Combinatorial Intersection Cohomology for Fans

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Abstract: We investigate *minimal extension sheaves* on arbitrary (possibly non-rational) fans as an approach toward a combinatorial "virtual" intersection cohomology. These are flabby sheaves of graded modules over a sheaf of polynomial rings, satisfying three relatively simple axioms that characterize the equivariant intersection cohomology sheaves on toric varieties. As in "classical" intersection cohomology, minimal extension sheaves are models for the pure objects of a "perverse category"; a Decomposition Theorem holds. – The analysis of the step from equivariant to non-equivariant intersection cohomology of toric varieties leads us to investigate "quasi-convex" fans (generalizing fans with convex or "co-convex" support), where our approach yields a meaningful virtual intersection cohomology. We characterize such fans by a topological condition and prove a version of Stanley's "Local-Global" formula relating the global intersection Poincaré polynomial to local data. Virtual intersection cohomology of quasi-convex fans is shown to satisfy Poincaré duality. To describe the local data in terms of the global data for lower-dimensional complete polytopal fans as in the rational case, one needs a "Hard Lefschetz" type result. It requires a vanishing condition that is valid for rational cones, but has not yet been proven in the general case.

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Introduction

A basic combinatorial invariant of a *complete simplicial fan* Δ in $\mathbf{R}^n$ is its *h-vector* $(h_1, \ldots, h_n)$: It encodes the numbers of cones of the different dimensions. By the classical *Dehn-Sommerville relations*, the equality $h_i = h_{n-i}$ holds, i.e., the vector is *palindromic*; furthermore, for a polytopal fan Δ, it is known to be *unimodal*, i.e., $h_i \leq h_{i+1}$ holds for $0 \leq i < n/2$. If Δ is even *rational*, then the h-vector admits a topological interpretation in terms of the associated compact $\mathbf{Q}$-smooth toric variety X_Δ: By the theorem of Jurkiewicz and Danilov, the real [1] cohomology ring $H^\bullet(X_\Delta)$ is a quotient of the Stanley-Reisner ring of Δ. In particular, this result implies that the Betti numbers of X_Δ are combinatorial invariants of the fan Δ (i.e., they are determined by the structure of Δ as a partially ordered set), they "live" only in even degrees, and the equality $h_i = \dim H^{2i}(X_\Delta)$ holds for $0 \le i \le n$.

Since every simplicial fan is combinatorially equivalent to a rational one, this interpretation allows to apply topological results about toric varieties to combinatorics. To give an example, we mention that the Dehn-Sommerville equations are just a combinatorial version of Poincaré duality. A deeper application is Stanley's proof of the necessity of McMullen's conditions that characterize the possible h-vectors of simplicial *polytopal* fans: To prove unimodality, it uses the "Hard" Lefschetz Theorem for the rational cohomology of the corresponding projective toric variety.

We now consider *complete non-simplicial fans*, looking first at the *rational* case. Unfortunately, the Betti numbers of the associated compact toric varieties no longer enjoy such good properties as in the simplicial case: Poincaré duality fails to hold, non-zero Betti numbers in odd degrees may occur, and worst of all, Betti numbers may fail to be combinatorial invariants. Replacing singular cohomology with intersection cohomology, however, yields invariants that share the essential properties of the classical h-vector in the simplicial case: Intersection Betti numbers satisfy Poincaré duality, they vanish in odd degrees, and they are determined by the combinatorics of the fan. The last property follows from the two "Local-Global Formulæ" that serve as a kind of *"Leitmotiv"*: For a complete rational fan Δ with associated toric variety X_Δ, one considers the global (intersection cohomology) Poincaré polynomial $P_\Delta := \sum_{q=0}^{2n} \dim_{\mathbf{R}} IH^q(X_\Delta)\cdot t^q$ and its local counterparts $P_\sigma := \sum_{q=0}^{2n} \dim_{\mathbf{R}} \mathcal{IH}^q_\sigma \cdot t^q$, where $\mathcal{IH}^\bullet_\sigma$ denotes the local intersection cohomology along the orbit corresponding to the cone $\sigma \in \Delta$. These polynomials are related by the first formula

$$P_\Delta(t) \;=\; \sum_{\sigma\in\Delta} (t^2-1)^{n-\dim\sigma} P_\sigma(t)\ .$$

By the second formula, each local polynomial P_σ in turn is readily obtained from the global one of a projective toric variety X_{Λ_σ} of strictly smaller dimension associated to the "flattened boundary fan" Λ_σ of the cone σ: One has

$$P_\sigma(t) \;=\; \tau_{\le d-1}\bigl((1-t^2)P_{\Lambda_\sigma}(t)\bigr) \ \text{ for } \ d := \dim\sigma\ ,$$

where $\tau_{\le d-1}$ denotes truncation.

Combined, these formulæ yield inductively that the global and local intersection cohomology Betti numbers

$h_i(\Delta) := \dim IH^{2i}(X_\Delta)$, for $i \le n$ and 0 elsewhere, and

$g_i(\sigma) := \dim \mathcal{IH}^{2i}_\sigma = h_i(\Lambda_\sigma) - h_{i-1}(\Lambda_\sigma)$, for $0 \le i < \dim\sigma/2$ and 0 elsewhere,

[1] Unless otherwise specified, we always use real coefficients for (intersection) cohomology.

are combinatorial invariants that can be computed recursively, starting from $g_i = h_i = \delta_{i0}$ in the case $n = 0$. This observation was used by Stanley in [St] to define generalized h- and g-vectors even for non-rational cones and fans. These invariants are computable via linear functions in the numbers of flags of cones with prescribed sequences of dimensions. —

In the case of a (complete) *simplicial* fan Δ, we may reverse the theorem of Jurkiewicz and Danilov and take the quotient of the Stanley-Reisner ring as definition of a "virtual cohomology algebra" $H^\bullet(\Delta)$ of the fan, thus obtaining virtual Betti numbers $\dim H^{2i}(\Delta)$ that coincide with $h_i(\Delta)$ for $0 \le i \le n$. Our main aim is to define a "virtual intersection cohomology" with analoguous properties for arbitrary fans.

Our approach toward such a theory builds on the previous study of *equivariant intersection cohomology of toric varieties* in [BBFK]. Coming back to complete rational simplicial fans for a moment, we recall that the Stanley-Reisner ring itself has a topological interpretation, namely, it is the *equivariant cohomology ring* of the toric variety defined by such a fan: For affine open toric subvarieties $X_\sigma \subset X_\Delta$, there are natural isomorphisms $H^\bullet_{\mathbf{T}}(X_\sigma) \cong A^\bullet_\sigma$ with the algebra $A^\bullet_\sigma$ of real-valued polynomial functions on σ. They induce an isomorphism between the associated sheaves $\mathcal{H}^\bullet_{\mathbf{T}}$ and $\mathcal{A}^\bullet$ on the "fan space" Δ, i.e., the fan Δ identified with the (non-Hausdorff) orbit space of the toric variety; its open subsets correspond to the subfans. Since $\mathcal{H}^\bullet_{\mathbf{T}}(\Delta) \cong H^\bullet_{\mathbf{T}}(X_\Delta)$ and $\mathcal{A}^\bullet(\Delta)$ constitutes the algebra of Δ-piecewise polynomial functions on the support of Δ, we only have to notice that the latter is nothing but the Stanley-Reisner ring of Δ. The theorem of Jurkiewicz and Danilov may then be restated as follows: A toric variety defined by a complete simplicial fan is *equivariantly formal*, i.e., equivariant and non-equivariant cohomology determine each other by Künneth type formulæ: Since the graded algebra $A^\bullet$ of real valued polynomial functions on $\mathbf{R}^n$ is canonically isomorphic to the cohomology ring $H^\bullet(B\mathbf{T})$ of the classifying space $B\,\mathbf{T} \cong (\mathbf{P}_\infty\mathbf{C})^n$ of the torus, the equivariant cohomology $H^\bullet_{\mathbf{T}}(X_\Delta)$ carries the structure of an $A^\bullet$-module, and X_Δ is called equivariantly formal if the natural map $H^\bullet_{\mathbf{T}}(X_\Delta) \longrightarrow H^\bullet(X)$ induces an isomorphism $A^\bullet/\mathfrak{m} \otimes_{A^\bullet} H^\bullet_{\mathbf{T}}(X_\Delta) \cong H^\bullet(X_\Delta)$, where $\mathfrak{m} := A^{>0}$ is the unique homogeneous maximal ideal of $A^\bullet$.

These observations led us to study the *equivariant intersection cohomology presheaf* $\mathcal{IH}^\bullet_{\mathbf{T}}$ in the case of a not necessarily simplicial rational fan Δ. This presheaf turns out to be very well behaved: In fact, it is a flabby *sheaf* of $\mathcal{A}^\bullet$-modules as has been proved in [BBFK], and it may be characterized by three relatively simple properties that determine it up to isomorphism. Its global sections yield the equivariant intersection cohomology $IH^\bullet_{\mathbf{T}}(X_\Delta)$, a graded $A^\bullet$-module, and in the compact case, we again have equivariant formality: The formula $IH^\bullet(X_\Delta) \cong A^\bullet/\mathfrak{m} \otimes_{A^\bullet} IH^\bullet_{\mathbf{T}}(X_\Delta)$ holds. The axiomatic characterization now allows to carry the whole construction over to the case of not necessarily rational fans and leads to the notion of a so-called "minimal extension sheaf" $\mathcal{E}^\bullet$ on Δ (such that $\mathcal{E}^\bullet \cong \mathcal{A}^\bullet$ is the sheaf of piecewise polynomial functions for simplicial Δ).

In particular, in the complete case, the role of the Stanley-Reisner ring is played by the $A^\bullet$-module $\mathcal{E}^\bullet(\Delta)$ of global sections, and the *virtual intersection cohomology* of a fan Δ is defined as $IH^\bullet(\Delta) := A^\bullet/\mathfrak{m} \otimes_{A^\bullet} \mathcal{E}^\bullet(\Delta)$ (where $\mathfrak{m} := A^{>0}$).

In the present article, we systematize the investigation of the algebraic theory of such minimal extension sheaves. We do hope that this will finally lead to a proof of the formula $h_i(\Delta) = \dim_{\mathbf{R}} IH^{2i}(\Delta)$ that would provide an interpretation of the components of the generalized h-vector in the case of a complete and possibly non-rational fan. In the first section, we recall and extend some results of [BBFK]; in particular, the virtual intersection Betti numbers of a complete rational fan Δ are seen to equal the intersection Betti numbers of X_Δ. The second section is devoted to combinatorially pure sheaves over the fan space Δ. These turn out to be direct sums of simple sheaves, which are generalized minimal extension sheaves: To each cone $\tau \in \Delta$, we associate a simple pure sheaf ${}_\tau\mathcal{L}^\bullet$, where $\mathcal{E}^\bullet$ coincides with the sheaf ${}_o\mathcal{L}^\bullet$ associated with the zero cone o, and prove a Decomposition Theorem (Theorem 2.4) for pure sheaves. As a corollary, we present a proof of Kalai's conjecture for virtual intersection cohomology Poincaré polynomials, as proposed by Tom Braden (see also [BrMPh].)

In the third section, we provide a main technical tool for the following sections in studying the cellular Čech cohomology of sheaves on the fan space. In the fourth section, we show that the acyclicity of that complex with coefficients in a minimal extension sheaf $\mathcal{E}^\bullet$ on a purely n-dimensional fan Δ has both a surprisingly easy algebraic and topological reformulation: It holds if and only if the $A^\bullet$-module $\mathcal{E}^\bullet(\Delta)$ of global sections is free resp. if and only if the support $|\partial\Delta|$ of the boundary fan $\partial\Delta$ is a real homology manifold, cf. Theorems 4.3 and 4.4. In particular that holds for fans with either convex or "co-convex" support, and that motivates to call such fans *quasi-convex*. For a rational fan Δ, quasi-convexity is a necessary and sufficient condition for the equality $IH^\bullet(\Delta) \cong IH^\bullet(X_\Delta)$ to hold, where X_Δ is the associated toric variety, i.e. Δ is quasi-convex iff X_Δ is IH-equivariantly formal. An equivalent formulation of that fact is the vanishing of the odd-dimensional intersection Betti numbers of X_Δ.

On the other hand, the freeness condition is used in order to have a satisfactory "Poincaré Duality" theory both on $\mathcal{E}^\bullet(\Delta)$ and $IH^\bullet(\Delta) = A^\bullet/\mathfrak{m} \otimes_{A^\bullet} \mathcal{E}^\bullet(\Delta)$. As a corollary we prove a conjecture of Bernstein and Lunts.

The fifth section deals with the computation of the virtual intersection Poincaré polynomials $P_\Delta := \sum \dim IH^{2j}(\Delta) \cdot t^{2j}$: For a quasi-convex fan Δ, the polynomial P_Δ can be expressed, as in the rational case, in terms of the virtual local intersection

Poincaré polynomials P_σ, see Theorem 5.3. That is a consequence of the above mentioned acyclicity of the cellular complex, and the fact that the global section modules $IH^\bullet(\Delta)$ and $\mathcal{E}^\bullet(\Delta)$ and their local counterparts $\mathcal{IH}^\bullet_\sigma := A^\bullet/\mathfrak{m} \otimes_{A^\bullet} \mathcal{E}^\bullet(\sigma)$ and $\mathcal{E}^\bullet(\sigma)$ are related by Künneth type formulae. To obtain a recursive computation algorithm for P_Δ as in the rational case, we relate the Poincaré polynomial P_σ to that of the "flattened boundary fan" Λ_σ of σ, the polytopal fan obtained by projecting the boundary of σ to V_σ/ℓ , where $V_\sigma := \mathrm{span}(\sigma)$ and $\ell \subset V_\sigma$ is a line meeting the relative interior of σ. To that end, we need the vanishing condition $\mathcal{IH}^q_\sigma = 0$ for $q \geq \dim \sigma > 0$, see 1.7. In the case of a *rational* cone, that condition holds because it is equivalent to the vanishing condition for the local intersection cohomology of X_σ along its closed orbit, and we expect it even to hold in the non-rational case. The above vanishing condition, together with Poincaré duality (see section 6), leads to a "Hard Lefschetz Theorem" for the virtual intersection cohomology $IH^\bullet(\Lambda_\sigma)$ of the polytopal fan Λ_σ, see Theorem 5.6, and that theorem provides the background for the description of P_σ in terms of P_{Λ_σ}. In particular, if all the cones in Δ satisfy the above vanishing condition, we have $h_i(\Delta) = \dim IH^{2i}(\Delta)$.

Finally, the last section is devoted to Poincaré duality: On a minimal extension sheaf $\mathcal{E}^\bullet$, a (non-canonical) internal "intersection product" $\mathcal{E}^\bullet \times \mathcal{E}^\bullet \to \mathcal{E}^\bullet$ and an evaluation map may be defined, leading to duality isomorphisms $\mathcal{E}^\bullet(\Delta) \cong \mathcal{E}^\bullet(\Delta, \partial\Delta)^*$ and $IH^\bullet(\Delta) \cong IH^\bullet(\Delta, \partial\Delta)^*$ for quasi-convex fans, see Theorem 6.3.

In order to make our results accessible to non-specialists, we have aimed at avoiding technical "machinery" and keeping the presentation as elementary as possible. Many essential results of the present article are contained in Chapters 7–10 of our Uppsala preprint [2]; the current version has been announced in the note [Fi_2]. Using the formalism of derived categories, closely related work has been done by Tom Braden in the rational case and by Paul Bressler and Valery Lunts in the polytopal case. Tom Braden sent us a manuscript presented at the AMS meeting in Washington, January 2000. Even more recently, Paul Bressler and Valery Lunts published their ideas in the e-print [BreLu_2].

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0. Preliminaries

0.A Cones and Fans: Let V be a real vector space of dimension n. A non-zero linear form $\alpha: V \to \mathbf{R}$ on V determines the *upper halfspace* $H_\alpha := \{v \in V; \alpha(v) \geq 0\}$. A (strictly convex polyhedral) *cone* in V is a finite intersection $\sigma = \bigcap_{i=1}^r H_{\alpha_i}$ of halfspaces with linear forms satisfying $\bigcap_{i=1}^r \ker \alpha_i = \{0\}$. Let $V_\sigma := \sigma + (-\sigma)$ denote

[2] *"Equivariant Intersection Cohomology of Toric Varieties"*, UUDM report 1998:34

the linear span of σ in V, and define $\dim\sigma := \dim V_\sigma$. A cone of dimension d is called a *d-cone*.

A cone also may be described as the set $\sigma = \sum_{j=1}^{s} \mathbf{R}_{\geq 0} v_j$ of all positive linear combinations of a finite set of non-zero vectors v_j in V. A cone spanned by a linearly independent system of generators is called *simplicial*. Cones of dimension $d \leq 2$ are always simplicial; in particular, this applies to the *zero cone* $o := \{0\}$ and to every *ray* (i.e., a one-dimensional cone $\mathbf{R}_{\geq 0} v$).

A *face* of a cone σ is any intersection $\tau = \sigma \cap \ker\beta$, where $\beta \in V^*$ is a linear form with $\sigma \subset H_\beta$. We then write $\tau \preceq \sigma$ (and $\tau \prec \sigma$ for a *proper* face). If in addition $\dim\tau = \dim\sigma - 1$, we call τ a *facet* of σ and write $\tau \prec_1 \sigma$.

A *fan* in V is a non-empty finite set Δ of cones such that each face of a cone in Δ also belongs to Δ and the intersection of two cones in Δ is a face of both. To a fan Δ, one associates its *support* $|\Delta| := \cup_{\sigma\in\Delta}\,\sigma$, a closed subset in V. The fan Δ is *generated* by cones $\sigma_1, \ldots, \sigma_r$ if Δ consists of all cones that are a face of some generating cone. In particular, a given cone σ generates the fan $\langle\sigma\rangle$ consisting of σ and its proper faces; such a fan is also called an *affine fan* and occasionally is simply denoted σ. Furthermore, we associate to σ its *boundary fan* $\partial\sigma := \langle\sigma\rangle \setminus \{\sigma\}$, and its *relative interior* $\mathring{\sigma} := \sigma \setminus |\partial\sigma|$.

Every fan is generated by the collection $\Delta^{\max}$ of its maximal cones. We define

$$\Delta^k := \{\sigma \in \Delta\ ;\ \dim\sigma = k\} \qquad \text{and} \qquad \Delta^{\leq k} := \bigcup_{r \leq k} \Delta^r\ ,$$

the latter being a subfan called the *k-skeleton*. The fan Δ is called *purely n-dimensional* if $\Delta^{\max} = \Delta^n$. In that case, we define its *boundary fan* $\partial\Delta$ as the subfan generated by those $(n-1)$-cones that are facets of precisely one n-cone in Δ. The boundary fan is supported by the topological boundary of $|\Delta|$. In contrast with the case of a single cone, we use $\mathring{\Delta}$ to denote the collection $\Delta \setminus \partial\Delta$ of *interior cones*.

A fan is called *simplicial* if all its cones are simplicial; this holds if and only if its maximal cones are simplicial. It is is called *complete* if it is supported by all of V.

A *subfan* Λ of a fan Δ is any subset that itself is a fan; we then write $\Lambda \preceq \Delta$ (and $\Lambda \prec \Delta$ if in addition Λ is a proper subfan). The collection of all subfans of Δ clearly satisfies the axioms for the open sets of a topology on Δ, the empty set being admitted as a fan. In the sequel, we always endow Δ with this *fan topology* and consider it as a topological space, the *fan space*.

A *refinement* of a fan Δ is a fan $\check{\Delta}$ with $|\check{\Delta}| = |\Delta|$ such that each cone $\check{\sigma} \in \check{\Delta}$ is contained in some cone $\sigma \in \Delta$. If σ is minimal with that property, we write $\sigma = \pi(\check{\sigma})$ and obtain in that way the associated refinement map $\pi\colon \check{\Delta} \longrightarrow \Delta$. Every purely n-dimensional fan Δ admits a refinement which can be embedded into a complete fan: For a cone $\sigma_i \in \Delta^n$, we fix a line ℓ meeting $\mathring{\sigma}_i$ and set $\varrho := -(\ell \cap \sigma_i)$; then the fan Δ_i

generated by σ_i and all cones $\varrho + \tau$ for $\tau \prec_1 \sigma_i$ is complete. For $\Delta^n = \{\sigma_1, \dots, \sigma_r\}$, the *fan-theoretic intersection*

$$\bigcap_{i=1}^{r} \Delta_i := \left\{ \bigcap_{i=1}^{r} \tau_i \; ; \; \tau_i \in \Delta_i \right\} .$$

is a complete fan including a refinement of Δ as a subfan.

A fan Δ in V is called *rational* (or, more precisely, *N-rational*) if there exists a lattice (i.e., a discrete additive subgroup) $N \subset V$ of maximal rank such that $\varrho \cap N \neq \{0\}$ for each ray $\varrho \in \Delta$.

0.B Graded $A^\bullet$-modules: Let $A^\bullet$ denote the symmetric algebra $S^\bullet(V^*)$ over the dual vector space V^* of V. Its elements are canonically identified with polynomial functions on V. In the case of a rational fan, $A^\bullet$ is isomorphic to the cohomology algebra $H^\bullet(B\mathbf{T})$ of the classifying space $B\mathbf{T} \cong (\mathbf{P}_\infty \mathbf{C})^n$ of the complex algebraic n-torus $\mathbf{T} \cong (\mathbf{C}^*)^n$ acting on the associated toric variety. Motivated by that topological considerations, we endow $A^\bullet$ with the positive even grading determined by setting $A^{2q} := S^q(V^*)$; in particular, $A^2 = V^*$ consists of all linear forms on V. Correspondingly, for a cone σ in V, we let $A^\bullet_\sigma$ denote the graded algebra $S^\bullet(V^*_\sigma)$; if σ is of dimension n, then $A^\bullet_\sigma = A^\bullet$ holds. The natural projection $V^* \to V^*_\sigma$ extends to an epimorphism $A^\bullet \to A^\bullet_\sigma$ of graded algebras. We usually consider the elements in $A^\bullet_\sigma$ as functions $f\colon \sigma \to \mathbf{R}$; the above epimorphism then corresponds to the restriction of polynomial functions.

For a graded $A^\bullet$-module $F^\bullet$, we write $\overline{F}^\bullet$ for its residue class module

$$\overline{F}^\bullet := F^\bullet/(\mathfrak{m}\cdot F^\bullet) \cong \mathbf{R}^\bullet \otimes_{A^\bullet} F^\bullet ,$$

where $\mathfrak{m} := A^{>0} \subset A^\bullet$ is the unique homogeneous maximal ideal of $A^\bullet$ and where $\mathbf{R}^\bullet := A^\bullet/\mathfrak{m} = \overline{A}^\bullet$ is the field $\mathbf{R}$, considered as graded algebra concentrated in degree zero. Obviously $\overline{F}^\bullet$ is a graded vector space over $\mathbf{R}$, which is finite dimensional if $F^\bullet$ is finitely generated over $A^\bullet$. If $F^\bullet$ is positively graded or, more generally, bounded from below, then the converse holds: A family $(f_1, \dots, f_r)$ of homogeneous elements in $F^\bullet$ generates $F^\bullet$ over $A^\bullet$ if and only if the system of residue classes $(\overline{f}_1, \dots, \overline{f}_r)$ modulo $\mathfrak{m}\cdot F^\bullet$ generates the vector space $\overline{F}^\bullet$. In that case, we have $\operatorname{rk}_{A^\bullet} F^\bullet \le \dim \overline{F}^\bullet$, where equality holds if and only if $F^\bullet$ is a free $A^\bullet$-module. The collection $(f_1, \dots, f_r)$ is part of a basis of the free $A^\bullet$-module $F^\bullet$ over $A^\bullet$ if and only if $(\overline{f}_1, \dots, \overline{f}_r)$ is linearly independent over $\mathbf{R}$. Furthermore, every homomorphism $\varphi\colon F^\bullet \to G^\bullet$ of finitely generated graded $A^\bullet$-modules induces a homomorphism $\overline{\varphi}\colon \overline{F}^\bullet \to \overline{G}^\bullet$ of graded vector spaces, which is surjective if and only if φ is so. If $F^\bullet$ is free, then every homomorphism $\psi\colon \overline{F}^\bullet \to \overline{G}^\bullet$ can be lifted to a homomorphism $\varphi\colon F^\bullet \to G^\bullet$ (i.e., $\overline{\varphi} = \psi$ holds); if $G^\bullet$ is free, then φ is an isomorphism if and only if that holds for $\overline{\varphi}$.

A finitely generated $A^\bullet$-module $F^\bullet$ is *free* if and only if $\operatorname{Tor}_1^{A^\bullet}(F^\bullet, \mathbf{R}^\bullet) = 0$. That condition is obviously necessary, so let us show that it is also sufficient: As we have seen above, there is a surjection $(A^\bullet)^d \to F^\bullet$ where $d := \dim \overline{F}^\bullet$; let $K^\bullet$ be its kernel. Since $\mathrm{Tor}_1^{A^\bullet}(F^\bullet, \mathbf{R}^\bullet) = 0$, the exact sequence

$$0 \longrightarrow K^\bullet \longrightarrow (A^\bullet)^d \longrightarrow F^\bullet \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \overline{K}^\bullet \longrightarrow (\overline{A}^\bullet)^d \longrightarrow \overline{F}^\bullet \longrightarrow 0\ .$$

By construction, $(\overline{A}^\bullet)^d \to \overline{F}^\bullet$ is an isomorphism, so we have $\overline{K}^\bullet = 0$ and thus also $K^\bullet = 0$, i.e., $F^\bullet \cong (A^\bullet)^d$ is free. □

By means of the restriction map $A^\bullet \to A^\bullet_\sigma$, an $A^\bullet_\sigma$-module $F^\bullet_\sigma$ is an $A^\bullet$-module, and there is a natural isomorphism $\overline{F}^\bullet_\sigma = F^\bullet_\sigma/(\mathfrak{m}{\cdot}F^\bullet_\sigma) \cong F^\bullet_\sigma/(\mathfrak{m}_\sigma{\cdot}F^\bullet_\sigma)$. Let us denote by $V_\sigma^\perp$ the orthogonal complement of $V_\sigma \subset V$ in the dual vector space V^*. We remark that, using the Koszul complex for the $A^\bullet$-module $I(V_\sigma) := A^\bullet \cdot V_\sigma^\perp \subset A^\bullet$, one finds a natural isomorphism of vector spaces

$$\mathrm{Tor}_i^{A^\bullet}(A^\bullet_\sigma, \mathbf{R}^\bullet) \ \cong\ \Lambda^i V_\sigma^\perp \tag{0.B.1}$$

over $\mathbf{R}^\bullet = A^\bullet/\mathfrak{m}$.

0.C Sheaves on a fan space: Sheaf theory on a fan space is particularly simple since the "affine" open sets $\langle\sigma\rangle \preceq \Delta$ form a basis of the fan topology whose elements can not be covered by strictly smaller open sets. In fact, let $(F_\sigma)_{\sigma\in\Delta}$ be a collection of abelian groups, say, together with "restriction" homomorphisms $\varrho^\sigma_\tau\colon F_\sigma \to F_\tau$ for $\tau \preceq \sigma$, i.e., we require $\varrho^\sigma_\sigma = \mathrm{id}$ and $\varrho^\tau_\gamma \circ \varrho^\sigma_\tau = \varrho^\sigma_\gamma$ for $\gamma \preceq \tau \preceq \sigma$. Then there is a unique sheaf $\mathcal{F}$ on the fan space Δ such that its group of sections $\mathcal{F}(\sigma) := \mathcal{F}\big(\langle\sigma\rangle\big)$ agrees with F_σ. The sheaf $\mathcal{F}$ is flabby if and only if each restriction map $\varrho^\sigma_{\partial\sigma}\colon \mathcal{F}(\sigma) \to \mathcal{F}(\partial\sigma)$ is surjective. – In the same spirit of ideas, sheaves on a fan space occur in the work of Bressler and Lunts [BreLu$_2$], Brion [Bri$_2$] and McConnell [MCo].

In particular, we consider the sheaf $\mathcal{A}^\bullet$ of graded polynomial algebras on Δ determined by $\mathcal{A}^\bullet(\sigma) := A^\bullet_\sigma$, the homomorphism $\varrho^\sigma_\tau\colon A^\bullet_\sigma \to A^\bullet_\tau$ being the restriction of functions on σ to the face $\tau \preceq \sigma$. The set of sections $\mathcal{A}^\bullet(\Lambda)$ on a subfan $\Lambda \preceq \Delta$ constitutes the algebra of (Λ-) piecewise polynomial functions on $|\Lambda|$ in a natural way.

If $\mathcal{F}^\bullet$ is a sheaf of $\mathcal{A}^\bullet$-modules, then every $\mathcal{F}^\bullet(\Lambda)$ also is an $A^\bullet$-module, and if $\mathcal{F}^\bullet(\sigma)$ is finitely generated for every cone $\sigma \in \Delta$, then so is $\mathcal{F}(\Lambda)$ for every subfan $\Lambda \preceq \Delta$: This is an immediate consequence of the facts that $A^\bullet$ is a noetherian ring and $\mathcal{F}^\bullet(\Lambda)$, a submodule of $\bigoplus_{\sigma\in\Lambda^{\max}} \mathcal{F}^\bullet(\sigma)$.

For notational convenience, we often write

$$F^\bullet_\Lambda := \mathcal{F}^\bullet(\Lambda) \qquad \text{and} \qquad F^\bullet_\sigma := \mathcal{F}^\bullet(\sigma)\ ;$$

more generally, for a pair of subfans (Λ, Λ_0), we define

$$F^\bullet_{(\Lambda,\Lambda_0)} := \ker(\varrho^\Lambda_{\Lambda_0}: F^\bullet_\Lambda \longrightarrow F^\bullet_{\Lambda_0}) ,$$

the submodule of sections on Λ vanishing on Λ_0. In particular, for a purely n-dimensional fan Δ, we obtain in that way the module

$$F^\bullet_{(\Delta,\partial\Delta)} := \ker(\varrho^\Delta_{\partial\Delta}: F^\bullet_\Delta \longrightarrow F^\bullet_{\partial\Delta})$$

of sections over Δ with "compact supports".

To a sheaf $\mathcal{F}^\bullet$ of $\mathcal{A}^\bullet$-modules, we may associate the *presheaf* of graded $\mathbf{R}^\bullet$-modules given by the assignment $\Lambda \mapsto \overline{\mathcal{F}^\bullet(\Lambda)}$. The associated sheaf $\overline{\mathcal{F}}^\bullet$ satisfies the equality $\overline{\mathcal{F}}^\bullet(\sigma) = \overline{\mathcal{F}^\bullet(\sigma)}$ on the basic open sets. This fact does not carry over to an arbitrary open set, i.e., the above presheaf need not be a sheaf. As an example, consider a complete simplicial rational fan Δ. Then $\overline{\mathcal{A}}^\bullet$ is the constant sheaf $\mathbf{R}^\bullet$ on Δ, so $\overline{\mathcal{A}}^\bullet(\Delta) \cong \mathbf{R}^\bullet$, while $\overline{A}^\bullet_\Delta := \overline{\mathcal{A}^\bullet(\Delta)} \cong H^\bullet(X_\Delta)$ has a non-vanishing weight subspace in degree $2n$ since the compact toric variety X_Δ satisfies $H^{2n}(X_\Delta) \neq 0$.

0.D Fan constructions associated with a cone: In addition to the *affine fan* $\langle\sigma\rangle$ and the *boundary fan* $\partial\sigma$ associated with a cone σ, we need two more constructions. Firstly, if σ belongs to a fan Δ, we consider the *star*

$$\mathrm{st}_\Delta(\sigma) := \{\gamma \in \Delta\ ;\ \sigma \preceq \gamma\}$$

of σ in Δ. This set is not a subfan of Δ – we note in passing that it is the *closure* of the one-point set $\{\sigma\}$ in the fan topology –, but its image

$$\Delta_\sigma \ :=\ p\big(\mathrm{st}_\Delta(\sigma)\big) \ =\ \{p(\gamma)\ ;\ \sigma \preceq \gamma\}$$

under the quotient projection $p: V \to V/V_\sigma$ is a fan in V/V_σ, called the *"transversal fan"* of σ in Δ.

Secondly, let σ be a non-zero cone. Fixing an auxiliary line ℓ in V passing through the relative interior $\mathring{\sigma}$, we consider the *"flattened boundary fan"* $\Lambda_\sigma = \Lambda_\sigma(\ell)$ that is obtained by projecting the boundary fan $\partial\sigma$ onto the quotient vector space V_σ/ℓ: If $\pi: V_\sigma \to V_\sigma/\ell$ is the quotient projection, then we set

$$\Lambda_\sigma := \pi(\partial\sigma) = \{\pi(\tau)\,;\ \tau \prec \sigma\} . \tag{0.D.1}$$

This fan is *complete*. Restricting the projection π to the support of $\partial\sigma$ yields a (piecewise linear) homeomorphism

$$\pi|_{|\partial\sigma|}: |\partial\sigma| \longrightarrow |\Lambda_\sigma| = V_\sigma/\ell$$

that in turn induces a homeomorphism $\partial\sigma \to \Lambda_\sigma$ of fan spaces; in particular, the combinatorial type of Λ_σ is independent of the choice of ℓ. Any linear function $T \in A^2_\sigma$ not identically vanishing on ℓ provides an isomorphism $\ell \xrightarrow{\cong} \mathbf{R}$; furthermore, it gives rise to a decomposition $V_\sigma = \ker(T) \oplus \ell$ and hence, to an isomorphism $\ker(T) \cong V_\sigma/\ell$. Identifying V_σ and $(V_\sigma/\ell) \times \mathbf{R}$ via these isomorphisms yields a natural identification

$$A^\bullet_\sigma \;=\; B^\bullet_\sigma[T]\,,$$

where

$$B^\bullet_\sigma \;:=\; \pi^*\big(S^\bullet\big((V_\sigma/\ell)^*\big)\big) \subset A^\bullet_\sigma \tag{0.D.2}$$

is the algebra of polynomial functions on V_σ that are constant along parallels to ℓ. Moreover, the support $|\partial\sigma|$ of the boundary fan is the graph of the strictly convex Λ_σ-piecewise linear function

$$f := T \circ (\pi|_{|\partial\sigma|})^{-1} \colon V_\sigma/\ell \to \mathbf{R}\,. \tag{0.D.3}$$

On the other hand, for a complete fan Λ in a vector space W and a strictly convex Λ-piecewise linear function $f\colon W \to \mathbf{R}$, the convex hull γ of the graph Γ_f in $W \times \mathbf{R}$ is a cone with boundary $\partial\gamma = \Gamma_f$.

1. Minimal Extension Sheaves

The investigation of a "virtual" intersection cohomology theory for arbitrary fans is couched in terms of a certain class of sheaves on fans called *minimal extension sheaves.* In this section, we introduce that notion and study some elementary properties of such sheaves.

1.1 Definition. *A sheaf $\mathcal{E}^\bullet$ of graded $\mathcal{A}^\bullet$-modules on a fan Δ is called a **minimal extension sheaf** (of $\mathbf{R}^\bullet$) if it satisfies the following conditions:*

(N) ***Normalization:*** *One has $E^\bullet_o \cong A^\bullet_o = \mathbf{R}^\bullet$ for the zero cone o.*

(PF) ***Pointwise Freeness:*** *For each cone $\sigma \in \Delta$, the module $E^\bullet_\sigma$ is free over $A^\bullet_\sigma$.*

(LME) ***Local Minimal Extension*** mod $\mathfrak{m}$*: For each cone $\sigma \in \Delta \setminus \{o\}$, the restriction mapping*

$$\varrho_\sigma := \varrho^\sigma_{\partial\sigma}\colon E^\bullet_\sigma \longrightarrow E^\bullet_{\partial\sigma}$$

induces an isomorphism

$$\overline{\varrho}_\sigma\colon \overline{E}^\bullet_\sigma \xrightarrow{\cong} \overline{E}^\bullet_{\partial\sigma}$$

of graded real vector spaces.

The above condition (LME) implies that $\mathcal{E}^\bullet$ is minimal in the set of all flabby sheaves of graded $\mathcal{A}^\bullet$-modules satisfying conditions (N) and (PF), whence the name "minimal extension sheaf".

1.2 Remark. *Let $\mathcal{E}^\bullet$ be a minimal extension sheaf on a fan Δ.*

i) The sheaf $\mathcal{E}^\bullet$ is flabby and vanishes in odd degrees.

ii) For each subfan $\Lambda \preceq \Delta$, the $A^\bullet$-module $E^\bullet_\Lambda$ is finitely generated.

iii) For each cone $\sigma \in \Delta$, there is an isomorphim of graded $A^\bullet_\sigma$-modules

$$E^\bullet_\sigma \;\cong\; A^\bullet_\sigma \otimes_{\mathbf{R}} \overline{E}^\bullet_\sigma\,. \tag{1.2.1}$$

Proof: (i) and (ii): By the results of 0.B, condition (LME) implies that ϱ_σ is surjective for each cone $\sigma \in \Delta$; hence, 0.C asserts flabbiness. To prove finite generation, we proceed by induction. Let us assume that $E^\bullet_\tau$ is finitely generated for $\dim \tau \leq k$, then so is $E^\bullet_\Lambda$ for each subfan $\Lambda \preceq \Delta^{\leq k}$, see 0.C. In particular, if σ is a cone of dimension $k+1$, then $E^\bullet_{\partial\sigma}$ is finitely generated, whence $\overline{E}^\bullet_\sigma \cong \overline{E}^\bullet_{\partial\sigma}$ is finite-dimensional, and thus the free $A^\bullet_\sigma$-module $E^\bullet_\sigma$ is finitely generated. Now an application of 0.C yields (ii). Since $A^\bullet$ only lives in even degrees, the obvious $\mathbf{R}^\bullet$-splitting $F^\bullet = F^{\mathrm{even}} \oplus F^{\mathrm{odd}}$ of a graded $A^\bullet$-module actually is a decomposition into graded $A^\bullet$-submodules. Hence, a finitely generated $A^\bullet$-module $F^\bullet$ vanishes in odd degrees if and only if $\overline{F}^\bullet$ does. Thus, we may achieve the proof of (i) by induction over the skeleta of Δ as above.

(iii) The isomorphism (1.2.1) is an immediate consequence of the results quoted in 0.B since the $A^\bullet_\sigma$-module $E^\bullet_\sigma$ is free and finitely generated. □

On every fan Δ, a minimal extension sheaf exists, it can be constructed recursively, and it is unique up to isomorphism; hence, we may speak of *the* minimal extension sheaf $\mathcal{E}^\bullet = {}_\Delta\mathcal{E}^\bullet$ of Δ:

1.3 Proposition (Existence and Uniqueness of Minimal Extension Sheaves): *On every fan Δ, there exists a minimal extension sheaf $\mathcal{E}^\bullet$; it is unique up to an isomorphism of graded $\mathcal{A}^\bullet$-modules. More precisely, for any two such sheaves $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ on Δ, every isomorphism $E^\bullet_o \cong F^\bullet_o$ extends to an isomorphism $\varphi\colon \mathcal{E}^\bullet \xrightarrow{\cong} \mathcal{F}^\bullet$ of graded $\mathcal{A}^\bullet$-modules.*

As to the uniqueness of φ, see Remark 1.8, (iii).

Proof: For the *existence*, we define the sheaf $\mathcal{E}^\bullet$ inductively on the k-skeleton subfans $\Delta^{\leq k}$, starting with $E^\bullet_o := \mathbf{R}^\bullet$ for $k = 0$. For $k > 0$, we assume that $\mathcal{E}^\bullet$ has been defined on $\Delta^{<k}$; in particular, $E^\bullet_{\partial\sigma}$ exists for every cone $\sigma \in \Delta^k$. It thus suffices to define $E^\bullet_\sigma$, together with a restriction homomorphism $E^\bullet_\sigma \to E^\bullet_{\partial\sigma}$. To that end, we fix an $\mathbf{R}^\bullet$-linear section $s\colon \overline{E}^\bullet_{\partial\sigma} \to E^\bullet_{\partial\sigma}$ of the residue class map $E^\bullet_{\partial\sigma} \to \overline{E}^\bullet_{\partial\sigma}$ that is homogeneous of degree zero. According to (1.2.1), we set

$$E^\bullet_\sigma \;:=\; A^\bullet_\sigma \otimes_{\mathbf{R}} \overline{E}^\bullet_{\partial\sigma} \quad \text{and} \quad \varrho^\sigma_{\partial\sigma}\colon E^\bullet_\sigma = A^\bullet_\sigma \otimes_{\mathbf{R}} \overline{E}^\bullet_{\partial\sigma} \xrightarrow{1\otimes s} A^\bullet_\sigma \otimes_{\mathbf{R}} E^\bullet_{\partial\sigma} \longrightarrow E^\bullet_{\partial\sigma}\,. \tag{1.3.1}$$

For the *uniqueness* of minimal extension sheaves up to isomorphism, we use the same induction pattern and show how a given isomorphism $\varphi\colon \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ of such sheaves on $\Delta^{<k}$ may be extended to $\Delta^{\leq k}$. It suffices to verify that, for each cone $\sigma \in \Delta^k$, there is a lifting of $\varphi_{\partial\sigma}\colon E^\bullet_{\partial\sigma} \xrightarrow{\cong} F^\bullet_{\partial\sigma}$ to an isomorphism $\varphi_\sigma\colon E^\bullet_\sigma \xrightarrow{\cong} F^\bullet_\sigma$. Using

the results recalled in section 0.B, the existence of such a lifting follows easily from the properties of graded $A^\bullet_\sigma$-modules: We choose a homogeneous basis $(e_1, \ldots, e_r)$ of the free $A^\bullet_\sigma$-module $E^\bullet_\sigma$. Since $\mathcal{F}^\bullet$ is a flabby sheaf, the images $\varphi_{\partial\sigma}(e_i|_{\partial\sigma})$ in $F^\bullet_{\partial\sigma}$ can be extended to homogeneous sections $f_1, \ldots, f_r$ in $F^\bullet_\sigma$ with $\deg e_j = \deg f_j$. The induced restriction isomorphism $\overline{F}^\bullet_\sigma \xrightarrow{\cong} \overline{F}^\bullet_{\partial\sigma}$ maps the residue classes $\overline{f}_1, \ldots, \overline{f}_r$ to a basis of $\overline{F}^\bullet_{\partial\sigma}$. It is immediate that these sections $f_1, \ldots, f_r$ form a basis of the free $A^\bullet_\sigma$-module $F^\bullet_\sigma$, and that $e_i \mapsto f_i$ defines a lifting $\varphi_\sigma \colon E^\bullet_\sigma \xrightarrow{\cong} F^\bullet_\sigma$ of $\varphi_{\partial\sigma}$. ▫

Simplicial fans are easily characterized in terms of minimal extension sheaves:

1.4 Proposition: *The following conditions for a fan Δ are equivalent:*

i) Δ is simplicial,

ii) $\mathcal{A}^\bullet$ is a minimal extension sheaf on Δ.

Proof: "(ii) $\Longrightarrow$ (i)" Assuming that $\mathcal{A}^\bullet$ is a minimal extension sheaf, we show by induction on the dimension d for each cone $\sigma \in \Delta^d$ that the number k of its rays equals d, i.e., that σ is simplicial. This is always true for $d \leq 2$. As induction hypothesis, we assume that the boundary fan $\partial\sigma$ is simplicial. On each ray of σ, we choose a non-zero vector v_i. Then there exist unique piecewise linear functions $f_i \in A^2_{\partial\sigma}$ with $f_i(v_j) = \delta_{ij}$ for $i, j = 1, \ldots, k$. These functions $f_1, \ldots, f_k$ are linearly independent over $\mathbf{R}$, whence $\dim_{\mathbf{R}} A^2_{\partial\sigma} \geq k$.

We proceed to prove the equality $\dim_{\mathbf{R}} A^2_{\partial\sigma} = \dim_{\mathbf{R}} A^2_\sigma = d$, thus obtaining the inequality $k \leq d$ that yields (i). Since $\mathcal{A}^\bullet$ is a minimal extension sheaf, the induced restriction homomorphism $\overline{A}^\bullet_\sigma \to \overline{A}^\bullet_{\partial\sigma}$ is an isomorphism. From $\overline{A}^\bullet_\sigma = \mathbf{R}^\bullet$, we conclude $\overline{A}^2_\sigma = 0$ and thus $\overline{A}^2_{\partial\sigma} = 0$, i.e., $A^2_{\partial\sigma}$ is the homogeneous component of degree 2 in the graded module $\mathfrak{m}A^\bullet_{\partial\sigma}$. That component obviously is nothing but $A^2 \cdot A^0_{\partial\sigma} = A^2|_{\partial\sigma} \cong A^2_\sigma|_{\partial\sigma}$. Hence, $k \leq \dim A^2_{\partial\sigma} = \dim A^2_\sigma|_{\partial\sigma} \leq \dim A^2_\sigma = d$, while $d \leq k$ is obvious.

"(i) $\Longrightarrow$ (ii)": We again proceed by induction on the dimension d, proving that for any simplicial cone σ with $\dim\sigma = d$, a minimal extension sheaf $\mathcal{E}^\bullet$ on $\langle\sigma\rangle$ in a natural manner is isomorphic to the sheaf $\mathcal{A}^\bullet$. The case $d = 0$ being immediate, let us first remark that a simplicial cone is the sum $\sigma = \varrho + \tau$ of any facet $\tau \prec_1 \sigma$ and the remaining ray ϱ. The decomposition $V_\sigma = V_\varrho \oplus V_\tau$ provides projections $p \colon V_\sigma \to V_\varrho$ and $q \colon V_\sigma \to V_\tau$ and thus subalgebras

$$D^\bullet_\varrho := p^*(S^\bullet(V^*_\varrho)) \qquad \text{and} \qquad D^\bullet_\tau := q^*(S^\bullet(V^*_\tau)) \tag{1.4.1}$$

of $A^\bullet_\sigma$, together with an isomorphism

$$A^\bullet_\sigma \cong D^\bullet_\varrho \otimes_{\mathbf{R}} D^\bullet_\tau \; . \tag{1.4.2}$$

As the facet τ is simplicial and thus $E^\bullet_\tau \cong D^\bullet_\tau$ holds by induction hypothesis, Lemma 1.5 below yields isomorphisms

$$E^\bullet_\sigma \;\cong\; A^\bullet_\sigma \otimes_{D^\bullet_\tau} E^\bullet_\tau \;\cong\; A^\bullet_\sigma \otimes_{D^\bullet_\tau} D^\bullet_\tau \;=\; A^\bullet_\sigma \ . \qquad \square$$

1.5 Lemma. *If a cone σ is the sum $\varrho+\tau$ of a facet τ and a ray ϱ, then the minimal extension sheaf $\mathcal{E}^\bullet$ on $\langle\sigma\rangle$ satisfies in a natural way*

$$E^\bullet_\sigma \;\cong\; A^\bullet_\sigma \otimes_{D^\bullet_\tau} E^\bullet_\tau \;\cong\; D^\bullet_\varrho \otimes_{\mathbf{R}} E^\bullet_\tau \ ,$$

using an isomorphism as in (1.4.2). *In particular, the restriction homomorphism $E^\bullet_\sigma \to E^\bullet_\tau$ induces an isomorphism $\overline{E}^\bullet_\sigma \cong \overline{E}^\bullet_\tau$ of graded vector spaces.*

Proof: We use induction on $\dim\sigma$. For a proper face $\gamma \prec \tau$, we write $\hat\gamma := \varrho+\gamma \prec \sigma$; furthermore, with the projection $q_\gamma\colon V_{\hat\gamma} = V_\varrho \oplus V_\gamma \to V_\gamma$ and the subalgebra $D^\bullet_\gamma := q^*_\gamma(A^\bullet_\gamma)$ of $A^\bullet_{\hat\gamma}$, we have $A^\bullet_{\hat\gamma} \cong D^\bullet_\varrho \otimes_{\mathbf{R}} D^\bullet_\gamma$. By induction hypothesis, there are natural isomorphisms $E^\bullet_{\hat\gamma} \cong A^\bullet_{\hat\gamma} \otimes_{D^\bullet_\gamma} E^\bullet_\gamma \cong D^\bullet_\varrho \otimes_{\mathbf{R}} E^\bullet_\gamma$. With a non-zero linear form $T \in A^2_\sigma$ that vanishes on V_τ, we may write $D^\bullet_\varrho = \mathbf{R}[T]$ and thus

$$A^\bullet_\sigma = D^\bullet_\tau[T]\ ,\ A^\bullet_{\hat\gamma} = D^\bullet_\gamma[T] \quad \text{and} \quad E^\bullet_{\hat\gamma} \cong A^\bullet_{\hat\gamma} \otimes_{D^\bullet_\gamma} E^\bullet_\gamma = E^\bullet_\gamma[T] = E^\bullet_\gamma \oplus T E^\bullet_\gamma[T]\ .$$

Since $\partial\sigma = \langle\tau\rangle \cup \{\hat\gamma;\ \gamma \prec \tau\}$, there is an isomorphism $E^\bullet_{\partial\sigma} \cong E^\bullet_\tau \oplus T E^\bullet_{\partial\tau}[T]$. To prove the isomorphism $E^\bullet_\sigma \cong A^\bullet_\sigma \otimes_{D^\bullet_\tau} E^\bullet_\tau$ of the assertion, we first note that the $A^\bullet_\sigma$-module on the right hand side is free. It thus suffices to show that the restriction homomorphism $A^\bullet_\sigma \otimes_{D^\bullet_\tau} E^\bullet_\tau \to E^\bullet_{\partial\sigma}$ induces an isomorphism modulo $\mathfrak{m}$. This homomorphism agrees with the natural map

$$A^\bullet_\sigma \otimes_{D^\bullet_\tau} E^\bullet_\tau \cong E^\bullet_\tau[T] = E^\bullet_\tau \oplus T E^\bullet_\tau[T] \longrightarrow E^\bullet_\tau \oplus T E^\bullet_{\partial\tau}[T]\ .$$

It is surjective, since $E^\bullet_\tau \to E^\bullet_{\partial\tau}$ is; hence, the restriction modulo $\mathfrak{m}$ is surjective, too; furthermore, it is injective since the composition $E^\bullet_\tau[T] \to E^\bullet_\tau \oplus T E^\bullet_{\partial\tau}[T] \to E^\bullet_\tau$ even is an isomorphism modulo $\mathfrak{m}$. $\square$

If Δ is an N-rational fan for a lattice $N \subset V$ of rank $n = \dim V$, one associates to Δ a toric variety X_Δ with the action of the algebraic torus $\mathbf{T} := N \otimes_{\mathbf{Z}} \mathbf{C}^* \cong (\mathbf{C}^*)^n$. Let $IH^\bullet_{\mathbf{T}}(X_\Delta)$ denote the equivariant intersection cohomology of X_Δ with real coefficients. The following theorem, proved in [BBFK], has been the starting point to investigate minimal extension sheaves:

1.6 Theorem. *Let Δ be a rational fan and $\mathcal{E}^\bullet$ a minimal extension sheaf on Δ.*

i) The presheaf

$$\mathcal{IH}^\bullet_{\mathbf{T}}\colon \Lambda \longmapsto IH^\bullet_{\mathbf{T}}(X_\Lambda)$$

is a minimal extension sheaf on the fan space Δ.

ii) For each cone $\sigma \in \Delta$, the (non-equivariant) intersection cohomology sheaf $\mathcal{IH}^\bullet$ of X_Δ is constant along the corresponding $\mathbf{T}$-orbit, and its stalks are isomorphic to $\overline{E}^\bullet_\sigma$.

iii) If Δ is complete or is affine of dimension n, then one has

$$IH^\bullet(X_\Delta) \;\cong\; \overline{E}^\bullet_\Delta \;.$$

Statement (iii) will be generalized in Theorem 4.3 to a considerably larger class of rational fans, called "quasi-convex".

For a non-zero *rational* cone σ, the vanishing axiom for intersection cohomology together with statement (ii) yields $\overline{E}^q_\sigma = 0$ for $q \geq \dim\sigma$. This fact turns out to be a cornerstone in the recursive computation of intersection Betti numbers in section 5. In the non-rational case, we have to state it as a condition; we conjecture that it holds in general:

1.7 Vanishing Condition $\mathbf{V}(\sigma)$: *A non-zero cone σ satisfies the condition $\mathbf{V}(\sigma)$ if*

$$\overline{E}^q_\sigma \;=\; 0 \quad \textit{for} \quad q \geq \dim\sigma$$

holds. A fan Δ satisfies the condition $\mathbf{V}(\Delta)$ if $\mathbf{V}(\sigma)$ holds for each non-zero cone $\sigma \in \Delta$.

We add some comments on that condition. Note that the statements (ii) and (iii) in the following remark are not needed for later results; in particular, the results cited in their proof do not depend on these statements. – Statement (iii) has been influenced by a remark of Tom Braden.

1.8 Remark. i) If a fan Δ is simplicial or rational, then condition $\mathbf{V}(\Delta)$ is satisfied.

ii) Condition $\mathbf{V}(\sigma)$ is equivalent to

$$E^q_{(\sigma,\partial\sigma)} \;=\; \{0\} \quad \text{for} \quad q \leq \dim\sigma \;.$$

iii) If Δ satisfies $\mathbf{V}(\Delta)$, then every homomorphism $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ between minimal extension sheaves on Δ is determined by the homomorphism $\mathbf{R}^\bullet \cong E^\bullet_o \to F^\bullet_o \cong \mathbf{R}^\bullet$.

Proof: (i) The rational case has been mentioned above; for the simplicial case, see Proposition 1.4.

(ii) Replacing V with V_σ if necessary, we may assume $\dim\sigma = n$; hence, the affine fan $\langle\sigma\rangle$ is "quasi-convex" (see Theorem 4.4). According to Corollary 6.9, there exists an isomorphism of vector spaces $\overline{E}^q_\sigma \cong \overline{E}^{2n-q}_{(\sigma,\partial\sigma)}$. Hence, condition $\mathbf{V}(\sigma)$ holds if and only if $\overline{E}^{\leq n}_{(\sigma,\partial\sigma)} = 0$. It remains to show that this is equivalent to the vanishing $E^{\leq n}_{(\sigma,\partial\sigma)} = 0$. To that end, we may apply the following fact: Let $F^\bullet \neq 0$ be a finitely generated $A^\bullet$-module; if $r < \infty$ is minimal with $F^r \neq 0$, then $\overline{F}^r \cong F^r$ and $\overline{F}^{<r} = 0$.

(iii) We use the terminology of the proof of Proposition 1.3: We have to show that a homomorphism $\varphi_{\partial\sigma}\colon E^\bullet_{\partial\sigma} \to F^\bullet_{\partial\sigma}$ extends in a unique way to a homomorphism $\varphi_\sigma\colon E^\bullet_\sigma \to F^\bullet_\sigma$. Statement (ii) implies that the restriction homomorphisms $E^q_\sigma \to E^q_{\partial\sigma}$

and $F_\sigma^q \to F_{\partial\sigma}^q$ are isomorphisms for $q \le \dim \sigma$. Since, as a consequence of $\mathbf{V}(\sigma)$, the $A^\bullet$-modules $E_\sigma^\bullet$ and $F_\sigma^\bullet$ can be generated by homogeneous elements of degree below $\dim \sigma$, the assertion follows. □

2. Combinatorial Pure Sheaves

In the case of a rational fan, "the" minimal extension sheaf is represented by the equivariant intersection cohomology sheaf (see Theorem 1.6) and thus can be considered as an object of a class of "pure" sheaves. This observation holds also for general minimal extension sheaves, regardless of rationality. The simple objects in this class are generalizations of minimal extension sheaves. We introduce such objects and prove an analogue to the decomposition theorem in intersection cohomology.

2.1 Definition: *A (combinatorially)* ***pure*** *sheaf on a fan space Δ is a flabby sheaf $\mathcal{F}^\bullet$ of graded $\mathcal{A}^\bullet$-modules such that, for each cone $\sigma \in \Delta$, the $A_\sigma^\bullet$-module $F_\sigma^\bullet$ is finitely generated and free.*

2.2 Remark: As a consequence of the results in section 0.B and 0.C, we may replace flabbiness with the following "local" requirement: For each cone $\sigma \in \Delta$, the restriction homomorphism $\varrho^\sigma_{\partial\sigma}\colon F_\sigma^\bullet \to F_{\partial\sigma}^\bullet$ induces a surjective map $\overline{F}_\sigma^\bullet \to \overline{F}_{\partial\sigma}^\bullet$.

Pure sheaves are built up from simple objects, which are generalized minimal extension sheaves:

(Combinatorially) Simple Sheaves: For each cone $\sigma \in \Delta$, we construct inductively a "simple" sheaf ${}_\sigma\mathcal{L}^\bullet$ on Δ as follows: For a cone $\tau \in \Delta$ with $\dim \tau \le \dim \sigma$, we set

$$ {}_\sigma L_\tau^\bullet \;:=\; {}_\sigma\mathcal{L}^\bullet(\tau) \;:=\; \begin{cases} A_\sigma^\bullet & \text{if } \tau = \sigma, \\ 0 & \text{otherwise.} \end{cases} $$

If ${}_\sigma\mathcal{L}^\bullet$ has been defined on $\Delta^{\le m}$ for some $m \ge \dim \sigma$, then for each $\tau \in \Delta^{m+1}$, we set

$$ {}_\sigma L_\tau^\bullet \;:=\; A_\tau^\bullet \otimes_{\mathbf{R}} {}_\sigma\overline{L}_{\partial\tau}^\bullet $$

and define the restriction map $\varrho^\tau_{\partial\tau}$ just as in (1.3.1).

Let us collect some useful facts about these sheaves.

2.3 Remark: i) The simple sheaf $\mathcal{F}^\bullet := {}_\sigma\mathcal{L}^\bullet$ is pure; it is determined by the following properties:

a) $\overline{F}_\sigma^\bullet \cong \mathbf{R}^\bullet$,

b) for each cone $\tau \ne \sigma$, the reduced restriction map $\overline{F}_\tau^\bullet \to \overline{F}_{\partial\tau}^\bullet$ is an isomorphism.

ii) The sheaf ${}_\sigma\mathcal{L}^\bullet$ vanishes outside of $\mathrm{st}_\Delta(\sigma)$ and can be obtained from a minimal extension sheaf ${}_{\Delta_\sigma}\mathcal{E}^\bullet$ on the transversal fan Δ_σ in the following way: We choose a decomposition $V = V_\sigma \oplus W$, and let $D^\bullet_W \subset A^\bullet$ denote the image of $S^\bullet\big((V/V_\sigma)^*\big)$ in $A^\bullet$ and $D^\bullet_\sigma$, the image of $S^\bullet(V^*_\sigma)$ with respect to the projection with kernel W. Then $A^\bullet \cong D^\bullet_\sigma \otimes_{\mathbf{R}} D^\bullet_W$, and on $\mathrm{st}(\sigma)$, there is a decomposition

$$_\sigma\mathcal{L}^\bullet \;\cong\; D^\bullet_\sigma \otimes_{\mathbf{R}} ({}_{\Delta_\sigma}\mathcal{E}^\bullet)\ ,$$

where we identify Δ_σ with $\mathrm{st}(\sigma)$.

iii) For the zero cone o, the simple sheaf ${}_o\mathcal{L}^\bullet$ is the minimal extension sheaf of Δ.

iv) If Δ is a *rational* fan and $Y \subset X_\Delta$ the orbit closure associated to a cone $\sigma \in \Delta$, then the presheaf

$${}_Y\mathcal{IH}^\bullet_{\mathbf{T}} \colon \Lambda \;\mapsto\; IH^\bullet_{\mathbf{T}}(Y \cap X_\Lambda)$$

on Δ is a sheaf isomorphic to ${}_\sigma\mathcal{L}^\bullet$.

As main result of this section, we provide a Decomposition Formula for pure sheaves.

2.4 Algebraic Decomposition Theorem: *Every pure sheaf $\mathcal{F}^\bullet$ on Δ admits a direct sum decomposition*

$$\mathcal{F}^\bullet \;\cong\; \bigoplus_{\sigma\in\Delta} \big({}_\sigma\mathcal{L}^\bullet \otimes_{\mathbf{R}} K^\bullet_\sigma\big) \tag{2.4.1}$$

with $K^\bullet_\sigma := K^\bullet_\sigma(\mathcal{F}^\bullet) := \ker\big(\overline{\varrho}^\sigma_{\partial\sigma}\colon \overline{F}^\bullet_\sigma \to \overline{F}^\bullet_{\partial\sigma}\big)$, a finite dimensional graded vector space.

Since a finite dimensional graded vector space $K^\bullet$ has a unique representation in the form $K^\bullet = \bigoplus_i \mathbf{R}^\bullet[-\ell_i]^{n_i}$, the decomposition (2.4.1) may be written as

$$\mathcal{F}^\bullet \;\cong\; \bigoplus_i {}_{\sigma_i}\mathcal{L}^\bullet[-\ell_i]^{n_i}\ ,$$

which is the "classical" version of the Decomposition Theorem.

Proof: The following result evidently allows an inductive construction of the decomposition (2.4.1):

Let $\mathcal{F}^\bullet$ be a pure sheaf on Δ. For each cone $\sigma \in \Delta$ of minimal dimension with $F^\bullet_\sigma \neq 0$, there is a decomposition $\mathcal{F}^\bullet = \mathcal{G}^\bullet \oplus \mathcal{H}^\bullet$ as a direct sum of pure $\mathcal{A}^\bullet$-submodules $\mathcal{G}^\bullet \cong {}_\sigma\mathcal{L}^\bullet \otimes_{\mathbf{R}} \overline{F}^\bullet_\sigma$ and $\mathcal{H}^\bullet$ (where $H^\bullet_\sigma = 0$).

We construct the decomposition recursively on each skeleton $\Delta^{\le m}$, starting with $m = \dim\sigma$: We set $K^\bullet_\sigma := \overline{F}^\bullet_\sigma$ and

$$\mathcal{G}^\bullet(\tau) := \begin{cases} F^\bullet_\sigma \cong A^\bullet_\sigma \otimes_{\mathbf{R}} K^\bullet_\sigma & \text{if } \tau = \sigma,\\ 0 & \text{otherwise,}\end{cases} \qquad \text{and} \quad \mathcal{H}^\bullet(\tau) := \begin{cases} 0 & \text{if } \tau = \sigma,\\ \mathcal{F}^\bullet(\tau) & \text{otherwise.}\end{cases}$$

We now assume that we have constructed the decomposition on $\Delta^{\le m}$. In order to extend it to $\Delta^{\le m+1}$, it suffices to fix a cone $\tau \in \Delta^{m+1}$ and to extend the decomposition from $\partial\tau$ to the affine fan $\langle\tau\rangle$. By induction hypothesis, there exists a commutative diagram

$$\begin{array}{ccccccc} & & F^\bullet_\tau & \twoheadrightarrow & F^\bullet_{\partial\tau} & \cong & G^\bullet_{\partial\tau} \oplus H^\bullet_{\partial\tau} \\ & & \downarrow & & \downarrow & & \downarrow \\ K^\bullet_\tau & \hookrightarrow & \overline{F}^\bullet_\tau & \twoheadrightarrow & \overline{F}^\bullet_{\partial\tau} & \cong & \overline{G}^\bullet_{\partial\tau} \oplus \overline{H}^\bullet_{\partial\tau} \end{array} .$$

We choose a first decomposition $\overline{F}^\bullet_\tau = K^\bullet_\tau \oplus N^\bullet \oplus M^\bullet$ satisfying $N^\bullet \cong \overline{G}^\bullet_{\partial\tau}$ and $M^\bullet \cong \overline{H}^\bullet_{\partial\tau}$. We may then lift it to a decomposition $F^\bullet_\tau = G^\bullet_\tau \oplus H^\bullet_\tau$ into free $A^\bullet_\tau$-submodules such that $\overline{G}^\bullet_\tau = N^\bullet$ and $\overline{H}^\bullet_\tau = K^\bullet_\tau \oplus M^\bullet$ as well as $G^\bullet_\tau|_{\partial\tau} = G^\bullet_{\partial\tau}$ and $H^\bullet_\tau|_{\partial\tau} = H^\bullet_{\partial\tau}$. ▫

2.5 Geometric Decomposition Theorem: *Let $\pi\colon \check\Delta \to \Delta$ be a refinement map of fans with minimal extension sheaves $\check{\mathcal{E}}^\bullet$ and $\mathcal{E}^\bullet$, respectively. Then there is a decomposition*

$$\pi_*(\check{\mathcal{E}}^\bullet) \;\cong\; \mathcal{E}^\bullet \oplus \bigoplus_{\sigma\in\Delta^{\ge 2}} {}_\sigma\mathcal{L}^\bullet \otimes_{\mathbf{R}} K^\bullet_\sigma$$

of $\mathcal{A}^\bullet$-modules, where the $K^\bullet_\sigma$ are (positively) graded vector spaces, and the "correction terms" are supported on $\Delta^{\ge 2}$.

Proof: For an application of the Algebraic Decomposition Theorem 2.4, we have to verify that the flabby sheaf $\pi_*(\check{\mathcal{E}}^\bullet)$ is pure. We still need to know that the $A^\bullet_\sigma$-modules $\pi_*(\check{\mathcal{E}}^\bullet)(\sigma)$ are free. If σ is an n-dimensional cone, then the affine fan $\langle\sigma\rangle$ is quasi-convex, see section 4. According to Corollary 4.7, the same holds true for the refinement $\check\sigma := \pi^{-1}(\langle\sigma\rangle) \preceq \check\Delta$; hence, by Theorem 4.3, $\check E_{\check\sigma}$ is a free $A^\bullet$-module. For a cone of positive codimension, we may go over to V_σ. – The fact that $\pi_*(\check{\mathcal{E}}^\bullet) \cong \mathcal{E}^\bullet \cong \mathcal{A}^\bullet$ on $\Delta^{\le 1}$ provides the condition $\dim\sigma \ge 2$, while $K^{<0}_\sigma = 0$ is an obvious consequence of the corresponding fact for $\pi_*(\check{\mathcal{E}}^\bullet)$. ▫

2.6 Corollary: *Let $\pi\colon \check\Delta \to \Delta$ be a simplicial refinement of Δ. Then the minimal extension sheaf $\mathcal{E}^\bullet$ on Δ can be embedded as a direct factor into the sheaf of functions on $|\Delta|$ that are $\check\Delta$-piecewise polynomial.*

Proof: According to Proposition 1.4, the sheaf $\check{\mathcal{A}}^\bullet$ is a minimal extension sheaf on $\check\Delta$. By the Geometric Decomposition Theorem 2.5, we know that $\mathcal{E}^\bullet$ is a direct subsheaf of $\pi_*(\check{\mathcal{A}}^\bullet)$, which is the sheaf of functions on $|\Delta|$ that are $\check\Delta$-piecewise polynomial. ▫

We conclude this section with an application of the Algebraic Decomposition Theorem 2.4 to Poincaré polynomials

$$P_\Delta(t) \;:=\; \sum_{q\ge 0}^{<\infty} \dim \overline{E}^{2q}_\Delta \cdot t^{2q} \;, \quad P_\sigma(t) \;:=\; \sum_{q\ge 0}^{<\infty} \dim \overline{E}^{2q}_\sigma \cdot t^{2q}$$

which has been communicated to us by Tom Braden (cf. also [BrMPh]):

2.7 Theorem (Kalai's Conjecture): *For an affine fan $\Delta := \langle\sigma\rangle$ and a face $\tau \preceq \sigma$ with transversal fan Δ_τ, there is a coefficientwise inequality of polynomials*

$$P_\sigma \;\geq\; P_\tau \cdot P_{\Delta_\tau}\,.$$

Proof. Let $\mathcal{E}^\bullet$ denote the minimal extension sheaf on Δ and $\mathcal{F}^\bullet$, the trivial extension of $\mathcal{E}^\bullet|_{\mathrm{st}(\tau)}$ by zero to Δ. For a subfan Λ of Δ, we have $F^\bullet_\Lambda = E^\bullet_{\Lambda_0}$, where $\Lambda_0 \preceq \Lambda$ is the subfan generated by the cones in $\Lambda \cap \mathrm{st}(\tau)$. In particular, we see that $\mathcal{F}^\bullet$ is a pure sheaf and hence, according to the Algebraic Decomposition Theorem 2.4, may be written in the form

$$\mathcal{F}^\bullet \;\cong\; ({}_\tau\mathcal{L}^\bullet \otimes K^\bullet_\tau) \oplus \ldots\;.$$

Thus, if we denote $P(K^\bullet)$ the Poincaré polynomial of the graded vector space $K^\bullet$, we obtain the inequality

$$P(\overline{F}^\bullet_\Delta) \;\geq\; P({}_\tau\overline{L}^\bullet_\Delta \otimes K^\bullet_\tau) \;=\; P({}_\tau\overline{L}^\bullet_\Delta)\cdot P(K^\bullet_\tau)\;.$$

The equalities $K^\bullet_\tau = \overline{E}^\bullet_\tau$ and $\overline{F}^\bullet_\Delta = \overline{E}^\bullet_\sigma$ are readily checked, so that $P(K_\tau) = P_\tau$ and $P(\overline{F}^\bullet_\Delta) = P_\sigma$ holds. In the notation of 2.3 (ii), we have

$$A^\bullet_\sigma \cong D^\bullet_\tau \otimes_{\mathbf{R}} D^\bullet_W\;, \quad {}_\tau L^\bullet_\Delta \cong D^\bullet_\tau \otimes_{\mathbf{R}} E^\bullet_{\Delta_\tau}\;,$$

i.e., the Poincaré polynomial of ${}_\tau\overline{L}^\bullet_\Delta \cong \overline{E}^\bullet_{\Delta_\tau}$ coincides with P_{Δ_τ}. ▫

3. Cellular Čech Cohomology

In this section, we introduce and discuss a "cellular" cochain complex associated with a sheaf on a fan and the corresponding cohomology. This theory will later be used as a principal technical tool to reach one of the main aims of the present article, namely, to characterize those fans Δ for which the $A^\bullet$-module $E^\bullet_\Delta$ of global sections of a minimal extension sheaf $\mathcal{E}^\bullet$ on Δ is free.

3.1 The cellular cochain complex. To a sheaf $\mathcal{F}$ of abelian groups on a fan space Δ, we associate a *cellular cochain complex* $C^\bullet(\Delta,\mathcal{F})$: The cochain groups are

$$C^k(\Delta,\mathcal{F}) \;:=\; \bigoplus_{\dim\sigma = n-k} \mathcal{F}(\sigma)\,.$$

To define the coboundary operator $\delta^k\colon C^k \to C^{k+1}$, we first fix, for each cone $\sigma \in \Delta$, an orientation $\mathrm{or}(\sigma)$ of V_σ such that $\mathrm{or}|_{\Delta^n}$ is a constant function. To each facet $\tau \prec_1 \sigma$, we then assign the orientation coefficient $\mathrm{or}^\sigma_\tau := 1$ if the orientation of V_τ, followed by some inward pointing vector, coincides with the orientation of V_σ, and $\mathrm{or}^\sigma_\tau := -1$ otherwise. We then set

$$\delta(f)_\tau := \sum_{\sigma \succ_1 \tau} \mathrm{or}^\sigma_\tau\, f_\sigma|_\tau \quad \text{for} \quad f = (f_\sigma) \in C^k(\Delta,\mathcal{F}) \quad \text{and} \quad \tau \in \Delta^{n-k-1}\,.$$

For a minimal extension sheaf $\mathcal{E}^\bullet$ on Δ, the complex $C^\bullet(\Delta, \mathcal{E}^\bullet)$ is, up to a rearrangement of indices, a *minimal complex* in the sense of Bernstein and Lunts. We shall come back to that at the end of section 4.

More generally, we have to consider relative cellular cochain complexes with respect to subfans.

3.2 Definition: *For a subfan Λ of Δ and a sheaf $\mathcal{F}$ on the fan space Δ, we set*

$$C^\bullet(\Delta, \Lambda; \mathcal{F}) := C^\bullet(\Delta; \mathcal{F})/C^\bullet(\Lambda; \mathcal{F}) \quad \textit{and} \quad H^q(\Delta, \Lambda; \mathcal{F}) := H^q(C^\bullet(\Delta, \Lambda; \mathcal{F}))$$

with the induced coboundary operator $\delta^\bullet := \delta^\bullet(\Delta, \Lambda; \mathcal{F})$. If Δ is purely n-dimensional, Λ a purely $(n-1)$-dimensional subfan of $\partial\Delta$, and Λ^c its "complementary" subfan generated by the cones in $(\partial\Delta)^{n-1} \setminus \Lambda$, then the restriction of sections induces an augmented complex

$$\tilde{C}^\bullet(\Delta, \Lambda; \mathcal{F}): \qquad 0 \to F_{(\Delta, \Lambda^c)} \xrightarrow{\delta^{-1}} C^0(\Delta, \Lambda; \mathcal{F}) \xrightarrow{\delta^0} \ldots \longrightarrow C^n(\Delta, \Lambda; \mathcal{F}) \to 0$$

with cohomology $\tilde{H}^q(\Delta, \Lambda; \mathcal{F}) := H^q\big(\tilde{C}^\bullet(\Delta, \Lambda; \mathcal{F})\big)$.

In fact, we need only the two cases $\Lambda = \partial\Delta$ and $\Lambda = \emptyset$, where the complementary subfan is $\Lambda^c = \emptyset$ resp. $\Lambda^c = \partial\Delta$. We mainly are interested in the case where $\mathcal{F}$ is an $\mathcal{A}^\bullet$-module. Then, the cohomology $\tilde{H}^q(\Delta, \Lambda; \mathcal{F})$ is an $A^\bullet$-module. – In the augmented situation described above, we note that $C^0(\Lambda; \mathcal{F}) = 0$ and hence $C^0(\Delta, \Lambda; \mathcal{F}) = C^0(\Delta; \mathcal{F})$ holds.

For the constant sheaf $\mathcal{F} = \mathbf{R}$, we want to compare the cohomology $\tilde{H}^\bullet(\Delta, \partial\Delta; \mathbf{R})$ with the usual real singular homology of a "spherical" cell complex associated with a purely n-dimensional fan Δ. To that end, we fix a euclidean norm on V (and hence on V/V_σ for every cone $\sigma \in \Delta$); let $S_V \subset V$ be its unit sphere, and for a subfan $\Lambda \preceq \Delta$, let

$$S_\Lambda := |\Lambda| \cap S_V .$$

For each non-zero cone σ in V, the subset $S_\sigma := \sigma \cap S_V$ is a closed cell of dimension $\dim\sigma - 1$. Hence, the collection $(S_\sigma)_{\sigma\in\Delta\setminus\{o\}}$ is a cell decomposition of S_Δ, and the corresponding (augmented) "homological" complex $C_\bullet(S_\Delta; \mathbf{R})$ of cellular chains with real coefficients essentially coincides with the cochain complex $C^\bullet(\Delta; \mathbf{R})$: We have $C^q(\Delta; \mathbf{R}) = C_{n-1-q}(S_\Delta; \mathbf{R})$ and $\delta^q = \partial_{n-1-q}$ for $q \le n-1$.

Let us call a *facet-connected component* of Δ each purely n-dimensional subfan Λ being maximal with the property that every two n-dimensional cones can be joined by a chain of n-dimensional cones in Λ where two consecutive ones meet in a facet.

3.3 Remark: Let Δ be a purely n-dimensional fan.

(i) If Δ is complete or $n \le 1$, then $\tilde{H}^\bullet(\Delta, \partial\Delta; \mathbf{R}) = 0$.

(ii) If Δ is not complete and $n \geq 2$, then

$$\tilde{H}^q(\Delta, \partial\Delta; \mathbf{R}) \cong H_{n-1-q}(S_\Delta, S_{\partial\Delta}; \mathbf{R}) \quad \text{for} \quad q > 0 \, ;$$

in particular, $\tilde{H}^q(\Delta, \partial\Delta; \mathbf{R}) = 0$ holds for $q \geq n-1$.

(iii) If s is the number of facet-connected components of Δ, then

$$\tilde{H}^0(\Delta, \partial\Delta; \mathbf{R}) \cong \mathbf{R}^{s-1} \, .$$

Proof: The case $n \leq 1$ is straightforward. For $n \geq 2$, the cohomology is computed via cellular homology; in the complete case, one has to use the fact that such a fan is facet-connected and that there is an isomorphism

$$\tilde{H}^q(\Delta; \mathbf{R}) \cong \tilde{H}_{n-1-q}(S_V; \mathbf{R}) \quad \text{for} \;\; n \geq 2 \;\; \text{and} \;\; q \geq 1 \, .$$

For iii), we note that Δ is connected; hence, the global sections of the constant sheaf $\mathbf{R}$ form a one-dimensional vector space. □

In order to study the cellular cohomology of a flabby sheaf $\mathcal{F}$ of real vector spaces on Δ, we want to write such a sheaf as a direct sum of simpler sheaves: To a cone σ in Δ, we associate its *characteristic sheaf* ${}_\sigma\mathcal{J}$, i.e.,

$${}_\sigma\mathcal{J}(\Lambda) := \begin{cases} \mathbf{R} & \text{if } \Lambda \ni \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

while the restriction homomorphisms are $\mathrm{id}_{\mathbf{R}}$ or 0.

The following lemma is an elementary analogue of the Algebraic Decomposition Theorem 2.4.

3.4 Lemma: *Every flabby sheaf $\mathcal{F}$ of real vector spaces on Δ admits a direct sum decomposition*

$$\mathcal{F} \cong \bigoplus_{\sigma \in \Delta} {}_\sigma\mathcal{J} \otimes_{\mathbf{R}} K_\sigma$$

with the vector spaces $K_\sigma := \ker\bigl(\varrho^\sigma_{\partial\sigma} \colon \mathcal{F}(\sigma) \to \mathcal{F}(\partial\sigma)\bigr)$.

Proof: The following arguments are analoguous to those in the proof of the Algebraic Decomposition Theorem 2.4. Evidently, it suffices to decompose such a flabby sheaf $\mathcal{F}$ as a direct sum

$$\mathcal{F} = \mathcal{G} \oplus \mathcal{H} \tag{3.4.1}$$

of flabby subsheaves $\mathcal{G}$ and $\mathcal{H}$, where $\mathcal{G} \cong {}_\sigma\mathcal{J} \otimes K_\sigma$ and $\mathcal{H}(\sigma) = 0$ for some cone $\sigma \in \Delta$. We then may use induction on the number of cones $\tau \in \Delta$, such that $\mathcal{F}(\tau) \neq 0$.

For (3.4.1), let σ be a cone of minimal dimension, say d, with $\mathcal{F}(\sigma) \neq 0$. We construct the subsheaves $\mathcal{G}$ and $\mathcal{H}$ on the skeleton $\Delta^{\leq d}$ as follows:

$$\mathcal{G}(\tau) := \begin{cases} \mathcal{F}(\sigma) = K_\sigma & \text{if } \tau = \sigma, \\ 0 & \text{otherwise} \end{cases}$$

while

$$\mathcal{H}(\tau) := \begin{cases} 0 & \text{if } \tau = \sigma, \\ \mathcal{F}(\tau) & \text{otherwise.} \end{cases}$$

We now suppose that the decomposition (3.4.1) has been constructed on $\Delta^{\leq m}$. Let τ be a cone of dimension $m+1$. In particular, there is a decomposition

$$\mathcal{F}(\partial\tau) \;=\; \mathcal{G}(\partial\tau) \oplus \mathcal{H}(\partial\tau) \;.$$

Since $\mathcal{F}$ is flabby, the restriction map $\varrho^{\tau}_{\partial\tau}\colon \mathcal{F}(\tau) \to \mathcal{F}(\partial\tau)$ is surjective. We can find a decomposition $\mathcal{F}(\tau) = U \oplus W$ into complementary subspaces $U, W \subset \mathcal{F}(\tau)$ such that $\varrho^{\tau}_{\partial\tau}$ induces an isomorphism $U \xrightarrow{\cong} \mathcal{G}(\partial\tau)$ and an epimorphism $W \twoheadrightarrow \mathcal{H}(\partial\tau)$. Now we set $\mathcal{G}(\tau) := U$ and $\mathcal{H}(\tau) := W$. In that manner, we can define $\mathcal{G}$ and $\mathcal{H}$ for all $(m+1)$-dimensional cones and thus on $\Delta^{\leq m+1}$. ◻

Cellular cohomology commutes with direct sums and the tensor product with a fixed vector space. Hence, from Lemma 3.4 stems an isomorphism of graded vector spaces

$$\tilde{H}^\bullet(\Delta, \partial\Delta; \mathcal{F}) \;\cong\; \bigoplus_{\sigma\in\Delta} \tilde{H}^\bullet(\Delta, \partial\Delta; {}_\sigma\mathcal{J}) \otimes_{\mathbf{R}} K_\sigma \;. \tag{3.4.2}$$

We thus are led to compute the cohomology of characteristic sheaves.

3.5 Remark: For a cone $\sigma \in \Delta$, its transversal fan Δ_σ, and the characteristic sheaf ${}_\sigma\mathcal{J}$, there are isomorphisms

$$\tilde{H}^\bullet(\Delta; {}_\sigma\mathcal{J}) \;\cong\; \tilde{H}^\bullet(\Delta_\sigma; \mathbf{R}) \quad \text{and} \quad \tilde{H}^\bullet(\Delta, \partial\Delta; {}_\sigma\mathcal{J}) \;\cong\; \tilde{H}^\bullet(\Delta_\sigma, \partial\Delta_\sigma; \mathbf{R}) \;.$$

In particular, Remark 3.3 ii) implies

$$\tilde{H}^q(\Delta, \partial\Delta; {}_\sigma\mathcal{J}) = 0 \qquad \text{for} \quad q \;\geq\; n - \dim\sigma - 1$$

for each cone $\sigma \in \Delta$.

4. Quasi-Convex Fans

In this section, we study those fans Δ for which the $A^\bullet$-module $E^\bullet_\Delta$ of global sections of a minimal extension sheaf $\mathcal{E}^\bullet$ on Δ is free. The great interest in that freeness condition is due to the "Künneth formula" $E^\bullet_\Delta \cong A^\bullet \otimes_{\mathbf{R}^\bullet} \overline{E}^\bullet_\Delta$, which holds in that case. The name "quasi-convex" introduced below for such fans is motivated by Theorem 4.4. Quasi-convexity allows us in sections 5 and 6 first to compute virtual intersection Betti numbers and Poincaré duality on the "equivariant" level $E^\bullet_\Delta$, and then to pass to "ordinary" (virtual) intersection cohomology $\overline{E}^\bullet_\Delta$. We give various characterizations of quasi-convex fans: We first formulate the main result of this section, then restate it in topological terms, and then proceed to the proof.

4.1 Definition: *A fan Δ is called **quasi-convex** if the $A^\bullet$-module $E^\bullet_\Delta$ is free.*

Quasi-convex fans are known to be purely n-dimensional, see [BBFK; 6.1]. In the rational case, quasi-convexity can be reformulated in terms of the associated toric variety:

4.2 Theorem: *A rational fan Δ is quasi-convex if and only if the intersection cohomology of the associated toric variety X_Δ vanishes in odd degrees:*

$$IH^{\mathrm{odd}}(X_\Delta;\mathbf{R}) \;:=\; \bigoplus_{q\geq 0} IH^{2q+1}(X_\Delta;\mathbf{R}) \;=\; 0\;.$$

In that case, there exists an isomorphism $IH^\bullet(X_\Delta)\cong \overline{E}^\bullet_\Delta$.

Proof: See Proposition 6.1 in [BBFK]. □

4.3 Theorem (Characterization of Quasi-Convex Fans): *For a purely n-dimensional fan Δ and its minimal extension sheaf $\mathcal{E}^\bullet$, the following statements are equivalent:*

(a) *The fan Δ is quasi-convex,*

(b) $\tilde{H}^\bullet(\Delta,\partial\Delta;\mathcal{E}^\bullet)=0$,

(c) $\tilde{H}^\bullet(\Delta_\sigma,\partial\Delta_\sigma;\mathbf{R})=0$ *for each cone $\sigma\in\Delta$.*

We put off the proof for a while, since we first want to deduce a *topological characterization* of quasi-convex fans. In its proof and in the subsequent lemma, we use the following notations:

For a cone σ in a fan Δ, we set $L_\sigma := S_{\Delta_\sigma}\subset (V/V_\sigma)$ and $\partial L_\sigma := S_{\partial\Delta_\sigma}$; in particular, we have $L_o = S_\Delta$. It is important to note that the cellular complex L_σ in the $(k-1)$-sphere S_{V/V_σ} (for $k:=n-\dim\sigma$) may be identified with the *link* at an arbitrary point of the $(n-k-1)$-dimensional stratum $S_\sigma\setminus S_{\partial\sigma}$ of the stratified space S_Δ, while its boundary ∂L_σ is the link of such a point in $S_{\partial\sigma}$.

4.4 Theorem: *A purely n-dimensional fan Δ is quasi-convex if and only if the support $|\partial\Delta|$ of its boundary fan is a real homology manifold. In particular, Δ is quasi-convex if Δ is complete or if S_Δ is a closed topological $(n-1)$-cell, e.g., if the support $|\Delta|$ or the complement of the support $V\setminus|\Delta|$ are convex sets.*

Proof: For a cone $\sigma\in\Delta\setminus\partial\Delta$, the transversal fan Δ_σ is complete; thus Remark 3.3, (i) implies $\tilde{H}^\bullet(\Delta_\sigma,\partial\Delta_\sigma;\mathbf{R})=0$, which means that condition (c) in Theorem 4.3 is satisfied for such a cone. In particular, Theorem 4.3 implies that a complete fan is quasi-convex. It remains to discuss the cones in $\partial\Delta$. If $\dim\sigma$ is at least $n-1$, then again 3.3 (i) implies the corresponding vanishing condition in 4.3 (c). Hence, it suffices to consider cones $\sigma \in (\partial\Delta)^{n-k}$ for $k \geq 2$. The proof in that case is achieved by Lemma 4.5. In fact, part (ii) of Remark 3.3 implies that

$$(4.4.1) \qquad \tilde{H}^q(\Delta_\sigma, \partial\Delta_\sigma; \mathbf{R}) \cong H_{k-1-q}(L_\sigma, \partial L_\sigma; \mathbf{R}) \qquad \text{for } q > 0 ,$$

while $\tilde{H}^0(\Delta_\sigma, \partial\Delta_\sigma; \mathbf{R}) = 0$ if and only if $H^0(\Delta_\sigma, \partial\Delta_\sigma; \mathbf{R}) \cong H_{k-1}(L_\sigma, \partial L_\sigma; \mathbf{R}) = \mathbf{R}$.
□

4.5 Lemma: *For a non-complete purely n-dimensional fan Δ, the following statements are equivalent:*

(i) *The fan Δ is quasi-convex.*

(ii) *Each cone σ in $\partial\Delta$ satisfies the following condition:*

(ii)$_\sigma$ *The pair $(L_\sigma, \partial L_\sigma)$ is a real homology $(k-1)$-cell modulo boundary for $k := n - \dim\sigma$.*

(iii) *Each cone σ in $\partial\Delta$ satisfies the following condition:*

(iii)$_\sigma$ *The link L_σ has the real homology of a point.*

(iv) *Each cone σ in $\partial\Delta$ satisfies the following condition:*

(iv)$_\sigma$ *The boundary of the link ∂L_σ has the real homology of a sphere of dimension $k-2$ for $k := n - \dim\sigma$.*

Proof: We already have seen in (4.4.1) that condition (c) of Theorem 4.3 and statement (ii) are equivalent; thus we have reduced the equivalence "(i) $\Longleftrightarrow$ (ii)" to Theorem 4.3.

In order to prove the equivalence of (ii), (iii), and (iv), we use induction on n. The case $n = 0$ is vacuous, and in case $n = 1$, it is trivial to check that (ii), (iii), and (iv) hold. We thus assume that the equivalence holds for every non-complete purely d-dimensional fan with $d \leq n-1$. If we apply that to the fans Δ_σ for $\sigma \in \partial\Delta \setminus \{o\}$, we see that the condition (ii)$_\sigma$ is satisfied for each cone $\sigma \in \partial\Delta \setminus \{o\}$, if and only if (iii)$_\sigma$ resp. (iv)$_\sigma$ is. Hence it suffices to derive the equivalence of (ii)$_o$, (iii)$_o$ and (iv)$_o$ under one of that assumptions. We need the following result:

4.6 Lemma: *Let $L := L_o$. If the condition* (iii)$_\sigma$ *is satisfied for each non-zero cone $\sigma \in \partial\Delta$, then*

$$(4.6.1) \qquad H_\bullet(L, L \setminus \partial L) = 0$$

holds.

Proof. For $i = -1, \ldots, n-2$, we set $U_i := L \setminus (\partial L)_i$, where $(\partial L)_i$ is the i-skeleton of $\partial L = S_{\partial\Delta}$. By induction on i, we show that $H_\bullet(L, U_i) = 0$ holds. This is evident for $i = -1$, and the case $i = n-2$ is what we are aiming at. For the induction step, we use the homology sequence associated to the triple (L, U_i, U_{i+1}) and show

$H_\bullet(U_i, U_{i+1}) = 0$. Let Δ' be the following "barycentric" subdivision of Δ: For each cone $\sigma \in \Delta \setminus \partial\Delta$, we choose an additional ray ϱ_σ meeting $\mathring{\sigma}$. Then Δ' consists of the cones

$$\tau + \varrho_{\tau_1} + \ldots + \varrho_{\tau_r}\ , \ \text{ where } \tau \in \partial\Delta \text{ and } \tau \prec_1 \tau_1 \prec_1 \ldots \prec_1 \tau_r \text{ with } \tau_i \in \mathring{\Delta}\ .$$

Let $\mathrm{st}'(\mathring{\sigma})$ denote the open star of $S_{\mathring{\sigma}}$ with respect to the cellular decomposition of L induced by Δ'. Then, by excision, the inclusion

$$\Big(\dot{\bigcup_{\sigma\in(\partial\Delta)^{i+2}}} \mathrm{st}'(\mathring{\sigma}),\ \dot{\bigcup_{\sigma\in(\partial\Delta)^{i+2}}} \big(\mathrm{st}'(\mathring{\sigma}) \setminus S_{\mathring{\sigma}}\big)\Big) \ \subset\ (U_i, U_{i+1})$$

induces an isomorphism in homology, while

$$H_\bullet\Big(\dot{\bigcup_{\sigma\in(\partial\Delta)^{i+2}}} \mathrm{st}'(\mathring{\sigma}),\ \dot{\bigcup_{\sigma\in(\partial\Delta)^{i+2}}} \big(\mathrm{st}'(\mathring{\sigma}) \setminus S_{\mathring{\sigma}}\big)\Big) \cong \bigoplus_{\sigma\in(\partial\Delta)^{i+2}} H_\bullet\big(\mathrm{st}'(\mathring{\sigma}), \mathrm{st}'(\mathring{\sigma}) \setminus S_{\mathring{\sigma}}\big)\ .$$

Furthermore, there is a homeomorphism $\mathrm{st}'(\mathring{\sigma}) \cong \mathring{c}(L_\sigma) \times S_{\mathring{\sigma}}$, where $\mathring{c}(L_\sigma)$ denotes the open cone over L_σ with vertex v. By the Künneth formula, we thus obtain the first isomorphism in the chain

$$H_\bullet\big(\mathrm{st}'(\mathring{\sigma}), \mathrm{st}'(\mathring{\sigma}) \setminus S_{\mathring{\sigma}}\big) \ \cong\ H_\bullet(\mathring{c}(L_\sigma), \mathring{c}(L_\sigma) \setminus \{v\}) \ \cong\ \tilde{H}_\bullet(L_\sigma)[-1] \ =\ 0\ ;$$

the second one follows from the homotopy equivalences $\mathring{c}(L_\sigma) \simeq v$ and $\mathring{c}(L_\sigma) \setminus \{v\} \simeq L_\sigma$, and the final equality from the assumption $(\mathrm{iii})_\sigma$. □

We now continue the proof of Lemma 4.5.

"$(\mathrm{ii})_o \iff (\mathrm{iii})_o$" With $\mathring{L} := L \setminus \partial L$, we conclude from (4.6.1) this chain of isomorphisms

$$H_q(L) \cong H_q(\mathring{L}) \cong H^{n-1-q}(S_V, S_V \setminus \mathring{L}) \cong H^{n-1-q}(L, \partial L) \cong H_{n-1-q}(L, \partial L)^*\ ,$$

where the first one follows from the above lemma, the second one, from relative Poincaré duality (see, e.g., [Sp: Thm. 6.2.17]), the third one is obtained by excision, and the fourth one is the obvious duality.

"$(\mathrm{iii})_o \Longrightarrow (\mathrm{iv})_o$": We may assume $n \geq 3$; we then have to show that ∂L has the same homology as an $(n-2)$-dimensional sphere. From (iii) together with the equivalent assumption (ii) and the exact homology sequence of the pair $(L, \partial L)$, we derive that $\tilde{H}_{j-1}(\partial L) \cong H_j(L, \partial L) = 0$ for $j \neq n-1$, and $H_{n-2}(\partial L) \cong H_{n-1}(L, \partial L) = \mathbf{R}$.

"$(\mathrm{iv})_o \Longrightarrow (\mathrm{iii})_o$": It remains to verify that the reduced homology $\tilde{H}_\bullet(L)$ vanishes. We set $C := S^{n-1} \setminus \mathring{L}$ and look at the Mayer-Vietoris sequence

$$\ldots \to H_{q+1}(S^{n-1}) \to H_q(\partial L) \to H_q(L) \oplus H_q(C) \to H_q(S^{n-1}) \to H_{q-1}(\partial L) \to \ldots$$

associated to $S^{n-1} = L \cup C$. The hypothesis immediately yields $\tilde{H}_q(L) \oplus \tilde{H}_q(C) = 0$ for $q \leq n-3$, which settles the claim for these values of q. The term $H_{n-1}(L) \oplus$

$H_{n-1}(C)$ vanishes since both, L and C, are $(n-1)$-dimensional cell complexes in S^{n-1} with non-empty boundary. That is obvious for L; for C, it is true since Δ has a refinement which can be embedded into a complete fan, see 0.A. The arrow $H_{n-1}(S^{n-1}) \to H_{n-2}(\partial L)$ in the exact sequence under consideration is thus injective; hence, it is even an isomorphism of one-dimensional vector spaces. This implies that the mapping $H_{n-2}(L) \oplus H_{n-2}(C) \to H_{n-2}(S^{n-1})$ is injective, too, and that yields $H_{n-2}(L) = 0$. ◻

As a consequence, we see that quasi-convexity of a purely n-dimensional fan depends only on the topology of its boundary:

4.7 Corollary: *Let Δ and Δ' be purely n-dimensional fans. If their boundaries have the same support $|\partial\Delta| = |\partial\Delta'|$, then Δ is quasi-convex if and only if Δ' is.*

In particular, that applies to the following special cases:

i) *Δ' is a refinement of Δ,*

ii) *Δ and Δ' are "complementary" subfans, i.e., $\Delta \cup \Delta'$ is a complete fan, and Δ and Δ' have no n-dimensional cones in common.*

We now come to the proof of Theorem 4.3:

4.8 Proof of Theorem 4.3: For convenience, we briefly recall that we have to prove the equivalence of the following three statements for a purely n-dimensional fan Δ and the minimal extension sheaf $\mathcal{E}^\bullet$ on Δ:

(a) *The $A^\bullet$-module $E^\bullet_\Delta = \mathcal{E}^\bullet(\Delta)$ is free;*

(b) $\tilde{H}^\bullet(\Delta, \partial\Delta; \mathcal{E}^\bullet) = 0$,

(c) $\tilde{H}^\bullet(\Delta_\sigma, \partial\Delta_\sigma; \mathbf{R}) = 0$ *for each cone $\sigma \in \Delta$.*

"$(b) \iff (c)$": If we write

$$\mathcal{E}^\bullet \;\cong\; \bigoplus_{\sigma\in\Delta} {}_\sigma\mathcal{J} \otimes K_\sigma$$

according to Lemma 3.4, we obtain the following direct sum decomposition

$$\tilde{H}^\bullet(\Delta, \partial\Delta; \mathcal{E}^\bullet) \;\cong\; \bigoplus_{\sigma\in\Delta} \tilde{H}^\bullet(\Delta_\sigma, \partial\Delta_\sigma; \mathbf{R}) \otimes K_\sigma$$

according to Remark 3.5 and the isomorphism (3.4.2). Hence it is sufficient to see that none of the vector spaces $K_\sigma = \ker(\varrho^\sigma_{\partial\sigma} \colon E^\bullet_\sigma \to E^\bullet_{\partial\sigma})$ is zero: Since $E^\bullet_\sigma$ is a non-zero free $A^\bullet_\sigma$-module and $E^\bullet_{\partial\sigma}$ is a torsion module (see [BBFK: 6.1]), the restriction homomorphism $\varrho^\sigma_{\partial\sigma}$ never is injective.

"$(b) \Longrightarrow (a)$": We shall use the abbreviations

$$C^r := C^r(\Delta, \partial\Delta; \mathcal{E}^\bullet)\ ,\ I^r := \operatorname{im}\delta^{r-1}\ , \quad \text{and} \quad \operatorname{Tor}_k := \operatorname{Tor}_k^{A^\bullet}\ .$$

By downward induction on r, we verifiy the vanishing statement

$$\text{(4.8.1)} \qquad \mathrm{Tor}_k(I^r, \mathbf{R}^\bullet) \;=\; 0 \quad \text{for} \quad k > r\,.$$

That yields the quasi-convexity: Since $I^0 = E^\bullet_\Delta$, we obtain $\mathrm{Tor}_1(E^\bullet_\Delta, \mathbf{R}^\bullet) = 0$; hence, according to (0.B), the graded $A^\bullet$-module $E^\bullet_\Delta$ is free.

Obviously (4.8.1) holds for $r = n+1$. By assumption, the complex $C^\bullet$ is acyclic; hence, each sequence

$$0 \longrightarrow I^r \longrightarrow C^r \longrightarrow I^{r+1} \longrightarrow 0$$

is exact and thus induces an exact sequence

$$\mathrm{Tor}_{k+1}(I^{r+1}, \mathbf{R}^\bullet) \longrightarrow \mathrm{Tor}_k(I^r, \mathbf{R}^\bullet) \longrightarrow \mathrm{Tor}_k(C^r, \mathbf{R}^\bullet)\,.$$

By induction hypothesis, its first term vanishes; thus, it suffices to verify the vanishing of the last term for $k > r$: The module $C^r = \bigoplus_{\dim\sigma = n-r} E^\bullet_\sigma$ actually is a direct sum of shifted modules $A^\bullet_\sigma$, so $\mathrm{Tor}_k(C^r, \mathbf{R}^\bullet) = 0$ for $k > r$, see (0.B.1).

"$(a) \Longrightarrow (b)$" In addition to the above, we use the abbreviations

$$K^r := \ker\delta^r \quad \text{and} \quad \tilde{H}^r := \tilde{H}^r(\Delta, \partial\Delta; \mathcal{E}^\bullet) = K^r/I^r\,.$$

In order to verify the vanishing of $\tilde{H}^\bullet := \tilde{H}^\bullet(\Delta, \partial\Delta; \mathcal{E}^\bullet)$, we choose an increasing sequence of subspaces $V_0 := 0 \subset V_1 \subset \ldots \subset V_n := V$ such that $V = V_r \oplus V_\sigma$ holds simultaneously for each $\sigma \in \Delta^{n-r}$. Then the algebras $D^\bullet_r := S^\bullet((V/V_r)^*)$ form a decreasing sequence of subalgebras of $A^\bullet$; moreover, there are isomorphisms $D^\bullet_r \cong A^\bullet_\sigma$ induced by the composed mappings $D^\bullet_r \subset A^\bullet \longrightarrow A^\bullet_\sigma$. In particular, each $C^r = \bigoplus_{\sigma\in\Delta^{n-r}} E^\bullet_\sigma$ is a free $D^\bullet_r$-module. In addition, we choose linear forms $T_1, \ldots, T_n$ in A^2 such that $D^\bullet_r = \mathbf{R}[T_1, \ldots, T_{n-r}]$.

By induction on r, we prove the stronger statement

$$\tilde{H}^q \;=\; 0 \;\text{ for }\; q < r\,, \quad \text{and} \quad I^r \text{ is a free } D^\bullet_r\text{-module.}$$

Since $I^0 = E^\bullet_\Delta$ is free by hypothesis, the assertion holds for $r = 0$. So let us proceed from r to $r+1$. The vanishing of $\tilde{H}^r$ is a consequence of the fact that its support in $\mathrm{Spec}(D^\bullet_r)$ is small: According to Lemma 4.10 below, the support of $\tilde{H}^r$ is of codimension at least $r+2$ in $\mathrm{Spec}(A^\bullet)$ and thus, considered as $D^\bullet_r$-module, of codimension at least 2 in $\mathrm{Spec}(D^\bullet_r)$. An application of Lemma 4.9 to the exact sequence

$$0 \longrightarrow I^r \longrightarrow K^r \longrightarrow \tilde{H}^r \longrightarrow 0$$

yields the vanishing $\tilde{H}^r = 0$.

It remains to prove that $I := I^{r+1} = \mathrm{im}\,\delta^r$ is a free module over $D^\bullet := D^\bullet_{r+1}$. By 0.B, this is equivalent to

$$\mathrm{Tor}_1^{D^\bullet}(I, \mathbf{R}) \;=\; 0\,.$$

Recall that $D^\bullet_r = D^\bullet[T]$ with $T := T_{n-r}$. Thus, the formula

$$\text{(4.8.2)} \qquad \mathrm{Tor}_k^{D^\bullet}(I, \mathbf{R}) \;\cong\; \mathrm{Tor}_k^{D^\bullet[T]}(I, \mathbf{R}[T])$$

provides the bridge to the induction hypothesis on the previous level r. The multiplication by T yields an endomorphism $\mu := \mu_T$ of $\mathbf{R}[T]$ that has degree two, providing exact sequences

$$0 \longrightarrow \mathbf{R}[T] \xrightarrow{\mu} \mathbf{R}[T] \longrightarrow \mathbf{R} \longrightarrow 0 \tag{4.8.3}$$

and

$$\mathrm{Tor}_2^{D^\bullet[T]}(I, \mathbf{R}) \longrightarrow \mathrm{Tor}_1^{D^\bullet[T]}(I, \mathbf{R}[T]) \xrightarrow{\vartheta} \mathrm{Tor}_1^{D^\bullet[T]}(I, \mathbf{R}[T]) \; .$$

The map ϑ is a homomorphism of degree two since it is induced by the multiplication μ_T. Moreover, it is *injective*: In the exact sequence of $D_r^\bullet$-modules

$$0 \longrightarrow K^r \longrightarrow C^r \xrightarrow{\delta^r} I \longrightarrow 0 \; , \tag{4.8.4}$$

the module K^r is isomorphic to I^r since $\tilde{H}^r$ vanishes. Hence, by induction hypothesis, the sequence (4.8.4) is a *free* $D_r^\bullet$-resolution of I, thus yielding $\mathrm{Tor}_2^{D^\bullet[T]}(I, \mathbf{R}^\bullet) = \mathrm{Tor}_2^{D_r^\bullet}(I, \mathbf{R}^\bullet) = 0$. – Eventually, since $\mathrm{Tor}_1^{D^\bullet[T]}(I, \mathbf{R}^\bullet[T]) \cong \mathrm{Tor}_1^{D^\bullet}(I, \mathbf{R}^\bullet)$ is a finite-dimensional graded vector space, the injective endomorphism ϑ of degree two is the zero map, whence $\mathrm{Tor}_1^{D^\bullet}(I, \mathbf{R}^\bullet) = 0$. ▫

We still have to state and prove the two lemmata referred to above. The first one is a general result of commutative algebra.

4.9 Lemma: *Let R be a noetherian normal integral domain and consider an exact sequence*

$$0 \longrightarrow R^s \longrightarrow M \longrightarrow H \longrightarrow 0$$

of finitely generated R-modules. If M is torsion free and H non-zero, then $\mathrm{supp}(H)$ *is of codimension at most* 1 *in* $\mathrm{Spec}\, R$.

Proof: We may assume that $Y := \mathrm{supp}\, H$ is a proper subset of $X := \mathrm{Spec}\, R$. Hence, H is a torsion module and thus M, of rank s. Let Q be the field of fractions of R. Since M is torsion-free, there is a natural monomorphism

$$M = M \otimes_R R \;\hookrightarrow\; M \otimes_R Q =: M_Q \cong Q^s \; .$$

We may interpret the given monomorphism $R^s \hookrightarrow M$ as an inclusion. Hence, an R-basis of R^s may be considered as a Q-basis of M_Q, thus providing an identification $M_Q = Q^s$.

We now fix a non-zero element $h \in H$ and an inverse image $m = (q_1, \ldots, q_s) \in M \subset Q^s$ of that element h. A prime ideal $\mathfrak{p}$ of R lies in $X \setminus Y$ if and only if the localized module $H_\mathfrak{p}$ vanishes, or equivalently – since localization is exact –, if and only if the localized inclusion $(R_\mathfrak{p})^s \hookrightarrow M_\mathfrak{p}$ is an isomorphism. Hence, $\mathfrak{p} \notin Y$ implies $q_1, \ldots, q_s \in R_\mathfrak{p}$. Since R is normal and noetherian, the stipulation $\mathrm{codim}_X(Y) \geq 2$ would yield $q_1, \ldots, q_s \in R$, i.e., $m \in R^s$, providing the contradiction $h = 0$. ▫

4.10 Lemma: *The support of the $A^\bullet$-module $\tilde H^q(\Delta,\partial\Delta;\mathcal{E}^\bullet)$ in* $\operatorname{Spec}(A^\bullet)$ *is of codimension at least $q+2$.*

Proof: For a prime ideal $\mathfrak{p}\subset A^\bullet$, let $\tilde H^q_{\mathfrak p}$ be the localization at $\mathfrak p$ of the $A^\bullet$-module $\tilde H^q_{\mathfrak p}$. We show that $\operatorname{supp}\tilde H^q := \{\mathfrak p\in\operatorname{Spec}(A^\bullet)\ ;\ \tilde H^q_{\mathfrak p}\neq 0\}$, the *support* of $\tilde H^q$, is contained in the union

$$\bigcup_{\sigma\in\Delta^{\le n-q-2}} \operatorname{Spec} A_\sigma \tag{4.10.1}$$

of the "linear subspaces" $\operatorname{Spec} A^\bullet_\sigma\subset\operatorname{Spec} A^\bullet$. To that end, we consider a prime ideal $\mathfrak p\in\operatorname{Spec}(A^\bullet)$. Since localization of $A^\bullet$-modules at $\mathfrak p$ is an exact functor, the localized cohomology module $\tilde H^q_{\mathfrak p}$ is the q-th cohomology of the complex

$$\tilde C^\bullet_{\mathfrak p} \;\cong\; \tilde C^\bullet(\Delta,\partial\Delta;\mathcal{E}^\bullet_{\mathfrak p})\ ,$$

where the "localized" sheaf $\mathcal{E}^\bullet_{\mathfrak p}$ is determined by

$$\mathcal{E}^\bullet_{\mathfrak p}(\tau) \;:=\; \mathcal{E}^\bullet(\tau)_{\mathfrak p}\ .$$

Let $k=k(\mathfrak p)$ be the minimal dimension of a cone $\tau\in\Delta$ such that $\mathfrak p$ belongs to $\operatorname{Spec}(A^\bullet_\tau)$. Then $\mathcal{E}^\bullet_{\mathfrak p}(\sigma)=0$ for a cone with $\dim\sigma<k$, whence in particular a decomposition

$$\mathcal{E}^\bullet_{\mathfrak p} \;\cong\; \bigoplus_{\dim\sigma\ge k} {}_\sigma\mathcal{J}\otimes K_\sigma\ ,$$

see Lemma 3.4. According to (3.4.2) and Remark 3.5,

$$\tilde H^q(\Delta,\partial\Delta;\mathcal{E}^\bullet_{\mathfrak p}) \;\cong\; \bigoplus_{\dim\sigma\ge k} \tilde H^q(\Delta,\partial\Delta;{}_\sigma\mathcal{J})\otimes K_\sigma$$

vanishes for $q\ge n-k-1$. Consequently, if $\mathfrak p$ belongs to $\operatorname{supp}\tilde H^q$, then $k(\mathfrak p)\le n-q-2$ holds, i.e., $\mathfrak p$ appears in the union (4.10.1). ◻

Theorem 4.3 provides a characterization of quasi-convex fans in terms of acyclicity of the relative cellular cochain complex. An analoguous statement holds also for the augmented absolute cellular cochain complex

$$0\longrightarrow F_{(\Delta,\partial\Delta)}\longrightarrow C^0(\Delta;\mathcal{F})\longrightarrow\ldots\longrightarrow C^n(\Delta;\mathcal{F})\longrightarrow 0 \tag{4.11.1}$$

for the sheaf $\mathcal{F}=\mathcal{E}^\bullet$ on Δ. Up to a shift, that complex turns out to be a minimal complex in the sense of Bernstein and Lunts: In [BeLu], a complex

$$Z^\bullet : 0\longrightarrow Z^{-n}\xrightarrow{\delta^{-n}} Z^{-n+1}\xrightarrow{\delta^{-n+1}}\ldots\xrightarrow{\delta^{-1}} Z^0\longrightarrow 0$$

of graded $A^\bullet$-modules is called *minimal* if it satisfies the following conditions:

(i) $Z^0\cong\mathbf{R}^\bullet[n]$, i.e., the $A^\bullet$-module $A^\bullet/\mathfrak m\cong\mathbf{R}^\bullet$ placed in degree $-n$;

(ii) there is a decomposition $Z^{-d}=\bigoplus_{\sigma\in\Delta^d} Z_\sigma$ for $0\le d\le n$;

(iii) each Z_σ is a free graded $A^\bullet_\sigma$-module;

(iv) for each cone $\sigma \in \Delta$, the differential δ maps Z_σ to $\bigoplus_{\tau \prec_1 \sigma} Z_\tau$, so for $\dim \sigma = d$, one obtains a subcomplex

$$0 \longrightarrow Z_\sigma \xrightarrow{\delta_\sigma^{-d}} \bigoplus_{\tau \prec_1 \sigma} Z_\tau \xrightarrow{\delta_\sigma^{-d+1}} \ldots \longrightarrow Z_o \longrightarrow 0 \; ;$$

(v) with $I_\sigma := \ker \delta_\sigma^{-d+1}$, the differential δ_σ^{-d} induces an isomorphism

$$\overline{\delta}_\sigma^{-d} : \overline{Z}_\sigma := Z_\sigma / \mathfrak{m} Z_\sigma \xrightarrow{\cong} \overline{I}_\sigma := I_\sigma / \mathfrak{m} I_\sigma$$

of real vector spaces.

If the fan Δ is purely n-dimensional, then the shifted cochain complex

$$Z^\bullet := C^\bullet(\Delta, \mathcal{E}^\bullet[n])[n] \quad \text{i.e.,} \quad Z^{-i} = C^{n-i}(\Delta, \mathcal{E}^\bullet[n])$$

is minimal: With $Z_\sigma := E_\sigma^\bullet[n]$, conditions (i) – (iv) are immediate; condition (v) follows from (LME) using the isomorphism $I_\sigma \cong \mathcal{E}^\bullet(\partial\sigma)[n] = E_{\partial\sigma}^\bullet[n]$ of $A_\sigma^\bullet$-modules.

The following result proves a conjecture of Bernstein and Lunts in [BeLu], p.129:

4.11 Theorem: *A purely n-dimensional fan Δ is quasi-convex if and only if the complex $C^\bullet(\Delta, \mathcal{E}^\bullet)$ is exact in degrees $q > 0$ and $H^0(\Delta, \mathcal{E}^\bullet) \cong E^\bullet_{(\Delta, \partial\Delta)}$. Specifically, for a complete fan Δ, a minimal complex in the sense of Bernstein and Lunts is exact except in degree $-n$.*

Proof: We use the fact that the sheaf $\mathcal{E}^\bullet$ is flabby, and we profit from the proof of the equivalence (c) $\Longleftrightarrow$ (b) in Theorem 4.3: By the absolute version of (3.4.2), the complex (4.11.1) is acyclic for the sheaf $\mathcal{E}^\bullet$ if and only if it is acyclic for each characteristic sheaf ${}_\sigma\mathcal{J}$ of $\sigma \in \Delta$ since none of the vector spaces K_σ vanishes, see the proof of Theorem 4.3, (c) $\Leftrightarrow$ (b). For $\sigma \notin \partial\Delta$, the characteristic sheaf ${}_\sigma\mathcal{J}$ has been treated at the beginning of the proof of Theorem 4.4. For $\sigma \in \partial\Delta$, we have ${}_\sigma\mathcal{J}(\Delta, \partial\Delta) = 0$, such that the absolute versions of Remark 3.5 and formula (4.4.1) yield isomorphisms

$$\tilde{H}^q(\Delta, {}_\sigma\mathcal{J}) \;\cong\; H^q(\Delta, {}_\sigma\mathcal{J}) \;\cong\; H^q(\Delta_\sigma, \mathbf{R}) \;\cong\; \tilde{H}_{k-1-q}(L_\sigma, \mathbf{R}) \; ,$$

where $k = \operatorname{codim} \sigma$ and L_σ is the link of some point $x \in S_\Delta \cap \mathring{\sigma}$. Eventually, statement (iii) of Proposition 4.5 gives $\tilde{H}_\bullet(L_\sigma, \mathbf{R}) = 0$. ◻

4.12 Corollary. *For a minimal extension sheaf $\mathcal{E}^\bullet$ on a quasi-convex fan Δ, the $A^\bullet$-submodule $E^\bullet_{(\Delta,\partial\Delta)}$ of $E^\bullet_\Delta$ is free.*

Proof: Since the absolute cellular cochain complex is acyclic, we may proceed as in the proof of Theorem 4.3. ◻

5. Poincaré Polynomials

In the present section, we discuss the virtual intersection Betti numbers $b_{2q}(\Delta) := \dim \overline{E}^{2q}_{\Delta}$ and $b_{2q}(\Delta, \partial\Delta) := \dim \overline{E}^{2q}_{(\Delta,\partial\Delta)}$ of a quasi-convex fan Δ, where $\mathcal{E}^{\bullet}$ is a minimal extension sheaf on Δ. It is convenient to use the language of Poincaré polynomials.

5.1 Definition: *The (equivariant) Poincaré series of a fan Δ is the formal power series*

$$Q_{\Delta}(t) \;:=\; \sum_{q\geq 0} \dim E^{2q}_{\Delta} \cdot t^{2q} \;,$$

its (intersection) Poincaré polynomial is the polynomial

$$P_{\Delta}(t) \;=\; \sum_{q\geq 0}^{<\infty} \dim \overline{E}^{2q}_{\Delta} \cdot t^{2q} \;=\; \sum_{q\geq 0}^{<\infty} b_{2q}(\Delta) t^{2q} \;.$$

For an affine fan $\langle\sigma\rangle$, we simply write

$$Q_{\sigma} := Q_{\langle\sigma\rangle} \quad \textit{and} \quad P_{\sigma} := P_{\langle\sigma\rangle} \;.$$

Furthermore, for a subfan $\Lambda \preceq \Delta$, the relative Poincaré polynomial $P_{(\Delta,\Lambda)}$ is defined in an analoguous manner.

We refer to P_{Δ} as the *global* Poincaré polynomial of Δ, while the polynomials P_{σ} for $\sigma \in \Delta$ are called its *local* Poincaré polynomials.

5.2 Remark. *If the fan Δ is quasi-convex, then*

$$Q_{\Delta}(t) \;=\; \frac{1}{(1-t^2)^n} \cdot P_{\Delta}(t) \;;$$

for a cone σ, that implies

$$Q_{\sigma}(t) \;=\; \frac{1}{(1-t^2)^{\dim\sigma}} \cdot P_{\sigma}(t) \;.$$

Proof. For a free graded $A^{\bullet}$-module $F^{\bullet}$, the Künneth formula $F^{\bullet} \cong A^{\bullet} \otimes_{\mathbf{R}} \overline{F}^{\bullet}$ holds, while the Poincaré series of a tensor product of graded vector spaces is the product of the Poincaré series of the factors. Since $Q_{A^{\bullet}} = 1/(1-t^2)^n$, the first formula follows immediately. Going over to the base ring $A^{\bullet}_{\sigma}$ yields the second one. □

The basic idea for the computation of the virtual intersection Betti numbers is to use a two-step procedure. In the first step, the global invariant is expressed as a sum of local terms. In the second step, these local invariants are described in terms of the global ones associated to lower-dimensional complete fans.

5.3 Theorem (Local-to-Global Formula): If Δ is a quasi-convex fan of dimension n and $\mathring{\Delta} := \Delta \setminus \partial\Delta$, then

$$P_\Delta(t) \;=\; \sum_{\sigma\in\mathring{\Delta}} (t^2-1)^{n-\dim\sigma} P_\sigma(t)$$

and

$$P_{(\Delta,\partial\Delta)}(t) \;=\; \sum_{\sigma\in\Delta} (t^2-1)^{n-\dim\sigma} P_\sigma(t)\,.$$

Proof. The augmented cellular cochain complex

$$0 \longrightarrow E^\bullet_{\mathring{\Delta}} \longrightarrow C^0(\Delta,\partial\Delta;\mathcal{E}^\bullet) \longrightarrow \ldots \longrightarrow C^n(\Delta,\partial\Delta;\mathcal{E}^\bullet) \longrightarrow 0$$

of 3.2 associated to the quasi-convex fan Δ is acyclic by Theorem 4.3. We set

$$Q_i(t) \;:=\; \sum_{q\geq 0} \dim\, C^i(\Delta,\partial\Delta;\mathcal{E}^{2q})\cdot t^{2q} \;=\; \sum_{\sigma\in\mathring{\Delta}\cap\Delta^{n-i}} Q_\sigma(t)\;.$$

Then we obtain the equality

$$Q_\Delta \;= \sum_{i=0}^{n}(-1)^i Q_i \;= \sum_{\sigma\in\mathring{\Delta}} (-1)^{n-\dim\sigma} Q_\sigma(t)\;.$$

The first assertion follows from Remark 5.2. The second formula is obtained in the same way using the acyclicity of the complex

$$0 \longrightarrow E^\bullet_{(\Delta,\partial\Delta)} \longrightarrow C^0(\Delta,\mathcal{E}^\bullet) \longrightarrow \ldots \longrightarrow C^n(\Delta,\mathcal{E}^\bullet) \longrightarrow 0\,,$$

see Theorem 4.11 and Corollary 4.12. ◻

For a non-zero cone σ, in order to reduce the computation of $\overline{E}^\bullet_\sigma$ to a problem in lower dimensions, we come back to section 0.D: We choose a line $\ell \subset V$ intersecting the relative interior $\mathring{\sigma}$ and consider the flattened boundary fan $\Lambda_\sigma := \pi(\partial\sigma)$, where $\pi\colon V_\sigma \to V_\sigma/\ell$ is the quotient map. Then the direct image sheaf

(5.3.1) $$\mathcal{G}^\bullet := \pi_*(\mathcal{E}^\bullet|_{\partial\sigma}) :\; \tau \mapsto \mathcal{E}^\bullet\big((\pi|_{\partial\sigma})^{-1}(\tau)\big)$$

is a minimal extension sheaf on Λ_σ. We use the identification $A^\bullet_\sigma = B^\bullet_\sigma[T]$ of (0.D.2) and the function $f \in \mathcal{A}^2(\Lambda_\sigma)$ of (0.D.3). If we form residue classes of the $A^\bullet_\sigma$-modules $E^\bullet_\sigma$ and $E^\bullet_{\partial\sigma}$ (with respect to $\mathfrak{m}_{A^\bullet_\sigma}$) and of the $B^\bullet_\sigma$-module $G^\bullet_{\Lambda_\sigma}$ (with respect to $\mathfrak{m}_{B^\bullet_\sigma}$), then we obtain isomorphisms of graded vector spaces

(5.3.2) $$\overline{E}^\bullet_\sigma \;\cong\; \overline{E}^\bullet_{\partial\sigma} \;\cong\; \overline{G}^\bullet_{\Lambda_\sigma}/(f\cdot\overline{G}^\bullet_{\Lambda_\sigma})\;.$$

A first result is an estimate for the degree of the Poincaré polynomials:

5.4 Corollary: *Let Δ be a quasi-convex fan and σ, a non-zero cone.*

i) The relative Poincaré polynomial $P_{(\Delta,\partial\Delta)}$ *is monic of degree* $2n$*; if* Δ *is not complete, then the absolute Poincaré polynomial* P_Δ *is of degree at most* $2n-2$.

ii) The "local" Poincaré polynomial P_σ *is of degree at most* $2\dim\sigma - 2$.

Proof. We proceed by induction on the dimension n of Δ: If (ii) holds up to dimension n, then so does (i), see Theorem 5.3. If (i) is valid up to dimension $n-1$, then (ii) holds for $\dim\sigma = n$. Since this is evident for $n = 1$, we may assume $n > 1$. Going over to the complete fan Λ_σ of dimension $n-1$, we use the isomorphism (5.3.2). Since $\overline{G}^q_{\Lambda_\sigma} = 0$ holds for $q > 2n-2$ according to the induction hypothesis, assertion (ii) follows. □

For the second step in the computation of Betti numbers, we have to relate the local Poincaré polynomial P_σ to the global Poincaré polynomial P_{Λ_σ} of the complete (and thus quasi-convex) fan Λ_σ of dimension $\dim\sigma - 1$. Here the vanishing condition $\mathbf{V}(\sigma)$ of 1.7 plays a decisive role:

5.5 Theorem (Local Recursion Formula): *Let* σ *be a cone.*

i) If σ *is simplicial, then* $P_\sigma \equiv 1$.

ii) If the condition $\mathbf{V}(\sigma)$ *is satisfied and* σ *is not the zero cone, then*

$$P_\sigma(t) \;=\; \tau_{<\dim\sigma}\big((1-t^2)P_{\Lambda_\sigma}(t)\big)\ .$$

The truncation operator $\tau_{<k}$ is defined by $\tau_{<k}(\sum_q a_q t^q) := \sum_{q<k} a_q t^q$. – Let us note that for $\dim\sigma = 1$ and 2, the statements (i) and (ii) agree.

Proof: Statement i) follows from the isomorphism $E^\bullet_\sigma \cong A^\bullet_\sigma$ for a simplicial cone σ, see 1.4. In order to prove statement ii), we use the isomorphism (5.3.2). We thus have to investigate the graded vector space $\overline{G}^\bullet_{\Lambda_\sigma}/f\overline{G}^\bullet_{\Lambda_\sigma}$, or equivalently the kernel and cokernel of the map

$$\overline{\mu}_f\colon \overline{G}^\bullet_{\Lambda_\sigma}[-2] \longrightarrow \overline{G}^\bullet_{\Lambda_\sigma}\ ,\ \overline{h} \mapsto \overline{fh}$$

induced by the multiplication $\mu_f\colon G^\bullet_{\Lambda_\sigma}[-2] \to G^\bullet_{\Lambda_\sigma}$. The formula ii) now is an immediate consequence of the "Hard Lefschetz" type theorem 5.6 below. □

5.6 Combinatorial Hard Lefschetz Theorem: *Let* Δ *be a complete fan and* $f \in \mathcal{A}^2(\Delta)$*, a strictly convex function. If the condition* $\mathbf{V}\big(\gamma(f)\big)$ *is satisfied, then multiplication with* f,

$$\overline{\mu}^{2q}_f\colon \overline{E}^{2q}_\Delta \longrightarrow \overline{E}^{2q+2}_\Delta\ ,\ \overline{h} \mapsto \overline{fh}\ ,$$

is injective for $2q \le n-1$ *and surjective for* $2q \ge n-1$.

Theorem 5.6 will be derived at the end of section 6 by means of the Poincaré Duality Theorem 6.3.

6. Poincaré Duality

Our aim in this section is to prove a "Poincaré Duality Theorem" for the virtual intersection cohomology of quasi-convex fans. The first step is to define a – non-canonical and not necessarily associative – $\mathcal{A}^\bullet$-bilinear "intersection product" $\mathcal{E}^\bullet \times \mathcal{E}^\bullet \to \mathcal{E}^\bullet$ on a minimal extension sheaf $\mathcal{E}^\bullet$ for an arbitrary fan Δ. On the level of global sections, it provides an $A^\bullet$-bilinear "product" $E^\bullet_\Delta \times E^\bullet_{(\Delta,\partial\Delta)} \to E^\bullet_{(\Delta,\partial\Delta)}$ for the "virtual equivariant intersection cohomology" of Δ. If Δ is quasi-convex, then in addition, there exists an evaluation mapping $\varepsilon\colon E^\bullet_{(\Delta,\partial\Delta)} \to A^\bullet[-2n]$. The crucial result is the "equivariant Poincaré Duality Theorem" 6.3 according to which the composition of the intersection product and the evaluation map is a dual pairing of $A^\bullet$-modules. Passing to the quotients modulo the maximal ideal $\mathfrak{m}$, we reach our aim.

In the case of a simplicial fan, where the sheaf $\mathcal{A}^\bullet$ of piecewise polynomial functions is a minimal extension sheaf, such an "interesection product" is simply given by the multiplication of functions. Hence, a possible approach to the general case is as follows: We choose a simplicial refinement $\hat{\Delta}$ of Δ. According to the Decomposition Theorem 2.5, we interpret $\mathcal{E}^\bullet$ as a direct factor of the sheaf $\hat{\mathcal{A}}$ of $\hat{\Delta}$-piecewise polynomial functions on Δ. Then we restrict the multiplication of functions from $\hat{\mathcal{A}}^\bullet$ to its direct factor $\mathcal{E}^\bullet$ and project onto it.

In order to keep track of the relation between the intersection product over the boundary of a cone and the cone itself, we apply the above idea repeatedly in a recursive extension procedure. The proof of Poincaré duality will follow the same pattern.

6.1 An Intersection Product: The 2-dimensional skeleton $\Delta^{\leq 2}$ is a simplicial subfan. Hence, up to scalar multiples, there is a canonical isomorphism $\mathcal{A}^\bullet \cong \mathcal{E}^\bullet$ on $\Delta^{\leq 2}$ (see 1.8). We thus define the intersection product on $\Delta^{\leq 2}$ to correspond via that isomorphism to the product of functions.

We now assume that the intersection product is defined on $\Delta^{\leq m}$ and consider a cone $\sigma \in \Delta^{m+1}$. So we are given a symmetric bilinear morphism $E^\bullet_{\partial\sigma} \times E^\bullet_{\partial\sigma} \to E^\bullet_{\partial\sigma}$ of $A^\bullet_\sigma$-modules. As in section 0.D, we fix a line $\ell \subset V_\sigma$ intersecting $\mathring{\sigma}$ and denote $B^\bullet_\sigma$ the subalgebra of $A^\bullet_\sigma$ consisting of the functions constant on parallels to ℓ. We recall that $E^\bullet_{\partial\sigma} \cong G^\bullet_{\Lambda_\sigma}$ is a free $B^\bullet_\sigma$-module, cf. Theorem 4.3 applied to the minimal extension sheaf $\mathcal{G}^\bullet$ on the flattened boundary fan Λ_σ. Since $E^\bullet_\sigma$ is a free $A^\bullet_\sigma$-module, the restriction homomorphism $E^\bullet_\sigma \to E^\bullet_{\partial\sigma}$ admits a factorization

$$E^\bullet_\sigma \xrightarrow{\alpha} A^\bullet_\sigma \otimes_{B^\bullet_\sigma} E^\bullet_{\partial\sigma} \xrightarrow{\beta} A^\bullet_\sigma \otimes_{A^\bullet_\sigma} E^\bullet_{\partial\sigma} = E^\bullet_{\partial\sigma}$$

through the *free* $A^\bullet_\sigma$-module

$$F^\bullet_\sigma := A^\bullet_\sigma \otimes_{B^\bullet_\sigma} E^\bullet_{\partial\sigma} . \tag{6.1.1}$$

Since the reduction of α modulo $\mathfrak{m}_\sigma \subset A^\bullet_\sigma$ is injective, the map $\alpha\colon E^\bullet_\sigma \to F^\bullet_\sigma$ is a "direct" embedding, i.e., there is a decomposition

$$F^\bullet_\sigma \cong \alpha(E^\bullet_\sigma) \oplus K^\bullet \tag{6.1.2}$$

of free $A^\bullet_\sigma$-modules. We may even assume that $K^\bullet$ is contained in the kernel of the natural map $\beta\colon F^\bullet_\sigma \to E^\bullet_{\partial\sigma}$: We fix a homogeneous basis $f_1, \ldots, f_r$ of $K^\bullet$. The images $\beta(f_i)$ of these elements in $E^\bullet_{\partial\sigma}$ are restrictions of elements $g_i \in E^\bullet_\sigma$; hence, we may replace $K^\bullet$ with the submodule generated by the elements $f_i - \alpha(g_i)$ for $1 \le i \le r$.

On the other hand, by scalar extension, there is an induced product

$$F^\bullet_\sigma \times F^\bullet_\sigma \longrightarrow F^\bullet_\sigma .$$

It provides the desired extension of the intersection product from $\partial\sigma$ to σ via the composition

$$E^\bullet_\sigma \times E^\bullet_\sigma \xrightarrow{\alpha\times\alpha} F^\bullet_\sigma \times F^\bullet_\sigma \longrightarrow F^\bullet_\sigma = \alpha(E^\bullet_\sigma) \oplus K^\bullet \longrightarrow \alpha(E^\bullet_\sigma) \cong E^\bullet_\sigma ,$$

where the last arrow is the projection onto $\alpha(E^\bullet_\sigma)$ with kernel $K^\bullet$. This ends the description of the extension procedure. To sum up, after a finite number of steps, we arrive at a symmetric bilinear morphism

$$\mathcal{E}^\bullet \times \mathcal{E}^\bullet \longrightarrow \mathcal{E}^\bullet$$

of sheaves of $\mathcal{A}^\bullet$-modules, called an *intersection product on the minimal extension sheaf* $\mathcal{E}^\bullet$. In particular, we thus have defined a product

$$E^\bullet_\Delta \times E^\bullet_\Delta \longrightarrow E^\bullet_\Delta$$

on the level of global sections that maps $E^\bullet_\Delta \times E^\bullet_{(\Delta,\partial\Delta)}$ to $E^\bullet_{(\Delta,\partial\Delta)}$ and thus induces a product

$$E^\bullet_\Delta \times E^\bullet_{(\Delta,\partial\Delta)} \longrightarrow E^\bullet_{(\Delta,\partial\Delta)} .$$

In order to obtain a dual pairing in the case of a quasi-convex fan Δ, we compose that induced product with an "evaluation" homomorphism

$$\varepsilon : E^\bullet_{(\Delta,\partial\Delta)} \longrightarrow A^\bullet[-2n]$$

that can be defined as follows: Firstly, as a consequence of Corollary 5.4, we have

$$\overline{E}^q_{(\Delta,\partial\Delta)} = \begin{cases} \mathbf{R} & \text{for } q = 2n, \\ 0 & \text{for } q > 2n. \end{cases}$$

Moreover, according to Corollary 4.12, the $A^\bullet$-module $E^\bullet_{(\Delta,\partial\Delta)}$ is free. Hence, there is a homogeneous base $v_1 \in E^{2n}_{(\Delta,\partial\Delta)}, v_2, ..., v_r \in E^{<2n}_{(\Delta,\partial\Delta)}$ of $E^\bullet_{(\Delta,\partial\Delta)}$.

Now set $\varepsilon(v_i) := \delta_{i1}$. In fact, ε is unique up to multiplication by a real scalar. If Δ is a simplicial fan, this homomorphism ε can be described quite explicitly:

Following [Bri$_2$, p.13], we fix a volume form ω on the vector space V. For each cone σ, we choose a basis $(e_1, \ldots, e_n)$ of vectors spanning the rays such that we have $\omega(e_1, \ldots, e_n) = 1$. Let $(e'_1, \ldots, e'_n)$ be the dual basis, and set $g_\sigma := e'_1 \cdots e'_n$. We then define the map ε as the composition

$$E^\bullet_{(\Delta,\partial\Delta)} \cong A^\bullet_{(\Delta,\partial\Delta)} \subset \bigoplus_{\sigma\in\Delta^n} A^\bullet_\sigma \longrightarrow Q(A^\bullet)\ , \quad f = (f_\sigma)_{\sigma\in\Delta^n} \longmapsto \sum_{\sigma\in\Delta^n} \frac{f_\sigma}{g_\sigma}\ ,$$

mapping to the homogeneous fractional ideal generated by the rational functions $1/g_\sigma$ (of degree $-2n$) in the quotient field $Q(A^\bullet)$. We indicate why the rational function $\varepsilon(f) = \sum_\sigma f_\sigma/g_\sigma$ is even regular. The denominators are products of linear forms h_τ vanishing on the facets $\tau \in \Delta^{n-1}$, and since such a factor h_τ does not appear in any denominator g_σ except for $\tau \prec \sigma$, it suffices to show that $\sum_{\sigma\succ_1\tau} f_\sigma/g_\sigma$ is regular along $\mathring{\tau}$. If $\tau \prec_1 \sigma$ lies in $\partial\Delta$, then the corresponding function f_σ vanishes on τ and hence is divisible by h_τ. Thus, we may assume that τ is the common facet of two cones σ^+, σ^- in Δ. It suffices to discuss the contribution $f^+/g^+ + f^-/g^-$ of these two cones to the sum. On τ, the linear form h_τ and $f^+ - f^-$ vanish; an explicit computation yields the result.

Since the intersection product $\mathcal{E}^\bullet \times \mathcal{E}^\bullet \to \mathcal{E}^\bullet$ is a homomorphism of sheaves, we may sum up the general situation as follows: For a quasi-convex fan Δ, there exists *homogeneous pairings* (i.e., a pair of elements of degree p and q is mapped to an element of degree $p+q$)

(6.1.3) $$E^\bullet_\Delta \times E^\bullet_{(\Delta,\partial\Delta)} \longrightarrow E^\bullet_{(\Delta,\partial\Delta)} \longrightarrow A^\bullet[-2n]$$

and

(6.1.4) $$\overline{E}^\bullet_\Delta \times \overline{E}^\bullet_{(\Delta,\partial\Delta)} \longrightarrow \overline{E}^\bullet_{(\Delta,\partial\Delta)} \longrightarrow \mathbf{R}^\bullet[-2n]\ .$$

Our aim is to prove that these are in fact both dual pairings. Fortunately, it suffices to verify that property for one of them: By the very definition of quasi-convexity and Corollary 4.12, the $A^\bullet$-modules $E^\bullet_\Delta$ and $E^\bullet_{(\Delta,\partial\Delta)}$ are both free. We thus may apply the following result.

6.2 Lemma. *Let $E^\bullet$ and $F^\bullet$ be two finitely generated free graded $A^\bullet$-modules. Then a homogeneous pairing*

$$E^\bullet \times F^\bullet \to A^\bullet[r]$$

is dual if and only if that holds for the induced pairing

$$\overline{E}^\bullet \times \overline{F}^\bullet \longrightarrow \overline{A}^\bullet[r] = \mathbf{R}^\bullet[r]\,.$$

Proof. Replacing $F^\bullet$ with $F^\bullet[-r]$, we may assume that $r = 0$. With respect to fixed homogeneous bases of $E^\bullet$ and $F^\bullet$, the pairing is represented by a square matrix M over $A^\bullet$. We claim that M is invertible if and only if that holds for its residue class $\overline{M}$ modulo $\mathfrak{m}_A$: The implication "⇒" is obvious, while for "⇐", it suffices to prove that $\det M$ lies in $A^0 = \mathbf{R}$. To that end, we arrange the basis for $E^\bullet$ in increasing order with respect to the degrees, and in decreasing order for $F^\bullet$. Since the induced pairing is dual, the homogeneous submodules of $E^\bullet$ and $F^\bullet$ generated by basis elements of fixed opposite degrees have the same rank. Hence, the matrix M is a lower triangular block matrix with square blocks along the diagonal all whose entries lie in A^0. Thus $\det M$ is the product of their respective determinants, so it lies in A^0, too. □

We come now to the central result of this section:

6.3 Theorem (Poincaré Duality): *For a quasi-convex fan Δ of dimension n, the composition*

$$E^\bullet_\Delta \times E^\bullet_{(\Delta,\partial\Delta)} \longrightarrow E^\bullet_{(\Delta,\partial\Delta)} \longrightarrow A^\bullet[-2n]$$

is a dual pairing of finitely generated free $A^\bullet$-modules.

Proof: For an affine simplicial fan Δ, Poincaré duality obviously holds. The general case follows by the next two lemmata 6.4 and 6.5, using a two step induction procedure. The proof of Lemma 6.5 will use the Lemmata 6.6 and 6.7.

6.4 Lemma. *If Poincaré duality holds for complete fans in dimensions $d < n$, then it holds for n-dimensional affine fans.*

6.5 Lemma. *If Poincaré duality holds for every affine fan $\langle\sigma\rangle$ of dimension at most n, then it also holds for every quasi-convex fan Δ of dimension n.*

Proof of Lemma 6.4: Let σ be an n-cone. As in (5.3.1) – (5.3.2), we identify $E^\bullet_{\partial\sigma}$ with the $B^\bullet_\sigma$-module $G^\bullet_{\Lambda_\sigma}$ of global sections of a minimal extension sheaf $\mathcal{G}^\bullet$ on the flattened boundary fan Λ_σ in V/ℓ. Since Λ_σ is $(n-1)$-dimensional, we obtain a dual pairing

$$E^\bullet_{\partial\sigma} \times E^\bullet_{\partial\sigma} \longrightarrow E^\bullet_{\partial\sigma} \longrightarrow B^\bullet_\sigma[2-2n]\ .$$

By extension of scalars as in (6.1), that induces dual pairings

$$\begin{aligned} F^\bullet_\sigma \times F^\bullet_\sigma &\longrightarrow F^\bullet_\sigma \xrightarrow{\ \eta\ } A^\bullet[2-2n]\ , \\ \overline{F}^\bullet_\sigma \times \overline{F}^\bullet_\sigma &\longrightarrow \overline{F}^\bullet_\sigma \longrightarrow \mathbf{R}^\bullet[2-2n] \end{aligned}$$

and, after a shift,

$$\begin{aligned} F^\bullet_\sigma \times F^\bullet_\sigma[-2] &\longrightarrow F^\bullet_\sigma[-2] \xrightarrow{\ \eta[-2]\ } A^\bullet[-2n]\ , \\ \overline{F}^\bullet_\sigma \times \overline{F}^\bullet_\sigma[-2] &\longrightarrow \overline{F}^\bullet_\sigma[-2] \longrightarrow \mathbf{R}^\bullet[-2n]\ . \end{aligned} \tag{6.4.1}$$

To achieve the proof, we show that there is a homomorphism $\vartheta\colon E^\bullet_{(\sigma,\partial\sigma)} \to F^\bullet_\sigma[-2]$ and a factorization of the induced pairing $\overline{E}^\bullet_\sigma \times \overline{E}^\bullet_{(\sigma,\partial\sigma)} \to \mathbf{R}^\bullet[-2n]$ obtained in (6.1.4) in the following form:

$$\overline{E}^\bullet_\sigma \times \overline{E}^\bullet_{(\sigma,\partial\sigma)} \stackrel{\overline{\alpha}\times\overline{\vartheta}}{\longrightarrow} \overline{F}^\bullet_\sigma \times \overline{F}^\bullet_\sigma[-2] \longrightarrow \overline{F}^\bullet_\sigma[-2] \longrightarrow \mathbf{R}^\bullet[-2n]\,. \tag{6.4.2}$$

We further show the existence of a homomorphism $\mu\colon \overline{F}^\bullet_\sigma[-2] \to \overline{F}^\bullet_\sigma$ such that α and ϑ induce isomorphisms

$$\overline{E}^\bullet_\sigma \cong \operatorname{coker}\mu \qquad \text{and} \qquad \overline{E}^\bullet_{(\sigma,\partial\sigma)} \cong \ker\mu.$$

Finally, forgetting about the shifts, the map μ is shown to be self-adjoint with respect to the dual pairing (6.4.1) on $\overline{F}^\bullet_\sigma$. Hence, the restriction to $\operatorname{coker}\mu \times \ker\mu$ is a dual pairing, too; and an application of 6.2 will finally complete the proof of the Lemma.

We interpret $F^\bullet_\sigma$ as the module of sections of a sheaf of $\mathcal{A}^\bullet$-modules on the affine fan $\langle\sigma\rangle$. To that end, we consider the subdivision

$$\Sigma \;:=\; \partial\sigma \cup \{\hat\tau := \varrho + \tau; \tau \in \partial\sigma\}$$

of $\langle\sigma\rangle$, where ϱ is the ray $\ell \cap \sigma$. As in (1.4.1), let $D^\bullet_\tau \subset A^\bullet_{\hat\tau}$ denote the subalgebra of functions constant on parallels to the line ℓ. Then, according to Lemma 1.5, the minimal extension sheaf $\mathcal{F}^\bullet$ on Σ is determined by

$$\tau \mapsto F^\bullet_\tau := E^\bullet_\tau\,,\ \hat\tau \mapsto F^\bullet_{\hat\tau} := A^\bullet_{\hat\tau} \otimes_{D^\bullet_\tau} E^\bullet_\tau \qquad \text{for } \tau \in \partial\sigma$$

and the obvious restriction homomorphisms; it satisfies $\mathcal{F}^\bullet(\Sigma) \cong A^\bullet \otimes_{B^\bullet_\sigma} E^\bullet_{\partial\sigma} = F^\bullet_\sigma$. Furthermore, the sheaf $\mathcal{F}^\bullet$ inherits an intersection product from $\mathcal{E}^\bullet|_{\partial\sigma} \cong \mathcal{F}^\bullet|_{\partial\sigma}$ as in 6.1.

For simplicity, we interpret the mapping α in 6.1 as an inclusion $E^\bullet_\sigma \subset F^\bullet_\sigma$ and identify $\mathcal{F}^\bullet$ with its direct image sheaf on the affine fan $\langle\sigma\rangle$ with respect to the refinement mapping $\Sigma \to \langle\sigma\rangle$. Then the decomposition $F^\bullet_\sigma = E^\bullet_\sigma \oplus K^\bullet$ of (6.1.2) corresponds to a decomposition of sheaves $\mathcal{F}^\bullet \cong \mathcal{E}^\bullet \oplus \mathcal{K}^\bullet$ with $\mathcal{E}^\bullet \cong {}_o\mathcal{L}^\bullet$ and the sheaf $\mathcal{K}^\bullet := {}_\sigma\mathcal{L}^\bullet \otimes \overline{K}^\bullet$ supported by the point σ, cf. section 2. In particular, there is an inclusion

$$E^\bullet_{(\sigma,\partial\sigma)} \;\subset\; F^\bullet_{(\sigma,\partial\sigma)} \;=\; E^\bullet_{(\sigma,\partial\sigma)} \oplus K^\bullet\,,$$

and $F^\bullet_{(\sigma,\partial\sigma)}$ is a free $A^\bullet$-module.

We thus obtain a natural commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E^\bullet_{(\sigma,\partial\sigma)} & \longrightarrow & E^\bullet_\sigma & \longrightarrow & E^\bullet_{\partial\sigma} & \longrightarrow & 0 \\
 & & \cap & & \cap & & \| \wr & & \\
0 & \longrightarrow & F^\bullet_{(\sigma,\partial\sigma)} & \stackrel{\lambda}{\longrightarrow} & F^\bullet_\sigma & \longrightarrow & F^\bullet_{\partial\sigma} & \longrightarrow & 0
\end{array}$$

consisting of free resolutions of the $A^\bullet$-module $E^\bullet_{\partial\sigma} \cong F^\bullet_{\partial\sigma}$.

Using the very definition of $\mathrm{Tor}^{A^\bullet}(*, \mathbf{R}^\bullet)$ and the fact that $\overline{E}^\bullet_{(\sigma,\partial\sigma)} \to \overline{E}^\bullet_\sigma$ is the zero map since $\overline{E}^\bullet_\sigma \to \overline{E}^\bullet_{\partial\sigma}$ is an isomorphism, we obtain identifications

$$\overline{E}^\bullet_\sigma \;\cong\; \mathrm{coker}(\overline{\lambda}) \;\cong\; \overline{E}^\bullet_{\partial\sigma} \quad \text{and} \quad \overline{E}^\bullet_{(\sigma,\partial\sigma)} \;\cong\; \mathrm{Tor}_1(E^\bullet_{\partial\sigma}, \mathbf{R}^\bullet) \;\cong\; \ker(\overline{\lambda})\,.$$

On the other hand, we may rewrite $F^\bullet_{(\sigma,\partial\sigma)} = gF^\bullet_\sigma \cong F^\bullet_\sigma[-2]$, where $g \in \mathcal{A}^2(\Sigma)$ is some piecewise linear function on Σ with $\partial\sigma$ as zero set: In the description $A^\bullet = B^\bullet_\sigma[T]$ of (0.D.2), we may assume that the kernel of $T \in A^2$ intersects σ in the point 0 only. Then, for $\tau \in (\partial\sigma)^{n-1}$, we set $g_{\hat\tau} = T - f_\tau$, where $f_\tau \in A^2_{\hat\tau} = A^2$ coincides with T on τ and is constant on parallels to ℓ, i.e., $f_\tau \in B^\bullet_\sigma$.

We note that

$$E^\bullet_{(\sigma,\partial\sigma)} \subset E^\bullet_{(\sigma,\partial\sigma)} \oplus K^\bullet = F^\bullet_{(\sigma,\partial\sigma)} = gF^\bullet_\sigma \cong F^\bullet_\sigma[-2]$$

determines the desired homomorphism

$$\vartheta\colon E^\bullet_{(\sigma,\partial\sigma)} \longrightarrow F^\bullet_\sigma[-2]$$

and leads to an evaluation map

$$E^\bullet_{(\sigma,\partial\sigma)} \longrightarrow F^\bullet_\sigma[-2] \overset{\eta[-2]}{\longrightarrow} A^\bullet[-2n]\,.$$

Moreover, we have $\overline{K}^{\geq 2n} = 0$ because of the isomorphism

$$\overline{E}^{2n}_{(\sigma,\partial\sigma)} \cong \mathbf{R} \cong \overline{F}^{2n}_{(\sigma,\partial\sigma)} \cong \overline{F}^{2n-2}_\sigma$$

and the vanishing $\overline{F}^{>2n}_{(\sigma,\partial\sigma)} = 0$, which yields that $K^\bullet \subset F^\bullet_{(\sigma,\partial\sigma)}$ is contained in the kernel of the map $F^\bullet_{(\sigma,\partial\sigma)} \to A^\bullet[-2n]$. Next we remark that the first part of the diagram

$$\begin{array}{ccccc} E^\bullet_\sigma \times E^\bullet_{(\sigma,\partial\sigma)} & \longrightarrow & E^\bullet_{(\sigma,\partial\sigma)} & \overset{\varepsilon}{\longrightarrow} & A^\bullet[-2n] \\ \cap & & \cap & & \| \\ F^\bullet_\sigma \times F^\bullet_{(\sigma,\partial\sigma)} & \longrightarrow & F^\bullet_{(\sigma,\partial\sigma)} & \overset{\eta[-2]}{\longrightarrow} & A^\bullet[-2n] \end{array} \tag{6.4.3}$$

need not be commutative, since $E^\bullet_\sigma$ is not necessarily closed under the intersection product in $F^\bullet_\sigma$. Nevertheless, commutativity holds after evaluation (where the two evaluation maps are scaled in such a way that the right square is commutative). This is true since the difference of the products in the first and second row is an element in $K^\bullet$, according to the construction.

As the intersection product in $F^\bullet_\sigma$ is $\mathcal{A}^\bullet(\Sigma)$-linear, we may replace $F^\bullet_{(\sigma,\partial\sigma)}$ in the diagram (6.4.3) with $F^\bullet_\sigma[-2]$ and, combining with (6.4.1), we arrive at the following pairing of $A^\bullet$-modules

$$E^\bullet_\sigma \times E^\bullet_{(\sigma,\partial\sigma)} \longrightarrow F^\bullet_\sigma \times F^\bullet_\sigma[-2] \longrightarrow F^\bullet_\sigma[-2] \longrightarrow A^\bullet[-2n]\,.$$

Passing to the quotients modulo $\mathfrak{m}_A$, we obtain (6.4.2), where $\mu\colon \overline{F}^\bullet_\sigma[-2] \to \overline{F}^\bullet_\sigma$ is induced by multiplication with the function $g \in \mathcal{A}^\bullet(\Sigma)$. ▫

Proof of Lemma 6.5: To simplify notation, we introduce the abbreviation $\widetilde{A}^\bullet := A^\bullet[-2n]$. We have to show that the "global" duality homomorphism

$$\Phi\colon E^\bullet_\Delta \longrightarrow \mathrm{Hom}_{A^\bullet}(E^\bullet_{(\Delta,\partial\Delta)}, \widetilde{A}^\bullet)$$

induced by the pairing (6.1.3) is an isomorphism. To that end, we embed it into a commutative diagram of the following form:

$$\begin{array}{ccccccc}
0 & \longrightarrow & E^\bullet_\Delta & \longrightarrow & C^0(\Delta,\partial\Delta;\mathcal{E}^\bullet) & \longrightarrow & C^1(\Delta,\partial\Delta;\mathcal{E}^\bullet) \\
(6.5.1) & & \Big\downarrow{\scriptstyle\Phi} & & \Big\downarrow{\scriptstyle\Psi} & & \Big\downarrow{\scriptstyle\Theta} \\
0 & \longrightarrow & \mathrm{Hom}(E^\bullet_{(\Delta,\partial\Delta)}, \widetilde{A}^\bullet) & \stackrel{\kappa}{\longrightarrow} & \bigoplus\limits_{\sigma\in\Delta^n} \mathrm{Hom}(E^\bullet_{(\sigma,\partial\sigma)}, \widetilde{A}^\bullet) & \stackrel{\lambda}{\longrightarrow} & \bigoplus\limits_{\tau\in\mathring{\Delta}^{n-1}} \mathrm{Hom}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet_\tau[2])\,.
\end{array}$$

Here Hom abbreviates $\mathrm{Hom}_{A^\bullet}$, and Ψ and Θ are the respective duality homomorphisms corresponding to the collections of dual pairings

$$E^\bullet_\sigma \times E^\bullet_{(\sigma,\partial\sigma)} \to E^\bullet_{(\sigma,\partial\sigma)} \to \tilde{A}^\bullet \qquad \text{resp.} \qquad E^\bullet_\tau \times E^\bullet_{(\tau,\partial\tau)} \to E^\bullet_{(\tau,\partial\tau)} \to \tilde{A}^\bullet_\tau[2]$$

with suitably chosen evaluation maps. The proof now will run along the following lines: The upper row of diagram (6.5.1) is exact, while the lower one is a complex with an injective map κ. The homomorphism Ψ and Θ are isomorphisms, and thus, a simple diagram chase yields that the same holds for Φ, which will end the proof of the Lemma 6.5.

The exactness of the upper row in (6.5.1) follows immediately from Theorem 4.3 since Δ is quasi-convex. We now describe the choice of the evaluation maps: The evaluation map $\varepsilon\colon E^\bullet_{(\Delta,\partial\Delta)} \to \widetilde{A}^\bullet$ induces a system $(\varepsilon_\sigma)_{\sigma\in\Delta^n}$ of maps $\varepsilon_\sigma\colon E^\bullet_{(\sigma,\partial\sigma)} \subset E^\bullet_{(\Delta,\partial\Delta)} \to \widetilde{A}^\bullet$. If we can show that each ε_σ is an evaluation map, then the direct sum of the corresponding duality homomorphisms $\Psi_\sigma\colon E^\bullet_\sigma \to \mathrm{Hom}(E^\bullet_{(\sigma,\partial\sigma)}, \widetilde{A}^\bullet)$ is an isomorphism, since Poincaré duality on σ holds by hypothesis. We thus have to show $\varepsilon_\sigma \neq 0$ for each σ. That follows immediately from the fact that the map $\mathbf{R} \cong \overline{E}^{2n}_{(\sigma,\partial\sigma)} \to \overline{E}^{2n}_{(\Delta,\partial\Delta)} \cong \mathbf{R}$ induced by the homomorphism $E^\bullet_{(\sigma,\partial\sigma)} \to E^\bullet_{(\Delta,\partial\Delta)}$ is an isomorphism, see Lemma 6.6. The system of duality isomorphisms $\Psi_\sigma\colon E^\bullet_\sigma \to \mathrm{Hom}(E^\bullet_{(\sigma,\partial\sigma)}, \widetilde{A}^\bullet)$ thus provides the isomorphism Ψ.

The map κ associates to a homomorphism $\varphi\colon E^\bullet_{(\Delta,\partial\Delta)} \to \widetilde{A}^\bullet$ its restrictions to the submodules $E^\bullet_{(\sigma,\partial\sigma)}$ of $E^\bullet_{(\Delta,\partial\Delta)}$. It is injective, since $\bigoplus_{\sigma\in\Delta^n} E^\bullet_{(\sigma,\partial\sigma)} \cong E^\bullet_{(\Delta,\Delta^{\leq n-1})}$ is a submodule of maximal rank in $E^\bullet_{(\Delta,\partial\Delta)}$: For $h := \prod_{\tau\in\mathring{\Delta}^{n-1}} h_\tau$, where $h_\tau \in A^2\setminus\{0\}$ vanishes on $V_\tau \subset V$, we have

$$hE^\bullet_{(\Delta,\partial\Delta)} \subset E^\bullet_{(\Delta,\Delta^{\leq n-1})}\ .$$

This ends the discussion of the first rectangle in (6.5.1).

The map λ will be composed of "restriction homomorphisms"

$$\lambda^\sigma_\tau\colon \mathrm{Hom}(E^\bullet_{(\sigma,\partial\sigma)}, \widetilde{A}^\bullet) \ \to\ \mathrm{Hom}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet_\tau[2]), \qquad \varphi \mapsto \varphi_\tau\ ,$$

where τ is a facet of $\sigma \in \Delta^n$. In order to define λ_τ^σ, we fix a euclidean norm on V and thus also on $V^* \cong A^2$. Let $h_\tau \in A^2$ be the unique linear form of norm 1 that vanishes on V_τ and is positive on $\mathring{\sigma}$. Then we use three exact sequences, starting with

$$0 \to E^\bullet_{(\sigma,\partial\sigma)} \to E^\bullet_\sigma \to E^\bullet_{\partial\sigma} \to 0\,.$$

The second one is composed of the multiplication with h_τ and the projection onto the cokernel:

$$0 \to \widetilde{A}^\bullet \xrightarrow{\mu(h_\tau)} \widetilde{A}^\bullet[2] \to \widetilde{A}^\bullet_\tau[2] \to 0\,.$$

Eventually the subfan $\partial_\tau\sigma := \partial\sigma \setminus \{\tau\}$ of $\partial\sigma$ yields the exact sequence

$$0 \to E^\bullet_{(\tau,\partial\tau)} \to E^\bullet_{\partial\sigma} \to E^\bullet_{\partial_\tau\sigma} \to 0\,. \tag{6.5.2}$$

The associated Hom-sequences provide a diagram

$$\begin{tikzcd}[column sep=small]
 & & \mathrm{Ext}(E^\bullet_{\partial_\tau\sigma}, \widetilde{A}^\bullet) \arrow[d] & \\
\mathrm{Hom}(E^\bullet_\sigma, \widetilde{A}^\bullet) \arrow[r] & \mathrm{Hom}(E^\bullet_{(\sigma,\partial\sigma)}, \widetilde{A}^\bullet) \arrow[r, "\alpha"] & \mathrm{Ext}(E^\bullet_{\partial\sigma}, \widetilde{A}^\bullet) \arrow[d, "\beta"] & \\
\mathrm{Hom}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet[2]) \arrow[r] & \mathrm{Hom}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet_\tau[2]) \arrow[r, "\gamma"] & \mathrm{Ext}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet) \arrow[r] & \mathrm{Ext}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet[2])
\end{tikzcd} \tag{6.5.3}$$

with $\mathrm{Ext} = \mathrm{Ext}^1_{A^\bullet}$. We show that γ is an isomorphism; we then may set

$$\lambda_\tau^\sigma := \gamma^{-1} \circ \beta \circ \alpha\,.$$

Indeed the rightmost arrow in the bottom row is the zero homomorphism, since it is induced by multiplication with h_τ, which annihilates $E^\bullet_{(\tau,\partial\tau)}$. On the other hand, the $A^\bullet_\tau$-module $E^\bullet_{(\tau,\partial\tau)}$ is a torsion module over $A^\bullet$, so that $\mathrm{Hom}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet[2])$ vanishes.

An explicit description of λ_τ^σ is as follows: For a homomorphism $\varphi\colon E^\bullet_{(\sigma,\partial\sigma)} \to \widetilde{A}^\bullet$, the "restriction" $\lambda_\tau^\sigma(\varphi) = \varphi_\tau\colon E^\bullet_{(\tau,\partial\tau)} \to \widetilde{A}^\bullet_\tau[2]$ is this: To $g \in E^\bullet_{(\tau,\partial\tau)}$, we associate a section $\hat{g} \in E^\bullet_\sigma$ such that $\hat{g}|_{\partial\sigma}$ is the trivial extension of g to $\partial\sigma$; then $\varphi_\tau(g) = \varphi(h_\tau\hat{g})|_\tau$.

For the definition of λ, we apply the standard Čech coboundary construction to the family (λ_τ^σ), making the lower row of diagram (6.5.1) a complex. We may do so since the following compatibility condition is satisfied: For two different cones $\sigma = \sigma_1, \sigma_2 \in \Delta^n$ with intersection $\tau \in \mathring{\Delta}^{n-1}$, the description of φ_τ implies that the compositions

$$\mathrm{Hom}(E^\bullet_{(\Delta,\partial\Delta)}, \widetilde{A}^\bullet) \to \mathrm{Hom}(E^\bullet_{(\sigma_i,\partial\sigma_i)}, \widetilde{A}^\bullet) \to \mathrm{Hom}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet_\tau[2])\ ,\ i = 1, 2 \tag{6.5.4}$$

coincide.

In particular, the homomorphisms

$$\varepsilon_\tau := \lambda_\tau^\sigma(\varepsilon_\sigma)\colon E^\bullet_{(\tau,\partial\tau)} \to \widetilde{A}^\bullet_\tau[2]\ ,\ \tau \in \mathring{\Delta}^{n-1}$$

do not depend on the choice of $\sigma \succ_1 \tau$. It remains to verify that ε_τ is not the zero homomorphism, i.e., we have to see that λ_τ^σ is injective in degree 0. In diagram (6.5.3), we have to show that α and β are injective in degree 0. By 5.4, the vector spaces $\overline{E}_\sigma^q$ vanish for $q \geq 2n$; hence, $E_\sigma^\bullet$ can be generated by elements of degree $< 2n$, and that yields the vanishing of $\mathrm{Hom}(E_\sigma^\bullet; \widetilde{A}^\bullet)$ in degree 0. According to Lemma 6.7, the exact sequence

$$0 \;\to\; E^\bullet_{(\sigma,\partial_\tau\sigma)} \;\to\; E^\bullet_\sigma \;\to\; E^\bullet_{\partial_\tau\sigma} \;\to\; 0$$

(see (6.5.2)) is a free resolution of $E^\bullet_{\partial_\tau\sigma}$, in particular, the module $\mathrm{Ext}^1(E^\bullet_{\partial_\tau\sigma}, \widetilde{A}^\bullet)$ is a quotient of $\mathrm{Hom}(E^\bullet_{(\sigma,\partial_\tau\sigma)}, \widetilde{A}^\bullet)$, which is trivial in degree 0, since according to Lemma 6.7,. we have $E^{\geq 2n}_{(\sigma,\partial_\tau\sigma)} = 0$.

— For $\tau \in \mathring{\Delta}^{n-1}$, the evaluation homomorphisms ε_τ induce isomorphisms

$$\Theta_\tau : E^\bullet_\tau \xrightarrow{\cong} \mathrm{Hom}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet_\tau[2]) \;,$$

which constitute the isomorphism Θ.

Finally the commutativity of the second square in the diagram (6.5.1) follows from the above explicit description of the restriction homomorphisms λ_τ^σ and the appropriate choice of the evaluation homomorphisms ε_τ. □

6.6 Lemma. *If $\Lambda \prec \Delta$ are quasi-convex fans, then the trivial extension of sections $E^\bullet_{(\Lambda,\partial\Lambda)} \subset E^\bullet_{(\Delta,\partial\Delta)}$ induces an isomorphism*

$$\mathbf{R} \cong \overline{E}^{2n}_{(\Lambda,\partial\Lambda)} \xrightarrow{\cong} \overline{E}^{2n}_{(\Delta,\partial\Delta)} \cong \mathbf{R} \;. \tag{6.6.1}$$

Proof. Let us first assume that Δ is complete. To the complementary fan $\Lambda^c \preceq \Delta$ generated by the cones in $\Delta^n \setminus \Lambda^n$ corresponds an exact sequence

$$0 \;\to\; E^\bullet_{(\Lambda,\partial\Lambda)} \cong E^\bullet_{(\Delta,\Lambda^c)} \;\longrightarrow\; E^\bullet_\Delta \;\to\; E^\bullet_{\Lambda^c} \;\to\; 0 \;,$$

which induces an exact sequence $\overline{E}^{2n}_{(\Lambda,\partial\Lambda)} \to \overline{E}^{2n}_\Delta \to \overline{E}^{2n}_{\Lambda^c}$. The fan Λ^c is quasi-convex according to Corollary 4.7, ii) and non-complete. Hence, the last term $\overline{E}^{2n}_{\Lambda^c}$ vanishes according to Corollary 5.4, and thus $\mathbf{R} \cong \overline{E}^{2n}_{(\Lambda,\partial\Lambda)} \to \overline{E}^{2n}_\Delta \cong \mathbf{R}$ is onto resp. an isomorphism.

We now assume that Δ admits a completion $\bar{\Delta}$. We consider the composed map

$$\overline{E}^{2n}_{(\Lambda,\partial\Lambda)} \;\longrightarrow\; \overline{E}^{2n}_{(\Delta,\partial\Delta)} \;\longrightarrow\; \overline{E}^{2n}_{\bar{\Delta}} \;.$$

Since it is an isomorphism, so is $\mathbf{R} \cong \overline{E}^{2n}_{(\Lambda,\partial\Lambda)} \to \overline{E}^{2n}_{(\Delta,\partial\Delta)} \cong \mathbf{R}$.

In the general case, we choose a refinement map $\pi\colon \check{\Delta} \to \Delta$ where $\check{\Delta}$ admits a completion, see 0.A. Hence, it suffices to verify that (6.6.1) is an isomorphism for the couple (Δ, Λ) as that holds for $(\check{\Delta}, \check{\Lambda})$ according to the second step. By the

Geometric Decomposition Theorem 2.5, we can write $\mathcal{F}^\bullet := \pi_*(\check{\mathcal{E}}^\bullet) \cong \mathcal{E}^\bullet \oplus \mathcal{G}^\bullet$. From Corollary 5.4 and the very definition of the direct image sheaf stem isomorphisms

$$\mathbf{R} \;\cong\; \overline{\check{E}}{}^{2n}_{(\check{\Delta},\partial\check{\Delta})} \;\cong\; \overline{F}{}^{2n}_{(\Delta,\partial\Delta)} \;\cong\; \overline{E}{}^{2n}_{(\Delta,\partial\Delta)} \oplus \overline{G}{}^{2n}_{(\Delta,\partial\Delta)} \;\cong\; \mathbf{R} \oplus \overline{G}{}^{2n}_{(\Delta,\partial\Delta)}\ ,$$

and thus, $\overline{G}{}^{2n}_{(\Delta,\partial\Delta)} = 0$ and $\overline{\check{E}}{}^{2n}_{(\check{\Delta},\partial\check{\Delta})} \cong \overline{F}{}^{2n}_{(\Delta,\partial\Delta)} \cong \overline{E}{}^{2n}_{(\Delta,\partial\Delta)}$. The corresponding isomorphisms also hold for Λ instead of Δ. Combining these isomorphisms, we obtain the isomorphism (6.6.1). ◻

The following result has been used in the proof of Lemma 6.5. For the notation, we refer to (0.D.1).

6.7 Lemma. *Let σ be a cone of dimension n and $\Lambda \subset \partial\sigma$ be a fan such that $\pi(\Lambda)$ is a quasi-convex subfan of Λ_σ. Then $E^\bullet_{(\sigma,\Lambda)}$ is a free $A^\bullet$-module, and, if in addition Λ is a proper subfan, $\overline{E}{}^q_{(\sigma,\Lambda)} = 0$ for $q \geq 2n$.*

Proof. As in (0.D.2), we write $A^\bullet = B^\bullet_\sigma[T]$ with a linear form $T \in A^2$. The exact sequence of $A^\bullet$-modules

$$0 \;\to\; E^\bullet_{(\sigma,\Lambda)} \;\to\; E^\bullet_\sigma \;\to\; E^\bullet_\Lambda \;\to\; 0$$

induces an exact Tor-sequence

$$\mathrm{Tor}_2^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet) \;\to\; \mathrm{Tor}_1^{A^\bullet}(E^\bullet_{(\sigma,\Lambda)}, \mathbf{R}^\bullet) \to 0 \to \mathrm{Tor}_1^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet) \;\to\; \overline{E}{}^\bullet_{(\sigma,\Lambda)} \to \overline{E}{}^\bullet_\sigma$$

since $E^\bullet_\sigma$ is a free $A^\bullet$-module. If $\mathrm{Tor}_2^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet)$ vanishes, then so does $\mathrm{Tor}_1^{A^\bullet}(E^\bullet_{(\sigma,\Lambda)}, \mathbf{R}^\bullet)$, and $E^\bullet_{(\sigma,\Lambda)}$ is a free $A^\bullet$-module by section 0.B. Since the fan $\langle\sigma\rangle$ is not complete, we have $\overline{E}{}^q_\sigma = 0$ for $q \geq 2n$ by 5.4; if the same vanishing holds for $\mathrm{Tor}_1^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet)$, then it follows for $\overline{E}{}^q_{(\sigma,\Lambda)}$ as well. It thus remains to determine $\mathrm{Tor}_i^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet)$. As in the proof of Theorem 4.3, we use the exact sequence

$$0 \;\to\; \mathbf{R}^\bullet[T][-2] \;\to\; \mathbf{R}^\bullet[T] \;\to\; \mathbf{R}^\bullet \;\to\; 0 \tag{6.7.1}$$

of $A^\bullet$-module homomorphisms of degree 0; there $\mathbf{R}^\bullet[T]$ is interpreted as the $A^\bullet$-module $A^\bullet/(\mathfrak{m}_{B^\bullet_\sigma} A^\bullet) = B^\bullet_\sigma/\mathfrak{m}_{B^\bullet_\sigma}[T]$ for the maximal homogeneous ideal $\mathfrak{m}_{B^\bullet_\sigma} := B^{>0}_\sigma$ of $B^\bullet_\sigma$. Coming back to the identity (4.8.2) with $E^\bullet_\Lambda$ instead of I, we obtain

$$\mathrm{Tor}_i^{B^\bullet_\sigma[T]}(E^\bullet_\Lambda, \mathbf{R}^\bullet[T]) \;\cong\; \mathrm{Tor}_i^{B^\bullet_\sigma}(E^\bullet_\Lambda, \mathbf{R}^\bullet) \;=\; 0 \quad \text{for } i \geq 1,$$

since $E^\bullet_\Lambda$ is a free $B^\bullet_\sigma$-module. Hence, from (6.7.1) stem exact sequences

$$0 \to \mathrm{Tor}_{i+1}^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet) \to \mathrm{Tor}_i^{B^\bullet_\sigma}(E^\bullet_\Lambda, \mathbf{R}^\bullet[-2]) \to 0 \quad \text{for } i \geq 1,$$

and

$$0 \to \mathrm{Tor}_1^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet) \to E^\bullet_\Lambda \otimes_{B^\bullet_\sigma} \mathbf{R}^\bullet[-2] \xrightarrow{\mu(T)} E^\bullet_\Lambda \otimes_{B^\bullet_\sigma} \mathbf{R}^\bullet\ .$$

This yields the desired description:

$$\mathrm{Tor}_i^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet) \;=\; \begin{cases} \ker\bigl(\mu(T)\colon E^\bullet_\Lambda \otimes_{B^\bullet_\sigma} \mathbf{R}^\bullet[-2] \;\to\; E^\bullet_\Lambda \otimes_{B^\bullet_\sigma} \mathbf{R}^\bullet\bigr), & \text{if } i = 1; \\ 0, & \text{if } i \geq 2\ . \end{cases}$$

Eventually, if $\pi(\Lambda) \subset \Lambda_\sigma$ is not complete, then the vector space $E^\bullet_\Lambda \otimes_{B^\bullet_\sigma} \mathbf{R}^\bullet[-2]$ vanishes in degrees $\geq 2n$; hence, the same vanishing holds for $\mathrm{Tor}_1^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet)$. $\square$

This ends the proof of the auxiliary Lemmata, hence the proof of the Poincaré duality theorem 6.3.

6.8 Remark. For every purely n-dimensional fan Δ, we can define an evaluation map $E^\bullet_{(\Delta,\partial\Delta)} \to A^\bullet[-2n]$ as the composition

$$E^\bullet_{(\Delta,\partial\Delta)} \subset \check{E}^\bullet_{(\check{\Delta},\partial\check{\Delta})} = E^\bullet_{(\Delta,\partial\Delta)} \oplus \ldots \longrightarrow \check{E}^\bullet_{\bar{\Delta}} \longrightarrow A^\bullet[-2n] \, ,$$

where $\check{\Delta}$ is a refinement of Δ admitting a completion $\bar{\Delta}$. It provides a homomorphism $E^\bullet_\Delta \to \mathrm{Hom}(E^\bullet_{(\Delta,\partial\Delta)}, A^\bullet[-2n])$ via the intersection pairing. In accordance with the proof of Lemma 6.5, that is an isomorphism whenever $\tilde{H}^0(\Delta, \partial\Delta; \mathcal{E}^\bullet) = 0$, or equivalently, if $\tilde{H}^0(\Delta_\sigma, \partial\Delta_\sigma; \mathbf{R}^\bullet) = 0$ holds for each cone $\sigma \in \Delta$ (see Remark 3.5). In more geometrical terms, Δ has to be both facet-connected and locally facet-connected, where we call a fan *locally facet-connected* if, for each non-zero cone $\sigma \in \Delta$, the transversal fan Δ_σ is facet-connected.

The smallest example of a three-dimensional fan that is both facet-connected and locally facet-connected, but not quasi-convex, is provided by the fan swept out by the "vertical" facets of a triangular prism.

Since the dual pairing $E^\bullet_\Delta \times E^\bullet_{(\Delta,\partial\Delta)} \to A^\bullet[-2n]$ of $A^\bullet$-modules induces a dual pairing of real vector spaces $\overline{E}^\bullet_\Delta \times \overline{E}^\bullet_{(\Delta,\partial\Delta)} \to \mathbf{R}^\bullet[-2n]$, we obtain the following consequence.

6.9 Corollary. *If Δ is a quasi-convex fan of dimension n, then we have*

$$b_q(\Delta) \ := \ \dim \overline{E}^q_\Delta \ = \ \dim \overline{E}^{2n-q}_{(\Delta,\partial\Delta)} \ := \ b_{2n-q}(\Delta, \partial\Delta) \, ;$$

rephrased in terms of Poincaré polynomials, we have the identity

$$P_{(\Delta,\partial\Delta)}(t) \ = \ t^{2n} P_\Delta(t^{-1}) \, . \qquad \square$$

We finally are prepared to prove the "Combinatorial Hard Lefschetz" Theorem 5.6.

Proof of the "Combinatorial Hard Lefschetz" Theorem 5.6: Since f is strictly convex, its graph Γ_f in $V \times \mathbf{R}$ is the support of the boundary fan $\partial\gamma$ of the $(n+1)$-dimensional cone $\gamma := \gamma(f)$ in $V \times \mathbf{R}$ as we have seen in 0.D. Let $\mathcal{F}^\bullet$ be a minimal extension sheaf on $\partial\gamma$ and $\varphi\colon \Delta \to \partial\gamma$, the map induced by $\mathrm{id}_V \times f\colon V \to V \times \mathbf{R}$. Then $\varphi_*(\mathcal{F}^\bullet)$ is a minimal extension sheaf on Δ, which we thus may identify with $\mathcal{E}^\bullet$. Analoguous to (5.3.2), the residue class module of the $A^\bullet[T]$-module $F^\bullet_{\partial\gamma}$ satisfies

$$\overline{F}^\bullet_{\partial\gamma} \cong \overline{E}^\bullet_\Delta / f \overline{E}^\bullet_\Delta = \mathrm{coker}(\overline{\mu}_f : \overline{E}^\bullet_\Delta[-2] \longrightarrow \overline{E}^\bullet_\Delta)$$

where $\overline{E}^\bullet_\Delta = (A^\bullet/\mathfrak{m}) \otimes_{A^\bullet} E^\bullet_\Delta$. Now the vanishing condition $\mathbf{V}(\gamma)$ yields the surjectivity of $\overline{\mu}_f^{2q}$ for $2q \geq n-1$. On the other hand, the map μ_f is selfadjoint with respect to the dual pairing $E^\bullet_\Delta \times E^\bullet_\Delta \to A^\bullet[-2n]$ as well as $\overline{\mu}_f$ with respect to $\overline{E}^\bullet_\Delta \times \overline{E}^\bullet_\Delta \to \mathbf{R}^\bullet[-2n]$. Hence by Poincaré duality the surjectivity of $\overline{\mu}_f^{2q}$ for $2q \geq n-1$ implies the injectivity of $\overline{\mu}_f^{2q}$ for $2q \leq n-1$. ▫

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