^3\}$$

is called Neil's parabola, in the origin it has a "cusp singularity".

Definition 1.5. Let $C = V(f) \subset k^2$ with a square free polynomial f. Then a point $(x, y) \in V(f)$ is called a singular point if $\nabla f(x, y) = (0, 0)$. A curve is called smooth if it has no singular points.

Example 1.6. 1. If $C = C_1 \cup ... \cup C_r$ with irreducible curves C_i then the intersection points, i.e. the points in $C_i \cap C_j, j \neq i$, are singular points.

2. Any curve has at most finitely many singular points.

3. The noose has the origin as its only singular point. If $a \neq 0$, it is a "self intersection point": This concept may be defined algebraically, but here we only give the definition for $k = \mathbb{C}$. A singular point $(b, c) \in C = V(f)$ (f being irreducible) is called a self intersection point, if there is a neighbourhood $U \subset \mathbb{C}^2$ of (b, c) together with holomorphic functions $g, h : U \longrightarrow \mathbb{C}$ with $g(b, c) = 0 = h(b, c)$, such that $f|_U = gh$ and $V(g) \neq \emptyset \neq V(h)$. In the case of the noose take $U = \{(x, y); |x| < |a|\}$, $g(x, y) = y + x\sqrt{x + a}, h(x, y) = y - x\sqrt{x + a}$, where $\sqrt{..} : \mathbb{C} \setminus \mathbb{R}_{<0}$ is a suitable branch of the square root.

4. A singular point of an irreducble curve which is not a self intersection point, is called a cusp singularity, e.g. the origin is a cusp singularity of Neil's parabola.

5. Any nonsingular point of a complex plane curve has a neighbourhood homeomorphic to a disc. This is an easy consequence of the holomorphic implicit function theorem.

The "generic" version of our noose is the curve $V(f)$ with

$$f = Y^2 - p(X),$$

where $p \in k[X]$ is a polynomial of degree 3 (resp. $2\ell + 1$) with only simple roots. In that case $V(f)$ is a smooth curve, it is called a plane elliptic curve (resp. hyperelliptic curve for $\ell \geq 2$). For such curves parametrizations $U \longrightarrow V(f)$ do not exist at all:

Theorem 1.7. *Let $C = V(f)$ be an elliptic resp. hyperelliptic plane curve over a field k of characteristic* $\mathrm{char}(k) \neq 2$. *Then there is no nonconstant map*

$$(g, h) : U \longrightarrow V(f),$$

where $U \subset k$ is the complement of a finite set and $g, h \in k(T) := Q(k[T])$ are rational functions with poles outside U.

Proof. We may assume that $g' \neq 0$ or $h' \neq 0$. (For $\mathrm{char}(k) = p > 0$ we have $g' = 0$ iff $g = \hat{g}(T^p)$ with some $\hat{g} \in k(T)$. Then every $g \in k(T)$ can be written $g = g_0(T^{p^r})$ with $g_0' \neq 0$. Now replace g with g_0. In order to see all that, use the partial fraction decomposition of g.) Furthermore that $U \subset k$ is the maximal domain of definition for (g, h), i.e. that at a point $a \in k \setminus U$ one of the functions g, h has a pole. The equality

$$h^2 = p(g)$$

shows that then the other function has a pole as well and that, if m, n are the pole orders of g, h at a, we have $2n = (2\ell + 1)m$. Formal differentiation of the above equality gives

$$2hh' = p'(g)g'.$$

Let us now consider the rational function

$$\varphi = \frac{2h'}{p'(g)} = \frac{g'}{h} \in k(T).$$

Since p has only simple zeros, $p'(g)$ and h have no common zeros in U, in particular the function φ is defined on U. We show that it can be extended to k, hence is a polynomial. Consider a point $a \notin U$ with m, n as above. Then the function $\varphi = g'/h$ has a removable singularity at a, since the pole order $m + 1$ of g' is $\leq$ the pole order n of h: Indeed $m + 1 \leq n \Longleftrightarrow 1 \leq (\ell - \frac{1}{2})m$, the latter inequality holds, since $m > 0$ is even.

Finally we want to show that $\varphi = 0$ - this implies $h' = 0 = g'$. We compute the degree of φ. First of all we can define the degree of a rational function - the difference of the degrees of numerator and denominator polynomial. Now denote m, n the degree of g, h. If $m > 0$, we have $\deg p(g) = (2\ell + 1)m$ and once again $2n = (2\ell + 1)m$. Since $\deg g' = m - 1 < m + 1 \leq n$, we see that φ has negative degree, hence $f = 0$ and $g' = 0$. If $m < 0$, we have $\deg p(g) = 0$ and then $\deg h = 0$ and $\deg g' \leq m - 1 < 0$, so $\deg(\varphi) < 0$. If $m = 0 = n$, we have $\deg(g') < 0$ and obtain again $\deg(\varphi) < 0$. Finally in the case $m = 0 > n$, we have $\deg p(g) < 0$ and thus $\deg p'(g) = 0$ - the polynomials p and p' don't have common zeros - and are done even then since $\deg h' \leq n - 1 < 0$. □

Holomorphic Parametrizations: For an elliptic curve $V(f)$ over $\mathbb{C}$ there exists nevertheless a holomorphic parametrization

$$(g, h) : U \longrightarrow V(f),$$

where $\Lambda := \mathbb{C} \setminus U$ is an infinite discrete set, indeed a lattice: A lattice $\Lambda \subset \mathbb{C}$ is a subset of the form

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,$$

where ω_1, ω_2 are linearly independent over $\mathbb{R}$, or, equivalently, $\omega_1/\omega_2 \notin \mathbb{R}$. A meromorphic function f on $\mathbb{C}$ is called Λ-periodic if $f(z+\omega) = f(z)$ holds for all $\omega \in \Lambda$. There are only constant holomorphic Λ-periodic functions: Such a function would be globally bounded by its supremum on the parallelogram $\{r\omega_1 + s\omega_2; 0 \le r, s \le 1\}$, and by Liouville's theorem it follows that it is a constant. But there are interesting meromorphic functions: The Weierstraß-$\wp$-function

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

where $\Lambda^* := \Lambda \setminus \{0\}$, is an example of a Λ-periodic function. It is holomorphic outside Λ and has poles of order 2 at the lattice points. For

$$a = 3 \sum_{\omega \in \Lambda} \frac{1}{\omega^4}, b = 5 \sum_{\omega \in \Lambda} \frac{1}{\omega^6}$$

we consider the function

$$(\wp')^2 - 4\wp^3 + 20a\wp + 28b.$$

It is holomorphic in U, the Laurent expansion at the origin shows, that it has a removable singularity there, indeed a zero. But then, being Λ-periodic, it has to be $\equiv 0$. Some more detailed investigations show that

$$(\wp, \wp') : U \longrightarrow V(f), \ f = Y^2 - 4X^3 + 20aX + 28b$$

is a surjective and locally homeomorphic parametrization; its fibres are exactly the cosets $z+\Lambda, z \in U$. Now, in order to get a more satisfactory picture, one compactifies $V(f)$ by adding a point ∞ to

$$\overline{V}(f) := V(f) \cup \{\infty\}.$$

and gets a parametrization

$$(\wp, \wp') : \mathbb{C} \longrightarrow \overline{V}(f),$$

sending the lattice points to ∞. Indeed it induces a homeomorphism

$$\mathbb{C}/\Lambda \longrightarrow \overline{V}(f),$$

of the torus with the one point compactification of $V(f)$. Indeed, traditionally one means the compactification $\overline{V}(f)$ when speaking about an elliptic curve. And it turns out that for every elliptic curve there is a corresponding lattice such that we have a homeomorphism of the above type.

There is a further interesting remark: The homeomorphism $\overline{V}(f) \cong \mathbb{C}/\Lambda$ endows on $\overline{V}(f)$ the structure of an (additive) abelian group with the point ∞ as unit element. The group multiplication, it turns out, can also be understood in a completely algebraic way: Take any line $L \subset \mathbb{C}^2$ and a point $a \in V(f) \cap L$. We can define an intersection multiplicity: It is the multiplicity of the polynomial function $f(\varphi(t))$ at 0 if $\varphi : \mathbb{C} \longrightarrow L$ is a linear parametrization with $\varphi(0) = a$. Now counting the intersection points in $L \cap V(f)$ with multiplicities, there are at most 3 intersection points - two for "vertical" lines $L = \{a\} \times \mathbb{C}$, and three otherwise - and their sum in $\overline{V}(f) \cong \mathbb{C}/\Lambda$ equals zero.

Hyperelliptic Curves: Now let us consider the case of a hyperelliptic curve: The fact that the polynomial $p(X)$ has odd degree guarantees, that the point $\infty \in \overline{V}(f)$ has a neighbourhood homeomorphic to a disc – if the degree is even, it is a self intersection point, has a neighbourhood homeomorphic to two copies of the unit disc $E \subset \mathbb{C}$ meeting at their centers, i.e. $(E \times \{0\}) \cup (\{0\} \times E) \subset \mathbb{C}^2$ with ∞ corresponding to $(0, 0)$.

In that situation we have to replace $\mathbb{C}$ with the open unit disc E: There is a surjective and locally homeomorphic parametrization

$$(g, h) : E \longrightarrow \overline{V}(f)$$

given by two meromorphic functions, where the fibre of the point at infinity consists of the points, where one of the functions g, h or both have a pole. Indeed, there is a subgroup $G \subset \mathrm{Aut}(E)$, acting freely on E, such that the parametrization is G-invariant and induces a homeomorphism

$$E/G \cong \overline{V}(f).$$

But for $\ell > 1$ the extended curve $\overline{V}(f)$ is not a group. Indeed $\overline{V}(f)$ is homeomorphic to a sphere with ℓ handles.

2 Algebraic Sets in k^n

Algebraic geometry deals with objects, called "*algebraic varieties*", which locally look like "*algebraic sets*", i.e. sets which can be described by a system of polynomial equations. For this one has first of all to fix a base field:

Notation: We denote k an algebraically closed field; vector spaces are always assumed to be vector spaces over k. In contrast to that, we use the letter K for a not necessarily algebraically closed field.

In this section we define algebraic sets and prove that they correspond to certain ideals in polynomial rings, the central result being "Hilberts Nullstellensatz", Th.2.17.

To begin with let us mention the following fact: Given any infinite field K the polynomial ring

$$K[T] := K[T_1, ..., T_n]$$

can be regarded as a subring

$$K[T] \subset K^{K^n}$$

of the ring K^{K^n} of all functions $K^n \longrightarrow K$.

Proof. For $n = 1$ this follows from the fact, that a polynomial $f \in K[T]$ has at most $\deg f$ zeros. For $n > 1$ write

$$f = \sum_{i=0}^{r} f_i(T_1, ..., T_{n-1})T_n^i \ \in \ (K[T_1, ..., T_{n-1}])\,[T_n]$$

and assume $f(x) = 0$ for all $x = (x', x_n) \in K^n = K^{n-1} \times K$. Hence for any $x' \in K^{n-1}$ we have $0 = f(x', T_n) \in K[T_n]$ resp. $f_i(x') = 0$ for all $x' \in K^{n-1}$. Using the induction hypothesis we conclude $f_i = 0 \in K[T_1, ..., T_{n-1}]$ for all $i = 0, ..., r$ and thus $f = 0 \in K[T_1, ..., T_n]$ as well. $\square$

Definition 2.1. A subset $X \subset k^n$ is called an *algebraic set*, if there is a subset $F \subset k[T]$ of the polynomial ring $k[T]$ with

$$X = N(F) := N(k^n; F) := \{x \in k^n;\ f(x) = 0, \forall f \in F\}.$$

We indicate this by writing

$$X \hookrightarrow k^n,$$

and for a finite set $F = \{f_1, ..., f_r\}$ we also simply write

$$N(f_1, ..., f_r) := N(k^n; f_1, ..., f_r) := N(k^n; F).$$

Example 2.2. 1. Both $k^n = N(k^n; 0)$ and the empty set $\emptyset = N(k^n; 1)$ are algebraic sets.

2. One point sets

$$\{a\} = N(k^n; T_1 - a_1, ..., T_n - a_n)$$

are algebraic sets.

3. Arbitrary intersections

$$\bigcap_{\lambda \in \Lambda} N(F_\lambda) = N(\bigcup_{\lambda \in \Lambda} F_\lambda)$$

and finite unions of algebraic sets are algebraic, since

$$N(FG) = N(F) \cup N(G)$$

with $FG := \{fg; f \in F, g \in G\}$. The inclusion "$\supset$" is obvious, while for a point $x \notin N(F) \cup N(G)$ we find $f \in F$ and $g \in G$ with $f(x) \neq 0 \neq g(x)$, so $fg \in FG$ does not vanish at x either.

4. All finite sets are algebraic; for $n = 1$ this are all the proper algebraic sets.

5. Vector subspaces $W \subset k^n$ are algebraic subsets.

6. Let $X := N(k^2; T_2^2 - p(T_1))$ with a polynomial $p(T_1) \in k[T_1]$. For $p(T_1) = T_1(T_1 - 1)(T_1 - \lambda)$ with $\lambda \neq 0, 1$ the algebraic set $X \hookrightarrow k^2$ is called an *elliptic curve*. For $k = \mathbb{C}$ and $\lambda \in \mathbb{R}$ draw a picture of $X \cap \mathbb{R}^2 \subset \mathbb{R}^2$! In the "degenerate case" $\lambda = 0, 1$ the real picture of X is a noose. For $p(T_1) = T_1^3$ it is also called *Neils parabola*, indeed, the latter can be viewed as degeneration of the nooses defined by $p(T_1) = T_1(T_1 - \lambda)^2$ with $\lambda \in k^*$.

7. The algebraic set $X := N(k^3; T_1T_3 - T_2^2)$ is also called a quadric cone.

8. The algebraic set $Z := N(k^4; T_1T_4 - T_2T_3)$ is called the *Segre cone*; if we identify k^4 with the vector space $k^{2\times 2}$ of 2×2 matrices with entries in k, then $Z = k^{2\times 2} \setminus GL_2(k)$ consists of all singular 2×2 matrices.

Remark 2.3. 1. According to Ex.2.2.1,3 and 4, the algebraic sets satisfy the axioms of the closed sets of a topology on k^n. The corresponding topology is called the Zariski topology. If we want to express that a subset $X \subset k^n$ is Zariski-closed we also write simply $X \hookrightarrow k^n$.

2. Two nonempty open sets $U, V \subset k^n$ have nonempty intersection. Or equivalently: The n-space k^n is not the union of two proper closed subsets $X, Y \hookrightarrow k^n$. Otherwise we may assume $X = N(f), Y = N(g)$ with $f, g \in k[T] \setminus \{0\}$. Then $fg \in k[T] \setminus \{0\}$ satisfies $k^n \neq N(fg) = N(f) \cup N(g)$.

3. If $k = \mathbb{C}$, there is yet another topology both on $\mathbb{C}^n$ and all its subsets, induced by the norm $||z|| := \sqrt{\sum_{i=1}^n z_i \overline{z}_i}$. We shall call it the strong topology on $\mathbb{C}^n$ resp. $X \subset \mathbb{C}^n$. Note that the strong topology is strictly finer than the Zariski topology: Polynomials are continuous functions with respect to both the Zariski and the strong topology.

4. A proper algebraic set $X \hookrightarrow \mathbb{C}^n$ has no interior points with respect to the strong topology: Take a polynomial $f \in \mathbb{C}[T] \setminus \{0\}$ vanishing on X. If $a \in X$ is an interior point, then $\frac{\partial^{\,\nu} f}{\partial T^\nu}(a) = 0$ for all $\nu \in \mathbb{N}^n$, and thus Taylor expansion gives

$$f = \sum_{\nu \in \mathbb{N}^n} \frac{1}{\nu!} \frac{\partial^{\,\nu} f}{\partial T^\nu}(a)(T - a)^\nu = 0,$$

a contradiction!

5. We remark without proof that an algebraic subset $X \subset \mathbb{C}^n$ is connected with respect to the Zariski topology iff it is with respect to the strong topology.

An algebraic subset can always be described by finitely many polynomial equations:

Proposition 2.4. *For every algebraic subset $X = N(F) \hookrightarrow k^n$ there is a finite subset $F_0 \subset F$ mit $X = N(F_0)$.*

In order to see that, we first enlarge the set of polynomials defining a given algebraic set:

Remark 2.5. The ideal

$$\mathfrak{a}_F := \sum_{f\in F} k[T]f := \left\{ \sum_{i=1}^{r} g_i f_i; g_i \in k[T], f_i \in F, r \in \mathbb{N} \right\}$$

of $k[T]$ generated by $F \subset k[T]$ satisfies $N(\mathfrak{a}_F) = N(F)$.

Proof. The inclusion $F \subset \mathfrak{a}_F$ implies obviously $N(F) \supset N(\mathfrak{a}_F)$; on the other hand every polynomial $h \in \mathfrak{a}_F$ is a finite sum $h = \sum_{i=1}^{r} g_i f_i$ with $f_i \in F$, whence $h(x) = \sum_{i=1}^{r} g_i(x) f_i(x) = 0$ for all $x \in N(F)$ and thus $N(F) \subset N(\mathfrak{a}_F)$. $\square$

Hence, in order to study algebraic subsets one could try to investigate first ideals $\mathfrak{a} \subset k[T]$ (or rather in $K[T]$, where K is any field). For $n = 1$ we know that every ideal $\mathfrak{a} \subset K[T]$ is a principal ideal $\mathfrak{a} = K[T] \cdot f$ with a polynomial $f \in K[T]$.

Digression to commutative algebra: Noetherian Rings

Definition 2.6. A commutative ring R with unity 1 is called *noetherian*, if one of the following three equivalent conditions is satisfied:

1. Every ideal $\mathfrak{a} \subset R$ is finitely generated, i.e., there is a finite set $F \subset R : \ |F| < \infty$, with $\mathfrak{a} = \mathfrak{a}_F$.
2. Every ascending chain $\mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_j \subset \dots$ of ideals in R terminates, i.e. there is a natural number $j_0 \in \mathbb{N}$ with $\mathfrak{a}_j = \mathfrak{a}_{j_0}$ for $j \geq j_0$.
3. Every nonempty set $A = \{\mathfrak{a}_i, i \in I\}$ of ideals $\mathfrak{a}_i \subset R$ contains a maximal element $\mathfrak{a}_0$, i.e. such that $\mathfrak{a}_0 \subset \mathfrak{a} \in A \Longrightarrow \mathfrak{a}_0 = \mathfrak{a}$.

Proof. Exercise! $\square$

Remark 2.7. We leave it to the reader to check that given an ideal $\mathfrak{a} = \mathfrak{a}_F \subset R$ in a noetherian ring R, we can choose a "finite basis" $F_0 \subset F$ of the ideal $\mathfrak{a}$, i.e. such that

$$\mathfrak{a}_{F_0} = \mathfrak{a}_F.$$

Note that the word "basis" is used here in an old fashioned way, meaning nothing but a system of generators, and not requiring linear independence of the generators, as one does in linear algebra.

Example 2.8. A PID (:=Principal Ideal Domain) is obviously noetherian.

Theorem 2.9. *The polynomial ring $R[T]$ in one variable over a noetherian ring R is again noetherian.*

Corollary 2.10 (Hilberts Basissatz)**.** *The polynomial ring $K[T_1, ..., T_n]$ over a field K is noetherian.*

Proof. $K[T_1, ..., T_n] = (K[T_1, ..., T_{n-1}])[T_n]$. □

Proof of 2.9. For an ideal $\mathfrak{a} \subset R[T]$ we define an ascending chain of ideals $\mathfrak{b}_0 \subset \cdots \subset \mathfrak{b}_j \subset \ldots$ in R by

$$\mathfrak{b}_j \ := \ \left\{ r \in R; \exists \ \ rT^j + \sum_{i<j} r_i T^i \ \in \ \mathfrak{a} \right\} .$$

For every ideal $\mathfrak{b}_j \hookrightarrow R$ we choose generators $(s_{j1}, \ldots, s_{j\ell_j})$ and corresponding polynomials

$$p_{jk} \quad = \quad s_{jk}T^j + \sum_{i<j} r_{ji}^k T^i \quad \in \quad \mathfrak{a}.$$

Since R is noetherian, the chain of ideals $(\mathfrak{b}_j)$ terminates at some $j_0 \in \mathbb{N}$. Our candidate for a "basis" is now

$$F := \{p_{jk}; 0 \le j \le j_0, 1 \le k \le \ell_j\} .$$

Obviously we have then $\mathfrak{a}_F \subset \mathfrak{a}$. It remains to show the opposite inclusion: For $p \in \mathfrak{a}$ we use induction on $d := \deg(p)$. For $d < 0$ we have $p = 0$. For "$d-1 \Longrightarrow d$" let $m := \min(d, j_0)$. The leading coefficient of p belongs to $\mathfrak{b}_m$, hence has the form $\sum_{k=1}^{\ell_m} r_k s_{mk}$ with suitable $r_k \in R$. Then the polynomial

$$q \quad := \quad p - T^{d-m} \sum_{k=1}^{\ell_m} r_k p_{mk} \in \mathfrak{a}$$

satisfies $\deg(q) < d$. So we know $q \in \mathfrak{a}_F$ and hence $p \in \mathfrak{a}_F$ as well. □

Prop. 2.4 now follows from Corollary 2.9 and the remarks 2.7 and 2.5.

Let us now come back to geometry: As a consequence of 2.5 every algebraic set is of the form $N(\mathfrak{a})$. In the following lemma we study how $N(\mathfrak{a})$ depends on the ideal $\mathfrak{a} \subset k[T]$.

Lemma 2.11. *Let $\mathfrak{a}, \mathfrak{b} \subset k[T]$ and $\mathfrak{a}_\lambda \subset k[T], \lambda \in \Lambda$, be ideals. Then*

1. $N(\mathfrak{a}) = k^n \iff \mathfrak{a} = \{0\}$; $\mathfrak{a} = k[T] \Longrightarrow N(\mathfrak{a}) = \emptyset$.
2. $\mathfrak{a} \subset \mathfrak{b} \implies N(\mathfrak{a}) \supset N(\mathfrak{b})$.
3. $N(\sum_{\lambda\in\Lambda} \mathfrak{a}_\lambda) = \bigcap_\lambda N(\mathfrak{a}_\lambda)$.
4. $N(\mathfrak{a}\mathfrak{b}) = N(\mathfrak{a}\cap\mathfrak{b}) = N(\mathfrak{a}) \cup N(\mathfrak{b})$.

Proof. Exercise! □

On the other hand any algebraic set $X \hookrightarrow k^n$ has an associated ideal, its vanishing ideal $I(X) \hookrightarrow k[T]$, as well:

Definition 2.12. The vanishing ideal $I(X) \subset k[T]$ of an algebraic set $X \hookrightarrow k^n$ is

$$I(X) := \{f \in k[T]; f|_X = 0\} \ .$$

The next lemma investigates $I(X)$ as a function of X:

Lemma 2.13. *For $X \hookrightarrow k^n$ and an ideal $\mathfrak{a} \subset k[T]$ we have:*

1. $I(X) = \{0\} \iff X = k^n$ *and* $I(X) = k[T] \iff X = \emptyset$;
2. $I(\bigcap_\lambda X_\lambda) \supset \sum_\lambda I(X_\lambda)$;
3. $I(X\cup Y) = I(X) \cap I(Y)$;
4. $N(I(X)) = X$;
5. $I(N(\mathfrak{a})) \supset \mathfrak{a}$.

Example 2.14. In general $\mathfrak{a} \subsetneqq I(N(\mathfrak{a}))$ and

$$\sum_{\lambda\in I} I(X_\lambda) \subsetneqq I(\bigcap_{\lambda\in I} X_\lambda).$$

Namely:

1. For $n = 1$ and $\mathfrak{a} := (T^2) \subset k[T]$ we have $N(\mathfrak{a}) = \{0\}$ and $I(N(\mathfrak{a})) = (T)$.

2. For $n = 2$ and $X_1 := k\times\{0\} = N(T_2)$, $X_2 := N(T_2 - T_1^2)$ we obtain $X_1 \cap X_2 = \{0\}$ and $I(X_1 \cap X_2) = (T_1, T_2)$, while

$$I(X_1) + I(X_2) \;=\; (T_2, T_2 - T_1^2) \;=\; (T_2, T_1^2) \;\subsetneqq\; (T_1, T_2)\ .$$

   The intuitive reason for this is, that the line X_1 and the parabola X_2 do not meet "transversally" at their common point 0: Both $f = T_2$ and $f = T_2 - T_1^2$ satisfy $\frac{\partial f}{\partial T_1}(0) = 0$ and hence this holds as well for all functions $f \in (T_2, T_2 - T_1^2)$.

What is wrong in the first example? Obviously vanishing ideals $I(X)$ have the property

$$f^m \in I(X) \implies f \in I(X)\ .$$

That leads us to the following:

Definition 2.15. An ideal $\mathfrak{a} \hookrightarrow R$ is called a radical ideal, if for all $f \in R$ one has:

$$f^m \in \mathfrak{a} \text{ for some } m \in \mathbb{N} \implies f \in \mathfrak{a}\ .$$

For every ideal $\mathfrak{a} \hookrightarrow R$ the ideal

$$\sqrt{\mathfrak{a}} \;:=\; \{f \in R\ ; \exists\ m \in \mathbb{N} \text{ with } f^m \in \mathfrak{a}\}$$

is a radical ideal; it is called the radical of $\mathfrak{a}$.

Example 2.16. For $n = 1$ we have $\sqrt{(T^2)} = (T)$ in $k[T]$.

Theorem 2.17 (Hilberts Nullstellensatz)**.** *For an ideal $\mathfrak{a} \subset k[T_1, \ldots, T_n]$ we have*

$$I\left(N(\mathfrak{a})\right) \;=\; \sqrt{\mathfrak{a}}\ .$$

In particular there is a bijection

$$\{\text{radical ideals in } k[T]\} \;\xrightarrow{N}\; \{\text{algebraic sets in } k^n\}$$

with the map $I\colon X \mapsto I(X)$ in the opposite direction.

Example 2.18. Consider the ideal $\mathfrak{a} := (T_1^2 + T_2^2) \subset \mathbb{R}[T_1, T_2]$. Since $T_1^2 + T_2^2$ is irreducible and $\mathbb{R}[T_1, T_2]$ factorial, we find $\sqrt{\mathfrak{a}} = \mathfrak{a}$. On the other hand $N(\mathfrak{a}) = \{(0.0)\}$ and thus

$$I(N(\mathfrak{a})) = (T_1, T_2) \supsetneqq \sqrt{\mathfrak{a}}.$$

For the proof of Hilberts Nullstellensatz Th.2.17 we need the following description of maximal ideals $\mathfrak{m} \subset k[T]$.

Theorem 2.19 (Weak Nullstellensatz)**.** *For every maximal ideal $\mathfrak{m} \subset k[T]$ there is a unique $a \in k^n$ with*

$$\mathfrak{m} = \mathfrak{m}_a := I(\{a\}) = (T_1 - a_1, ..., T_n - a_n) .$$

In particular $N(\mathfrak{a}) \neq \emptyset$ for every proper ideal $\mathfrak{a} \subsetneqq k[T]$.

Indeed, it is a consequence of the following purely algebraic result, to be shown in the next section:

Theorem 2.20. *Let K be any field and $\mathfrak{m} \subset K[T]$ a maximal ideal. Then $K \hookrightarrow L := K[T]/\mathfrak{m}$ is a finite field extension.*

Proof of Th.2.19. Since k is algebraically closed, we find even $k \cong k[T]/\mathfrak{m}$, hence have a natural ring homomorphism

$$\varphi : k[T] \longrightarrow k[T]/\mathfrak{m} \cong k,$$

which is nothing but evaluation at $a = (a_1, ..., a_n) \in k^n$ with $a_i := \varphi(T_i)$. $\square$

Corollary 2.21. *If the polynomials $f_1, ..., f_r \in k[T] = k[T_1, ..., T_n]$ do not have a common zero, then there are polynomials $g_1, ..., g_r$ with*

$$\sum_{i=1}^{r} g_i f_i = 1.$$

Proof. According to Th.2.19 we have $(f_1, ..., f_r) = k[T]$. $\square$

Proof of Th. 2.17. The equality

$$\sqrt{\mathfrak{a}} \ = \ I\left(N(\mathfrak{a})\right)$$

for an ideal $\mathfrak{a} \subset k[T]$ is a consequence of the weak Nullstellensatz Th. 2.19: The inclusion "$\subset$" is immediate: From $\mathfrak{a} \subset I\left(N(\mathfrak{a})\right)$ we obtain

$$\sqrt{\mathfrak{a}} \ \subset \ \sqrt{I\left(N(\mathfrak{a})\right)} \ = \ I\left(N(\mathfrak{a})\right),$$

since vanishing ideals are radical ideals.

For the inclusion "$\supset$" we consider a function $g \in I(N(\mathfrak{a})) \setminus \{0\}$. We introduce an additional indeterminate S and consider in the ring $k[T,S] := k[T_1, ..., T_n, S]$ the ideal

$$\mathfrak{b} := \mathfrak{a}[S] + (1{-}gS) \subset k[T,S] = (k[T])[S]$$

with $\mathfrak{a}[S] := \sum_{i=0}^{\infty} \mathfrak{a} \cdot S^i$. Its set of zeros is

$$N(k^n{\times}k; \mathfrak{b}) \quad = \quad N(k^n \times k; \mathfrak{a}[S]) \cap N(k^n{\times}k; 1{-}gS)$$

with

$$N(k^n \times k; \mathfrak{a}[S]) = N(k^n; \mathfrak{a}) \times k$$

and

$$N(k^n{\times}k; 1{-}gS) = \Gamma_{1/g} \;:=\; \left\{(x, g(x)^{-1}); x \in k_g^n\right\} \;\subset\; k_g^n{\times}k \;\subset\; k^n \times k,$$

the graph of the function $g^{-1} : k_g^n := k^n \setminus N(g) \longrightarrow k$. Hence $N(\mathfrak{b}) \hookrightarrow k^n \times k$ is empty and thus $\mathfrak{b} = k[T,S]$ according to Th.2.19. So there are elements $f_0, ..., f_m \in \mathfrak{a}$ and $h \in k[T,S]$ with

$$1 \quad = \quad \sum_{i=0}^{m} f_i S^i + h{\cdot}(1{-}gS).$$

Now we apply the algebra homomorphism

$$k[T,S] \;\to\; k(T) := Q(k[T]), \quad T_j \mapsto T_j \;\text{ for }\; j = 1, \ldots, n, \quad S \mapsto 1/g\ ,$$

and obtain

$$1 \quad = \quad \sum_{i=0}^{m} f_i g^{-i}.$$

So, eventually

$$g^m \quad = \quad \sum_{i=0}^{m} g^{m-i} f_i \in \mathfrak{a}$$

resp. $g \in \sqrt{\mathfrak{a}}$. □

Remark 2.22. If K is any field and $\overline{K}$ its algebraic closure, Hilberts Nullstellensatz $I(N(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ holds for ideals $\mathfrak{a} \hookrightarrow K[T_1, ., , , T_n]$ if the zero set $N(\mathfrak{a})$ is taken in $\overline{K}^n$ instead of in K^n. In that situation one has to consider "K-algebraic sets" only, i.e. algebraic sets which are invariant under the (componentwise) action of the automorphism group $\mathrm{Aut}_K(\overline{K})$.

We end the first section with some easy topological considerations which in particular apply to algebraic sets, endowed with the Zariski topology:

Definition 2.23. The Zariski topology on an algebraic set $X \hookrightarrow k^n$ is the topology induced by the Zariski topology of k^n.

Definition 2.24.
1. A topological space X is called noetherian, if every decreasing chain $X_1 \supset X_2 \supset \dots$ of closed subsets terminates.
2. A noetherian topological space X is called irreducible, if it is non-empty and does not admit a decomposition $X = X_1 \cup X_2$ with proper closed subsets $X_i \subsetneqq X$.
3. A maximal irreducible subspace of a noetherian space is called an irreducible component.

Example 2.25.
1. "Affine n-space" k^n is noetherian: Since $k[T]$ is noetherian, the increasing sequence $I(X_1) \subset I(X_2) \subset ...$ terminates and hence the sequence of the $X_i = N(I(X_i))$ does so as well.
2. A noetherian space is quasicompact, i.e. every open cover admits a finite subcover.
3. A subspace of a noetherian space is again noetherian. In particular
   (a) Algebraic sets are noetherian.
   (b) Any Zariski open set $U \subset X$ of an algebraic set X is quasicompact.
4. An algebraic set $X \hookrightarrow k^n$ is irreducible iff $I(X) \hookrightarrow k[T]$ is a prime ideal.
5. Irreducible spaces are connected, but not vice versa: $X := N(k^2; T_1T_2) = (k \times 0) \cup (0 \times k)$ is connected, but not irreducible.

Proof. We comment only on 4). If $X = X_1 \cup X_2$ is a nontrivial decomposition, choose polynomials $f_i \in I(X_i) \setminus I(X)$. Then $f_1 f_2 \in I(X)$, so $I(X)$ is not a prime ideal. On the other hand if $I(X)$ is not a prime ideal, take $f_1, f_2 \notin I(X)$ with $f_1 f_2 \in I(X)$. Then $X = X_1 \cup X_2$ with $X_i := X \cap N(f_i) \subsetneqq X$. $\square$

Proposition 2.26. *Let X be a noetherian topological space.*

1. *If X is irreducible, any non-empty open subset is dense.*

2. *If $Z \subset X$ is an irreducible subspace, so is its closure $\overline{Z}$. In particular the irreducible components of X are closed.*

3. *X is the irredundant union of its finitely many irreducible components.*

4. *If $X = \bigcup_{i=1}^r U_i$ with irreducible open sets $U_i \subset X$, such that $U_i \cap U_j \neq \emptyset$ for all i, j, then X is irreducible.*

Proof. If $X = X_1 \cup X_2$ and the $X_i \hookrightarrow X$ are the union of their irreducible components, then so is X. Hence, if X is not the union of its irreducible components, we can construct a strictly decreasing sequence of closed subspaces $Y_0 \supsetneqq Y_1 \supsetneqq Y_2 \supsetneqq .....$ enjoying the same property: Take $Y_0 := X$. Since Y_n is not the union of its irreducible components, it is not irreducible, so $Y_n = Z_1 \cup Z_2$ with proper closed subspaces Z_1, Z_2. W.l.o.g. then we may assume that Z_1 is not the union of its irreducible components and set $Y_{n+1} := Z_1$. — Finally assume that $X_1, ..., X_r$ are the (pairwise different) irreducible components of X, and that already $X = X_1 \cup ... \cup X_{r-1}$. Then, $X_r = (X_r \cap X_1) \cup ... \cup (X_r \cap X_{r-1})$ being irreducible, we find $X_r \subset X_i$ for some $i, 1 \leq i < r$. But then necessarily $X_r = X_i$, a contradiction. □

Example 2.27. $X := N(k^3; T_3T_1, T_3T_2) = (k^2 \times 0) \cup (0 \times 0 \times k)$.

3 Integral Ring Extensions

Though the immediate aim of this section is to prove Th.2.20, the notions we use are very important also later on. So we give a more general presentation, starting with the following class of rings, which is basic for algebraic geometry:

Definition 3.1. A K-algebra A is called *affine*, if it is a finitely generated K-algebra:

$$A = K[t_1, ..., t_n]$$

with suitable elements $t_1, ..., t_n \in A$, i.e., any element in A is a polynomial (with coefficients in the field K) in the elements $t_1, ..., t_n$, or equivalently, if A is isomorphic to a factor algebra of a polynomial ring:

$$A \cong K[T_1, ..., T_n]/\mathfrak{a}$$

with some ideal $\mathfrak{a} \hookrightarrow K[T_1, ..., T_n]$.

The essential point now is to find for a given affine K-algebra A a subalgebra $B \subset A$ (Th.3.10), such that

1. B is isomorphic to a polynomial algebra $B \cong K[S_1, ..., S_d]$.
2. A is a finite (:=finitely generated) B-module.

For the construction of the subalgebra $B \subset A$ we have to introduce the notion of an *integral ring extension*. We characterize first *integral elements* in a ring extension $R' \supset R$ (of commutative rings R, R') by three equivalent conditions. For that we need the notion of a faithful module:

Definition 3.2. An R-module M is called faithful, if $rM = 0$ with $r \in R$ implies $r = 0$.

Example 3.3.

1. If $R \hookrightarrow R'$ is a ring extension, then R' is a faithful R-module.
2. The ideal (2) in $R = \mathbb{Z}/(4)$ is not a faithful R-module.

Theorem 3.4. *Let $R \hookrightarrow R'$. For an element $s \in R'$ the following three conditions are equivalent:*

1. *There is a monic polynomial $p \in R[S]$ with $p(s) = 0$.*
2. *$R[s]$ is a finitely generated R-module.*
3. *There is a faithful $R[s]$-module, which as an R-module(!) is finitely generated.*

Definition 3.5. Let $R \hookrightarrow R'$.

1. An element $s \in R'$ is called integral over R, if one of the three equivalent conditions in Th. 3.4 is satisfied.
2. The ring extension $R' \supset R$ is called integral, if every $s \in R'$ is integral over R. Any equation $p(s) = 0$ with a monic polynomial $p \in R[S]$ is then called an integral equation for s.

Remark 3.6. In general there is no unique integral equation of minimal degree for a given element $s \in R' \supset R$: Take a field K and consider the ring

$$R := K[T^2, T^3] = K \oplus \bigoplus_{i=2}^{\infty} K \cdot T^i \subset R' := K[T] \subset Q(R),$$

with $K[T]$ the polynomial ring in one variable. Then $s := T \in R'$ has $p := S^2 - T^2 \in R[S]$ as an integral equation of minimal degree, but the linear, non-monic polynomial $q := T^2 \cdot S - T^3 \in R[S]$ has s as its zero as well, so $\tilde{p} := p + q$ is another integral equation of minimal degree.

Proof of 3.4. "1) $\Longrightarrow$ 2)": Take an element $f(s) \in R[s]$, where $f \in R[S]$. The euclidian algorithm for monic polynomials in $R[S]$ provides a decomposition $f = qp + r$ with $\deg(r) < m := \deg(p)$. Hence $f(s) = r(s) \in \sum_{i=0}^{m-1} Rs^i$. Thus

$$R[s] = \sum_{i=0}^{m-1} Rs^i.$$

"2) $\Longrightarrow$ 3)": The ring $R[s]$ is a faithful $R[s]$-module.
"3) $\Longrightarrow$ 1)": If $M = \sum_{i=1}^{d} Rm_i$ is the faithful $R[s]$-module, we may apply Lemma 3.7. We obtain a monic polynomial $p \in R[S]$ with $p(s)M = 0$. Since M is faithful, we find $p(s) = 0$. $\square$

Lemma 3.7. *Let M be an $R[s]$-module with a generator system $m_1, \dots, m_d$ over R. For every $i = 1, \dots, d$ we choose elements $r_{ij} \in R$, $j = 1, \dots, d$, with*

$$sm_i \;=\; \sum_{j=1}^{d} r_{ij} m_j \; .$$

Then the determinant

$$p := \det B \in R[S]$$

of the matrix

$$B := S \cdot E_d - (r_{ij}) \;\in\; (R[S])^{d \times d}$$

(with the d-dimensional unit matrix E_d) is a monic polynomial of degree d satisfying $p(s)M = 0$.

Proof. The Leibniz formula for determinants implies, that $p \in R[S]$ is a monic polynomial of degree d. Denoting $B^{\#}$ the complementary matrix of B, i.e.

$$B^{\#} = \big((-1)^{i+j}(\det B_{ji})\big) \;\in\; (R[S])^{d \times d} ,$$

where $B_{ji} \in (R[S])^{d-1 \times d-1}$ is the matrix obtained from B by deleting the j-th row and the i-th column, we have

$$B^{\#}B = p \cdot E_d \;\in\; R[S] \cdot E_d \;\subset\; \big(R[S]\big)^{d \times d} .$$

Now substitute s for S and multiply the equality from the right hand side with the column vector $m := (m_1, \dots, m_d)^t \in M^d$. The choice of the entries r_{ij} implies $B(s)m = 0$. Hence, in M^d we obtain

$$0 = B^{\#}(s)B(s)m = p(s)m \; ,$$

respectively $p(s)m_i = 0$ for all i, i.e., $p(s)M = 0$. □

Note that for a field $R = K$ and $M := K^d, m_i := e_i, i = 1, ..., d$, the above argument is a proof for the Cayley Hamilton Theorem: An endomorphism of K^d is annihilated by its characteristic polynomial. Take $s : K^d \longrightarrow K^d$ as the endomorphism given by the matrix (r_{ij}), such that $K[s]$ becomes a commutative subring of $\mathrm{End}(K^d)$.

Remark 3.8.

1. If R' is integral over R and R' finitely generated as R-algebra, then R' is even a finitely generated R-module. On the other hand, if R' is a finitely generated R-module, then R' is also integral over R.

2. If $R \hookrightarrow R'$ and $R' \hookrightarrow R''$ are integral ring extensions, so is $R \hookrightarrow R''$.

3. For every ring extension $R \hookrightarrow R'$ the integral closure
$$\widehat{R} \ := \ \{s \in R'; \, s \text{ is integral over } R\}$$
of R in R' is a subring of R'.

4. Let $R \hookrightarrow R'$ be an integral extension of integral domains. Then R is a field iff R' is.

Proof. 1) The second statement follows from condition 3) in 3.4 with $M = R'$. For the first there is a chain of integral ring extensions
$$R \ \hookrightarrow \ R[s_1] \ \hookrightarrow \ R[s_1, s_2] \ \hookrightarrow \ \ldots \ \hookrightarrow \ R[s_1, \ldots, s_t] = R',$$
which are finite modules over its predecessors. Since that property is "transitive" the claim follows.
2) For $c \in R''$ let $c^n + \sum_{j=0}^{n-1} s_j c^j = 0$ be an integral equation over R'. Then $R[s_0, \ldots, s_{n-1}]$ is, according to 1) a finitely generated R-module and $R[s_0, \ldots, s_{n-1}, c]$ a finitely generated $R[s_0, \ldots, s_{n-1}]$-module, so altogether a finitely generated R-module. As extension of $R[c]$ it is a faithful $R[c]$-module, hence condition 3) in 3.4 is satisfied.
3) Assume s and s' to be integral over R. We have to show, that $s+s'$ and ss' are as well. For the integral extensions $R \hookrightarrow R[s] \hookrightarrow R[s, s']$ the R-algebra $R[s, s']$ is according to 2) a finitely generated R-module as well as a faithful module over $R[s+s']$ resp. over $R[ss']$. Again 3.4 3) is satisfied.
4) Let $K := R'$ be a field. The inverse $s^{-1} \in K$ of an element $s \in R \setminus \{0\}$ satisfies an integral equation
$$s^{-n} + \sum_{j=0}^{n-1} r_j s^{-j} \ = \ 0$$
over R with some $n \geq 1$. Multiplication with s^{n-1} yields
$$s^{-1} = -\sum_{j=0}^{n-1} r_j s^{n-1-j} \in R.$$
If, on the other hand $K := R$ is a field and $s \in R' \setminus \{0\}$, then, R' being an integral domain, there is an integral equation $p(s) = 0$ with a polynomial $p \in R[S]$ not divisible by S. So $p = Sq(S) + a$ with some $a \in K^* := K \setminus \{0\}$. But then $s(-a^{-1}q(s)) = 1$. $\square$

Remark 3.9. An integral domain R is called *integrally closed in its field of fractions* (or *normal*) if the ring extension $Q(R) \supset R$ satisfies

$$Q(R) \supset \widehat{R} = R.$$

Indeed, if the integral domain $R' \supset R$ is any extension of such a normal ring the minimal polynomial of an integral element $s \in R' \subset Q(R')$ over the field of fractions $Q(R)$ has coefficients in R and is the unique integral equation of minimal degree. We mention furthermore, that UFDs are normal rings. The proof is left as an exercise.

Here is the construction of the subalgebra $B \subset A$ for a given affine K-algebra A.

Theorem 3.10 (Noether normalization lemma)**.** *Let K be an infinite field, $A := K[t_1, \ldots, t_n]$ an affine K-algebra and*

$$\varphi_n : K[T_1, \ldots, T_n] \longrightarrow A, \quad T_i \mapsto t_i$$

the corresponding surjective homomorphism of K-algebras. Then there exists a surjective linear map (but not necessarily a coordinate projection) $\pi \colon K^n \longrightarrow K^d$, such that the pull back of functions

$$\pi^* : K[S_1, ..., S_d] \longrightarrow K[T_1, ..., T_n], f \mapsto f \circ \pi,$$

provides a finite injective homomorphism of K-algebras

$$\varphi_d := \varphi_n \circ \pi^* : K[S_1, \ldots, S_d] \xrightarrow{\pi^*} K[T_1, \ldots, T_n] \xrightarrow{\varphi_n} A\ ,$$

in particular the ring extension $K[S_1, \ldots, S_d] \hookrightarrow A$ is integral.

Remark 3.11. We shall see later on that the number $d \in \mathbb{N}$ is uniquely determined. If one admits pull backs with respect to "nonlinear projections", the above theorem is even valid for finite fields K. But then polynomials should be considered as $\overline{K}$-valued functions on $\overline{K}^n$, where $\overline{K}$ is the algebraic closure of K.

Proof of Th.2.20. For $A := K[T]/\mathfrak{m}$ we choose $B = \pi^*(K[S_1, ..., S_d])$ according to Th.3.10. Since A is a field, so is $B \cong K[S_1, ..., S_d]$, hence $d = 0$ and A is a finite field extension of K. $\square$

Proof of Th.3.10. We do induction on n, the number of generators of the K-algebra A. For $n = 0$ we have $A = K$, so there is nothing to prove. Now let $A \cong K[T]/\mathfrak{a}$ with $T = (T_1, ..., T_n)$. If $\mathfrak{a} = 0$, we are done. Otherwise we take some polynomial $f \in \mathfrak{a} \setminus \{0\}$. Indeed, we may assume that f is T_n-monic:

Lemma 3.12. *Let K be an infinite field. For any $f \in K[T] \setminus \{0\}$ there is a linear automorphism $F : K^n \longrightarrow K^n$ such that $g := F^*(f)$ is an "almost T_n-monic" polynomial*

$$g \;=\; cT_n^m + \sum_{i<m} g_i(T')T_n^i \,,$$

where $c \in K^*$ and with $T' := (T_1, ..., T_{n-1})$.

Proof. Let $f = \sum_{i=0}^m p_i$ be the decomposition of f as sum of homogeneous polynomials p_i of degree i and $p_m \neq 0$. Since K is infinite, there is a point $x \in K^n \setminus \{0\}$ with $p_m(x) \neq 0$. We may even assume $x = (x', 1) = (x_1, \ldots, x_{n-1}, 1) \in K^n$. Now with

$$F : K^n \longrightarrow K^n, t = (t', t_n) \mapsto (t' + t_n x', t_n)$$

we obtain

$$\begin{aligned} F^*(f) = f(T' + T_n x', T_n) \;&=\; \sum_{i=0}^{m} p_i(T_1 + x_1T_n, \ldots, T_{n-1} + x_{n-1}T_n, T_n) \\ &=\; cT_n^m + \sum_{i<m} g_i(T')T_n^i =: g \end{aligned}$$

with $c := p_m(x', 1) \neq 0$. □

Let us now continue with the proof of Th.3.10: If f has T_n-degree m, we have

$$K[T]/(f) = \bigoplus_{\nu=0}^{m-1} K[T'] \cdot T_n^\nu$$

and choose $\pi_1 : K^n \longrightarrow K^{n-1}, x = (x', x_n) \mapsto x'$. Now consider

$$A' := \varphi_n(K[T']) \subset A.$$

Then A, being a factor module of $K[T]/(f)$, is a finite A'-module. By induction hypothesis, there is a linear projection $\pi_2 : K^{n-1} \longrightarrow K^d$, such that

$$K[S_1, ..., S_d] \xrightarrow{\pi_2^*} K[T'] \xrightarrow{\varphi_{n-1}} A'$$

with $\varphi_{n-1} := \varphi_n|_{K[T']}$ is finite and injective. Finally take $\pi := \pi_2 \circ \pi_1$. □

Remark 3.13. We briefly indicate how the above proof has to be modified if K is a finite field: In that case the pull back automorphism $F^* : K[T] \longrightarrow K[T]$ in the proof of the auxiliary lemma 3.12 has to be replaced with $\sigma : K[T] \longrightarrow K[T]$, where

$$\sigma : T_i \mapsto T_i + T_n^{m_i} \text{ for } i \leq n-1, \text{ and } \; T_n \mapsto T_n$$

with exponents $m_i := b^{n-i}$, where b is bigger than all the exponents of indeterminates T_i in the monomials of the polynomial f. Then the maximal T_n-power in

$$\sigma(T^\alpha) = T_n^m + \sum_{i<m} g_i(T')T_n^i$$

is $m := m(\alpha) = \sum_{i=1}^n \alpha_i b^{n-i}$. Since different multi-exponents α give different maximal T_n-exponents $m(\alpha)$, it follows that the maximal T_n-exponent in $\sigma(f)$ is the maximal $m(\alpha)$ for a monomial T^α appearing in f and thus $\sigma(f)$ is T_n-monic.

Finally, for use later on, we prove the following important result in the theory of integral extensions:

Theorem 3.14. *Let $R \hookrightarrow R'$ be an integral ring extension. Then*

1. *given a prime ideal $\mathfrak{p}$ in R, there is a prime ideal $\mathfrak{p}' \hookrightarrow R'$ lying above $\mathfrak{p}$, i.e. $\mathfrak{p}' \cap R = \mathfrak{p}$.*
2. *Furthermore, $\mathfrak{p} \hookrightarrow R$ is a maximal ideal iff $\mathfrak{p}' \hookrightarrow R'$ is, and*
3. *if $\mathfrak{p}_1' \hookrightarrow \mathfrak{p}_2' \hookrightarrow R'$ lie above $\mathfrak{p} \hookrightarrow R$, we have $\mathfrak{p}_1' = \mathfrak{p}_2'$.*

Proof. 1.) First we assume that R is a local ring and $\mathfrak{p}$ its (unique) maximal ideal $\mathfrak{m}$. If then $\mathfrak{m}' \hookrightarrow R'$ is any maximal Ideal, the intersection $\mathfrak{m}' \cap R \hookrightarrow R$ is a proper ideal, since it does not contain 1. In the commutative diagram

$$\begin{array}{ccc} R & \hookrightarrow & R' \\ \downarrow & & \downarrow \\ R/(R \cap \mathfrak{m}') & \hookrightarrow & R'/\mathfrak{m}' \end{array} .$$

the ring $R'/\mathfrak{m}'$ is a field, which is integral over $R/(R \cap \mathfrak{m}')$. According to Rem.3.8.4 $R/(R \cap \mathfrak{m}')$ is a field as well, such that $R \cap \mathfrak{m}'$ is a maximal ideal in R and therefore $\mathfrak{m} = R \cap \mathfrak{m}'$, the ring R being local.

For general R and $\mathfrak{p} \subset R$ we consider the multiplicative system $S := R \setminus \mathfrak{p}$. Then $\mathfrak{m} := S^{-1}\mathfrak{p}$ is the only maximal ideal in the local ring $S^{-1}R$, and the ring extension $S^{-1}R \hookrightarrow S^{-1}R'$ is integral. The first part provides a maximal ideal $\mathfrak{m}' \hookrightarrow S^{-1}R'$ over $\mathfrak{m}$. Thus we obtain a commutative diagram

$$\begin{array}{ccccccc} \mathfrak{p} & \hookrightarrow & R & \stackrel{\iota}{\hookrightarrow} & R' & \hookleftarrow & \mathfrak{p}' \\ & & \downarrow & & \downarrow & & \downarrow \\ \mathfrak{m} & \hookrightarrow & S^{-1}R & \stackrel{S^{-1}\iota}{\hookrightarrow} & S^{-1}R' & \hookleftarrow & \mathfrak{m}' \end{array} .$$

The inverse image $\mathfrak{p}' \subset R'$ of $\mathfrak{m}'$ with respect to the natural map $R' \longrightarrow S^{-1}R'$ is again a prime ideal, lying actually over $\mathfrak{p}$: If $j_R : R \longrightarrow S^{-1}R$ and $j_{R'} : R' \longrightarrow S^{-1}R'$ denote the vertical maps, then

$$\mathfrak{p}' \cap R \; = \; \iota^{-1}\left(j_{R'}^{-1}(\mathfrak{m}')\right) \; = \; j_R^{-1}\left((S^{-1}\iota)^{-1}(\mathfrak{m}')\right)$$

$$= \; j_R^{-1}(\mathfrak{m}' \cap S^{-1}R) \; = \; j_R^{-1}(\mathfrak{m}) \; = \; \mathfrak{p} \; .$$

2.) This remark follows from Rem.3.8.4 applied to the integral ring extension $R/\mathfrak{p} \hookrightarrow R'/\mathfrak{p}'$.
3.) We may replace R with $R/\mathfrak{p}$ and R' with $R'/\mathfrak{p}'_1$ resp. assume $\mathfrak{p} = 0 = \mathfrak{p}'_1$ and that R' (as well as R) is an integral domain. So we have to show that for integral domains $R \subset R'$ and a prime ideal $\mathfrak{p}' \hookrightarrow R'$ we have $\mathfrak{p}' \cap R = 0 \Longrightarrow \mathfrak{p}' = 0$. Finally replacing $R \subset R'$ with $S^{-1}R \subset S^{-1}R'$ with the multiplicative subset $S = R \setminus \{0\}$, Rem. 3.8.4 yields once again that in the integral ring extension $S^{-1}R \subset S^{-1}R'$ not only $S^{-1}R$, but also $S^{-1}R'$ is a field, so $S^{-1}\mathfrak{p}' = 0$ resp. $\mathfrak{p}' = 0$.

$\square$

4 Affine Varieties

In this section we treat all the algebraic subsets $X \hookrightarrow k^n$ (with the "embedding dimension" $n = n(X)$ depending on X) simultaneously as objects of a category $\mathcal{TA}$ (no standard terminology!), thereby getting rid of any specific inclusion into some affine space. From a given embedding $X \hookrightarrow k^n$ we only keep the information about the functions obtained as restrictions of a polynomial on k^n. These *regular functions* provide the morphisms $X \longrightarrow k$, and they are associated to the topological space X as an additional datum.

Definition 4.1. Let $X \hookrightarrow k^n$ be an algebraic set. A function $f : X \longrightarrow k$ is called *regular* if there is a polynomial $g \in k[T]$ with $f = g|_X$.

A first observation is

Proposition 4.2. *The regular functions on an algebraic set $X \hookrightarrow k^n$ form a k-algebra $\mathcal{O}(X)$. Indeed there is an exact sequence*

$$0 \longrightarrow I(X) \longrightarrow k[T] \xrightarrow{\varrho_X} \mathcal{O}(X) \longrightarrow 0$$

with the inclusion $I(X) \hookrightarrow k[T]$ and the restriction homomorphism

$$\varrho_X : \ k[T] \longrightarrow \mathcal{O}(X), \ g \mapsto g|_X.$$

Proof. Exercise! □

Example 4.3. Here we shall represent the ring $\mathcal{O}(X)$ for some examples either as subalgebra or as extension of some polynomial ring.

1. The Neil parabola $X := N(k^2; T_2^2 - T_1^3)$ admits a polynomial parametrization $\varphi : k \longrightarrow X, s \mapsto (s^2, s^3)$, hence there is an induced pull back map $\varphi^* : \mathcal{O}(X) \longrightarrow k[S]$, an injection indeed, the map φ being onto (or rather bijective!). Composing it with the restriction homomorphism $\varrho_X : k[T_1, T_2] \longrightarrow \mathcal{O}(X)$ leads to the homomorphism $k[T] \longrightarrow k[S], T_1 \mapsto S^2, T_2 \mapsto S^3$, whence

$$\mathcal{O}(X) \cong k[S^2, S^3] \cong k \cdot 1 \oplus \bigoplus_{i=2}^{\infty} k \cdot S^i.$$

2. For $\mathrm{char}(k) \neq 2$ the parametrization

$$k \longrightarrow X := N(k^2; T_2^2 - T_1^2(T_1 + 1)), s \mapsto (s^2 - 1, s(s^2 - 1))$$

of the noose X induces an isomorphism $\mathcal{O}(X) \cong \{f \in k[S]; f(1) = f(-1)\}$.

3. For the reducible algebraic set $X = (k \times 0) \cup (0 \times k)$ take disjoint copies Z_1, Z_2 of the affine line k and look at

$$\varphi : Z_1 \cup Z_2 \longrightarrow X, Z_1 \in s_1 \mapsto (s_1, 0), Z_2 \in s_2 \mapsto (0, s_2).$$

It induces an isomorphism

$$\mathcal{O}(X) \cong \varphi^*(\mathcal{O}(X)) = \{(f_1, f_2) \in k[S_1] \oplus k[S_2]; f_1(0) = f_2(0)\}.$$

Here $k[S_1] \oplus k[S_2]$ denotes the direct sum of the two polynomial rings $k[S_i]$ (in one variable S_i!), the ring operations are componentwise.

4. If a parametrization is not available one can look at a projection: Consider

$$\psi : X = N(k^2, T_2^2 - p(T_1)) \hookrightarrow k^2 \xrightarrow{\mathrm{pr}_1} k.$$

Since ψ is surjective, the corresponding pull back of functions is injective, so we may identify $\psi^*(k[T_1])$ with $k[T_1]$ and thus obtain

$$\mathcal{O}(X) = k[T_1] \cdot 1 \oplus k[T_1] \cdot \overline{T}_2 \cong (k[T_1])[\sqrt{p}]$$

with $\overline{T}_2^2 = p(T_1)$.

With Hilberts Nullstellensatz Th. 2.17 we can characterize all the algebras $\mathcal{O}(X)$ up to isomorphy:

Remark 4.4. A k-algebra A is isomorphic to the algebra $\mathcal{O}(X)$ of regular functions of some algebraic subset $X \hookrightarrow k^n$, i.e.

$$A \cong \mathcal{O}(X),$$

iff it is affine and reduced.

Here "*reduced*" means:

Definition 4.5. A ring R is called reduced, if it does not admit non-zero nilpotent elements, i.e. if

$$\sqrt{0} = 0$$

holds in R.

Proof of 4.4. An affine algebra $A \cong k[T]/\mathfrak{a}$ is reduced iff $\mathfrak{a} = \sqrt{\mathfrak{a}}$, the latter being equivalent to $\mathfrak{a} = I(X)$ for $X = N(\mathfrak{a})$. $\square$

Remark 4.6. An algebraic set $X \hookrightarrow k^n$ is irreducible iff $\mathcal{O}(X)$ is an integral domain iff any two non-empty open subsets have points in common.

Let us now consider the following category $\mathcal{TA}$ of **T**opological spaces with a distinguished **A**lgebra of regular functions:

1. The objects are the pairs (X, A), where X is a noetherian topological space and $A \subset \mathcal{C}(X)$ a subalgebra of the algebra $\mathcal{C}(X)$ of all k-valued continuous functions on X (where k is endowed with the Zariski topology, so a function $f : X \longrightarrow k$ is continuous iff all the level sets $f^{-1}(c), c \in k$, are closed.) satisfying in addition

   $$A^* = \{f \in A; N(f) = \emptyset\},$$

   i.e. a function $f \in A$ is invertible iff it has no zeros. Functions $f \in A$ are also referred to as *regular functions* on X.

2. A morphism from (X, A) to (Y, B) is a continuous map

   $$\varphi : X \longrightarrow Y\ ,$$

   such that the pull back homomorphism

   $$\varphi^* : \mathcal{C}(Y) \longrightarrow \mathcal{C}(X), \quad f \ \mapsto \ f \circ \varphi\ ,$$

   maps $B \subset \mathcal{C}(Y)$ into $A \subset \mathcal{C}(X)$, i.e.:

   $$\varphi^*(B) \quad \subset \quad A\ .$$

Then the pair $(X, \mathcal{O}(X))$ belongs to this category for any algebraic set $X \hookrightarrow k^n$ as a consequence of

Remark 4.7. 1. Any maximal ideal $\mathfrak{m} \subset \mathcal{O}(X)$ is of the form

$$\mathfrak{m} = \mathfrak{m}_a := \{f \in \mathcal{O}(X);\ f(a) = 0\}$$

with a unique point $a \in X$: Consider the restriction map $\varrho := \varrho_X : k[T] \longrightarrow \mathcal{O}(X)$. Then $\varrho^{-1}(\mathfrak{m}) \hookrightarrow k[T]$ is a maximal ideal as well, the map ϱ being onto, hence

$$\varrho^{-1}(\mathfrak{m}) = I(\{a\}) \hookrightarrow k[T]$$

for some $a \in k^n$, according to Th.2.19. Since $I(\{a\}) \supset I(X)$, we have $a \in X$ and thus $\mathfrak{m} = \varrho(I(\{a\})) = \mathfrak{m}_a$.

2. If $f_1, ..., f_r \in \mathcal{O}(X)$ are functions without a common zero, they are not contained in any maximal ideal, hence generate the unit ideal, i.e. there are functions $g_1, ..., g_r \in \mathcal{O}(X)$ with $\sum_{i=1}^r g_i f_i = 1$. In particular regular functions without zeros are invertible.

For convenience of notation we shall from now on denote the objects in $\mathcal{TA}$ simply by capital letters $X, Y, ...$ and, in analogy to algebraic sets, $\mathcal{O}(X), \mathcal{O}(Y), ...$ their associated algebras of regular functions.

Definition 4.8. An object $Y \in \mathcal{TA}$ is called an affine variety (over the field k) if $Y \cong X \hookrightarrow k^n$ with some Zariski closed set X in some affine n-space k^n. We denote $\mathcal{AV} \subset \mathcal{TA}$ the full subcategory with the affine varieties as objects.

Remark 4.9. 1. For an affine variety X the regular functions separate points, i.e. given two different points $x, y \in X$ there is a function $f \in \mathcal{O}(X)$ with $f(x) \neq f(y)$.

2. The algebra $\mathcal{O}(X)$ is a finite dimensional k-vector space iff $|X| < \infty$. In that case we have a ring isomorphism $\mathcal{O}(X) \cong k^r$, where $r := |X|$.

Example 4.10. Here are two examples of affine varieties X, which are given independent of any explicit embedding into some k^n:

1. Closed subsets of an affine variety are again affine varieties, also called "*closed subvarieties*": A closed subset $X \hookrightarrow Y$ of an affine variety Y inherits the structure of an affine variety: Take

$$\mathcal{O}(X) := \mathcal{O}(Y)|_X \cong \mathcal{O}(Y)/I(X)$$

with the ideal

$$\mathcal{O}(Y) \hookleftarrow I(X) := \{f \in \mathcal{O}(Y); f|_X = 0\}.$$

2. An abstract version of k^n: Every finite dimensional k-vector space V is in a natural way a to $k^{\dim V}$ isomorphic affine variety: The algebra $\mathcal{O}(V)$ is generated by the linear forms $V \longrightarrow k$, indeed

$$\mathcal{O}(V) \cong S(V^*)$$

is isomorphic to the symmetric algebra over the dual vector space V^*. (Recall that $S(W)$ for a vector space W is a factor algebra of the tensor algebra $\bigoplus_{q=0}^{\infty} W^{\otimes q}$: Divide by the two sided ideal generated by the elements $w_1 \otimes w_2 - w_2 \otimes w_1 \in W^{\otimes 2} = W \otimes W$.)

The topology on V is the coarsest topology such that all functions $f \in \mathcal{O}(V)$ become continuous.

Remark 4.11. Here are some morphisms in $\mathcal{TA}$:

1. The inclusion $j : X \hookrightarrow k^n$ of an algebraic subset into k^n is a morphism.

2. Let $X \in \mathcal{TA}$. A map $\varphi = (\varphi_1, ..., \varphi_n) : X \longrightarrow k^n$ is a morphism iff $\varphi_1, ..., \varphi_n \in \mathcal{O}(X)$.

3. Let $X \in \mathcal{TA}$ and $j : Y \hookrightarrow k^n$ be the inclusion of an algebraic set. Then a map $\varphi : X \longrightarrow Y$ is a morphism iff $j \circ \varphi$ is.

4. A map $\varphi : X \longrightarrow Y$ between embedded affine varieties $X \hookrightarrow k^m$ and $Y \hookrightarrow k^n$ is a morphism iff it is the restriction $\varphi = \widehat{\varphi}|_X$ of a polynomial map $\widehat{\varphi} : k^m \longrightarrow k^n$ (then necessarily satisfying $\widehat{\varphi}(X) \subset Y$).

5. Let $X \in \mathcal{TA}$ and $Y \hookrightarrow k^n$ be an affine variety. Then the morphisms $\varphi : X \longrightarrow Y$ correspond bijectively to the algebra homomorphisms $\sigma : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$. For $Y = k^n$ that follows from the fact that algebra homomorphisms $\sigma : k[T_1, ..., T_n] \longrightarrow \mathcal{O}(X)$ correspond to n-tuples $(\varphi_1, ..., \varphi_n) \in \mathcal{O}(X)^n$ (with $\varphi_i = \sigma(T_i)$). In the general case consider an embedding $Y \hookrightarrow k^n$ and note that $k[T] \longrightarrow \mathcal{O}(X), T_i \mapsto \varphi_i$, factors through $\mathcal{O}(Y) \cong k[T]/I(Y)$ iff the map $\varphi = (\varphi_1, ..., \varphi_n) : X \longrightarrow k^n$ satisfies $\varphi(X) \subset Y$.

So it seems, all information about an affine variety X is contained in its algebra $\mathcal{O}(X)$ of regular functions. Indeed:

Theorem 4.12. *Denote $\mathcal{FRA}$ the category of affine reduced k-algebras. The functor*

$$\mathcal{O} : \mathcal{AV} \longrightarrow \mathcal{FRA}$$

given by

$$X \mapsto \mathcal{O}(X), \varphi \mapsto \varphi^*$$

defines an anti-equivalence of categories.

Proof. According to Rem.4.4 we know that every algebra $A \in \mathcal{FRA}$ is isomorphic to the regular function algebra $\mathcal{O}(X)$ of some affine variety X. It remains to show that $\mathcal{O} : \mathrm{Mor}(X, Y) \longrightarrow \mathrm{Hom}(\mathcal{O}(Y), \mathcal{O}(X))$ is bijective for all affine varieties X and Y, but here we may refer to Rem.4.11.5. $\square$

Let us now describe a functor

$$\mathrm{Sp} : \mathcal{FRA} \longrightarrow \mathcal{AV}$$

inverse to $\mathcal{O} : \mathcal{AV} \longrightarrow \mathcal{FRA}$. For a reduced affine k-algebra A we set

$$\mathrm{Sp}(A) := \{\mathfrak{m} \hookrightarrow A, \text{max. ideal}\},$$

the "(maximal) spectrum" of the ring A. We want to endow $\mathrm{Sp}(A)$ with a topology and a distinguished algebra $\mathcal{O}(\mathrm{Sp}(A))$ of regular functions: Fix a surjective homomorphism $k[T_1, ..., T_n] \longrightarrow A$, denote $\mathfrak{a} \hookrightarrow k[T]$ its kernel and $X := N(\mathfrak{a}) \hookrightarrow k^n$ its zero set. Then

$$X \longrightarrow \mathrm{Sp}(A), x \mapsto \mathfrak{m}_x$$

is a bijection and hence can be used to define a topology and regular functions on $\mathrm{Sp}(A)$. Indeed these data depend only on A: The closed sets are the sets

$$N(\mathfrak{b}) := \{\mathfrak{m} \in \mathrm{Sp}(A); \mathfrak{b} \subset \mathfrak{m}\}$$

with an ideal $\mathfrak{b} \hookrightarrow A$. It remains to determine $\mathcal{O}(\mathrm{Sp}(A))$: For every $\mathfrak{m} \in \mathrm{Sp}(A)$ the map $\imath : k = k{\cdot}1 \hookrightarrow A \longrightarrow A/\mathfrak{m}$ is an isomorphism. Hence every

$f \in A$ provides a continuous function $\widehat{f} : \mathrm{Sp}(A) \to k, \mathfrak{m} \mapsto \imath^{-1}(f+\mathfrak{m})$. If we then declare these functions to be regular:

$$\mathcal{O}(\mathrm{Sp}(A)) := \{\widehat{f} : \mathrm{Sp}(A) \longrightarrow k; f \in A\},$$

we obtain, due to the fact that

$$\bigcap_{\mathfrak{m} \in \mathrm{Sp}(A)} \mathfrak{m} = \bigcap_{x \in X} \mathfrak{m}_x = \{0\},$$

an isomorphism

$$\Phi_A : A \xrightarrow{\cong} \mathcal{O}(\mathrm{Sp}(A)), f \mapsto \widehat{f}.$$

A comment on notation: Though the elements in the set $\mathrm{Sp}(A)$ are ideals, one usually prefers a geometric notation and denotes $x \in \mathrm{Sp}(A)$ its points, such that, strictly speaking, $x = \mathfrak{m}_x$.

Finally, given an algebra homomorphism $\sigma\colon B \to A$ between reduced affine algebras B and A, the inverse image $\sigma^{-1}(\mathfrak{m})$ of a maximal ideal $\mathfrak{m} \hookrightarrow A$ is again maximal, since $k \hookrightarrow B/\sigma^{-1}(\mathfrak{m}) \hookrightarrow A/\mathfrak{m} \cong k$ implies $B/\sigma^{-1}(\mathfrak{m}) \cong k$. Even better, the map

$$\mathrm{Sp}(\sigma) : \mathrm{Sp}(A) \longrightarrow \mathrm{Sp}(B)\ , \quad \mathfrak{m} \ \mapsto \ \sigma^{-1}(\mathfrak{m})$$

satisfies obviously

$$\widehat{g} \circ \mathrm{Sp}(\sigma) = \widehat{\sigma(g)}$$

for any $g \in B$ and thus

$$\mathrm{Sp}(\sigma)^*(\mathcal{O}(\mathrm{Sp}(B)) \subset \mathcal{O}((\mathrm{Sp}(A)),$$

so it really is a morphism in our category $\mathcal{TA}$ of topological spaces with distinguished algebra of regular functions.

In order to see that $\mathcal{O}$ and Sp are inverse one to the other we have to construct natural equivalences

$$\Phi : \mathrm{id}_{\mathcal{FRA}} \xrightarrow{\cong} \mathcal{O} \circ \mathrm{Sp}.$$

as well as

$$\Psi : \mathrm{id}_{\mathcal{AV}} \xrightarrow{\cong} \mathrm{Sp} \circ \mathcal{O}.$$

The first one is defined for $A \in \mathcal{FRA}$ by the above

$$\Phi_A : A \longrightarrow \mathcal{O}(\mathrm{Sp}(A)), f \mapsto \widehat{f},$$

while

$$\Psi_X : X \longrightarrow \mathrm{Sp}(\mathcal{O}(X)), x \mapsto \mathfrak{m}_x$$

is the natural equivalence over an affine variety $X \in \mathcal{AV}$: We need to check that given a morphism $\varphi : X \longrightarrow Y$, we have

$$\Psi_Y \circ \varphi = \mathrm{Sp}(\mathcal{O}(\varphi)) \circ \Psi_X.$$

We evaluate the LHS and the RHS at a point $x \in X$ and obtain

$$\mathfrak{m}_{\varphi(x)} \stackrel{?}{=} (\varphi^*)^{-1}(\mathfrak{m}_x),$$

an equality readily verified.

Let us now briefly discuss some particular morphisms:

Definition 4.13. A morphism $\varphi : X \longrightarrow Y$ is called

1. *dominant*, if $Y = \overline{\varphi(X)}$;
2. a *closed embedding*, if it admits a factorization $X \stackrel{\cong}{\longrightarrow} Z \hookrightarrow Y$, where $Z \hookrightarrow Y$ is a closed subvariety of Y;
3. *finite*, if $\mathcal{O}(X)$ is a finite(ly generated) $\mathcal{O}(Y)$-module.

Proposition 4.14. *Let $\varphi : X \longrightarrow Y$ be a morphism of affine varieties.*

1. *The morphism φ is dominant, iff $\varphi^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ is injective.*
2. *The morphism φ is a closed embedding, iff $\varphi^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ is surjective.*
3. *A finite dominant morphism $\varphi : X := \mathrm{Sp}(A) \longrightarrow Y := \mathrm{Sp}(B)$ is onto (:= surjective) and has finite fibers: For every $y \in Y$ we have $|\varphi^{-1}(y)| \le r$, if the finite B-module A can be generated by r elements.*
4. *A finite morphism is closed.*

Proof. The first statement as well as the implication "$\Longrightarrow$" of the second statement is obvious. For the reversed one take $Z := \overline{\varphi(X)}$. Then $X \longrightarrow Z$ is an isomorphism, since the pull back homomorphism $\mathcal{O}(Z) \longrightarrow \mathcal{O}(X)$ is. For the third part take a point $y \in Y$. Since φ^* is injective and $\mathcal{O}(X) \supset \mathcal{O}(Y)$ an integral extension, we may apply Th.3.14 and obtain a maximal ideal $\mathfrak{m}' \hookrightarrow \mathcal{O}(X)$ with $\mathfrak{m}' \cap \mathcal{O}(Y) = \mathfrak{m}_y$. But $\mathfrak{m}' = \mathfrak{m}_x$ for some $x \in X$ and thus $y = \varphi(x)$. If $A = \sum_{i=1}^r Ba_i$, then $A/\mathfrak{m}_y \cdot A$ is generated by the residue classes $a_i + \mathfrak{m}_y \cdot A$ over $B/\mathfrak{m}_y \cong k$, the same holds for $\mathcal{O}(\varphi^{-1}(y)) \cong A/I(\varphi^{-1}(y))$, a factor ring of $A/\mathfrak{m}_y \cdot A$. So $\mathcal{O}(\varphi^{-1}(y))$ is a finite dimensional k-vector space of a dimension $\leq r$, hence the affine variety $\varphi^{-1}(y)$ is finite with at most r points. Finally let $\varphi : X \longrightarrow Y$ be finite and $Z \hookrightarrow X$ be a closed subset. Then $Z \longrightarrow \overline{\varphi(Z)}$ is a finite dominant map, hence surjective and thus $\varphi(Z) = \overline{\varphi(Z)}$. □

We include here a geometric reformulation of the Noether normalization theorem 3.10:

Corollary 4.15. *For any irreducible variety X there is a finite surjective morphism $\varphi : X \longrightarrow k^d$. If $X = N(k^n; f) \hookrightarrow k^n$ with a nonconstant polynomial f we may choose $d = n - 1$.*

Remark 4.16. Let us explain for $k = \mathbb{C}$ the geometry behind the notion of a finite map:

1. First of all we note that a complex affine variety X is compact (w.r.t. the strong topology) iff it is finite. The nontrivial implication follows with the above geometric version of the Noether normalization lemma: For compact X we have $d = 0$.

2. A morphism $\varphi : X \longrightarrow Y$ of complex affine varieties is finite if and only if it is (w.r.t. the strong topologies) proper, i.e. if for any compact set $K \subset Y$ the inverse image $\varphi^{-1}(K) \subset X$ is compact as well. Obviously that implies that φ has finite fibers, but that is definitely not sufficient: The map
$$N(T_1T_2 - 1) \hookrightarrow \mathbb{C}^2 \xrightarrow{\mathrm{pr}_1} \mathbb{C}$$
has finite fibers, but is not proper.

3. The most suggestive characterization of properness is as follows: A morphism $\varphi : X \longrightarrow Y$ between complex affine varieties is proper if and only if a point sequence $(x_n) \subset X$ has points of accumulation in X iff the sequence $(\varphi(x_n)) \subset Y$ has in Y. The implication "$\Longrightarrow$" makes use of the continuity of φ, while for "$\Longleftarrow$" consider the compact set K consisting of the points y_n and the accumulation points of that sequence. One could thus say, that $\varphi : X \longrightarrow Y$ is proper iff the space X has "no holes over Y".

A closed subset $Z \hookrightarrow X$ of an affine variety X is again an affine variety with the regular function algebra

$$\mathcal{O}(Z) := \mathcal{O}(X)|_Z \cong \mathcal{O}(X)/I(Z).$$

But what can we do in order to give open subsets $U \subset X$ also some structure, say, to make them objects in the category $\mathcal{TA}$? We have to look for a good notion of regular functions on U. We could once again try with the following algebra

$$\mathcal{O}(U) := \mathcal{O}(X)|_U.$$

But there is some embarrassing fact: If U is dense in X - this is always the case for irreducible X and non-empty U - we get

$$\mathcal{O}(X)|_U \cong \mathcal{O}(X).$$

In particular for the subalgebra $\mathcal{O}(X)|_U \subset \mathcal{C}(U)$ it need not be true that functions without zeros are invertible. So we have to enlarge that algebra. For certain open subsets that is easily done:

Definition 4.17. Let X be an affine variety.

1. A principal open subset $U \subset X$ is an open subset U of the form

$$U = X_f := X \setminus N(f)$$

with $N(f) := N(X; f)$ for a suitable function $f \in \mathcal{O}(X)$.

2. For any $f \in \mathcal{O}(X) \setminus \{0\}$ we define

$$\mathcal{O}(X)_f := \left\{ \frac{h}{f^m} : X_f \longrightarrow k; h \in \mathcal{O}(X), m \in \mathbb{N} \right\}.$$

Remark 4.18. 1. Any Zariski open set $U \subset X$ is a finite union of principal open sets: We can write $X \setminus U = N(f_1, ..., f_r)$, hence $U = \bigcup_{i=1}^{r} X_{f_i}$.

2. For an ideal $\mathfrak{a} \hookrightarrow \mathcal{O}(X)$ in the algebra of regular functions on an affine variety X Hilberts Nullstellensatz holds as well:

$$I(N(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

That follows immediately from Th.2.17 by embedding $X \hookrightarrow k^n$ into some affine space.

3. $\mathcal{O}(X)_f$ is a reduced affine k-algebra: If $\sigma : k[T] \longrightarrow \mathcal{O}(X)$ is onto, so is the extension $k[T, S] \longrightarrow \mathcal{O}(X)_f, T_i \mapsto \sigma(T_i), S \mapsto f^{-1}$.

4. For principal open sets $X_f \supset X_g$ we have

$$\mathcal{O}(X)_f|_{X_g} \subset \mathcal{O}(X)_g.$$

The inclusion $X_g \subset X_f$ tells us that $g \in I(N(f))$ and thus $g^\ell \in (f)$ for some $\ell \in \mathbb{N}$ by Hilberts Nullstellensatz. If, say, $g^\ell = \tilde{g}f$ we have

$$\frac{h}{f^m} = \frac{\tilde{g}^m h}{g^{\ell m}}$$

on X_g.

Since in particular $\mathcal{O}(X)_f = \mathcal{O}(X)_g$ for $X_f = X_g$ we may define:

Definition 4.19. The algebra of regular functions $\mathcal{O}(U)$ on a principal open subset $U = X_f$ of an affine variety X is defined as

$$\mathcal{O}(U) := \mathcal{O}(X)_f.$$

Remark 4.20. 1. A regular function on $U = X_f$ without zeros is invertible: We may assume that it is of the form $h|_U$ with $h \in \mathcal{O}(X)$. But already the restriction of h to $X_h \supset X_f$ is invertible!

2. The restriction $\mathcal{O}(X) \longrightarrow \mathcal{O}(X_f)$ is an injection if $f \in \mathcal{O}(X)$ is not a zero divisor. In particular that holds true automatically if $\mathcal{O}(X)$ is an integral domain.

3. If $X = \mathrm{Sp}(A)$ is irreducible, we obtain an inclusion $\mathcal{O}(X_f) = A_f \subset Q(A)$.

So a principal open set $U \subset X$ defines an object $(U, \mathcal{O}(U)) \in \mathcal{TA}$. Indeed, it is again an affine variety:

Proposition 4.21. *For a principal open subset $X_f \subset X$ of an affine variety X we have*

$$X_f \cong \mathrm{Sp}(\mathcal{O}(X)_f).$$

Before we prove Prop.4.21 we need cartesian products of affine varieties:

Remark 4.22. If $X \hookrightarrow k^m$ and $Y \hookrightarrow k^n$ are affine varieties, so is

$$X \times Y \hookrightarrow k^m \times k^n = k^{m+n}.$$

Note that the homomorphism

$$\mathcal{O}(X) \otimes \mathcal{O}(Y) \longrightarrow \mathcal{O}(X \times Y)$$

induced by the bilinear map

$$\mathcal{O}(X) \times \mathcal{O}(Y) \longrightarrow \mathcal{O}(X \times Y), (f, g) \mapsto \mathrm{pr}_X^*(f) \cdot \mathrm{pr}_Y^*(g)$$

is an isomorphism (since linear independent functions $f_i \in \mathcal{O}(X)$ and $g_j \in \mathcal{O}(Y)$ give linear independent functions $\mathrm{pr}_X^*(f_i) \cdot \mathrm{pr}_Y^*(g_j)$); hence the affine variety $X \times Y$ depends only on the affine varieties X and Y and not on the chosen embeddings. But note that the Zariski topology on $X \times Y$ is, except in trivial cases, strictly finer that the product topology of the Zariski topologies on the factors X and Y.

Proof of 4.21. Let $X = \mathrm{Sp}(A)$. It suffices to show that X_f is affine, indeed isomorphic to the affine variety

$$\Gamma_{1/f} := N(X \times k; 1 - fS) \hookrightarrow X \times k \cong \mathrm{Sp}(A[S]),$$

the graph of the function $1/f : X_f \longrightarrow k$ considered as subset of $X \times k$, the map

$$X_f \longrightarrow \Gamma_{1/f}, x \mapsto (x, \frac{1}{f(x)})$$

being an isomorphism. Note that we even have

$$\mathcal{O}(X_f) \cong A[S]/(1 - fS).$$

□

Finally, as a preparation of the next section, we prove that a locally regular function on an affine variety is regular:

Proposition 4.23. *Let* $X = \bigcup_{i=1}^r U_i$ *be an open cover of an affine variety by principal open subset* $U_i = X_{f_i}$. *Then a function* $h : X \longrightarrow k$ *is regular iff all the restrictions* $h|_{U_i}$ *are regular for* $i = 1, ..., r$.

Proof. Write $h|_{U_i} = h_i/f_i^{m_i}$ with functions $h_i \in \mathcal{O}(X)$. Then we have $f_i^{m_i}h = h_i$ on U_i and $f_i^{m_i+1}h = f_ih_i$ even on all of X. Since the functions $f_i^{m_i+1}$ have no common zero, there are functions $g_i \in \mathcal{O}(X)$ with $\sum_{i=1}^r g_if_i^{m_i+1} = 1$. Then

$$h = (\sum_{i=1}^r g_if_i^{m_i+1})h = \sum_{i=1}^r g_if_ih_i \in \mathcal{O}(X).$$

□

Remark 4.24. 1. Let $\varphi : X \longrightarrow Y$ be a morphism of affine varieties. The inverse image of a principal open subset $V = Y_g$ is obviously again a principal open subset: $U := \varphi^{-1}(V) = X_{\varphi^*(g)}$.

2. Let $Y = \bigcup_{j=1}^s V_j$ be an open cover of an affine variety by principal open subset V_j. Then $\varphi : X \longrightarrow Y$ is a closed embedding resp. finite resp. an isomorphism iff all the maps $\varphi|_{U_j} : U_j := \varphi^{-1}(V_j) \longrightarrow V_j, j = 1, ..., s$, are.

   Proof. Exercise! □

5 Prevarieties

Affine varieties are special objects in the category $\mathcal{TA}$ of topological spaces with a distinguished algebra of regular functions. In order to define algebraic varieties we have to replace $\mathcal{TA}$ with the category of ringed spaces where one has not only a distinguished subalgebra $\mathcal{O}(X)$ on the entire space X, but for every open set $U \subset X$. That leads to the notion of a sheaf of functions:

Definition 5.1. Let X be a topological space.

1. A sheaf $\mathcal{F}$ of (continuous) k-valued functions on X consists of a choice of a subalgebra $\mathcal{F}(U) \subset \mathcal{C}(U)$ for every open subset $U \subset X$ such that (calling functions $\in \mathcal{F}(U)$ regular functions)

   (a) regular functions restrict to regular functions: $\mathcal{F}(U)|_V \subset \mathcal{F}(V)$, whenever $V \subset U$ are open sets, and

   (b) locally regular functions are regular: If $f \in \mathcal{C}(U)$ for $U = \bigcup_{i\in I} U_i$ and $f|_{U_i} \in \mathcal{F}(U_i)$ for all $i \in I$, then already $f \in \mathcal{F}(U)$.

2. A (k-)structure sheaf on X is a sheaf $\mathcal{F}$ of (continuous) k-valued functions on X, such that

$$\mathcal{F}(U)^* = \mathcal{C}(U)^* \cap \mathcal{F}(U)$$

   for all $U \subset X$, i.e. regular functions without zeros are invertible.

Example 5.2. 1. $\mathcal{F}(U) := \mathcal{C}(U)$ is a sheaf of functions, denoted simply $\mathcal{C}$.

2. $\mathcal{L}(U) := \{f : U \longrightarrow k \text{ locally constant}\}$ is called the sheaf of locally constant functions.

3. Let X be an affine variety. For principal open subsets $U \subset X$ we have already defined $\mathcal{O}(U)$. For arbitrary open $U \subset X$ we set then

   $$\mathcal{O}(U) := \{f : U \longrightarrow k; f|_V \in \mathcal{O}(V) \text{ for all principal open subsets } V \subset U\}$$

   and obtain a structure sheaf on X: Let $V \subset U = \bigcup_{i\in I} U_i$ be a principal open subset and $f : U \longrightarrow k$ with $f|_{U_i} \in \mathcal{O}(U_i)$. Since V is quasi-compact, we may refine $V = \bigcup_{i\in I} V \cap U_i$ by a finite covering $V = V_1 \cup ... \cup V_r$ with principal open subsets $V_j \subset V$. Then we have $f|_{V_j} \in \mathcal{O}(V_j)$ for $j = 1, ..., r$ and Prop.4.23 tells us that $f|_V \in \mathcal{O}(V)$. Since that holds for any principal open subset $V \subset U$, we get $f \in \mathcal{O}(U)$.

4. Take $k = \mathbb{C}$ and endow it with its strong topology. Then we define on $X = \mathbb{C}^n$ (with the strong topology as well) two sheaves of functions $\mathcal{H}$ and $\mathcal{E}$, namely

$$\mathcal{H}(U) := \{f : U \longrightarrow \mathbb{C} \text{ holomorphic}\}$$

and

$$\mathcal{E}(U) := \{f : U \longrightarrow \mathbb{C} \text{ differentiable}\},$$

where a differentiable function means a C^∞-function in the "real sense", i.e. if we use $\mathbb{C} \cong \mathbb{R}^2$ and $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Note that for Zariski open $U \subset \mathbb{C}^n$ we have inclusions

$$\mathcal{O}(U) \subsetneqq \mathcal{H}(U) \subsetneqq \mathcal{E}(U).$$

Remark 5.3. Let $U \subset X$ be an open subset of the affine variety $X = \mathrm{Sp}(A)$. Then

$$\mathcal{O}(U) \subset \mathcal{O}(X)_h|_U,$$

whenever $U \subset X_h$, i.e. $h \in \mathcal{O}(X)$ does not vanish on U, but in general a regular function $f \in \mathcal{O}(U)$ need not admit a representation $f = g/h$ on the entire set U. If $U = V_1 \cup ... \cup V_r$ with special open sets $V_i = X_{h_i}$, we can write $f = g_i/h_i$ on each V_i, such that for irreducible X, we may write

$$\mathcal{O}(U) = \bigcap_{i=1}^{r} \mathcal{O}(V_i) \subset Q(A).$$

Only for a UFD A, that can be simplified to

$$\mathcal{O}(U) = \bigcap_{i=1}^{r} A_{h_i} = A_h,$$

where $h = \gcd(h_1, .., h_r)$. In particular $U \subset X_h$.

Example 5.4. 1. With the above remark 5.3 we find for $U := (k^2)^* := k^2 \setminus \{0\}$ that $\mathcal{O}(U) = \mathcal{O}(k^2)|_U \cong k[T_1, T_2]$. Note that $(U, \mathcal{O}(U)) \in \mathcal{TA}$ is not an affine variety, since the maximal ideal $\mathfrak{m} := (T_1, T_2)$ is not of the form $\mathfrak{m}_a$ for some $a \in U$.

2. Let us consider the Segre cone $X := N(k^4; T_1T_4 - T_2T_3) \hookrightarrow k^4$. Take $X_i := X_{g_i}$ with $g_i := T_i|_X$. Consider now the open set

$$U := X \setminus \big(0 \times 0 \times k^2\big) = X_1 \cup X_2.$$

Since $g_1g_4 = g_2g_3$ we may define a regular function $f \in \mathcal{O}(U)$ by

$$f|_{X_1} := \frac{g_3}{g_1} \ , \ f|_{X_1} := \frac{g_4}{g_2},$$

but there is no representation $f = g/h$ with regular functions $g, h \in \mathcal{O}(X)$ with a denominator nowhere vanishing on U, or equivalently, there is no proper principal open subset $X_h \supset U$. Indeed $\mathcal{O}(U)^* = k^*$. For the proof consider the morphism $\varphi : X \longrightarrow k^2, (t_1, ..., t_4) \mapsto (t_1, t_2)$. It has fibers $\varphi^{-1}(t_1, t_2) = k(t_1, t_2) \cong k$ for $(t_1, t_2) \in k^2 \setminus \{0\}$ and $\varphi^{-1}(0, 0) = 0 \times 0 \times k^2 \cong k^2$, furthermore a section $\sigma : U \longrightarrow X$, namely $\sigma(t_1, t_2) := (t_1, t_2, 0, 0)$. We claim that $\varphi^* : \mathcal{O}(k^2 \setminus \{0\})^* \longrightarrow \mathcal{O}(U)^*$ is an isomorphism. A regular function on k without zero is a constant, hence f is constant along the fibers of φ and $f = \varphi^*(g)$ with $g = \sigma^*(f)$. Finally we know from the first point, that $\mathcal{O}(k^2 \setminus \{0\}) \cong k[T_1, T_2]$. Indeed $U \cong (k^2 \setminus \{0\}) \times k \subset k^3$ holds in $\mathcal{TA}$.

3. There are even examples of affine varieties X admitting open subsets $U \subset X$, such that $\mathcal{O}(U)$ is not even an affine algebra!

Here is the analogue of the category $\mathcal{TA}$:

Definition 5.5. A k-ringed space $(X, \mathcal{O} = \mathcal{O}_X)$ is a topological space together with a k-structure sheaf $\mathcal{O}$. A morphism between k-ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a continuous map $\varphi : X \longrightarrow Y$, such that

$$\varphi^*(\mathcal{O}_Y(V)) \subset \mathcal{O}_X(\varphi^{-1}(V))$$

holds for all open subsets $V \subset Y$. We denote $\mathcal{RS} = \mathcal{RS}_k$ the category of k-ringed spaces.

Proposition 5.6. *The category $\mathcal{AV}$ of affine varieties is equivalent to a full subcategory of $\mathcal{RS}_k$.*

Proof. Given an affine variety $X \in \mathcal{AV}$ define its structure sheaf $\mathcal{O}_X$ as in the above example 5.2.3. Now assume that $\varphi : X \longrightarrow Y$ is a morphism of affine varieties. We have to show that φ is also a morphism in $\mathcal{RS}_k$. For a principal open subset $V = Y_g$ we obviously have $U := \varphi^{-1}(V) = X_{\varphi^*(g)}$ and

$$\varphi^*(\mathcal{O}(V)) = \varphi^*(\mathcal{O}(Y)_g) \subset \mathcal{O}(X)_{\varphi^*(g)} = \mathcal{O}(U).$$

Here we have, for convenience of notation, written $\mathcal{O}$ in order to denote $\mathcal{O}_X$ and $\mathcal{O}_Y$, and φ^* denotes the pull back $\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(\varphi^{-1}(V))$ for any open $V \subset Y$.

In the general case write the open set $V \subset Y$ as a finite union $V = V_1 \cup ... \cup V_s$ of principal open subsets $V_j \subset Y$. Given $f \in \mathcal{O}(V)$, we see that $\varphi^*(f)$ is regular on $U_j := \varphi^{-1}(V_j)$ and hence, using condition b) of Def.5.1.1., regular on $U = \varphi^{-1}(V) = U_1 \cup ... \cup U_s$ as well.

Finally Th.4.12 gives that any morphism $X \longrightarrow Y$ in $\mathcal{AV}$ is induced by exactly one morphism $X \longrightarrow Y$ in $\mathcal{RS}_k$. $\square$

Remark 5.7.

1. Let X be a k-ringed space and Y be an affine variety. Then $\varphi \mapsto \varphi^*$ provides a bijection between the morphisms $X \longrightarrow Y$ and the algebra homomorphisms $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$.

2. Every open subset $U \subset X$ of a k-ringed space $(X, \mathcal{O}_X)$ is again a k-ringed space with the structure sheaf $\mathcal{O}_U$ satisfying $\mathcal{O}_U(V) = \mathcal{O}_X(V)$ for open $V \subset U$, making the inclusion $U \longrightarrow X$ a morphism.

3. An arbitrary subset $Y \hookrightarrow X$ of a k-ringed space $(X, \mathcal{O}_X)$ is again a k-ringed space as well: We want again that the inclusion $Y \hookrightarrow X$ becomes a morphism in $\mathcal{RS}_k$, but the definition of the structure sheaf $\mathcal{O}_Y$ is more complicated: Call a function $f : V \longrightarrow k$ on an open subset $V \subset Y$ *locally extendible to* X if any point $y \in V$ admits an open neighborhood $U \subset X$ with $U \cap Y \subset V$ and a function $g \in \mathcal{O}_X(U)$ extending $f|_{U \cap Y}$. Now set

$$\mathcal{O}_Y(V) := \{f : V \longrightarrow k \text{ locally extendible to } X\}.$$

Warning: If $U \subset X$ is open and $V := U \cap Y$, the restriction homomorphism $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(V)$ need not be surjective!

Definition 5.8. An algebraic prevariety is a k-ringed space X admitting an open cover $X = U_1 \cup ... \cup U_r$ by open subsets U_i which are affine varieties.

Example 5.9. 1. Open subsets $U \subset X$ of an affine variety X are prevarieties, U being a finite union of principal open subsets, but not necessarily affine (they are called instead "quasi-affine"): The prevariety $U := k^n \setminus \{0\} \subset X = k^n$ is not affine for $n > 1$.

2. A closed subspace $Y \hookrightarrow X$ of a prevariety X is again a prevariety. This is an immediate consequence of:

Lemma 5.10. *Let $X = \mathrm{Sp}(A)$ be an affine variety and $Y \hookrightarrow X$ be a closed subspace, endowed with the induced structure sheaf $\mathcal{O}_Y$. Then $Y \cong \mathrm{Sp}(B)$ with $B = \mathcal{O}(X)/I(Y)$.*

Proof. The restriction homomorphism $A = \mathcal{O}_X(X) \longrightarrow \mathcal{O}_Y(Y)$ factors through

$$\mathcal{O}_X(X) \longrightarrow B := \mathcal{O}_X(X)/I(Y) \longrightarrow \mathcal{O}_Y(Y)$$

respectively $j : Y \longrightarrow X$ as

$$Y \longrightarrow Z := \mathrm{Sp}(B) \longrightarrow X.$$

As topological subspaces of X we have $Y = Z$, and we have to show that, for all open subsets $V \subset Y = Z$, the inclusion $\mathcal{O}_Z(V) \subset \mathcal{O}_Y(V)$ is an equality. Because of condition b) in Def.5.1.1 we may assume that $V \subset Z$ is a special open set; and writing it as $V = Z \cap U$ with a special open subset $U \subset X$, we see that we may assume $U = X$ and $V = Z = Y$. Take a function $f \in \mathcal{O}_Y(Y)$. By definition, we find principal open subsets $U_1, ..., U_r \subset X$ covering $Y = Z$ and functions $g_i \in \mathcal{O}_X(U_i)$ with $g_i|_{V_i} = f|_{V_i}$, where $V_i := U_i \cap Z$. Hence $f|_{V_i} \in \mathcal{O}_Z(V_i)$ for $i = 1, ..., r$ resp. $f \in \mathcal{O}_Z(Z)$. □

Corollary 5.11. *For a morphism $\varphi : X \longrightarrow Y$ of prevarieties the following statements are equivalent:*

1. *It is a closed embedding, i.e. admits a factorization*

$$X \xrightarrow{\cong} Z \hookrightarrow Y,$$

where $Z \hookrightarrow Y$ is the inclusion of a closed subspace.

2. *For any affine open subspace $V \subset Y$ the inverse image $U := \varphi^{-1}(V) \subset X$ is affine and $\varphi^* : \mathcal{O}(V) \longrightarrow \mathcal{O}(U)$ surjective.*

3. *There is an open affine cover* $Y = \bigcup_{i=1}^{s} V_i$*, such that* $U_i := \varphi^{-1}(V_i) \subset X$ *is affine and* $\varphi^* : \mathcal{O}(V_i) \longrightarrow \mathcal{O}(U_i)$ *surjective for* $i = 1, ..., s$.

Let us now come back to algebraic geometry and present some important examples of prevarieties:

Quotient Constructions: A general idea how to produce new prevarieties X from a given one Z is to look at quotient spaces $X = Z/\sim$ of Z modulo an equivalence relation "$\sim$" on Z, or, with other words, to take any surjective map $\pi : Z \longrightarrow X$ from Z to some set X. Then X is endowed with its π-quotient topology:

$$U \subset X \text{ open } :\Longleftrightarrow \pi^{-1}(U) \subset Z \text{ open.}$$

The construction of the structure sheaf $\mathcal{O}_X$ uses the same idea:

$$\mathcal{O}_X(U) := \{f \in \mathcal{C}(U); \pi^*(f) \in \mathcal{O}_Z(\pi^{-1}(U))\}.$$

Unfortunately, there is in general no cover of X by open affine subspaces, so the resulting object is a priori just a k-ringed space. Nevertheless, sometimes one has good luck:

Projective n-space: Projective n-space is a prevariety

$$\mathbb{P}_n = \mathbb{P}_n(k) = k^n \cup \{\infty_L; L \subset k^n\},$$

which is obtained from k^n by adding to every line, i.e. one-dimensional subspace, $L \subset k^n$ a separate "point at infinity", denoted ∞_L. Indeed, we take

$$Z := (k^{n+1})^* := k^{n+1} \setminus \{0\}$$

and

$$\mathbb{P}_n = \mathbb{P}_n(k) := \left\{L \hookrightarrow k^{n+1} \text{ one dimensional subspace}\right\}$$

with the map

$$\pi : (k^{n+1})^* \longrightarrow \mathbb{P}_n(k), (z_0, ..., z_n) \mapsto [z_0, ..., z_n] := k(z_0, ..., z_n).$$

The numbers $z_0, ..., z_n$ are called the homogeneous coordinates of the point $[z_0, ..., z_n] \in \mathbb{P}_n$, they are unique up to a common nonzero multiple $\lambda \in k^*$. In order to see that the k-ringed space $\mathbb{P}_n$ is a prevariety we consider the open cover

$$\mathbb{P}_n = \bigcup_{i=0}^{n} U_i$$

with the open subspaces

$$U_i := \{[z]; z_i \neq 0\} \cong k^n,$$

the latter isomorphism being

$$U_i \longrightarrow k^n, [z] \mapsto \left(\frac{z_0}{z_i}, ..., \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, ..., \frac{z_n}{z_i}\right).$$

In particular we can identify

$$k^n \xrightarrow{\cong} U_0, z = (z_1, ..., z_n) \mapsto [1, z],$$

the point ∞_L for $L = kz$ being the point $[0.z]$. For $n = 1$ we thus get the "*projective line*"

$$\mathbb{P}_1 = k \cup \{\infty\}$$

with $\infty := [0, 1]$. For $k = \mathbb{C}$ the projective line $\mathbb{P}_1$ is (with respect to the strong topology) homeomorphic to the 2-sphere $\mathbb{S}^2$, the "Riemann sphere" of complex analysis. But note that $\mathbb{P}_n$ is not homeomorphic to $\mathbb{S}^{2n}$ for $n > 1$.

We finish this section with the remark that the quotient map $\pi : (k^{n+1})^* \longrightarrow \mathbb{P}_n$ belongs to a particularly interesting class of morphisms:

Definition 5.12. A morphism $\pi : E \longrightarrow X$ between prevarieties E and X is called a k^*-*principal bundle* if there is an open affine cover $X = \bigcup_{i=1}^r U_i$ together with isomorphisms ("trivializations")

$$\tau_i : \pi^{-1}(U_i) \longrightarrow U_i \times k^*$$

satisfying

1. $\mathrm{pr}_{k^*} \circ \tau_i = \pi$ with the projection $\mathrm{pr}_{k^*} : U_i \times k^* \longrightarrow k^*$, giving rise to

2. transitions
$$\tau_j \circ \tau_i^{-1} : U_{ij} \times k^* \longrightarrow U_{ij} \times k^*$$
looking as follows
$$(x, \lambda) \mapsto (x, f_{ij}(x)\lambda)$$
with (necessarily nowhere vanishing) functions $f_{ij} \in \mathcal{O}(U_{ij})$.

Example 5.13. The quotient map $\pi : (k^{n+1})^* \longrightarrow \mathbb{P}_n$ is a k^*-principal bundle: Consider the standard open cover $\mathbb{P}_n = \bigcup_{i=0}^n U_i$ with the open subspaces $U_i := \{[z]; z_i \neq 0\} \cong k^n$ and define

$$\tau_i : \pi^{-1}(U_i) \longrightarrow U_i \times k, (z_0, ..., z_n) \mapsto ([z_0, ..., z_n], z_i).$$

Determine the corresponding transition functions f_{ij}!

6 Algebraic Varieties

In "real" (i.e. non Zariski) topology reasonable spaces are Hausdorff, and also in algebraic geometry one needs a corresponding condition: Reasonable prevarieties, to be called later on simply (algebraic) varieties, should be "*separated*". But since even affine varieties, the local models for prevarieties, are very far from being Hausdorff, we can not use literally the same definition as in topology. First of all here is an example of the anomalies we want to avoid:

Example 6.1. Starting with the disjoint union

$$Z \ := \ N\left(k^2; T_2(T_2-1)\right) \ = \ k\times\{0\} \cup k\times\{1\} \ \cong \ \mathrm{Sp}\,(k[T] \oplus k[S])$$

of two lines we consider the ringed space

$$X \ := \ Z/\sim\ ,$$

where

$$(x,i) \sim (y,j) :\Longleftrightarrow x = y \in k^* \text{ or } i = j.$$

Then

$$X = V_0 \cup V_1 = V_{01} \cup \{o_0, o_1\}$$

with the open subsets $V_i := \pi(k \times \{i\}) \cong k$, so X is a prevariety, and $V_{01} = V_0 \cap V_1 \cong k^*$. So X is an affine line with the two different origins $o_i := \pi(0,i), i = 0,1$.

Now a topological space X is Hausdorff iff the diagonal

$$\Delta := \{(x.x); x \in X\} \subset X \times X$$

is a closed subspace of $X \times X$, the product of the topological space X with itself. So we could try to use that formulation: As we already have seen for affine varieties X, Y, the Zariski topology on $X \times Y$ is in general strictly finer than the product topology, and thus, the above condition is in the context of prevarieties much weaker than the usual Hausdorff property. But first of all we have to define the product $X \times Y$ for prevarieties X, Y.

Proposition 6.2. *In the category of prevarieties (i.e. the full subcategory of $\mathcal{RS}_k$ with the prevarieties as objects) there exists products.*

Proof. For affine X, Y we have already defined $X \times Y$. In the general case we have to endow the cartesian product $X \times Y$ with the structure of a prevariety. In order to do so, we take open affine coverings $X = \bigcup_{i=1}^{r} U_i$ and $Y = \bigcup_{\mu=1}^{s} V_\mu$ and "patch together" the affine varieties $U_i \times V_\mu$: Consider the prevariety Z defined as the disjoint union

$$Z := \bigcup_{i,\mu} Z_{i\mu},$$

of the affine varieties

$$Z_{i\mu} := U_i \times V_\mu.$$

Now endow $X \times Y$ with the π-quotient structure with respect to the natural map $\pi : Z \longrightarrow X \times Y$, which on $Z_{i\mu}$ is just the inclusion into $X \times Y$. If we can show that $\pi(Z_{i\mu}) \subset X \times Y$ is open and

$$\pi|_{Z_{i\mu}} : Z_{i\mu} \longrightarrow \pi(Z_{i\mu})$$

an isomorphism, we are done. To that end let

$$(Z_{i\mu})_{j\nu} := U_{ij} \times V_{\mu\nu} \subset Z_{i\mu},$$

considered as open (ringed) subspace of $Z_{i\mu}$, where $U_{ij} := U_i \cap U_j$ and $V_{\mu\nu} := V_\mu \cap V_\nu$. We have to check that the identity map

$$(Z_{i\mu})_{j\nu} \xrightarrow{\text{id}} (Z_{j\nu})_{i\mu}$$

is an isomorphism. That can be done locally: Take a point $(x, y) \in U_{ij} \times V_{\mu\nu}$. There are neighborhoods $U \subset U_{ij}$ of x and $V \subset V_{\mu\nu}$ of y, which are simultaneously principal open subsets of U_i and U_j resp. of V_μ and V_ν: Take for U the intersection of a principal open neighbourhood of x in U_i and a principal open neighbourhood of x in U_j (using 6.3.1), and for V apply the same recipe. Then the cartesian product $U \times V$ is an open subset of both $Z_{i\mu}$ and $Z_{j\nu}$ and the respective induced ringed space structures coincide with that one belonging to $U \times V$ as the product of the affine varieties U and V, as follows from Lemma 6.3, part 2.

Lemma 6.3. *1. Let X be an affine variety, $U = X_f$ and $V = U_g$ be principal open subsets of X resp. U. Then V is a principal open subset of X.*

2. Let X, Y be affine varieties, $f \in \mathcal{O}(X), g \in \mathcal{O}(Y)$. Then

$$(X \times Y)_{f \otimes g} \cong X_f \times Y_g.$$

Proof. Do the first part yourself! For the second part, note that both the RHS and the LHS are affine varieties with the same underlying set, hence it suffices to check that the regular function algebras agree. □

To finish the proof of Prop 6.2 we have to check that the prevariety $X \times Y$ is a product of X and Y in the category $\mathcal{RS}_k$. That follows from the fact, that given morphisms $\varphi : Z \longrightarrow X$ and $\psi : Z \longrightarrow Y$, the map $(\varphi, \psi) : Z \longrightarrow X \times Y$ is a morphism as well, since the restrictions $(\varphi, \psi)|_{(\varphi,\psi)^{-1}(U_i \times V_\mu)} : (\varphi, \psi)^{-1}(U_i \times V_\mu) \longrightarrow U_i \times V_\mu$ are. □

So, eventually, we can define algebraic varieties:

Definition 6.4. A prevariety X is called *separated* or an *algebraic variety (over k)* if the diagonal $\Delta \subset X \times X$ is closed in $X \times X$.

Remark 6.5.

1. Affine n-space k^n is separated, since
   $\Delta = N(k^n \times k^n; S_1 - T_1, ..., S_n - T_n)$.
2. If $\varphi : X \longrightarrow Y$ is an injective morphism from a prevariety X into an algebraic variety Y, then X is an algebraic variety as well: We have $\Delta_X = (\varphi \times \varphi)^{-1}(\Delta_Y)$.
3. Open and closed subspaces of an algebraic variety are separated.
4. Affine and more generally quasi-affine varieties are separated.
5. In the general case of a prevariety X with an affine open cover $X = \bigcup_{i=1}^r U_i$ we thus get that

   $$\Delta \hookrightarrow \bigcup_{i=1}^{r} U_i \times U_i$$

   is a closed subspace of the open set

$\bigcup_{i=1}^r U_i \times U_i \subset X \times X$ and that the diagonal map

$$\delta : X \longrightarrow \Delta, x \mapsto (x, x),$$

is an isomorphism of prevarieties. Hence X is separated iff $\Delta_{ij} := \Delta \cap (U_i \times U_j)$ is a closed subset of $U_i \times U_j$ for $i \neq j$. That leads to the following reformulation of the "diagonal criterion":

Proposition 6.6. *Assume $X = \bigcup_{i=1}^r U_i$ is an open affine cover of the prevariety X. Then X is separated iff the intersections $U_{ij} := U_i \cap U_j \subset X$ are affine for all i, j and*

$$\mathcal{O}(U_i) \otimes \mathcal{O}(U_j) \longrightarrow \mathcal{O}(U_{ij}), f \otimes g \mapsto f|_{U_{ij}} \cdot g|_{U_{ij}},$$

is surjective for all $1 \leq i, j \leq r$.

Proof. Let us first remark that the above ring homomorphism is nothing but the pull back morphism

$$\delta^* : \mathcal{O}(U_i \times U_j) \longrightarrow \mathcal{O}(U_{ij})$$

belonging to the diagonal map $\delta|_{U_{ij}} : U_{ij} \longrightarrow U_i \times U_j$.
"$\Longrightarrow$": If $\Delta \subset X$ is closed, then $U_{ij} \cong \Delta \cap (U_i \times U_j) \hookrightarrow U_i \times U_j$ is affine and the pullback of the inclusion surjective.
"$\Longleftarrow$": Since the intersection U_{ij} is affine, the given condition means that the morphisms

$$\delta|_{U_{ij}} : U_{ij} = \delta^{-1}(U_i \times U_j) \longrightarrow U_i \times U_j$$

are closed embeddings. Since the $U_i \times U_j$ cover $X \times X$, the morphism $\delta : X \longrightarrow X \times X$ is itself a closed embedding. $\square$

Proposition 6.7. *Projective n-space $\mathbb{P}_n$ is an algebraic variety.*

Proof. Denote $\mathbb{P}_n = \bigcup_{i=0}^n U_i$ the standard open cover of $\mathbb{P}_n$. The intersection

$$U_{ij} \cong k^i \times k^* \times k^{j-i-1} \times k^* \times k^{n-j}$$

is affine - we assume $j > i$. We treat the injections

$$\mathcal{O}(U) \xrightarrow{\pi^*} \mathcal{O}(\pi^{-1}(U))$$

as inclusions and thus

$$\mathcal{O}(U) = \mathcal{O}(\pi^{-1}(U))_0 := \{f \in \mathcal{O}(\pi^{-1}(U)); f(\lambda t) = f(t), \forall \lambda \in k^*\}.$$

In particular we have

$$\mathcal{O}(U_i) = k\left[\frac{T_0}{T_i}, ..., \frac{T_n}{T_i}\right],$$

while

$$\mathcal{O}(U_{ij}) = (k[T]_{T_iT_j})_0 = k\left[\frac{T_\mu T_\nu}{T_iT_j}; 0 \le \mu, \nu \le n\right].$$

Hence the homomorphism

$$k\left[\frac{T_0}{T_i}, ..., \frac{T_n}{T_i}\right] \otimes k\left[\frac{T_0}{T_j}, ..., \frac{T_n}{T_j}\right] \longrightarrow k\left[\frac{T_\mu T_\nu}{T_iT_j}; 0 \le \mu, \nu \le n\right],$$

$$\frac{T_\mu}{T_i} \otimes \frac{T_\nu}{T_j} \mapsto \frac{T_\mu T_\nu}{T_iT_j}$$

is surjective. □

6.1 Digression: Complex Analytic Spaces

This section is a (non-obligatory) supplement about the relationship between complex algebraic and complex analytic geometry. No proofs are presented.

We consider on $\mathbb{C}^n$ (endowed with the strong topology) the structure sheaf

$$\mathcal{H} : U \mapsto \mathcal{H}(U) := \{f : U \longrightarrow \mathbb{C} \text{ holomorphic}\}$$

of holomorphic functions. Following the recipe of Rem. 5.7 we then may consider any subset of $\mathbb{C}^n$ as a ringed space. Actually we only need open subsets $W \subset \mathbb{C}^n$ and *analytic subsets* $Z \hookrightarrow W$ of an open set $W \subset \mathbb{C}^n$, i.e. (relatively) closed subsets, such that for any point $a \in Z$ there is an open neighbourhood $U \subset W$ together with finitely many holomorphic functions $f_1, ..., f_r \in \mathcal{H}(U)$ such that

$$Z \cap U := N(U; f_1, ..., f_r) := \{z \in U; f_1(z) = ... = f_r(z) = 0\}.$$

In particular, for a (relatively) open subset $V \subset Z$ the algebra $\mathcal{H}(V)$ consists of all functions $V \longrightarrow \mathbb{C}$, which locally can be extended to a holomorphic function on some open subset of $\mathbb{C}^n$.

Definition 6.8. 1. A *complex analytic space* (or *complex analytic variety*) X is a $\mathbb{C}$-ringed Hausdorff space admitting an open cover $X = \bigcup_{i \in I} U_i$ with open subspaces $U_i \cong Z_i \hookrightarrow W_i$, where Z_i is an analytic subset of the open subset $W_i \subset \mathbb{C}^{n_i}$.

2. A complex n-manifold is a complex analytic space $X = \bigcup_{i \in I} U_i$ with open subspaces $U_i \cong W_i$, the W_i being open subspaces $W_i \subset \mathbb{C}^n$.

3. A Riemann surface is a connected complex 1-manifold.

Remark 6.9. 1. An affine variety $X \hookrightarrow \mathbb{C}^n$ is an analytic subset of $\mathbb{C}^n$, hence a complex analytic space, denoted X_h. So as a set $X_h = X$, while the topology is "strengthened" and regular functions are replaced with holomorphic functions.

2. To a complex algebraic variety X we can, using an open affine cover $X = \bigcup_{i=1}^{r} U_i$, associate a complex analytic variety X_h with $X_h = X$ as sets. (Here separatedness of X gives the Hausdorff property for X_h). We thus obtain a functor $X \mapsto X_h$ from the category of complex algebraic varieties to the category of complex analytic spaces.

3. A complex algebraic variety X is connected iff X_h is.

4. As a consequence of the (holomorphic) inverse function theorem an algebraic subset
$$X := N(\mathbb{C}^n; f_1, ..., f_r)$$
with polynomials $f_1, ..., f_r \in \mathbb{C}[T]$ provides a complex $(n-r)$-manifold X_h if
$$\operatorname{rank}\left(\frac{\partial f}{\partial T}(a)\right) = r$$
holds with the Jacobi matrix
$$\frac{\partial f}{\partial T}(a) := \left(\frac{\partial f_i}{\partial T_j}(a)\right) \in \mathbb{C}^{r,n}$$
for all points $a \in X$. In particular the isomorphisms $\varphi : U_i \longrightarrow W_i$, the "local coordinates" or "local charts" in classical terminology, can be chosen always as restrictions of linear functions, but the U_i are in general not Zariski open nor are the inverse local coordinates polynomials.

A complex manifold is a complex analytic space. Here are some remarks how far the latter are from the former:

Definition 6.10. A point $a \in X$ of a complex analytic space is called a *singular point* (or a *singularity*) if it does not admit an open neighbourhood isomorphic to an open subset in some $\mathbb{C}^n$. A complex analytic space without singular points is called *nonsingular* or *smooth*; its connected components form complex manifolds.

Remark 6.11. The *singular locus* $S(X)$ of a complex analytic space X is a nowhere dense closed subset of X, locally given as the zero locus (:= set of zeros) of finitely many holomorphic functions. Its complement $X \setminus S(X)$ is called the *regular locus* of X.

Example 6.12. For the Neil parabola $X := N(\mathbb{C}^2; T_2^2 - T_1^3)$ and the noose $Y := N(\mathbb{C}^2; T_2^2 - T_1^2(T_1 + 1))$ the only singular point of X_h resp. Y_h is the origin. The affine curves $Z := N(\mathbb{C}^2; T_2^2 - p(T_1))$ induces a Riemann surface Z_h.

Remark 6.13.

1. The singular locus $S(X_h)$ of a complex algebraic variety X is even a Zariski closed subset and can be defined without the passage to complex analytic spaces. Indeed, the definition works for any field k, but it is not literally the analogue of the definition for complex analytic spaces.

2. For a connected complex affine variety X the theorem of Liouville holds: A bounded holomorphic function on X_h is constant. In particular a bounded open set $W \subset \mathbb{C}^n$ is never isomorphic to the complex analytic space X_h associated to a complex algebraic variety X.

7 Projective Varieties

Projective space $\mathbb{P}_n$ plays a rôle similar to that of affine space k^n as an ambient space for interesting algebraic varieties:

Definition 7.1. A *projective* variety is an algebraic variety isomorphic to a closed subspace $X \hookrightarrow \mathbb{P}_n$.

Remark 7.2. There are two ways to relate affine varieties to projective varieties:

1. To every embedded affine variety
$$Y \hookrightarrow k^n \cong U_0 \subset \mathbb{P}_n$$
we can associate its *projective closure*
$$\overline{Y} \hookrightarrow \mathbb{P}_n = U_0 \cup \mathbb{P}(0 \times k^n).$$

2. The *affine cone* over a projective variety $X \hookrightarrow \mathbb{P}_n$ is the affine variety
$$C(X) := \overline{\pi^{-1}(X)} = \pi^{-1}(X) \cup \{0\} \subset k^{n+1},$$
where $\pi : (k^{n+1})^* \longrightarrow \mathbb{P}_n$ denotes the quotient morphism. A warning: The affine cone $C(X)$ does not only depend on X as algebraic variety, but as well on the chosen embedding $X \hookrightarrow \mathbb{P}_n$!

3. The above two cases are related as follows
$$C(\overline{Y}) = \overline{k^* \cdot (\{1\} \times Y)},$$
where $k^* \cdot A := \{\lambda x; \lambda \in k^*, x \in A\}$ for a subset $A \subset k^{n+1}$.

The abstract definition of an affine cone is as follows:

Definition 7.3. An affine cone $Z \subset k^n$ (or for short an affine n-cone) is a nonempty Zariski closed subset, such that $\lambda Z \subset Z$ for all $\lambda \in k$. It is called nontrivial, if it contains points $\neq 0$.

Remark 7.4.

1. A nontrivial affine cone is a (finite or infinite) union of lines through the origin.

2. There is a bijection between the non-empty closed subsets $X \hookrightarrow \mathbb{P}_n$ and the nontrivial affine $(n+1)$-cones:
$$X \mapsto C(X)$$
with the inverse
$$k^{n+1} \supset Z \ \mapsto \ \pi(Z^*) \subset \mathbb{P}_n$$
with $Z^* := Z \setminus \{0\}$.

The counterparts to affine n-cones on the level of ideals are the homogeneous ideals:

Definition 7.5. Denote $k[T]_q := \bigoplus_{|\alpha|=q} k \cdot T^\alpha$ the vector subspace of $k[T]$ of all homogeneous polynomials of degree $q \in \mathbb{N}$. An ideal $\mathfrak{a} \hookrightarrow k[T]$ is called homogeneous if one of the two following equivalent conditions is satisfied:

1. The ideal $\mathfrak{a}$ is generated by homogeneous polynomials.

2. The ideal $\mathfrak{a}$ is the direct sum

$$\mathfrak{a} = \bigoplus_{q=0}^{\infty} \mathfrak{a}_q$$

   of its homogeneous subspaces $\mathfrak{a}_q := \mathfrak{a} \cap k[T]_q$.

The second condition means that with a polynomial $g = \sum_q g^{(q)} \in \mathfrak{a}$ also its homogeneous terms $g^{(q)}$ belong to $\mathfrak{a}$. Hence we may replace a system $f_1, ..., f_r$ of generators with the $f_i^{(q)}, i = 1, ..., r, q \in \mathbb{N}$. On the other hand, if $\mathfrak{a} = (f_1, ..., f_r)$ with polynomials $f_i \in k[T]_{q_i}$, then any $g = h_1 f_1 + ... + h_r f_r \in \mathfrak{a}$ satisfies

$$g^{(q)} = \sum_{i=1}^{r} (h_i)^{(q-q_i)} f_i \in \mathfrak{a}_q.$$

Proposition 7.6. *1. The zero set $N(\mathfrak{a}) \hookrightarrow k^n$ of a homogeneous ideal is an affine n-cone.*

2. An algebraic set $Z \hookrightarrow k^n$ is an affine cone iff its vanishing ideal $I(Z)$ is a homogeneous ideal.

Proof. The first part as well as the implication "$\Longleftarrow$" of the second part follow from the fact that $\mathfrak{a}$ can be generated by homogeneous polynomials. Finally for an affine cone Z the assumption $f = \sum_q f^{(q)} \in I(Z)$ implies $f_\lambda = \sum_q \lambda^q f^{(q)} \in I(Z)$ for all $\lambda \in k$, where $f_\lambda(x) := f(\lambda x)$. Evaluating that equality at any point $x \in Z$ gives $f^{(q)}(x) = 0$ for all $q \in \mathbb{N}$ resp. $f^{(q)} \in I(Z)$. $\square$

Corollary 7.7. *Any projective variety $X \hookrightarrow \mathbb{P}_n$ can be described as the set of zeros*

$$X = N(\mathbb{P}_n; f_1, ..., f_r) := \{[t]; f_1(t) = ... = f_r(t) = 0\},$$

of finitely many homogeneous polynomials $f_1, ..., f_r \in k[T_0, ..., T_n]$.

Proof. Apply Prop. 7.6 to the affine cone $C(X) \hookrightarrow k^{n+1}$. □

Note that homogeneous polynomials are functions only on k^{n+1} and not on $\mathbb{P}_n$, but since such a polynomial either vanishes identically on a punctured line $k^* \cdot t$ ($t \neq 0$) or has no zeros there, the above description of a projective variety makes sense nevertheless.

Remark 7.8. If $Y = N(\mathfrak{a}) \hookrightarrow k^n \cong U_0$, then its projective closure can be described as follows:

For a polynomial $f \in k[T_1, ..., T_n] \backslash \{0\}$ its homogeneization $\widehat{f} \in k[T_0, ..., T_n]$ is the polynomial

$$\widehat{f} := T_0^{\deg(f)} f(T_0^{-1}T_1, ..., T_0^{-1}T_n) \in k[T_0, ..., T_n].$$

Then the ideal

$$\widehat{\mathfrak{a}} := \sum_{f \in \mathfrak{a}} k[T_0, ..., T_n] \cdot \widehat{f} \quad \hookrightarrow \quad k[T_0, ..., T_n]$$

satisfies

$$N(\widehat{\mathfrak{a}}) = C(\overline{Y}).$$

Proof. Since $\mathbb{P}_n$ carries the quotient topology w.r.t. $\pi : k^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}_n$, we have

$$C(\overline{Y}) = \overline{k^* \cdot (\{1\} \times Y)}.$$

Consider the projection

$$k^* \times k^n \longrightarrow k^n, (t_0, ..., t_n) \mapsto \left(\frac{t_1}{t_0}, ..., \frac{t_n}{t_0}\right).$$

The functions $f(\frac{t_1}{t_0}, ..., \frac{t_n}{t_0}), f \in \mathfrak{a}$, then have

$$k^* \cdot (\{1\} \times Y)$$

as their common set of zeros in $k^* \times k^n$, and the same holds for the $\hat{f}, f \in \mathfrak{a}$. Since $\hat{f} \in k[T_0, ..., T_n]$, it follows

$$\overline{k^* \cdot (\{1\} \times Y)} \subset N(\widehat{\mathfrak{a}}).$$

For the reverse inclusion we show

$$I(\overline{k^* \cdot (\{1\} \times Y)}) \subset I(N(\widehat{\mathfrak{a}}))$$

Take any $g \in k[T_0, ..., T_n]$ vanishing on $\overline{k^* \cdot (\{1\} \times Y)}$. Since both ideals are homogeneous, we may assume, that g is homogeneous (of degree q) and not divisible by T_0 - a factor T_0 does not contribute to the zeros on $k^* \times k^n$. Then $g_1 := g(1, T_1, ..., T_n)$ vanishes on Y, so $(g_1)^\ell \in \mathfrak{a}$ for some $\ell > 0$. But then $g = \widehat{g}_1$ resp. $g \in I(N(\widehat{\mathfrak{a}}))$. $\square$

Example 7.9. If $\mathfrak{a} = (f)$ is a principal ideal, then $\widehat{\mathfrak{a}} = (\widehat{f})$, because of the multiplicativity $\widehat{gf} = \widehat{f} \cdot \widehat{g}$ of the homogeneization. But it is not linear, and thus a generator system $f_1, ..., f_r$ for $\mathfrak{a}$ does not necessarily provide a generator system $\widehat{f_1}, ..., \widehat{f_r}$ for $\widehat{\mathfrak{a}}$.

Given an algebraic variety Y, an "*extension*" (no standard terminology!) of Y is an algebraic variety X together with an isomorphism $Y \cong U \subset X$, where $U \subset X$ is a dense open subset. For an affine variety

$$Y \hookrightarrow k^n \cong U_0 \subset \mathbb{P}_n.$$

such an extension is given by its projective closure:

$$X := \overline{Y} \hookrightarrow \mathbb{P}_n = U_0 \cup \mathbb{P}(0 \times k^n).$$

Indeed the projective closure yields a *maximal* extension, since, as we shall see in a moment, projective varieties do not admit any further proper extensions: They are "*complete*".

Definition 7.10. An algebraic variety X is called *complete* if it satisfies the following property: For any algebraic variety Z the projection

$\mathrm{pr}_Z : X \times Z \longrightarrow Z$ is a closed map, i.e. maps closed sets to closed sets.

Remark 7.11. 1. In order to check completeness of an algebraic variety X, it obviously suffices to considers affine "test varieties" Z.

2. A closed subvariety $Y \hookrightarrow X$ of a complete variety is complete, the induced map $Y \times Z \longrightarrow X \times Z$ being a closed embedding as well.

3. The image $\varphi(X) \subset Y$ of a morphism $\varphi : X \longrightarrow Y$ with a complete source X is closed in Y and complete: The graph $\Gamma_\varphi \subset X \times Y$ of any morphism $\varphi : X \longrightarrow Y$ is the inverse image

$$\Gamma_\varphi = (\mathrm{id}_X \times \varphi)^{-1}(\Delta)$$

of the diagonal $\Delta \hookrightarrow Y \times Y$ with respect to the morphism $\mathrm{id}_X \times \varphi : X \times Y \longrightarrow Y \times Y$, hence closed, and $\varphi(X) = \mathrm{pr}_Y(\Gamma_\varphi)$. Finally, if $C \hookrightarrow \varphi(X) \times Z$, consider the inverse image $(\varphi \times \mathrm{id}_Z)^{-1}(C) \hookrightarrow X \times Z$ and use the completeness of X.

4. A complete algebraic variety X has no nontrivial extensions: The image U of an extension homomorphism $\varphi : X \xrightarrow{\cong} U \subset Y$ is closed and dense, hence $U = Y$.

5. A regular function on a connected complete variety X is constant, i.e. $\mathcal{O}(X) = k$: A regular function can be regarded as a morphism $f : X \longrightarrow \mathbb{P}_1 = k \cup \{\infty\}$ avoiding the value ∞. Thus the closed set $f(X)$ is finite.

6. An affine variety X is complete iff it is finite.

7. A complex algebraic variety X is complete iff X_h is compact.

Theorem 7.12. *Projective varieties are complete.*

Proof. According to the above Remark 7.11.2 it suffices to show that $\mathbb{P}_n$ is complete. So let $Z := \mathrm{Sp}(C)$ be a w.l.o.g. affine variety and $Y \hookrightarrow \mathbb{P}_n \times Z$ a closed set. With $B := \mathrm{pr}_Z(Y)$ we have the commutative diagram

$$\begin{array}{ccc} Y & \hookrightarrow & \mathbb{P}_n \times Z \\ \downarrow & & \downarrow \\ B & \overset{?}{\hookrightarrow} & Z \end{array}$$

and want to see, why we are allowed to remove the question mark. For $z \in Z$ we consider the sectional variety

$$Y_z := \{y \in \mathbb{P}_n; (y, z) \in Y\},$$

such that

$$B = \{z \in Z; Y_z \neq \emptyset\}.$$

Denote

$$C(Y) \hookrightarrow k^{n+1} \times Z$$

the closure of

$$(\pi \times \mathrm{id}_Z)^{-1}(Y) \hookrightarrow (k^{n+1} \setminus \{0\}) \times Z,$$

where $\pi \times \mathrm{id}_Z : (k^{n+1} \setminus \{0\}) \times Z \longrightarrow \mathbb{P}_n \times Z$. Indeed

$$(\pi \times \mathrm{id}_Z)^{-1}(Y) = \bigcup_{z \in B} (C(Y_z)^* \times \{z\})$$

and

$$C(Y) = \left(\bigcup_{z \in B} C(Y_z) \times \{z\} \right) \cup (\{0\} \times \overline{B}).$$

We want to show $\overline{B} = B$, i.e. the second term is not really needed. The below lemma assures that $V = Z \setminus B$ is open:

Lemma 7.13. *Let $Z = \mathrm{Sp}(C)$ be an affine variety and $X \hookrightarrow k^{n+1} \times Z$ be a relative cone, i.e.*

$$X_z := \{t \in k^{n+1}; (t, z) \times X\} \hookrightarrow k^{n+1}$$

is a (possibly empty) affine cone. Then the set

$$V := \{z \in Z; X_z \subset \{0\}\}$$

is open.

Proof. We consider the vanishing ideal

$$\mathfrak{a} := I(X) \hookrightarrow C[T] := C[T_0, ..., T_n],$$

a graded ideal

$$\mathfrak{a} = (f_1, ..., f_r)$$

generated by homogeneous polynomials $f_i \in C[T]$ of degree d_i as well as its "sectional ideals"

$$\mathfrak{a}_z := \{f(z,T); f \in \mathfrak{a}\} = (f_1(z,T), ..., f_r(z,T)) \hookrightarrow k[T],\ z \in Z.$$

Then

$$z \in V \Longleftrightarrow N(\mathfrak{a}_z) \subset \{0\}$$

and furthermore

$$N(\mathfrak{a}_z) \subset \{0\} \Longleftrightarrow \mathfrak{a}_z \supset \mathfrak{m}^\ell \text{ for some } \ell \in \mathbb{N}$$

with the ideal

$$\mathfrak{m} := (T_0, ..., T_n) \hookrightarrow k[T_0, ..., T_n].$$

The implication "$\Longleftarrow$" is clear, while for the opposite direction we first note that Hilberts Nullenstellensatz provides an exponent $s \in \mathbb{N}$ with $T_i^s \in \mathfrak{a}_z$ for $i = 0, ..., n$. Now take $\ell := (n+1)s$: The ideal $\mathfrak{m}^\ell$ is generated by the monomials T^α with $|\alpha| = \ell$, and any such monomial divisible by some T_i^s. As a consequence we obtain that

$$V = \bigcup_{\ell=1}^{\infty} V_\ell$$

is the ascending union of the sets

$$V_\ell := \left\{z \in Z; \mathfrak{a}_z \supset \mathfrak{m}^\ell\right\}.$$

Thus we are done if we can show that $V_\ell \subset Z$ is open for any $\ell \in \mathbb{N}$. In order to see that we consider the linear maps

$$F_z \ :\ \ E := \bigoplus_{i=1}^{r} k[T]_{\ell-d_i} \longrightarrow L := k[T]_\ell$$

$$(g_1, ..., g_r) \mapsto g_1 f_1(z,T) + ... + g_r f_r(z,T).$$

Since the homogeneous ideal $\mathfrak{m}^\ell$ is generated by $k[T]_\ell$, the inclusion $\mathfrak{a}_z \supset \mathfrak{m}^\ell$ is equivalent to the surjectivity of the linear map $F_z : E \longrightarrow L$:

$$V_\ell = \{z \in Z; F_z : E \longrightarrow L \text{ is onto }\}.$$

Finally, the map $Z \longrightarrow \mathrm{Hom}(E,L), z \mapsto F_z$, is continuous ($\mathrm{Hom}(E,L)$ is a k-vector space and the subset consisting of the surjective maps open, namely using matrix representations, described by the nonvanishing of at least one of the minors of F_z of size $\dim L$). Thus $V_\ell \subset Z$ is open, as desired. $\square$

$\square$

8 Dimension

A first basic invariant of an algebraic variety X is its dimension:

Definition 8.1. The dimension of an algebraic variety X at a point $a \in X$ is defined as

$$\dim_a X := \max\{n \in \mathbb{N}; \exists\ X_0 = \{a\} \subsetneqq X_1 \subsetneqq ... \subsetneqq X_n, X_i \hookrightarrow X \text{ irreducible}\},$$

while

$$\dim X := \max_{a \in X} \dim_a X.$$

Remark 8.2.

1. Since a maximal strictly increasing chain of irreducible subvarieties starts with a point, we have

$$\dim X = \max\{n \in \mathbb{N}; \exists\ X_0 \subsetneqq X_1 \subsetneqq ... \subsetneqq X_n, X_i \hookrightarrow X \text{ irreducible}\}.$$

2. If $a \in U \subset X$ with an open subset $U \subset X$, then $\dim_a U = \dim_a X$. This follows from the fact that $Y \mapsto Z := \overline{Y}$ defines a bijection between the irreducible subvarieties $Y \hookrightarrow U$ containing a and the irreducible subvarieties $Z \hookrightarrow X$ containing a, the inverse map being given by $Z \mapsto Z \cap U$.

3. $\dim X = 0$ iff $|X| < \infty$.

4. If $X_1, ..., X_r \hookrightarrow X$ are the irreducible components of X containing a, then

$$\dim_a X = \max_{i=1,...,r} \dim_a X_i.$$

5. If $X_1, ..., X_r \hookrightarrow X$ are the irreducible components of X, then

$$\dim X = \max_{i=1,...,r} \dim X_i.$$

   We call X purely n-dimensional, if $n = \dim X_i$ for all irreducible components $X_i \hookrightarrow X$.

6. A pure 1-dimensional algebraic variety is called a *curve*.

7. A pure 2-dimensional algebraic variety is called a *surface*.

8. An irreducible variety is a curve iff the proper closed subsets are the finite sets.

9. $\dim k = 1$ and $\dim_a k^n \geq n$ for all $a \in k^n$: Take $X_i := a + (k^i \times 0)$ with the origin $0 \in k^{n-i}$.

In order to determine the dimensions of varieties the following comparison result is a standard tool:

Proposition 8.3. *Let $\varphi : X \longrightarrow Y$ be a finite surjective morphism of affine varieties, $b \in Y$. Then we have*

$$\dim_b Y = \max_{a \in \varphi^{-1}(b)} \dim_a X,$$

and in particular

$$\dim X = \dim Y.$$

Proof. First note that for any dominant morphism $\varphi : X \longrightarrow Y$ and closed $Z \hookrightarrow X$ we have

$$I(\overline{\varphi(Z)})) = I(Z) \cap \mathcal{O}(Y),$$

where we treat $\varphi^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ as inclusion.
"$\dim_b Y \geq \max_{a \in \varphi^{-1}(b)} \dim_a X$": Consider a strictly increasing sequence of irreducible subvarieties $X_0 = \{a\} \subsetneqq X_1 \subsetneqq ... \subsetneqq X_n$ on X. Then the (closed) images $Y_i := \varphi(X_i)$ are as well irreducible closed subvarieties of Y and $Y_i \subsetneqq Y_{i+1}$: Otherwise we would have

$$I(X_i) \cap \mathcal{O}(Y) = I(Y_i) = I(Y_{i+1}) = I(X_{i+1}) \cap \mathcal{O}(Y),$$

a contradiction to Th.3.14.3, since $\mathcal{O}(Y) \subset \mathcal{O}(X)$ is an integral extension.
"$\dim_b Y \leq \max_{a \in \varphi^{-1}(b)} \dim_a X$": Given any irreducible closed subvariety $Z \hookrightarrow Y$, there is an irreducible component of $\varphi^{-1}(Z)$ projecting onto Z, i.e. the restriction of φ to that component has image Z. Now consider a strictly increasing sequence of irreducible subvarieties $Y_0 = \{b\} \subsetneqq Y_1 \subsetneqq ... \subsetneqq Y_n$ on Y. By downward induction we construct a strictly increasing sequence $X_0 \subsetneqq X_1 \subsetneqq ... \subsetneqq X_n$ on X lying above the Y_i. Take X_n as an irreducible component of $\varphi^{-1}(Y_n)$ projecting onto Y_n. If X_i is found, take $X_{i-1} \hookrightarrow \varphi^{-1}(Y_i) \cap X_i$ as an irreducible component projecting onto Y_{i-1}. In particular $X_0 = \{a\}$ with some $a \in \varphi^{-1}(b)$. □

Corollary 8.4. *We have $\dim k^n = n = \dim_a k^n$, where $a \in k^n$, and*

$\dim X < \infty$ *for any affine variety* X.

Proof. The second part of the statement follows from Prop. 8.3 together with the Noether normalization theorem. The first part is obtained by induction, the case $n = 1$ being obvious. Assume now $X_0 \subsetneqq X_1 \subsetneqq ... \subsetneqq X_m = k^n$ is a strictly increasing sequence of irreducible subvarieties in $k^n, n > 1$. If we can show $\dim X_{m-1} < n$, we have necessarily $m - 1 \leq n - 1$ resp. $m \leq n$: Since $X_{m-1} \neq k^n$, there is a polynomial $f \in I(X_{m-1}) \setminus \{0\}$. As a consequence of the lemmata 3.12 and 8.3, we see that

$$\dim X_{m-1} \leq \dim N(k^n; f) = n - 1.$$

□

Remark 8.5. 1. If $U \subset X$ is a non-empty open subvariety of the irreducible variety X, then we have $\dim U = \dim X$. Write $X = \bigcup_{i=1}^r V_i$ with affine open sets $V_i \subset X$. Since $\dim X = \max_{i=1,...,r} \dim V_i$ and $\dim U = \max_{i=1,...,r} \dim U \cap V_i$, we may assume that X is affine. For $X = k^n$ the claim is obvious because of $\dim_a k^n = n$ for all $a \in k^n$. For arbitrary affine X consider again a finite surjective map $\varphi : X \longrightarrow k^d$. Take $V := k^d \setminus \varphi(X \setminus U)$ - a non-empty open subset, since $\dim \varphi(X \setminus U) = \dim(X \setminus U) < \dim X = n$. But $\varphi^{-1}(V) \longrightarrow V$ being finite and surjective, the inclusion $U \supset \varphi^{-1}(V)$ implies $\dim U \geq \dim \varphi^{-1}(V) = \dim V = n$.

2. Let $E \supset K$ be a field extension. The transcendence degree $\mathrm{trdegr}_K E \in \mathbb{N} \cup \{\infty\}$ is defined as the maximal number n of over K algebraically independent elements $x_1, ..., x_n \in E$, i.e. such that there is an injective K-algebra homomorphism $K[T_1, ..., T_n] \hookrightarrow E$, and if such a maximal number does not exist, then one sets $\mathrm{trdegr}_K E = \infty$. As a consequence of the geometric Noether normalization theorem 4.15 we see that for an irreducible affine variety $X = \mathrm{Sp}(A)$, we have $\dim X = \mathrm{trdegr}_k Q(A)$.

A minimal irreducible subvariety of a variety X is a point. But what can we say about maximal proper irreducible subvarieties $Z \hookrightarrow X$ of an irreducible variety X, i.e. such that $\dim Z = \dim X - 1$? If X is affine, there is a function $f \in \mathcal{O}(X) \setminus \{0\}$ vanishing on Z. Hence $Z \hookrightarrow N(f)$ is an irreducible component of the "hypersurface" $N(f) \hookrightarrow X$ - but in general one can't expect equality! On the other hand all the irreducible components $Z \hookrightarrow N(f)$ of a given hypersurtface $N(f)$ have indeed codimension 1:

Theorem 8.6 (Krulls Hauptidealsatz)**.** *Let X be an n-dimensional irreducible algebraic variety. Then the zero locus $N(f) \hookrightarrow X$ of a regular function $f \in \mathcal{O}(X) \setminus \{0\}$ is empty or purely $(n-1)$-dimensional, i.e. all its irreducible components $Z \hookrightarrow N(f)$ satisfy* $\dim Z = n-1$.

Proof. By Rem. 8.5.1 we may assume that $X = \mathrm{Sp}(A)$ is affine as well as $N(f) \hookrightarrow X$ irreducible: Given an irreducible component $Z \hookrightarrow N(f)$ of $N(f)$, replace X with an open affine subset $U \subset X$ with $U \cap N(f) = U \cap Z$. For $X = k^n$ we may assume $f \in k[T]$ to be T_n-monic, hence the restricted projection

$$N(f) \hookrightarrow k^n \longrightarrow k^{n-1}$$

onto the first $n-1$ coordinates is finite and surjective and we may refer to Prop. 8.3.

In order to obtain the result in the general case we choose a finite surjective morphism $\varphi : X \longrightarrow k^n$ and look for a polynomial $g \in k[T]$ with $\varphi(N(f)) = N(g)$: Then we may apply Proposition 8.3 to the morphism

$$\varphi|_{N(f)} : N(f) \longrightarrow N(g).$$

We take $g := \mathfrak{N}(f)$ with the norm function

$$\mathfrak{N} : Q(A) \longrightarrow k(T) := Q(k[T])$$

of the field extension $Q(A) \supset k(T)$, see the below remark with $B = K[T]$ for a short presentation. Since $f|\varphi^*(g)$ we have

$$N(f) \subset \varphi^{-1}(N(g)) \hookrightarrow X$$

and thus $\varphi(N(f)) \subset N(g)$. We consider the diagram

$$\begin{array}{ccccc} X & \supset & N(f) & \cup & \varphi^{-1}(V) \\ \downarrow & & \downarrow & & \downarrow \\ k^n & \supset & N(g) & \cup & V \end{array}$$

with a principal open subset $V = k^n_h \subset k^n$. Since $\varphi(N(f)) \hookrightarrow k^n$ is closed, it suffices to show that the lower union is disjoint if the upper one is. But in that case we have $f \in \mathcal{O}(\varphi^{-1}(V))^*$ and thus $g = \mathfrak{N}(f) \in \mathcal{O}(V)^*$. $\square$

The norm function associated to a field extension: We consider a finite field extension $E \supset K$ fitting in a diagram

$$\begin{array}{ccccccc} Q(A) & = & E & \supset & K & = & Q(B) \\ & & \cup & & \cup & & \cup \\ & & A & \supset & B & = & \widehat{B} \end{array}$$

with rings A, B and the integral closure $\widehat{B}$ of B in $Q(B)$. We define the norm function $\mathfrak{N} = \mathfrak{N}_{[E:K]} : E \longrightarrow K$ of the extension $E \supset K$ by

$$\mathfrak{N}(a) := \det(\mu_a : E \longrightarrow E),$$

where $\mu_a \in \mathrm{End}_K(E)$ is the multiplication with a, i.e. $\mu_a(x) := ax$. Then we have

1. $\mathfrak{N}(1) = 1$,
2. $\mathfrak{N}(ac) = \mathfrak{N}(a) \cdot \mathfrak{N}(c)$,
3. $\mathfrak{N}(A) \subset B$,
4. $\mathfrak{N}(A^*) \subset B^*$, and
5. $a|\mathfrak{N}(a)$ in A.

The first two statements are obvious, we comment on 3) and 5), while 4) follows from the multiplicativity 2) and statement 3). We use:

Lemma 8.7. *Denote $p_a \in K[S]$ the minimal polynomial of an element $a \in E$. Then, for $a \in A$ we have:*

$$p_a \in B[S]$$

and the characteristic polynomial $\chi_a \in K[S]$ of $\mu_a \in \mathrm{End}_K(E)$ satisfies

$$\chi_a = (p_a)^r \in B[S]$$

with $r := [E : K[a]]$.

Since $(-1)^n \mathfrak{N}(a) = \chi_a(0)$ (with $n := [E : K]$) is the constant term of the characteristic polynomial χ_a and $\chi_a(a) = 0$, that implies both 3) and 5).

Proof. The minimal polynomial $p_a \in K[T]$ of an element $a \in A \subset E$ lies in $B[S]$: We may assume - by enlarging E if necessary - that p_a splits into linear factors over E. Then $p_a|f$, if $f \in B[S]$ is an integral equation for $a \in A$ (i.e., a monic polynomial in $B[S]$ with a as zero). As a consequence, all zeros of p_a are integral over R, hence its coefficients - being obtained from these zeros by additions and multiplications - are both in $K = Q(B)$ and integral over B, but B is integrally closed in $Q(B)$, so they are already in B.

Now let us consider the characteristic polynomial $\chi_a \in K[S]$ of $\mu_a \in \mathrm{End}_K(E)$. The minimal polynomial p_a of a is also the minimal polynomial of μ_a, a divisor of the characteristic polynomial of μ_a. If $E = K[a]$, both polynomials are monic of the same degree, so they coincide. In the general case, choosing a basis $x_1, ..., x_r$ of E as $K[a]$-vector space provides a μ_a-invariant decomposition $E = \bigoplus_{i=1}^{d} K[a]x_i$, and the characteristic polynomial of $\mu_a|_{K[a]x_i}$ is p_a. So, altogether we obtain $(p_a)^r$ as the characteristic polynomial of μ_a. □

Finally, in order to see that we may apply the above reasoning with $B = K[T]$ we add

Proposition 8.8. *A UFD B is integrally closed in its field of fractions.*

Proof. Assume that $a := p/q \in Q(B)$ with relatively prime $p, q \in B$ has an integral equation

$$a^n + \sum_{i=0}^{n-1} r_i a^i = 0.$$

Multiplication with q^n yields

$$p^n + \sum_{i=0}^{n-1} r_i p^i q^{n-i} = 0,$$

in particular $q|p^n$ resp. $q|p$. But that implies $q \in B^*$ resp. $a \in B$. □

Proposition 8.9. *For any point $a \in X$ of an irreducible affine variety X we have* $\dim_a X = \dim X$.

Proof. We may assume that X is affine, since for $a \in U \subset X$ with an open subspace $U \subset X$ we have both $\dim U = \dim X$ and $\dim_a U = \dim_a X$, and use induction on $\dim X$. The case $\dim X = 0$ is trivial, since then X is finite. Now assume $\dim X > 0$. Take a regular function $f \in \mathfrak{m}_a \setminus \{0\}$ and let $Z \hookrightarrow N(f)$ be an irreducible component of $N(f) \hookrightarrow X$ containing a. Then, by induction hypothesis, $\dim_a Z = \dim Z$, and thus

$$\dim X \geq \dim_a X > \dim_a Z = \dim Z = \dim X - 1,$$

so necessarily $\dim_a X = \dim X$. $\square$

Corollary 8.10. *1. Let $X := N(k^n; f_1, ..., f_r) \hookrightarrow k^n$ be a subvariety determined by r equations. Then either X is empty or every irreducible component $X_0 \hookrightarrow X$ of X has dimension $\dim X_0 \geq n - r$.*

2. Let $X := N(\mathbb{P}_n; f_1, ..., f_r) \hookrightarrow \mathbb{P}_n$ be a projective variety determined by r nonconstant homogeneous polynomials. If $r \leq n$, then X is nonempty and every irreducible component $X_0 \hookrightarrow X$ of X has dimension $\dim X_0 \geq n - r$.

Proof. 1.) The first part is proved by induction on the number r of equations, the case $r = 0$ being trivial. Now for $r > 0$ every irreducible component $X_0 \hookrightarrow X$ is contained in an irreducible component Z_0 of $Z := N(k^n; f_1, ..., f_{r-1})$. By induction hypothesis we know that $\dim Z_0 \geq n-r+1$, while X_0 is an irreducible component of $N(Z_0; f_r)$, so, according to 8.6, has dimension at least $n - r$.
2.) With $\pi : (k^{n+1})^* \longrightarrow \mathbb{P}_n$ denoting the quotient morphism we have

$$X = \pi(N(k^{n+1}; f_1, ..., f_r) \setminus \{0\}) \neq \emptyset,$$

since $f_i(0) = 0$ for $i = 1, ..., r$ and, according to the first part, all the irreducible components of $N(k^{n+1}; f_1, ..., f_r)$ have dimension $\geq n + 1 - r \geq 1$. Finally use that $\dim C(X) = \dim X + 1$, since

$$C(X)_{T_i} \cong X_i \times k^*$$

holds for $X_i := X \cap U_i$ (with $U_i := \mathbb{P}_n \setminus N(\mathbb{P}_n; T_i)$). $\square$

Let us now study the dimensions of fibres of a dominant morphism $\varphi : X \longrightarrow Y$ between algebraic varieties X and Y, where we use the convention

$$\dim \emptyset := -\infty.$$

Simultaneously we investigate the sets $\varphi(X)$ obtained as images of morphisms. Here is an easy example showing some possible phenomena:

Example 8.11. The morphism $\varphi : k^2 \longrightarrow k^2, (s_1, s_2) \mapsto (s_1, s_1 s_2)$ maps a line $s_2 = a$ parallel to the s_1-axis to a the line $t_2 = at_1$ through the origin. In particular it satisfies

$$\varphi^{-1}(t_1, t_2) = \begin{cases} \{(t_1, t_1^{-1} t_2)\} & , \text{ if } (t_1, t_2) \in k^* \times k \\ \{0\} \times k & , \text{ if } (t_1, t_2) = (0, 0) \\ \emptyset & , \text{ otherwise} \end{cases}$$

and thus

$$\dim \varphi^{-1}(t_1, t_2) = \begin{cases} 0 & , \text{ if } (t_1, t_2) \in k^* \times k \\ 1 & , \text{ if } (t_1, t_2) = (0, 0) \\ -\infty & , \text{ otherwise} \end{cases} .$$

Furthermore for $V := k^* \times k$, we get an isomorphism

$$\varphi|_{\varphi^{-1}(V)} : \varphi^{-1}(V) \xrightarrow{\cong} V.$$

Note furthermore that $\varphi(k^2) = V \cup \{(0, 0)\}$ is neither open nor closed in k^2, not even locally closed and thus does not inherit in a natural way the structure of an algebraic variety.

Proposition 8.12. *Let $\varphi : X \longrightarrow Y$ be a dominant morphism between the irreducible varieties X and Y. Then*

1. $\dim X \geq \dim Y$,
2. *the irreducible components of a nonempty fibre $\varphi^{-1}(y)$ have at least dimension $d := \dim X - \dim Y$, and*
3. *there is a nonempty open subset $W \subset Y$, such that all fibers $\varphi^{-1}(y)$ of points $y \in W$ are pure-d-dimensional.*

Proof. 2.) We may assume that Y is affine and take a surjective finite map $\pi : Y \longrightarrow k^r$ (so $r = \dim Y$). Look at $\psi := \pi \circ \varphi : X \longrightarrow k^r$. Since w.l.o.g. $\psi(y) = 0$, we see with the first part of Cor.8.10, that all irreducible components of $\psi^{-1}(0) \hookrightarrow X$ have dimension at least $\dim X - r$. On the other hand, if $\pi^{-1}(0) = \{y_1 = y, y_2, ..., y_s\}$, then

$$\psi^{-1}(0) = \bigcup_{j=1}^{s} \varphi^{-1}(y_j),$$

a disjoint union, so the fiber $\varphi^{-1}(y)$ is a union of irreducible components of $\psi^{-1}(0)$, and we are done.

In order to prove the remaining points we need the following auxiliary lemma:

Lemma 8.13. *Let $\varphi : X = \mathrm{Sp}(A) \longrightarrow Y = \mathrm{Sp}(B)$ be a dominant morphism of irreducible affine varieties. Then, with $d := \dim X - \dim Y$, there is a non-empty open subset $W \subset Y$ and a factorization*

$$\varphi|_{\varphi^{-1}(W)} = \mathrm{pr}_W \circ \psi : \varphi^{-1}(W) \xrightarrow{\psi} W \times k^d \xrightarrow{\mathrm{pr}_W} W$$

with the projection $\mathrm{pr}_W : W \times k^d \longrightarrow W$ and a surjective finite morphism $\psi : \varphi^{-1}(W) \longrightarrow W \times k^d$. In particular the fibers $\varphi^{-1}(y), y \in W$, are pure d-dimensional.

Proof. We have already seen that an irreducible component $Z \hookrightarrow \varphi^{-1}(y)$ has dimension at least d; on the other hand the morphism

$$Z \hookrightarrow \varphi^{-1}(y) \longrightarrow \{y\} \times k^d$$

is finite, whence $\dim Z \leq d$.

The pull back $\varphi^* : B \longrightarrow A$ being injective, we treat it as an injection and create the following diagram:

$$\begin{array}{ccccccccccc}
B[a_1, ..., a_r] & = & A & \subset & (B \setminus \{0\})^{-1}A & =: & C & = & K[a_1, ..., a_r] & \supset & A_g \\
 & & \cup & & \cup & & \cup & & \cup & & \cup \\
 & & B & \subset & Q(B) & =: & K & \subset & K[T] & \supset & B_g[T] \\
 & & & & & & & & & & \cup \\
 & & & & & & & & & & B_g
\end{array}.$$

The algebra C is an affine K-algebra. So we may apply the Noether normalization theorem 3.10 and obtain a finite injective homomorphism

$$K[T] := K[T_1, ..., T_d] \hookrightarrow C,$$

where we even may assume $T_j \in A$ for $j = 1, ..., d$. If the polynomials $q_i \in (K[T])[S], i = 1, ..., r$, are integral equations over $K[T]$ for the (B-algebra) generators $a_i \in A$, we have $q_i \in (B_g[T])[S]$ for a suitable
$g \in B \setminus \{0\}$: Take g as a common denominator of the coefficients of the polynomials $f_{ij} \in K[T]$, such that $q_i = S^{m_i} + \sum_{j<m_i} f_{ij} S^j$. Then obviously, A_g is a finite $B_g[T]$-module, and we may take $W = Y_g$, with the above factorization corresponding to the algebra homomorphisms

$$B_g \hookrightarrow B_g[T] \hookrightarrow A_g.$$

Since $\dim X = \dim \varphi^{-1}(W) = \dim W \times k^d = \dim W + d = \dim Y + d$, we obtain $d = \dim X - \dim Y$. $\square$

We go on with the proof of 8.12:
1.) Take any affine open subset $V \subset Y$ and $U \subset \varphi^{-1}(V)$ and apply the lemma 8.13 to $\varphi|_U : U \longrightarrow V$, finally remember that $\dim Y = \dim V$ as well as $\dim X = \dim U$.
3.) We may again assume that Y is affine and write $X = \bigcup_{i=1}^r U_i$ with affine open subsets U_i. Then we may find open subsets $W_i \subset Y$ such that $\varphi^{-1}(y) \cap U_i$ is pure-d-dimensional for all $y \in W_i$. Now set $W := \bigcap_{i=1}^r W_i$. $\square$

It is natural to ask how $\dim \varphi(y)$ depends on $y \in Y$. We may for example consider the sets

$$T_m(\varphi) := \{y \in Y; \dim \varphi(y) \geq m\}$$

for $m \in \mathbb{N} \cup \{-\infty\}$. Then one could hope that the $T_m(\varphi) \hookrightarrow Y$ are closed subsets, but $T_0(\varphi) = \varphi(X)$ need not be closed. Instead we try it with the sets

$$S_m(\varphi) := \{x \in X; \dim_x \varphi^{-1}(\varphi(x)) \geq m\}.$$

In any case $S_0(X) = X$ is closed, and

$$T_m(\varphi) = \varphi(S_m(\varphi)).$$

Since the definition of $S_m(\varphi)$ is local in X, we succeed showing

Proposition 8.14. *For any morphism $\varphi : X \longrightarrow Y$ between algebraic varieties X and Y the sets*

$$S_m(\varphi) := \{x \in X; \dim_x \varphi^{-1}(\varphi(x)) \geq m\}.$$

are closed:

$$S_m(\varphi) \hookrightarrow X.$$

Equivalently: The function $X \longrightarrow \mathbb{N}, x \mapsto \dim_x \varphi^{-1}(\varphi(x))$ *is upper semicontinuous.*

Proof. We do induction on $\dim X$, the case $\dim X = 0$ being trivial. If $X = \bigcup_{i=1}^r X_i$ is the decomposition of X into irreducible components, then

$$S_m(\varphi) = \bigcup_{i=1}^r S_m(\varphi|_{X_i}).$$

Hence we may assume that X is irreducible, and, obviously, that φ is dominant and thus Y irreducible as well. According to Prop.8.12 we have $S_m(\varphi) = X$ for $m \le d := \dim X - \dim Y$. So let $m > d$. But 8.12 assures that there is a nonempty open set $W \subset Y$, where all fibres $\varphi^{-1}(y), y \in W$, are pure-d-dimensional. Consequently $S_m(\varphi) \subset Z := X \setminus \varphi^{-1}(W)$ with $\dim Z < \dim X$. Since $S_m(\varphi) = S_m(\varphi|_Z)$, we may use the induction hypothesis. $\square$

Finally we use Lemma 8.13 in order to characterize image sets $\varphi(X)$ of morphisms $\varphi : X \longrightarrow Y$ between algebraic varieties X and Y. First of all we call a subset $Y \subset X$ of a topological space *locally closed* if it is of the form $Y = Z \cap U$ with a closed subset $Z \hookrightarrow X$ and an open subset $U \subset X$, i.e. Y is closed in a suitable open neighborhood $U \subset X$ or, the other way around, Y is open in its closure $\overline{Y}$.

Definition 8.15. A subset of an algebraic variety X is called *constructible* if it is a finite union of locally closed subsets.

Lemma 8.16. *The constructible sets of an algebraic variety X form an algebra of sets $\mathfrak{C}(X)$, i.e. $\mathfrak{C}(X) \subset \mathfrak{P}(X)$ is closed with respect to finite unions and intersections and with respect to taking the complement.*

Proof. By definition $\mathfrak{C}(X)$ is closed with respect to finite unions, and if we can show $A \in \mathfrak{C}(X) \Longrightarrow X \setminus A \in \mathfrak{C}(X)$, the de Morgan rules give that $\mathfrak{C}(X)$ is closed with respect to finite intersections as well. So let $A = \bigcup_{i=1}^r A_i \cap U_i \in \mathfrak{C}(X)$ with closed $A_i \hookrightarrow X$ and open $U_i \subset X$. Then

$$X \setminus A = X \setminus \left(\bigcup_{i=1}^r A_i \cap U_i \right) = \bigcap_{i=1}^r (X \setminus A_i \cap U_i)$$

$$= \bigcap_{i=1}^{r} (X \setminus A_i) \cup (X \setminus U_i)$$

$$= \bigcup_{I \in \mathfrak{P}(\{1,...,r\})} \left(\bigcap_{i \in I} (X \setminus A_i) \cap \bigcap_{i \notin I} (X \setminus U_i) \right) \in \mathfrak{C}(X).$$

□

Example 8.17. 1. $\mathfrak{C}(X) \subset \mathfrak{P}(X)$ is the smallest algebra of sets containing all open subsets.

2. On any set X the sets which either are finite or cofinite (i.e. have a finite complement) form an algebra of sets. From this and the previous point it follows that for an irreducible curve constructible sets are either finite or cofinite resp. either closed or open. If X is a reducible curve, any constructible set is still locally closed: If $X = \bigcup_{i=1}^{r} X_i$ is the decomposition into irreducible components, then $A \in \mathfrak{C}(X) \Longleftrightarrow A_i := A \cap X_i \in \mathfrak{C}(X_i)\ \forall i = 1, ..., r$, so A is constructible iff every $A_i \subset X_i$ either is closed or open in X_i. Then it follows that A is open in its closure $\overline{A}$, which is the union of certain irreducible components of X and finitely many isolated points.

3. The set $A = (k^* \times k) \cup \{0\} \subset k^2$ is constructible, but not locally closed, since it is not open in its closure k^2.

Proposition 8.18. *The image $\varphi(X) \subset Y$ of a morphism $\varphi : X \longrightarrow Y$ between algebraic varieties X and Y is a constructible set:*

$$\varphi(X) \in \mathfrak{C}(Y).$$

Proof. We use induction on $\dim X$, the case $\dim X = 0$, i.e. $|X| < \infty$, being obvious. Furthermore we may assume X to be irreducible and φ to be dominant and hence Y is irreducible as well. Then, according to Prop. 8.12 there is a non-empty open subset $W \subset \varphi(X)$, so $\varphi(X) = W \cup \varphi(Z)$ with $Z := Y \setminus \varphi^{-1}(W)$. Since then $\dim Z < \dim X$ the induction hypothesis gives that $\varphi(Z)$ is constructible, hence so is $W \cup \varphi(Z)$. □

9 Rational functions and local rings

On a complete algebraic variety X there are only locally constant regular functions; so they do not provide any information about the geometry of X. Instead we have to allow "poles" and "points of indeterminacy": We consider pairs (U, f), where $U \subset X$ is a dense open subset and $f \in \mathcal{O}(U)$ a regular function on U. Two such pairs are identified using the equivalence relation

$$(U, f) \sim (V, g) \iff f|_{U \cap V} = g|_{U \cap V}.$$

In order to see that this really provides an equivalence relation, note that a finite intersection of dense open subsets is again a dense open subset.

Definition 9.1. A rational function on an algebraic variety X is an equivalence class of pairs (U, f) with a dense open subset $U \subset X$ and $f \in \mathcal{O}(U)$. We denote $\mathcal{R}(X)$ the set of all rational functions on X.

Remark 9.2.

1. Since the restriction $\mathcal{O}(U) \longrightarrow \mathcal{O}(W)$ of functions is injective for dense open subsets $W \subset U$, a rational function represented by (U, f) has a unique representative $(U_{\max}, f_{\max})$ with a maximal open subset $U_{\max} \subset X$: Set

$$U_{\max} := \bigcup_{(V,g) \sim (U,f)} V \ , \ f_{\max}|_V := g.$$

   In order to simplify notation, we denote rational functions f and $\mathbb{D}_f := U_{\max}$ their domain of definition.

2. The rational functions form a k-algebra, the sum $f+g$ resp. the product fg having a domain of definition containing the (dense) intersection $\mathbb{D}_f \cap \mathbb{D}_g$.

3. A morphism $\varphi : X \longrightarrow Y$, such that $\overline{\varphi(X_0)} \hookrightarrow Y$ is an irreducible component of Y for every irreducible component $X_0 \hookrightarrow X$, induces a pull-back homomorphism $\varphi^* : \mathcal{R}(Y) \longrightarrow \mathcal{R}(X)$. In particular the inclusion of an open subset $U \subset X$ induces a a restriction map $\mathcal{R}(X) \longrightarrow \mathcal{R}(U)$; indeed it is an isomorphism $\mathcal{R}(X) \cong \mathcal{R}(U)$ if $U \subset X$ is dense.

4. If $X = \mathrm{Sp}(A)$ is affine, then

$$\mathcal{R}(X) \cong S^{-1}A$$

with the localization of A with respect to the multiplicative set

$$S := \{f \in A; f|_{X_0} \notin I(X_0) \ \forall \text{ irreducible components } X_0 \hookrightarrow X\}$$

of all nonzero divisors in A. In particular

$$\mathcal{R}(X) \cong Q(A),$$

if X is irreducible.

5. For an irreducible variety X the ring $\mathcal{R}(X)$ is a field

$$\mathcal{R}(X) \cong Q(\mathcal{O}(U))$$

isomorphic to the field of fractions of the ring of regular functions on any (non-empty) open affine subset $U \subset X$.

6. The morphism

$$f : \mathbb{D}_f \longrightarrow k$$

representing a nonzero rational function on an irreducible variety X can be extended to a morphism

$$\widehat{f} : U := \mathbb{D}_f \cup \mathbb{D}_{1/f} \longrightarrow \mathbb{P}_1 = k \cup \{\infty\},$$

by defining

$$\widehat{f}(x) := [(1/f)(x), 1] \in U_1 \subset \mathbb{P}_1$$

for $x \in \mathbb{D}_{1/f}$. The points outside $\mathbb{D}_f \cup \mathbb{D}_{1/f}$ are called "*points of indeterminacy*" of the rational function f. If $X = \mathrm{Sp}(A)$ is affine, a point $a \in X$ is a point of indeterminacy of $f \in Q(A)$, iff $g(a) = 0 = h(a)$ for all representations of f as fraction $f = g/h$ with $g, h \in A$.

For example take $X := k^2$ and $f := T_1/T_2$. Then $\mathbb{D}_f = k \times k^*$ and $\mathbb{D}_{1/f} = k^* \times k$, while the origin is the only point of indeterminacy. Indeed the morphism $\widehat{f} : (k^2)^* \longrightarrow \mathbb{P}_1$ turns out to be the quotient morphism. In particular there is no extension to a morphism $k^2 \longrightarrow \mathbb{P}_1$, since the restriction $\widehat{f}|_{k^*t}$ of $\widehat{f}$ to a punctured line k^*t can be extended to the entire line kt only with $0 \mapsto [t]$, that value depending on the chosen line.

In the sequel we shall use also the letter f in order to denote $\widehat{f}$.

7. If $X = \bigcup_{i=1}^{r} X_i$ is the decomposition of X into irreducible components, then the component covering

$$\pi : \widetilde{X} := \coprod_{i=1}^{r} X_i \longrightarrow X$$

induces an isomorphism

$$\mathcal{R}(X) \cong \bigoplus_{i=1}^{r} \mathcal{R}(X_i),$$

since, with $U := X \setminus \bigcup_{i \neq j} X_i \cap X_j$ and $\widetilde{U} := \pi^{-1}(U) \cong U$, we have $\mathcal{R}(\widetilde{X}) \cong \mathcal{R}(\widetilde{U}) \cong \mathcal{R}(U) \cong \mathcal{R}(X)$.

In particular $\mathcal{R}(X)$ is a finite direct sum of fields.

Remembering the fact that affine varieties form a category anti-equivalent to the category of affine reduced k-algebras, it is natural to ask whether the category of k-algebras, which are finite sums of (as field) finitely generated k-extensions, has a geometric counterpart as well. The point here is to find the correct notion of a morphism. In analogy to rational functions there is also the notion of a rational morphism:

Definition 9.3. A rational morphism between two algebraic varieties X, Y is an equivalence class of pairs (U, φ) with a dense open subset $U \subset X$ and a morphism $\varphi : U \longrightarrow Y$, where the equivalence relation is

$$(U, \varphi) \sim (V, \psi) \iff \varphi|_{U \cap V} = \psi|_{U \cap V}.$$

As with rational functions every rational morphism $\varphi : X \longrightarrow Y$ has a a maximal domain of definition $\mathbb{D}_\varphi$, where it is an "ordinary" morphism of algebraic varieties. A rational morphism $\varphi : X \longrightarrow Y$ between irreducible varieties X, Y is called dominant if $\varphi(U) \subset Y$ is dense in Y for some (and thus all) dense open subset(s) $U \subset \mathbb{D}_\varphi$.

Theorem 9.4. *The functor $\mathcal{R} : X \mapsto \mathcal{R}(X)$ from the category of irreducible algebraic varieties together with the dominant rational morphisms as morphisms to the category of finitely generated field extensions of k is an anti-equivalence of categories.*

Proof. If $K = k(a_1, ..., a_r)$, we have $K = \mathcal{R}(X)$ with $X := \mathrm{Sp}(k[a_1, ..., a_r])$. Assume now that $\sigma : \mathcal{R}(Y) \longrightarrow \mathcal{R}(X)$ is a k-algebra homomorphism. We may replace X and Y with (nonempty) open affine subsets, i.e., we may assume $X = \mathrm{Sp}(A), Y = \mathrm{Sp}(B)$. If $B = k[b_1, ..., b_s]$, choose a common denominator $g \in A$ for the elements $\sigma(b_1), ..., \sigma(b_s) \in Q(A)$. So $\sigma(B) \subset A_g$ and the morphism $X_g \longrightarrow Y$ corresponding to the k-algebra homomorphism $\sigma|_B : B \longrightarrow A_g$ defines the rational morphism $X \longrightarrow Y$ we are looking for – the easy proof of uniqueness is left to the reader. $\square$

Corollary 9.5. *For irreducible algebraic varieties X, Y the following statements are equivalent:*

1. $\mathcal{R}(X) \cong \mathcal{R}(Y)$
2. *There are nonempty open sets $U \subset X$ and $V \subset Y$ such that $U \cong V$ as algebraic varieties.*

In that case we say also, that the varieties X and Y are birationally equivalent.

Proof. "$\Longleftarrow$": $\mathcal{R}(X) \cong \mathcal{R}(U) \cong \mathcal{R}(V) \cong \mathcal{R}(Y)$.
"$\Longrightarrow$": Let $\varphi : U_1 \longrightarrow Y$ and $\psi : V_1 \longrightarrow X$ be morphisms inducing the above isomorphism in the respective directions. Take $U := \varphi^{-1}(V_1) \subset U_1$ and $V := \varphi(U) \subset V_1$. $\square$

Definition 9.6. An n-dimensional irreducible algebraic variety X is called rational, if it is birationally equivalent to k^n, or equivalently, if it admits a non-empty open subset $U \subset X$ isomorphic to an open subset $V \subset k^n$.

For the study of local properties of an algebraic variety resp. of rational functions one needs even non-affine subrings of $\mathcal{R}(X)$, which are sufficiently big in order to be understandable:

Definition 9.7. Let $Y \hookrightarrow X$ be an irreducible subvariety and $X_0 \hookrightarrow X$ the union of the irreducible components of X containing Y. The local ring $\mathcal{O}_{X,Y}$ of X at Y is defined as

$$\mathcal{O}_{X,Y} := \{f \in \mathcal{R}(X_0); Y \cap \mathbb{D}_f \neq 0\}.$$

Note that $\mathcal{O}_{X,Y}$ is a subring, since any two nonempty open subsets of the irreducible Y have nonempty intersection and thus

$$\mathbb{D}_{f+g}, \mathbb{D}_{fg} \supset \mathbb{D}_f \cap \mathbb{D}_g \hookleftarrow (\mathbb{D}_f \cap Y) \cap (\mathbb{D}_g \cap Y) \neq \emptyset.$$

Furthermore:

Remark 9.8. 1. To justify its name, we remark that $\mathcal{O}_{X,Y}$ is a local ring with maximal ideal

$$\mathfrak{m}_{X,Y} := \{f \in \mathcal{R}(X_0); f|_{Y \cap \mathbb{D}_f} = 0\}.$$

2. If $U \subset X$ is an open subset intersecting Y, we have $\mathcal{O}_{X,Y} \cong \mathcal{O}_{U,U\cap Y}$.

3. If $X = \mathrm{Sp}(A)$ with $Y \subset X$ being contained in all irreducible components of X, the ring $\mathcal{O}_{X,Y}$ is isomorphic to the localization

$$\mathcal{O}_{X,Y} \cong S^{-1}A$$

of A with respect to the multiplicative subset $S := A \setminus I(Y)$.

4. The ring $\mathcal{O}_{X,Y}$ is noetherian: If $\mathfrak{b} \hookrightarrow \mathcal{O}_{X,Y} \cong S^{-1}A$ is an ideal, then A being noetherian, we have $\mathfrak{a} := \mathfrak{b} \cap A = Ag_1 + ... + Ag_r$, and that implies $\mathfrak{b} = S^{-1}\mathfrak{a} = S^{-1}Ag_1 + ... + S^{-1}Ag_r$.

5. The ring $\mathcal{O}_{X,Y}$ is never an affine algebra: The only local reduced affine algebra is the base field k.

The extremal choices of Y are of particular importance: For minimal $Y \neq \emptyset$, i.e. a one point set $\{a\}$, we obtain *the local ring $\mathcal{O}_{X,a}$ of X at the point a*:

Definition 9.9. The local ring $\mathcal{O}_{X,a}$ of X at $a \in X$ is defined as

$$\mathcal{O}_{X,a} := \mathcal{O}_{X,\{a\}} = \{f \in \mathcal{R}(X_0); \mathbb{D}_f \ni a\},$$

where $X_0 \hookrightarrow X$ is the union of all irreducible components of X containing a. The ring $\mathcal{O}_{X,a}$ is called the "*local ring of X at the point $a \in X$*" or the "*stalk of the structure sheaf $\mathcal{O}_X$ at the point $a \in X$*".

Remark 9.10. 1. We can also describe $\mathcal{O}_{X,a}$ as the direct limit

$$\mathcal{O}_{X,a} := \lim_{U \ni a} \mathcal{O}_X(U),$$

taken with respect to the (by inclusion) partially ordered set of open neighbourhoods of a. With other words the elements in $\mathcal{O}_{X.a}$ are equivalence classes of pairs (U, f), where $a \in U$ and $f \in \mathcal{O}_X(U)$ with $(U, f) \sim (V, g)$ iff $f|_W = g|_W$ for some open neighbourhood $W \subset U \cap V$ of the point a.

2. If $k = \mathbb{C}$ we can as well consider the stalk

$$\mathcal{H}_a := \lim_{U \ni a} \mathcal{H}(U),$$

of the structure sheaf $\mathcal{H}$ of X_h, where now U ranges over all strongly open neighbourhoods of a. We have

$$\mathcal{H}_a \cong \mathbb{C}\{T_1, ..., T_n\}$$

with the ring $\mathbb{C}\{T_1, ..., T_n\}$ of convergent power series on the RHS iff $a \notin S(X_h)$. Furthermore we now can also define rigorously the notion of a self intersection point: A point $a \in X$ is a self intersection point iff $\mathcal{H}_a$ is not an integral domain.

The local ring $\mathcal{O}_{X,a}$ of an algebraic variety X at a point $a \in X$ is an important tool in the study of the local geometry of X near a; indeed there is an analogue to Th. 9.4: We define the category of "*germs of algebraic varieties*" to have as objects the pairs (X, a), where X is an algebraic variety and $a \in X$ a point. Furthermore a morphism from (X, a) to (Y, b) is an equivalence class of morphisms $\varphi : U \longrightarrow Y$ of algebraic varieties, where $U \subset X$ is an open neighborhood of $a \in X$ and $\varphi(a) = b$, two morphisms being equivalent if they agree on some open neighborhood of a. Obviously such a morphism φ induces a pullback homomorphism $\varphi_a^* : \mathcal{O}_{Y,\varphi(a)} \longrightarrow \mathcal{O}_{X.a}$ depending only on the equivalence class of φ.

Theorem 9.11. *The functor*

$$(X, a) \mapsto \mathcal{O}_{X,a}, \ [\varphi] \mapsto \varphi_a^*$$

is an anti-equivalence between the category of germs of algebraic varieties and the full subcategory of the category of all k-algebras, whose objects are local rings of algebraic varieties.

The proof is analogous to that of Th. 9.4 and thus left as an exercise to the reader. In particular we have as well the following corollary:

Corollary 9.12. *For points $a \in X, b \in Y$ in the algebraic varieties X, Y the following statements are equivalent:*

1. *$\mathcal{O}_{X,a} \cong \mathcal{O}_{Y,b}$ as k-algebras*
2. *There are open neighborhoods U of $a \in X$ and V of $b \in Y$, such that $U \cong V$ with an isomorphism of algebraic varieties sending a to b.*

Remark 9.13.
1. The local ring $\mathcal{O}_{X,a}$ is an integral domain iff a is contained in exactly one irreducible component of X. So all the local rings of an algebraic variety X are integral domains iff X is the disjoint union of its irreducible components.
2. For an irreducible variety we have $\mathcal{O}_{X,a} \subset \mathcal{R}(X)$ for all $a \in X$ and
$$\mathcal{O}(X) = \bigcap_{a \in X} \mathcal{O}_{X,a}.$$

10 Normal Varieties

Definition 10.1. A point $a \in X$ in an algebraic variety X is called a normal point, iff the local ring $\mathcal{O}_{X,a}$ of X at a is an integral domain, which is integrally closed in its field of fractions $Q(\mathcal{O}_{X,a}) = \mathcal{R}(X)$. An algebraic variety X is called normal iff all its points are normal.

So a normal variety is the disjoint union of its irreducible components.

Proposition 10.2.
1. *An irreducible affine variety is normal iff its algebra of regular functions A is integrally closed in its field of fractions $Q(A)$.*
2. *The normal points of an algebraic variety X form a dense open subset.*

In the proof we use the following basic facts:

Remark 10.3.
1. If A is normal, i.e. integrally closed in its field of fractions, so is $S^{-1}A$ for any multiplicative subset $S \subset A$: If $f \in S^{-1}A$ has integral equation $p(f) = 0$ with a monic polynomial $p \in S^{-1}A[T]$, then hf has integral equation $h^n p(h^{-1}T), n = \deg p$. But for a suitable choice of $h \in S$ that polynomial has coefficients in A, and A being normal, we get $hf \in A$ resp. $f = hf/h \in S^{-1}A$.

2. The integral closure $\widetilde{A} \subset Q(A)$ of an affine k-algebra without zero divisors in its field of fractions $Q(A)$ is a finitely generated A-module, thus in particular an affine algebra.

Proof. The first part follows from the fact, that a localization of a ring integrally closed in its field of fractions enjoys the same property, as well as any intersection – see Rem.9.13 – of such rings does.
For the second part consider a normal point $a \in X$. Since the irreducible points of X, i.e. the points lying in only one of the irreducible components of X, form a dense open subset, we may replace X with an irreducible affine open neighborhood of a resp. assume that $X = \mathrm{Sp}(A)$ is irreducible and affine. Then we have $\widetilde{A} \subset \mathcal{O}_{X,a}$ for the integral closure $\widetilde{A}$ of A in its field of fractions $Q(A)$. But on the other hand it is a finitely generated A-module, say $\widetilde{A} = Af_1 + ... + Af_r$ with $f_1, ..., f_r \in Q(A)$. Hence there is a common denominator $g \in A$ for $f_1, ..., f_r$ satisfying $g(a) \neq 0$ and thus $A_g = \widetilde{A}_g$ is integrally closed in $Q(A_g) = Q(A)$, so the open neighborhood X_g of a contains only normal points. $\square$

Remark 10.4. The affine variety $\widetilde{X} := \mathrm{Sp}(\widetilde{A})$ is called the normalization of X, the morphism $\pi : \widetilde{X} \longrightarrow X$ induced by the inclusion $A \subset \widetilde{A}$ is finite and surjective; it fits into a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X \\ \cup & & \cup \\ \pi^{-1}(U) & \stackrel{\cong}{\longrightarrow} & U \end{array},$$

where $U \subset X$ is the open subset of all normal points of X.

Let us now pass to the case of a maximal proper irreducible subvariety $Y \hookrightarrow X$ of an irreducible normal variety X, i.e. Y has codimension 1 in X: The ring $\mathcal{O}_{X,Y}$ then is a "*discrete valuation ring*":

Definition 10.5. A discrete valuation ring R is a noetherian local integral domain, whose maximal ideal $\mathfrak{m} \hookrightarrow R$ is a principal ideal:

$$\mathfrak{m} = (\pi)$$

with an element $\pi \in R \setminus \{0\}$.

Remark 10.6. Every nonzero element $x \in R$ in a discrete valuation ring R can be written uniquely

$$x = e\pi^n \tag{1}$$

with a unit $e \in R^*$ and $n \in \mathbb{N}$. In particular R is a PID, the ideals being the zero ideal and the ideals $(\pi^n), n \in \mathbb{N}$. This is a consequence of the fact, that an element in R is either a unit or divisible with π. We can write $x = a_\nu \pi^\nu$ so long as possible; then the chain of ideals (a_ν) is strictly increasing, whence it follows that there is a final $e := a_n$, it is necessarily a unit. Furthermore the field of fractions is a disjoint union

$$Q(R) = R_\pi = \{0\} \cup \bigcup_{n=-\infty}^{\infty} R^* \, \pi^n.$$

Example 10.7. Let $p \in R$ be a prime element of the UFD R. Then $R_{(p)} := \{a/b \in Q(R); a, b \in R, \gcd(p, b) = 1\}$ is a discrete valuation ring.

Indeed there are more rings R, such that the localization $R_\mathfrak{p}$ with respect to a prime ideal $\mathfrak{p} \subset R$ is a discrete valuation ring. Of course $\mathfrak{p}$ has to be a minimal nonzero prime ideal. Here we shall prove:

Theorem 10.8. *For an irreducible normal variety X its local ring $\mathcal{O}_{X,Y}$ at a one codimensional irreducible subvariety $Y \hookrightarrow X$ is a discrete valuation ring.*

Proof. Since R is a noetherian local ring, the lemma of Nakayama applies to $M := \mathfrak{m} \subset R$ and tells us that $\mathfrak{m} \neq 0$ implies $\mathfrak{m}^2 \neq \mathfrak{m}$. We take any element $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ and claim

$$\mathfrak{m} = (f).$$

First of all we prove:

$$\mathfrak{m}^n \subset (f)$$

for $n \gg 0$. Namely: We have $f \in I(\mathbb{D}_f \cap Y) \subset \mathcal{O}(\mathbb{D}_f)$. Then $Y \hookrightarrow \overline{N(f)}$, and thus Y, being of codimension 1 in X, is an irreducible component of $\overline{N(f)} \hookrightarrow X$. Hence we may choose a principal open subset $U \subset \mathbb{D}_f$ with $U \cap Y = U \cap N(f)$. Replacing X with the affine variety U, we see that we may assume $X = \mathrm{Sp}(A)$ and $f \in A$ with $Y = N(f)$.

Now, if $I(Y) = (g_1, ..., g_r)$, then $g_i \in I(N(f))$, so Hilberts Nullstellensatz provides a number $m \in \mathbb{N}$ with $g_i^m \in (f)$ for $i = 1, ..., r$. Then we have

$$I(Y)^n \subset (f) \hookrightarrow A$$

with $n := mr$ and as well

$$\mathfrak{m}^n \subset (f) \hookrightarrow R.$$

We now show that the assumption $\mathfrak{m} \not\subset (f)$ leads to a contradiction. Choose $n \in \mathbb{N}$ minimal with $\mathfrak{m}^n \subset (f)$. Obviously $n \geq 2$ and there is an element $g \in \mathfrak{m}^{n-1} \setminus (f) \subset \mathfrak{m}$. We investigate how the element $h := g/f \in Q(A) \setminus A$ acts on $\mathfrak{m} \hookrightarrow R$:

1. We have
$$h \cdot \mathfrak{m} = \frac{g \cdot \mathfrak{m}}{f} \subset \frac{\mathfrak{m}^n}{f} \subset A,$$
because of $g \in \mathfrak{m}^{n-1}$ and $\mathfrak{m}^n \subset (f)$,

2. but
$$h \cdot \mathfrak{m} \not\subset \mathfrak{m}.$$
Namely: Otherwise $M := \mathfrak{m}$ is a faithful $R[h]$-module, which is finitely generated as an R-module. But that means that $h \in Q(R)$ is integral over R, hence, R being integrally closed in its field of fractions $Q(R)$, we conclude that $h \in R$, a contradiction.

Hence the ideal $h \cdot \mathfrak{m} \subset R$ contains elements in $R \setminus \mathfrak{m} = R^*$, and consequently $h \cdot \mathfrak{m} = R$ resp. $h^{-1} \in \mathfrak{m}$ and

$$f = f \cdot h \cdot h^{-1} = g \cdot h^{-1} \in \mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}^2,$$

a contradiction, since $f \in \mathfrak{m} \setminus \mathfrak{m}^2$.

$\square$

As a consequence of Th.10.8 we can define multiplicities: Given a normal irreducible variety X we can for any codimension one irreducible subvariety $Y \hookrightarrow X$ define an order function

$$\mathrm{ord}_Y : \mathcal{R}(X) \longrightarrow \mathbb{Z} \cup \{\infty\}$$

associating to a rational function $h \in \mathcal{R}(X)$ its "*order*" (or "*multiplicity*") along Y as follows: If $\mathcal{O}_{X,Y} \supset \mathfrak{m} = (f)$, then

$$\mathrm{ord}_Y(h) := \begin{cases} n & , \quad \text{if } h = ef^n, \text{ with } e \in \mathcal{O}_{X,Y}^* \\ \infty & , \quad \text{if } h = 0 \end{cases} .$$

The order function satisfies

$$\mathrm{ord}_Y(gh) = \mathrm{ord}_Y(g) + \mathrm{ord}_Y(h)$$

as well as

$$\mathrm{ord}_Y(g + h) \geq \min \{\mathrm{ord}_Y(g), \mathrm{ord}_Y(h)\} .$$

Note that in particular for irreducible $Y \hookrightarrow X$ of codimension 1 and a nonzero $f \in \mathcal{R}(X)$ we have either $Y \cap \mathbb{D}_f \neq \emptyset$ or $Y \cap \mathbb{D}_{1/f} \neq \emptyset$ depending on whether $\mathrm{ord}_Y(f) \geq 0$ or $\mathrm{ord}_Y(f) < 0$. So the points of indeterminacy of a rational function f form a closed set of codimension at least 2. For normal curves we thus find:

Remark 10.9.

1. A rational function $f \in \mathcal{R}(X)$ on an irreducible normal curve has no points of indeterminacy, hence defines a morphism $f : X \longrightarrow \mathbb{P}_1 = k \cup \{\infty\}$.
2. Indeed, we may replace $\mathbb{P}_1$ with any complete variety Y and obtain that any morphism $\varphi : U \longrightarrow Y$ defined on a non-empty open subset $U \subset X$ of an irreducible normal curve into a complete variety Y can be extended to a morphism $\hat{\varphi} : X \longrightarrow Y$.
3. As a consequence we obtain that the category of irreducible normal complete curves is anti-equivalent to the category of finitely generated field extensions of our base field k of transcendence degree 1.
4. Finally let us mention, that for $k = \mathbb{C}$ the functor $X \mapsto X_h$ is an equivalence between the category of irreducible normal complete curves and the category of compact Riemann surfaces.

Definition 10.10. For an irreducible affine variety X we denote $\mathfrak{D}(X)$ the free abelian group generated by the codimension one irreducible subvarieties of X, i.e.

$$\mathfrak{D}(X) := \left\{ \sum_{i=1}^{r} n_i Y_i ; n_i \in \mathbb{Z}, Y_i \hookrightarrow X \text{ irreducible}, \dim X = \dim Y_i + 1 \right\} .$$

The elements $D = \sum_{i=1}^{r} n_i Y_i$ in $\mathfrak{D}(X)$ are called Weil divisors on X.

Now given a rational function $f \in \mathcal{R}(X) \setminus \{0\}$ on an irreducible normal variety X we can associate to it a divisor

$$D_f := \sum_{\text{irred. 1-codim } Y \hookrightarrow X} \operatorname{ord}_Y(f) \cdot Y.$$

Note that the above sum is finite, indeed $\operatorname{ord}_Y(f) \neq 0$, iff Y is an irreducible component of $\overline{N(f)}$ or of $\overline{P(f)}$, where $N(f) = f^{-1}(0)$ is the set of the zeros of f and $P(f) = f^{-1}(\infty)$ the set of poles of f. Then

$$\mathcal{R}(X)^* \longrightarrow \mathfrak{D}(X), f \mapsto D_f,$$

is a homomorphism of abelian groups: $D_{fg} = D_f + D_g$. The divisors of the form D_f are called *principal divisors*; its cokernel is the divisor class group of X.

11 Regular Points

In order to study local properties of an affine algebraic variety $X \hookrightarrow k^n$ one could - as in real analysis - think of “approximating“ X near a point $a \in X$ by an affine linear subspace $a + T_aX$, where $T_aX \hookrightarrow k^n$ is a vector subspace of k^n: For a polynomial $f \in k[T]$ and $a \in k^n$ denote

$$d_af := \sum_{i=1}^{n} \frac{\partial f}{\partial T_i}(a) \cdot T_i \ \in \ k]T]_1$$

its "differential“ at $a \in k^n$, i.e. d_af is the linear term in the Taylor expansion of $f(T + a) \in k[T]$ at 0. Then we define provisionally(!) the tangent space of X at the point $a \in X$ as

$$T_aX := N(k^n; d_af, f \in I(X)).$$

Indeed, since $d_a(fg) = f(a)d_ag + g(a)d_af$ we see that

$$T_aX = N(k^n; d_af_1, ..., d_af_r),$$

where the polynomials $f_1, ..., f_r$ generate the ideal $I(X)$.

Example 11.1. For a hypersurface $X = N(k^n; f)$ with squarefree $f \in k[T]$ (such that $I(X) = (f)$) we have $T_a(X) = N(k^n; d_a f)$. So with the "*singular set*" $S(X) := \{a \in X; d_a f = 0\} \hookrightarrow k^n$ we have

$$\dim T_a X = \begin{cases} \dim X & , \quad \text{if } a \in X \setminus S(X) \\ \dim X + 1 & , \quad \text{if } a \in S(X) \end{cases} .$$

For example for $f = T_2^2 - T_1^3$ and $\operatorname{char}(k) \neq 2, 3$ we have $d_a f = 2a_2 T_2 - 3a_1^2 T_1 = 0$ iff $a = (a_1, a_2) = 0$, so the Neil parabola $X = N(k^2; T_2^2 - T_1^3)$ has one singular point, the origin.

Though the geometric information encoded in the tangent space is essentially related to $T_a X$ as a subspace of k^n, it is nevertheless useful to define it in a functorial way as an abstract vector space, depending only on the local ring $\mathcal{O}_{X.a}$: While the only isomorphy invariant of a vector space is its dimension, the functoriality enables us to rediscover embedded versions of the tangent space.

To begin with let us consider a point $a \in X$ in an affine variety $X = \operatorname{Sp}(A)$. We call a linear map $\delta : A \longrightarrow k$ a derivation at a if the Leibniz rule

$$\delta(fg) = \delta(f)g(a) + f(a)\delta(g)$$

holds for all $f, g \in A$. Denote $\operatorname{Der}_a(A)$ the vector space of all derivations of A at $a \in X$.

Example 11.2. For $t \in k^n$ denote $\delta_t^a : k[T] \longrightarrow k, f \mapsto d_a f(t)$, the formal derivative in the direction $t \in k^n$. Then

$$\operatorname{Der}_a(k[T]) = \{\delta_t^a; t \in k^n\} \cong k^n.$$

If $\varrho : k[T] \longrightarrow A$ denotes the restriction map onto $A = k[T]/I(X)$, then

$$\operatorname{Der}_a(A) \longrightarrow \operatorname{Der}_a(k[T]), \delta \mapsto \delta \circ \varrho$$

is an injection with image

$$\operatorname{Der}_a(A) \circ \varrho = \{\delta_t^a; t \in T_a X\}.$$

Indeed, both, the left and the right hand side consist of the derivations of $k[T]$ at a vanishing on $I(X)$.

Lemma 11.3. *There is a natural isomorphism*

$$(\mathfrak{m}_a/\mathfrak{m}_a^2)^* \longrightarrow \mathrm{Der}_a(A), \alpha \mapsto \alpha \circ q,$$

where $q : A = k \oplus \mathfrak{m}_a \longrightarrow \mathfrak{m}_a/\mathfrak{m}_a^2$ *is the composition of the projection onto* $\mathfrak{m}_a$ *with the quotient map* $\mathfrak{m}_a \longrightarrow \mathfrak{m}_a/\mathfrak{m}_a^2$.

Proof. Obviously the given map is injective, and one checks immediately that $\alpha \circ q$ really is a derivation at $a \in X$. On the other hand for any derivation $\delta \in \mathrm{Der}_a(A)$ we have $\delta|_{\mathfrak{m}_a^2} = 0$ as a consequence of the Leibniz rule, so δ induces a linear form $\overline{\delta} : \mathfrak{m}_a/\mathfrak{m}_a^2 \longrightarrow k, f + \mathfrak{m}_a^2 \mapsto \delta(f)$, and since $\delta|_k = 0$ anyway, we obtain $\delta = \overline{\delta} \circ q$. □

In the next step we replace $\mathfrak{m}_a \subset A$ with the maximal ideal $\mathfrak{m}_{X,a} \subset \mathcal{O}_{X,a}$ of the local ring of X at a.

Remark 11.4. For $a \in X = \mathrm{Sp}(A)$ the natural map $A \longrightarrow \mathcal{O}_{X,a}$ induces an isomorphism

$$\mathfrak{m}_a/\mathfrak{m}_a^2 \longrightarrow \mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2.$$

To see this remember that for any multiplicative set the functor $M \mapsto S^{-1}M$ is exact. Here we take $S := A \setminus \mathfrak{m}_a$. Then $\mathfrak{m}_{X,a} = S^{-1}\mathfrak{m}_a$ as well as $\mathfrak{m}_{X,a}^2 = S^{-1}\mathfrak{m}_a^2$, while $S^{-1}(\mathfrak{m}_a/\mathfrak{m}_a^2) = \mathfrak{m}_a/\mathfrak{m}_a^2$, since $f \in S$ acts on $\mathfrak{m}_a/\mathfrak{m}_a^2$ by multiplication with $f(a) \in k^*$.

Definition 11.5. The Zariski tangent space of an algebraic variety X at a point $a \in X$ is defined as

$$T_aX := (\mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2)^*.$$

Remark 11.6.

1. For any morphism $\varphi : X \longrightarrow Y$ of algebraic varieties and $a \in X$ there is an induced homomorphism $T_a\varphi : T_aX \longrightarrow T_{\varphi(a)}Y$. Let $b := \varphi(a)$. The pull back $\varphi^* : \mathcal{O}_{Y,b} \longrightarrow \mathcal{O}_{X,a}$ maps $\mathfrak{m}_{Y,b}$ to $\mathfrak{m}_{X,a}$ as well as $\mathfrak{m}_{Y,b}^2$ to $\mathfrak{m}_{X,a}^2$, so there is an induced homomorphism $\mathfrak{m}_{Y,b}/\mathfrak{m}_{Y,b}^2 \longrightarrow \mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2$,. Its dual is our homomorphism $T_a\varphi$.

2. There is a covariant functor

$$T : (X, a) \mapsto T_aX, \varphi \mapsto T_a\varphi$$

from the category of germs of algebraic varieties to the category of k-vector spaces.

3. The tangent map $T_a\iota : T_aZ \longrightarrow T_aX$ induced by a closed embedding $\iota : Z \hookrightarrow X$ of a closed subvariety Z containing a is injective: The homomorphisms $\mathcal{O}_{X,a} \longrightarrow \mathcal{O}_{Z,a}$ as well $\mathfrak{m}_{X,a} \longrightarrow \mathfrak{m}_{Z,a}$ and $\mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2 \longrightarrow \mathfrak{m}_{Z,a}/\mathfrak{m}_{Z,a}^2$ are surjective. Hence $T_a\iota$, being the dual of the last map, is injective.

Proposition 11.7. *For the tangent space T_aX of an algebraic variety we have*

$$\dim T_aX \geq \dim_a X.$$

Proof. If $X_1, ..., X_r$ are the irreducible components of X passing through a, we have $\dim_a X = \max(\dim_a X_1, ..., \dim_a X_r)$ and $T_aX_i \hookrightarrow T_aX$. So it suffices to prove the claim in the case $r = 1$. Furthermore we may shrink X, hence assume that $X = \mathrm{Sp}(A)$ is affine and that the vector space $\mathfrak{m}_{X.a}/\mathfrak{n}_{X,a}^2$ has a basis $f_1 + \mathfrak{m}_{X,a}^2, ..., f_n + \mathfrak{m}_{X,a}^2$ with regular functions $f_i \in \mathcal{O}(X)$. Then, according to the lemma of Nakayama we have even $\mathfrak{m}_{X,a} = (f_1, ..., f_n)$ resp. after a further shrinking $\mathfrak{m}_a = (f_1, ..., f_n) \hookrightarrow A$. But according to Cor.8.10 every irreducible component of $N(X; f_1, ..., f_n) = \{a\}$ has at least dimension $\dim X - n$, whence $n \geq \dim X$. □

Definition 11.8. A point of an algebraic variety is called a *regular* or *smooth* point if $\dim T_aX = \dim_a X$, otherwise a *singular* point or a *singularity*. We denote $S(X) \subset X$ the set of all singular points of X, also called its *singular locus*. The complement $X \setminus S(X)$ then is the *regular locus* of X.

Example 11.9. 1. If $X = N(k^n; f_1, ..., f_r)$ is of pure dimension d, then a point $a \in X$ is a smooth point iff the Jacobi matrix $\left(\frac{\partial f_i}{\partial T_j}(a)\right)_{1\leq i\leq r, 1\leq j\leq n}$ has a non-vanishing $(n-d) \times (n-d)$-minor.

2. The local ring $\mathcal{O}_{X,a}$ of a curve is a discrete valuation ring iff $a \in X$ is a normal point iff $a \in X$ is a smooth point: A discrete valuation ring R is a UFD, hence integrally closed in its field of fractions, and the quotient $\mathfrak{m}/\mathfrak{m}^2$ is a one-dimensional vector space over $R/\mathfrak{m}$, since the maximal ideal $\mathfrak{m} \hookrightarrow R$ is a principal ideal. On the other hand Th. 10.8 ensures that the local ring at a normal point of a curve is a discrete valuation ring. For a smooth point $a \in X$ we have $\dim_k \mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2 = 1$, hence, according to Nakayamas lemma, $\mathfrak{m}_{X,a}$ is a principal ideal, while $\mathcal{O}_{X,a}$ is an integral domain according to Th.11.10 below and noetherian anyway.

For the next result we need a statement which we shall prove later on:

Theorem 11.10. *The local ring $\mathcal{O}_{X,a}$ of an algebraic variety at a smooth point is an integral domain. Equivalently, a smooth point of a regular variety is contained in exactly one irreducible component of X.*

Theorem 11.11. *The regular locus $X \setminus S(X)$ of an algebraic variety is a dense open set.*

Proof. Let us first show that the singular locus is closed: $S(X) \hookrightarrow X$, resp. that the regular locus $X \setminus S(X)$ is open. Since according to 11.10 the reducible points of X, i.e. those which are contained in several irreducible components of X, are singular we may assume that X is irreducible, indeed even affine. But in that case the description of $S(X)$ of Ex. 11.9.1 by minors gives the result. It remains to show that any irreducible variety X admits smooth points. We start with irreducible hypersurfaces:

Proposition 11.12. *For an irreducible polynomial $f \in k[T_1, ..., T_n]$ there are points $a \in N(k^n; f)$ with $d_a f \neq 0$. In particular the regular locus of $X = N(k^n; f)$. is nonempty.*

Proof. If $d_a f = 0$ for all $a \in X$, we have $\frac{\partial f}{\partial T_i} \in I(X)$ for $i = 1, ..., n$ resp. $f | \frac{\partial f}{\partial T_i}$. Hence all partial derivatives, having a total degree less than that of f, have to vanish. And that means that our base field k has characteristic $p > 0$ and $f \in k[T^p] = k[T_1^p, ..., T_n^p]$, but k being algebraically closed we have $k[T^p] = (k[T])^p$, in particular, f is not irreducible, a contradiction. □

In order to apply that in the general case we need

Proposition 11.13. *If $L = k(a_1, ..., a_r) \subset k$ is a finitely generated field extension, then there is a subset $I \subset \{1, ..., r\}$, such that the $a_i, i \in I$, are algebraically independent and $L \supset E := k(a_i; i \in I)$ is a finite separable field extension.*

Proof. We use induction on r. We may assume that $a_1, ..., a_\ell$ is a maximal set of over k algebraic independent elements. If $\ell = r$ there is nothing to be shown. Otherwise denote $f \in k[T_1, ...T_\ell, T_r] \setminus \{0\}$ a polynomial with $f(a_1, ..., a_\ell, a_r) = 0$. We may assume that f is irreducible. Hence, according to Prop.11.12 some partial derivative $\frac{\partial f}{\partial T_i}, i = 1, ..., \ell, r$, does not vanish, indeed we may assume $i = r$. So a_r is separable over $k(a_1, ..., a_\ell)$ as well as the field extension $L \supset k(a_1, ..., a_{r-1})$. Now apply the induction hypothesis to

$F := k(a_1, ..., a_{r-1})$ and use the fact that two successive separable extensions give again a separable extension. $\square$

Finally we are done with the following

Lemma 11.14. *Any irreducible algebraic variety X (of dimension n, say) contains a non-empty open subset U isomorphic to an open subset V of a hypersurface $N(k^{n+1}; f)$ in affine $n+1$-space:*

$$X \supset U \cong V \subset N(k^{n+1}; f).$$

Proof of Lemma 11.14. According to Prop.11.13 there is an intermediate field $E \cong k(T_1, ..., T_n)$ of the field extension $\mathcal{R}(X) \supset k$, such that $\mathcal{R}(X) \supset E$ is a finite separable extension. The primitive element theorem now provides an element $\vartheta \in \mathcal{R}(X)$ with $\mathcal{R}(X) = E[\vartheta]$; indeed we may even (after possibly multiplying with some polynomial in $k[T_1, ..., T_n]$) assume that ϑ is integral over $k[T_1, ..., T_n]$. Then we have $f \in (k[T_1, ..., T_n])[T_{n+1}]$ for the minimal polynomial f of ϑ over $k(T_1, ..., T_n)$. Thus $N(k^{n+1}; f)$ and X have isomorphic rational function fields, and we may apply 9.5. $\square$

$\square$

Remark 11.15. Let us comment on smooth points in complex algebraic varieties. We may assume that $X \hookrightarrow \mathbb{C}^n$ is affine, let $I(X) = (f_1, ..., f_r)$. Consider a smooth point $a \in X$. W.l.o.g. $d_a f_1, ..., d_a f_\ell$ form a basis of $\sum_{i=1}^r \mathbb{C} d_a f_i \hookrightarrow \mathbb{C}[T]_1$, and the Jacobi matrix $\left(\frac{\partial f_i}{\partial T_j}(a)\right)_{1 \le i,j \le \ell}$ is nonsingular. Then

$$F : t \mapsto (z_1, ..., z_n) := (f_1(t), ..., f_\ell(t), t_{\ell+1}, ..., t_n)$$

maps an open neighborhood U of a biholomorphically to a ball V centered at $F(a)$ and

$$F(X \cap U) \subset N(V; z_1, ..., z_\ell) \hookrightarrow V.$$

But we may assume that $\dim_b X = \dim_a X$ for all $b \in U$, hence the functions $g_{\ell+1} := f_{\ell+1} \circ F^{-1}, ..., g_r := f_r \circ F^{-1}$ do not depend on $z_{\ell+1}, ..., z_n$ - otherwise we would obtain $\dim T_b X < n - \ell = \dim_a X = \dim_b X$ for some points $b \in U$. So $g_i(0, ..., 0, z_{\ell+1}, ..., z_n) = g_i(F(a)) = 0$ even for $i \ge \ell + 1$ and thus

$$F(X \cap U) = N(V; z_1, ..., z_\ell) \hookrightarrow V,$$

i.e., the point $a \in X$ admits with $X \cap U$ a strong neighborhood, which is, as a complex analytic variety, isomorphic to an open ball in $\mathbb{C}^{n-\ell}$.

In contrast to that it is quite far from being true – even for $k = \mathbb{C}$ – that a smooth point $a \in X$ admits a Zariski-neighborhood isomorphic (as algebraic variety) to an Zariski-open subset of k^n (where $n = \dim_a X$). Indeed, this would imply that (w.l.o.g. X being irreducible) $\mathcal{R}(X) \cong k(T_1, ..., T_n)$ is a purely transcendent extension of k.

In order to investigate the local structure of a variety near a point $a \in X$ a little bit closer we replace the tangent space T_aX with the tangent cone $TC_a(X) \hookrightarrow T_aX$. As before we start with an embedded description for $X \hookrightarrow k^n$ at $a = 0$. While the tangent space T_0X is described by the linear parts of the polynomials vanishing on X, every nonzero polynomial $f \in I(X)$ contributes to the description of the tangent cone $TC_0(X)$ with its (non-vanishing) homogeneous component $f^{\min}$ of lowest degree: For such a polynomial $f = \sum_{i=q}^{r} f_i \in k[T] = \bigoplus_{i=0}^{\infty} k[T]_i$ we define

$$f^{\min} := f_q, \text{ if } f_q \neq 0.$$

Then

$$TC_0(X) := N(I_0(X)) \subset T_0X$$

with the homogeneous ideal

$$I_0(X) := \sum_{f \in I(X)^*} k[T] \cdot f^{\min}.$$

and $I(X)^* := I(X) \setminus \{0\}$.

Example 11.16. Since $(fg)^{\min} = f^{\min} \cdot g^{\min}$, the tangent cone of a hypersurface $X = N(k^n; f)$ with squarefree f is a hypersurface as well:

$$I_0(X) = (f^{\min}) \text{ and } TC_0(X) = N(k^n; f^{\min}).$$

For example the noose $X = N(k^2; T_2^2 - T_1^2(T_1 + 1)$ has tangent cone

$$TC_0(X) = N(k^2; T_2^2 - T_1^2) = k(1,1) \cup k(1,-1) \hookrightarrow k^2 = T_0X$$

a union of two lines (if $\mathrm{char}(k) \neq 2$). In particular, the tangent cone can detect the two branches of X at 0 in a purely algebraic way!

On the other hand for the Neil parabola $X = N(T_2^2 - T_1^3)$ we have

$$I_0(X) = (T_2^2)$$

and hence

$$TC_0(X) = k(1,0).$$

So we see, that $I_0(X)$ need not be a radical ideal!

Let us now present a description of $TC_a(X)$ depending only on the local ring $\mathcal{O}_{X,a}$ of X at a. We start with an algebraic construction, the associated graded algebra:

Definition 11.17. Let R be a ring and $\mathfrak{m} \hookrightarrow R$ a maximal ideal. The associated graded algebra $\mathrm{gr}_{\mathfrak{m}}(R)$ is the direct sum

$$\mathrm{gr}_{\mathfrak{m}}(R) := \bigoplus_{q=0}^{\infty} \mathfrak{m}^q/\mathfrak{m}^{q+1},$$

the multiplication being the bilinear extension of

$$\mathfrak{m}^q/\mathfrak{m}^{q+1} \times \mathfrak{m}^s/\mathfrak{m}^{s+1} \longrightarrow \mathfrak{m}^{q+s}/\mathfrak{m}^{q+s+1},$$

$$(a + \mathfrak{m}^{q+1}, b + \mathfrak{m}^{s+1}) \mapsto ab + \mathfrak{m}^{q+s+1}.$$

If R is a local ring, then $\mathrm{gr}(R) := \mathrm{gr}_{\mathfrak{m}}(R)$ with the unique maximal ideal $\mathfrak{m} \hookrightarrow R$.

Remark 11.18.

1. Note that there is no natural ring homomorphism $R \longrightarrow \mathrm{gr}(R)$. Nevertheless for $f \in R$ we can define

   $$\mathrm{gr}(f) := \begin{cases} f + \mathfrak{m}^{q+1} \in \mathrm{gr}(R)_q & , \text{ if } f \in \mathfrak{m}^q \setminus \mathfrak{m}^{q+1} \text{ for some } q \in \mathbb{N} \\ 0 & , \text{ otherwise} \end{cases},$$

   but that does not yield a homomorphism. We note that $\mathrm{gr}(fg) = \mathrm{gr}(f) \cdot \mathrm{gr}(g)$ holds, if $\mathrm{gr}(R)$ is an integral domain and that $\mathrm{gr}(f) = 0$ iff $f \in \bigcap_{q=1}^{\infty} \mathfrak{m}^q$.

2. The ideal $\mathfrak{m}$ gives rise to a filtration $(\mathfrak{m}^q)_{q \in \mathbb{N}}$ of the ring R, but only if we can find for every q a complementary additive subgroup $R_q \subset \mathfrak{m}^q$, i.e. satisfying $\mathfrak{m}^q = R_q \oplus \mathfrak{m}^{q+1}$, such that $R_i \cdot R_j \subset R_{i+j}$, then $\mathrm{gr}(R)$ can be identified with the subring $\bigoplus_{q=0}^{\infty} R_q$ of R. For example this is true for $R = k[T]$ and $\mathfrak{m} = \mathfrak{m}_0$: Take simply $R_q = k[T]_q$, the vector subspace of polynomials homogeneous of degree q.

3. For the local ring $S^{-1}R$ with $S := R \setminus \mathfrak{m}$ we have

$$\mathrm{gr}(S^{-1}R) \cong \mathrm{gr}_{\mathfrak{m}}(R)$$

due to the fact that $S^{-1}(\mathfrak{m}^q/\mathfrak{m}^{q+1}) \cong \mathfrak{m}^q/\mathfrak{m}^{q+1}$, the elements in S already acting as isomorphisms on the $R/\mathfrak{m}$-vector spaces $\mathfrak{m}^q/\mathfrak{m}^{q+1}$.

Proposition 11.19. *For $X = \mathrm{Sp}(A) \hookrightarrow k^n$ we have*

$$k[T]/I_0(X) \cong \mathrm{gr}(\mathcal{O}_{X,0}).$$

Proof. According to Rem.11.18.3 we have

$$\mathrm{gr}(\mathcal{O}_{X,0}) \cong \mathrm{gr}_{\mathfrak{m}_0}(A).$$

On the other hand the quotient homomorphism

$$\varrho : k[T] \longrightarrow A, f \mapsto f|_X$$

induces a surjective homomorphism

$$\mathrm{gr}(\varrho) : k[T] \cong \mathrm{gr}(k[T]) \longrightarrow \mathrm{gr}(A),$$

where the graded algebras are taken with respect to $\mathfrak{m}_0 \hookrightarrow A$ and $\mathfrak{m} := \varrho^{-1}(\mathfrak{m}_0) \hookrightarrow k[T]$. We have to show that $\ker(\mathrm{gr}(\varrho)) = I_0(X)$. For a homogeneous polynomial $f \in k[T]_q$ we have

$$\mathrm{gr}(\varrho)(f) = 0 \Longleftrightarrow f|_X \in \mathfrak{m}_0^{q+1} \Longleftrightarrow f \in \mathfrak{m}^{q+1} + I(X),$$

i.e. $f = g^{\min}$ for some $g \in I(X)$. $\square$

Definition 11.20. The reduction $\mathrm{Red}(R)$ of a ring R is defined as

$$\mathrm{Red}(R) := R/\sqrt{0},$$

where $\sqrt{0} \hookrightarrow R$ denotes the nilradical of R.

Remark 11.21. 1. The reduction $\mathrm{Red}(A)$ of a graded algebra $A = \bigoplus_{q=0}^{\infty} A_q$ is again a graded algebra, its nilradical $\sqrt{0} \hookrightarrow A$ being a graded ideal.

2. A graded reduced affine algebra $A = \bigoplus_{q=0}^{\infty} A_q$ is (as graded algebra) isomorphic to the algebra of regular functions on an affine cone iff A, as a k-algebra, is generated by the elements of degree 1.

Definition 11.22. The tangent cone of an algebraic variety X at a point $a \in X$ is defined as the affine cone

$$TC_a(X) := \mathrm{Sp}\left(\mathrm{Red}(\mathrm{gr}(\mathcal{O}_{X,a}))\right).$$

Remark 11.23. There is a natural embedding

$$TC_a(X) \hookrightarrow T_aX,$$

since the linear map

$$\mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2 = \mathrm{gr}(\mathcal{O}_{X,a})_1 \hookrightarrow \mathrm{gr}(\mathcal{O}_{X,a}) \longrightarrow \mathrm{Red}(\mathrm{gr}(\mathcal{O}_{X,a}))$$

extends uniquely to a graded epimorphism

$$S^*(\mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2) \longrightarrow \mathrm{Red}(\mathrm{gr}(\mathcal{O}_{X,a})) \cong \mathcal{O}(TC_a(X))$$

and $S^*(\mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2) \cong \mathcal{O}(T_aX)$ because of $(T_aX)^* \cong \mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2$.

In contrast to the tangent space, the tangent cone has always the same dimension as the original variety:

Theorem 11.24. *For a pure d-dimensional variety X all the tangent cones $TC_a(X)$ at its points $a \in X$ are pure d-dimensional as well.*

Proof. The idea of the proof is as follows: First of all we may assume that $X = \mathrm{Sp}(A) \hookrightarrow k^n$ and $a = 0$. We then construct a morphism $\pi : Z \longrightarrow k^n$, called "*the blow up of k^n at the origin*", such that

1. Z is irreducible and can be covered by n open subsets $W_i \cong k^n$,
2. the morphism $\pi : Z \longrightarrow k^n$ induces an isomorphism

$$\pi|_{Z^*} : Z^* \longrightarrow (k^n)^*$$

with $Z^* := \pi^{-1}((k^n)^*)$, and

3. the "*exceptional fiber*" $\pi^{-1}(0) \hookrightarrow Z$ is a local hypersurface isomorphic to projective $(n-1)$-space $\mathbb{P}_{n-1}$.

The "*strict transform*" $\widehat{X}$ of $X \hookrightarrow k^n$ with respect to π is obtained from $X^* := X \setminus \{0\}$ as the closure

$$\widehat{X} := \overline{\pi^{-1}(X^*)} \hookrightarrow Z,$$

the inclusion $\widehat{X} \hookrightarrow \pi^{-1}(X)$ being proper in general. We shall prove that the isomorphism

$$\pi^{-1}(0) \cong \mathbb{P}_{n-1}$$

induces an isomorphism

$$\widehat{X} \cap \pi^{-1}(0) \cong N(\mathbb{P}_{n-1}; I_0(X)).$$

Hence, the projective variety $N(\mathbb{P}_{n-1}; I_0(X))$, being isomorphic to a local hypersurface with dense complement in the pure-d-dimensional variety $\widehat{X}$, is pure of codimension 1 in $\widehat{X}$. The tangent cone being the affine cone over it, then is of pure dimension $\dim X$.

Blow up: Let us give first a heuristic description of Z: It is obtained from k^n by a sort of surgery, one removes the origin and replaces it with a copy of $\mathbb{P}_{n-1}$ in such a way, that the closure of $k^* \cdot t$ in Z is obtained by adding the limit point $[t] \in \mathbb{P}_{n-1}$.

We realize the variety Z as the closure

$$Z := \overline{\Gamma}_\varphi \hookrightarrow k^n \times \mathbb{P}_{n-1}$$

of the graph

$$\Gamma_\varphi \hookrightarrow (k^n)^* \times \mathbb{P}_{n-1}$$

of the quotient morphism

$$\varphi : (k^n)^* \longrightarrow \mathbb{P}_{n-1}, t \mapsto [t]$$

in $k^n \times \mathbb{P}_{n-1}$. We have

$$Z = N(k^n \times \mathbb{P}_n; S_iT_j - S_jT_i, 1 \le i < j \le n\},$$

where the points in $k^n \times \mathbb{P}_{n-1}$ are denoted $(t, [s])$ with $t, s \in k^n$ and $S_iT_j - S_jT_i \in k[T_1, ..., T_n, S_1, ..., S_n]$. In order to understand the geometry of the variety Z we consider the restricted projection

$$p : Z \hookrightarrow k^n \times \mathbb{P}_{n-1} \xrightarrow{\mathrm{pr}_2} \mathbb{P}_{n-1}.$$

Above the standard open subset $V_i := \mathbb{P}_{n-1} \setminus N(\mathbb{P}_{n-1}; S_i)$ there is a trivialization

$$\tau_i : V_i \times k \xrightarrow{\cong} W_i := p^{-1}(V_i),$$

$$([s], \lambda) \mapsto (\lambda s_i^{-1} s, [s]).$$

Hence $Z = \bigcup_{i=1}^{n} W_i$ with the open subsets $W_i \cong V_i \times k \cong k^{n-1} \times k = k^n$. Since each of the subsets $W_i \subset Z$ is dense because of $W_i \cap W_j \cong (V_i \cap V_j) \times k$, we get the first part of our claim about Z, and the exceptional fiber is a local hypersurface, since

$$\pi^{-1}(0) \cap W_i = \tau_i(V_i \times 0) \hookrightarrow \tau_i(V_i \times k) \cong W_i.$$

Strict transform: We have an isomorphism

$$\pi \circ \tau_i : V_i \times k^* \longrightarrow (k^n)_{T_i}$$

given by

$$([s], \lambda) \mapsto \lambda s_i^{-1} s.$$

In order to determine $\widehat{X}_i := \widehat{X} \cap W_i$ we look at $V_i \times k^* \subset V_i \times k \cong W_i$ and compute the closure of

$$(\pi \circ \tau_i)^{-1}(X_{T_i}) \hookrightarrow V_i \times k^* \subset V_i \times k$$

in $V_i \times k$. A regular function in $k[T]_{T_i}$ vanishing on X_{T_i} is of the form $T_i^{-\ell} f$ with a polynomial $f \in I(X) \hookrightarrow k[T]$. If $f = \sum_q f_q \in I(X) \subset k^n$ with homogeneous polynomials f_q of degree q, we find

$$f \circ \pi \circ \tau_i = \Lambda^{-\ell} f\left(\Lambda \frac{S}{S_i}\right) = \sum_q f_q\left(\frac{S}{S_i}\right) \Lambda^{q-\ell}.$$

Now $\tau_i^{-1}(\widehat{X} \cap W_i) \hookrightarrow V_i \times k$ is the zero locus of all regular functions on $V_i \times k$, whose restriction to $V_i \times k^*$ vanishes on $(\pi \circ \tau_i)^{-1}(X_{T_i})$. These functions being of the above form with only $q \geq \ell$, either vanish on $V_i \times 0$ or yield $f^{\min}(S_i^{-1} S)$. Since that holds for $i = 1, ..., n$, we see that

$$(0 \times \mathbb{P}_{n-1}) \cap \widehat{X} = 0 \times N(\mathbb{P}_{n-1}; I_0(X))$$

as desired. □

Let us now draw some conclusions:

Remark 11.25. If $a \in Y \hookrightarrow X$, then obviously $TC_a(Y) \hookrightarrow TC_a(X)$. Furthermore, the closure of a finite union being the finite union of the separate closures, we see that, if $X_1, ..., X_r$ are the irreducible components of X passing through a, then

$$\varphi(TC_a(X)^*) = \bigcup_{i=1}^{r} \varphi(TC_a(X_i)^*)$$

with the quotient morphism $\varphi : (k^n)^* \longrightarrow \mathbb{P}_{n-1}$. It follows

$$TC_a(X) = \bigcup_{i=1}^{r} TC_a(X_i).$$

Note that the corresponding statement does not hold for the tangent spaces: The union of the subspaces $T_a(X_i) \hookrightarrow T_aX$ will in general be no subspace anymore - e.g. consider $X = N(k^2; T_1T_2)$ at $a = 0$, but it is not even true that $T_aX = \sum_{i=1}^{r} T_a(X_i)$, as the example $X = N(k^2; T_2(T_2 - T_1^2))$ shows.

Theorem 11.26. *A point $a \in X$ in a variety X is a smooth point iff $\mathrm{gr}(\mathcal{O}_{X,a})$ is a polynomial ring.*

Proof. Let $g_1, ..., g_n$ be a basis of $\mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2$. Then

$$\sigma : k[S_1, ..., S_n] \longrightarrow \mathrm{gr}(\mathcal{O}_{X,a}), S_i \mapsto g_i,$$

is an epimorphism. With other words

$$TC_a(X) = \mathrm{Sp}\left(\mathrm{Red}(\mathrm{gr}(\mathcal{O}_{X,a}))\right) \cong N\left(k^n; \sqrt{\ker(\sigma)}\right) \hookrightarrow k^n.$$

"$\Longleftarrow$": If $\mathrm{gr}(\mathcal{O}_{X,a})$ is a polynomial ring, then $TC_a(X) \cong k^n$ and $\dim_k \mathrm{gr}(\mathcal{O}_{X,a})_1 = \dim TC_a(X)$, whence

$$\dim_k T_aX = \dim_k \mathrm{gr}(\mathcal{O}_{X,a})_1 = \dim TC_a(X) = \dim_a X,$$

so X is smooth at a.
"$\Longrightarrow$": If $a \in X$ is a smooth point and $\dim_a X = n$, the n-dimensional subvariety $TC_a(X) \hookrightarrow k^n$ coincides with k^n. In particular $\ker(\sigma) = \{0\}$ and $\sigma : k[S] \longrightarrow \mathrm{gr}(\mathcal{O}_{X,a})$ is an isomorphism. ∎

Remark 11.27. Note that for a smooth point $a \in X$ with $\dim_a X = n$ we have $TC_a(X) \cong k^n$, but $TC_a(X) \cong k^n$ does not imply that $a \in X$ is a smooth point. For example for the Neil parabola $X = N(k^2; T_2^2 - T_1^3)$ we have $TC_0(X) = k \times 0 \cong k$, while $\mathrm{gr}(\mathcal{O}_{X,0}) \cong k[T_1, T_2]/(T_2^2)$.

In order to prove Th, 11.10, we need a further result of commutative algebra:

Theorem 11.28 (Krull Intersection Theorem)**.** *Let M be a finitely generated module over the local noetherian ring R with maximal ideal $\mathfrak{m}$. Then*

$$\bigcap_{q=1}^{\infty} \mathfrak{m}^q M = \{0\}.$$

Proof of Th.11.10. Take $f, g \in \mathcal{O}_{X,a} \setminus \{0\}$. The Krull intersection theorem gives $\mathrm{gr}(f) \neq 0 \neq \mathrm{gr}(g)$, hence $\mathrm{gr}(\mathcal{O}_{X,a})$ being an integral domain, we obtain $0 \neq \mathrm{gr}(f) \cdot \mathrm{gr}(g) = \mathrm{gr}(fg)$, in particular $fg \neq 0$. □

Proof of 11.28. Since R is noetherian, the module M is noetherian as well: So the submodule

$$D := \bigcap_{q=1}^{\infty} \mathfrak{m}^q M$$

is finitely generated. If we knew that $\mathfrak{m}D = D$, the lemma of Nakayama would give the result, but we do not and have to use a more sophisticated argument. The module M being noetherian, there is a maximal submodule $F \subset M$ with $D \cap F = 0$. Furthermore, according to Nakayama's lemma $D = 0$ iff $D/\mathfrak{m}D = 0$. Hence we may replace M with $M/\mathfrak{m}D$ resp. assume that $\mathfrak{m}D = 0$. Now we are looking for some $q \in \mathbb{N}$ satisfying $\mathfrak{m}^q M \subset F$. Namely, if we succeed with that, then

$$D = D \cap \mathfrak{m}^q M \subset D \cap F = 0.$$

We show that for every $a \in \mathfrak{m}$ there is some exponent $s \in \mathbb{N}$ with $a^s M \subset F$. Hence, if $\mathfrak{m} = (a_1, ..., a_r)$ and $s_i \in \mathbb{N}$ for a_i as above, then we have $\mathfrak{m}^q M \subset F$ with $q := r \max(s_1, ..., s_r)$. So, given an element $a \in \mathfrak{m}$ let us look for an appropriate exponent $s \in \mathbb{N}$. Indeed, if $(a^s M + F) \cap D = 0$, it already follows from the maximality of F, that $F = a^s M + F$ resp. $a^s M \subset F$. Now the

$$M_k := \{u \in M; a^k u \in F\}$$

form an increasing sequence of submodules $M_k \subset M$; hence M being noetherian, we have $M_s = M_{s+1}$ for some $s \in \mathbb{N}$. For $b = a^s m + f \in (a^s M + F) \cap D$ we then obtain

$$ab = a^{s+1} m + af \in aD \subset \mathfrak{m} D = 0,$$

i.e. $ab = 0$ and $a^{s+1} m = -af \in F$ resp. $m \in M_{s+1} = M_s$ and already $a^s m \in F$. Hence $b \in F$ and thus $b \in F \cap D = 0$, i.e. $b = 0$ as desired. $\square$

Eventually we want to discuss a little bit more in detail the structure of a local ring at a regular point.

Definition 11.29. Let R be a noetherian local ring. The $\mathfrak{m}$-adic completion $\widehat{R}$ of R is the ring

$$\widehat{R} := \varprojlim R/\mathfrak{m}^n \subset \prod_{n=1}^{\infty} R/\mathfrak{m}^n$$

consisting of the sequences $(a_n + \mathfrak{m}^n)_{n \geq 1}$ satisfying $a_{n+1} - a_n \in \mathfrak{m}^n$ for all $n \in \mathbb{N}_{>0}$. With other words, a sequence in $\widehat{R}$ is obtained recursively as follows: Start with any element in $R/\mathfrak{m}$, and given the n-th element take as its immediate successor any inverse image of it with respect to the natural map $R/\mathfrak{m}^{n+1} \longrightarrow R/\mathfrak{m}^n$.

Example 11.30.

1. For $R = \mathcal{O}_{X.a}$ with $X = \mathrm{Sp}(A)$ we have

   $$\widehat{\mathcal{O}}_{X,a} \cong \varprojlim A/(\mathfrak{m}_a)^n,$$

   to be seen as follows: Since the elements in $S := A \setminus \mathfrak{m}_a$ become units in the local ring $A/(\mathfrak{m}_a)^n$, we have

   $$\mathcal{O}_{X,a}/(\mathfrak{m}_{X,a})^n \cong S^{-1}(A/(\mathfrak{m}_a)^n) \cong A/(\mathfrak{m}_a)^n.$$

2. For $a \in k^n$ we have

   $$\widehat{\mathcal{O}}_{k^n,a} \cong k[[T_1, ..., T_n]],$$

   where $k[[T_1, ..., T_n]]$ denotes the ring of formal power series in n indeterminates with coefficients in k. Assuming $a = 0$ apply the previous point to $X = k^n$.

Proposition 11.31. *A point $a \in X$ is a simple point iff the completion $\widehat{\mathcal{O}}_{X,a}$ is isomorphic to a formal power series ring.*

Proof. Let $a \in X$ be a smooth point. We may assume that $X = \mathrm{Sp}(A)$. Take functions $f_1, ..., f_n \in \mathfrak{m}_a$ providing a basis of $\mathfrak{m}_a/(\mathfrak{m}_a)^2$ and consider the map

$$k[T_1, ..., T_n] \longrightarrow A, T_i \mapsto f_i.$$

It induces surjections $k[T] \longrightarrow A/(\mathfrak{m}_a)^n$ for every $n \in \mathbb{N}$. On the other hand, $\mathrm{gr}(\mathcal{O}_{X,a})$ being a polynomial ring in the indeterminates $\mathrm{gr}(f_i)$, we conclude that $k[T]/(\mathfrak{m}_0)^n$ and $A/(\mathfrak{m}_a)^n$ have the same dimension as a k-vector space, so the induced map

$$k[T_1, ..., T_n]/(\mathfrak{m}_0)^n \longrightarrow A/(\mathfrak{m}_a)^n$$

is an isomorphism for all $n \in \mathbb{N}$. Now apply Ex.11.30.2. On the other hand

$$\mathrm{gr}(\mathcal{O}_{X,a}) \cong \mathrm{gr}(\widehat{\mathcal{O}}_{X,a})$$

gives with Th.11.26 the reverse implication. □

Theorem 11.32. *The local ring $\mathcal{O}_{X,a}$ of an algebraic variety X at a smooth point a is factorial.*

A possibility to prove Th.11.32, is to use the fact that the formal power series ring $\widehat{\mathcal{O}}_{X,a} \cong k[[T_1, ..., T_d]]$ is known to be factorial, but we prefer here a geometric proof of 11.32. Of course we may assume that $X = \mathrm{Sp}(A)$ is an affine irreducible variety. To any subvariety $Z \hookrightarrow X$ passing through a we associate the ideal

$$I(Z)_a := \{f \in \mathcal{O}_{X,a}; f \in I(\mathbb{D}_f \cap Y) \subset \mathcal{O}(\mathbb{D}_f)\}.$$

Theorem 11.33. *Let $a \in X$ be a smooth point of the irreducible affine variety $X = \mathrm{Sp}(A)$. Then for every irreducible subvariety $Z \hookrightarrow X$ of codimension 1 passing through a, the ideal $I(Z)_a \subset \mathcal{O}_{X,a}$ is a principal ideal.*

Proof of Th.11.32. Since $I(Z)_a = (f)$ for $Z \hookrightarrow X$ irreducible of codimension 1 is a prime ideal, the generator f is a prime element. It remains to show that every non-unit is a product of such generators. To see that we consider the set of proper principal ideals $(g) \subset \mathcal{O}_{X,a}$ with a generator g not admitting a factorization. Take a maximal such ideal (g). Assuming $g \in A$ we have $a \in N(g) \hookrightarrow X$. Let $Z \hookrightarrow N(g)$ be an irreducible component passing through a. Then Z has codimension 1 according to Krull's principal ideal theorem, Th.8.6, and thus $I(Z)_a = (f)$. Hence $g = hf$ with $(h) \supsetneqq (g)$. So h admits a factorization and thus g as well: Contradiction! □

Proof of Th.11.33. First let us reduce the problem to the case where

1. $X \hookrightarrow k^{d+1}$ is a hypersurface and $a = 0$, and
2. the restriction $\psi : X \longrightarrow k^d$ of the projection $k^{d+1} \longrightarrow k^d$ onto the first d coordinates is finite and $Z \cap \psi^{-1}(0) = \{0\}$.

The following lemma describes how to reduce successively the embedding dimension of X:

Lemma 11.34. *Let $X \hookrightarrow k^n$ be a d-dimensional irreducible variety, $0 \in X$ a regular point, and $Z \hookrightarrow X$ a 1-codimensional irreducible subvariety passing through 0. Then there is a projection $\pi_{n-1} : k^n \longrightarrow k^{n-1}$ such that for $\psi = \psi_{n-1} := \pi_{n-1}|_X$ the following holds:*

1. *ψ is finite,*
2. *the tangent map $T_0\psi : T_0X \longrightarrow T_0k^{n-1}$ is injective,*
3. *for $n \geq d+2$ we have $\psi^{-1}(0) = \{0\}$, and ψ induces an isomorphism of germs $(X, 0) \cong (\psi(X), 0)$, finally*
4. *for $n = d+1$ we have still $\psi^{-1}(0) \cap Z = \{0\}$.*

Proof. A projection (:= surjective linear map) $\pi : k^n \longrightarrow k^{n-1}$ is up to a coordinate change in k^{n-1} determined by its kernel $L := \ker(\pi)$, a line. In order to find the good choice of L we consider $k^n \cong U_0 \subset \mathbb{P}_n$ as a subset of projective n-space. Denote $\widehat{X}$ the projective closure of X and $X_\infty := \widehat{X} \setminus X$ its part at infinity, furthermore

$$q : \mathbb{P}_n \setminus \{[1, 0, ..., 0]\} \longrightarrow N(\mathbb{P}_n; T_0), [t] \mapsto [0, t_1, ..., t_n]$$

the projection onto $N(\mathbb{P}_n; T_0) \cong \mathbb{P}_{n-1}$. For $n \geq d+2$ consider the set

$$E := \overline{q(X \setminus \{0\})} \cup q(T_0X \setminus \{0\}) \cup X_\infty$$

with the identification $T_0X \hookrightarrow T_0k^n \cong k^n$. It is a proper subset of $N(\mathbb{P}_n; T_0) \cong \mathbb{P}_{n-1}$ for dimensional reasons: $\dim X_\infty = d-1$, since $X_\infty \hookrightarrow \widehat{X}$ is a local hypersurface, and $q(T_0X \setminus \{0\}) \cong \mathbb{P}_{d-1}$, while $\dim \overline{q(X \setminus \{0\})} \leq \dim X = d$. For $n = d+1$ let

$$E := \overline{q(Z \setminus \{0\})} \cup q(T_0X \setminus \{0\}) \cup X_\infty.$$

Now for $L = (q^{-1}(a) \cap U_0) \cup \{0\}$ with $a \notin E$ the conditions 2), 4) and the first one of 3) are obviously satisfied. It remains to show the finiteness. After a linear change of coordinates we may assume that $L = ke_n$ resp. that π_{n-1} is the projection onto the first $n-1$ coordinates. Then we have to find an integral equation for $T_n|_X$ over $k[T_1, ..., T_{n-1}]$. Since $[0, ..., 0, 1] \notin \widehat{X}$, there is a polynomial $f \in I(X) \subset k[T_1, ..., T_n]$, such that its homogeneization $\widehat{f} = T_0^r f(T_0^{-1}T)$ satisfies $\widehat{f}(0, .., 0, 1) = 1$ and thus $\widehat{f} \in (k[T_0, ..., T_{n-1}])[T_n]$ as well as $f = \widehat{f}(1, T_1, ..., T_n)$ are T_n-monic - this gives the desired integral equation. Finally let us show the second part of 3): Set $Y := \psi(X) \hookrightarrow k^{n-1}$. Since ψ is finite and $\psi^{-1}(0) = \{0\}$, the sets $\psi^{-1}(V)$ form a neighbourhood basis of $0 \in X$. As a consequence, if $\mathcal{O}(X) = \mathcal{O}(Y)f_1 + ... + \mathcal{O}(Y)f_r$, we have as well $\mathcal{O}_{X,0} = \mathcal{O}_{Y,0}f_1 + ... + \mathcal{O}_{Y,0}f_r$. Since $T_0\psi : T_0X \longrightarrow T_0Y$ is injective, $\psi^* : \mathfrak{m}_{Y,0}/\mathfrak{m}_{Y,0}^2 \longrightarrow \mathfrak{m}_{X,0}/\mathfrak{m}_{X,0}^2$ is surjective. Now we apply Nakayama's lemma to the finite $\mathcal{O}_{X,0}$-module $\mathfrak{m}_{X,0}$ and obtain $\mathfrak{m}_{X,0} = \mathcal{O}_{X,0} \cdot \psi^*(\mathfrak{m}_{Y,0})$. Hence

$$\mathcal{O}_{X,0}/\mathcal{O}_{X,0}\psi^*(\mathfrak{m}_{Y,0}) = \mathcal{O}_{X,0}/\mathfrak{m}_{X,0} \cong k.$$

So 1 is a generator of the finite $\mathcal{O}_{Y,0}$-module $\mathcal{O}_{X,0}$ since it is mod $\mathfrak{m}_{Y,0}$, i.e. $\psi^* : \mathcal{O}_{Y,0} \longrightarrow \mathcal{O}_{X,0}$ is surjective, while injectivity follows from the fact that $\psi : X \longrightarrow Y$ is dominant and $\psi^{-1}(0) = \{0\}$. $\square$

The special situation as described in the beginning is treated with the tools of the proof of Krull's principal ideal theorem. Let $X = \mathrm{Sp}(A)$ and treat the pull back homomorphism $\psi^* : k[T] \hookrightarrow A$ as an inclusion. We write $I(\psi(Z)) = k[T] \cdot h$ with some polynomial $h \in k[T]$ and show that $I(Z)_0 = \mathcal{O}_{X,0} \cdot h$. If $\psi^{-1}(0) = \{x_1 = 0, x_2, ..., x_r\}$, we are done, if we succeed in showing

1. Any function $g \in I(Z)$ with $g(x_i) \neq 0$ for $i = 2, ..., r$ lies in $(h) \hookrightarrow \mathcal{O}_{X,0}$.
2. The ideal $I(Z)_0$ is generated by functions $g \in I(Z)$ as above.

1.) For the characteristic polynomial $\chi(T, S) \in (k[T_1, ..., T_d])[S]$ of the multiplication $\mu_g \in \mathrm{End}_{k(T)}(Q(A))$ we have

$$\chi(T, S) = p(T, S)S + (-1)^\ell\ \mathfrak{N}(g)\ \in (k[T])[S]$$

with the extension degree $\ell := [Q(A) : k(T)]$. Due to the Hamilton-Cayley theorem we have $\chi(\mu_g) = 0$ resp. $\chi(g) = 0$, hence

$$\mathfrak{N}(g) = (-1)^{\ell-1}p(T, g)g$$

yields $\mathfrak{N}(g) \in I(\psi(Z)) = k[T]\cdot h$; so it suffices to show that $\widehat{g} := p(T, g) \in A$ is invertible in $\mathcal{O}_{X,0} \supset A$ or, equivalently, that $\widehat{g}(0) = p(0, g(0)) = p(0, 0) \neq 0$. To see that we consider the induced multiplication map

$$\mu_{\overline{g}} : A/(T_1, ..., T_d) \longrightarrow A/(T_1, ..., T_d)$$

over the fiber $\psi^{-1}(0) = N(X; T_1, ..., T_d)$ with $\overline{g} := g+(T_1, ..., T_d) \in A/(T_1, ..., T_d)$. Note that the algebra $A/(T_1, ..., T_d)$ need not be reduced, but in any case we have

$$\mathrm{Red}(A/(T_1, ..., T_d)) \cong \mathcal{O}(\psi^{-1}(0)) \cong k^r.$$

If now $f(T, S) \in (k[T])[S]$ denotes the minimal polynomial of $T_{d+1}|_X$ over $k(T_1, ..., T_d)$ (since $k[T_1, ..., T_d]$ is integrally closed in its field of fractions $k(T_1, ..., T_d)$, it has coefficients in $k[T_1, ..., T_d]$), we have

$$A \cong k[T_1, ..., T_{d+1}]/(f(T, T_{d+1})).$$

Hence

$$A/(T_1, ..., T_d) \cong k[T_{d+1}]/(f(0, T_{d+1})),$$

and

$$f(0, T_{d+1}) = T_{d+1} \prod_{i=2}^{r} (T_{d+1} - a_i)^{m_i},$$

where $x_i = (0, a_i) \in k^d \times k, i = 2, ..., r$, are the points $\neq 0$ in the fiber $\psi^{-1}(0)$. The polynomial $f(0, T_{d+1})$ has a simple zero at the origin, since otherwise $\frac{\partial f}{\partial T_{d+1}}(0) = 0$ gives $0 \times k \subset T_0 X$, which contradicts the injectivity of the tangent map

$$T_0\psi : T_0 X = \ker d_0 f \longrightarrow T_0 k^d = k^d.$$

With the Chinese remainder theorem we decompose

$$A/(T_1, ..., T_d) \cong k \oplus \bigoplus_{i=2}^{r} k[T_{d+1}]/((T_{d+1} - a_i)^{m_i})$$

and, correspondingly, $\overline{g} = (\overline{g}_1, ..., \overline{g}_r)$. Since $g(x_i) \neq 0, i = 2, ..., r$, the element $\overline{g}_i \in k[T_{d+1}]/((T_{d+1} - a_i)^{m_i})$ is a unit, and thus the characteristic polynomial of the isomorphism $\mu_{\overline{g}_i}$ has no zero for $i = 2, ..., r$, while that of $\mu_{\overline{g}_1} = 0$ is $S \in k[S]$. Hence

$$\chi(0, S) = Sq(S) = p(0, S)S$$

with $p(0, 0) = q(0) \neq 0$.

2.) Assume $I(Z) = (h_1, ..., h_s)$. Choose $c \in k$, such that $c^2 \neq h_j(x_i)$ for $i = 1, ..., r, j = 1, ..., s$ and $f \in I(Z)$ with $f(x_i) = c$ for $i = 2, ..., r$ - remember that the restriction $A = \mathcal{O}(X) \longrightarrow \mathcal{O}(Z \cup \{x_2, ..., x_r\})$ is surjective. Take now $g_j := h_j + f^2$: According to Nakayama's lemma they generate $I(Z)_0$, since their residue classes $\overline{g}_j$ mod $\mathfrak{m}_{X,0} I(Z)_0$ coincide with $\overline{h}_j$, hence generate $I(Z)_0 / \mathfrak{m}_{X,0} I(Z)_0$. □

Remark 11.35. Let $R := \mathcal{O}_{X,a}$ be the local ring of a regular point $a \in X$ of the algebraic variety X. According to the above argument we can embed R as an intermediate ring

$$R_0 := k[T]_{\mathfrak{m}_0} \subset R \subset k[[T]].$$

More precisely, R looks as follows: Take a power series $f \in k[[T]]$ integral over R_0. Then

$$R_0[f] \supset R_0$$

is a semilocal ring, i.e. it contains only finitely many maximal ideals, and

$$R = Q(R_0[f]) \cap k[[T]] \hookrightarrow Q(k[[T]]),$$

is the localization of $R_0[f]$ with respect to the maximal ideal $\mathfrak{m} := R_0[f] \cap (T_1, ..., T_n)$.

Trevlig Sommar!