Riemannian Geometry

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1 Introduction

Let $M \hookrightarrow \mathbb{R}^3$ be a surface, i.e.

$$M = \{\mathbf{x} \in \mathbb{R}^3; f(\mathbf{x}) = 0\}$$

with a smooth function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, s.th. $\nabla f(\mathbf{x}) \neq 0$ for $\mathbf{x} \in M$, e.g. a sphere or a torus. in particular M is a metric space with the restriction of the euclidean distance. But, unfortunately, though it induces the topology of M, it doesn't reflect the "geometry" of M very well: The distance of two points may be small, nevertheless it takes a lot of time to travel within M from one of them to the other.

Take instead

$$d(\mathbf{x}, \mathbf{y}) := \inf\{L(\gamma); \gamma : \mathcal{I} = [a, b] \stackrel{\text{smooth}}{\longrightarrow} M, \gamma(0) = \mathbf{x}, \gamma(1) = \mathbf{y}\},$$

where

$$L(\gamma) = \int_0^1 ||\dot{\gamma}(t)||dt,$$

with the euclidean norm $||.|| : \mathbb{R}^3 \longrightarrow \mathbb{R}$.

Question: Given $\mathbf{x}, \mathbf{y} \in M$, is there a (length-)minmizing path from $\mathbf{x}$ to $\mathbf{y}$, i.e. s.th.

$$d(\mathbf{x}, \mathbf{y}) = L(\gamma)?$$

Example 1.1.

1. $M = P \subset \mathbb{R}^3$ a plane: Minimizing paths are line segments.
2. $M = \mathbb{S}^2 \subset \mathbb{R}^3$ the unit sphere. Minimizing paths are segments of great circles $= \mathbb{S}^2 \cap P$ with planes $P \ni 0$.

Strategy: Look for locally minimizing paths $\gamma : [a, b] \longrightarrow M$, i.e. such that for sufficiently small $\varepsilon > 0$

$$d(\gamma(t + \varepsilon), \gamma(t)) = L(\gamma|_{[t+\varepsilon,t]})$$

for all $t \in \mathcal{I}$. Indeed

Theorem 1.2. *Let*

$$\mathbf{n} : M \longrightarrow \mathbb{R}^3, \mathbf{x} \mapsto \mathbf{n}_{\mathbf{x}},$$

be s.th. $\mathbf{n_x} \perp M$ *at* $\mathbf{x} \in M$*, a "normal vector field", e.g.*

$$\mathbf{n_x} = \frac{\nabla f(\mathbf{x})}{||\nabla f(\mathbf{x})||}.$$

Then $\gamma : \mathcal{I} \longrightarrow M \subset \mathbb{R}^3$ *is locally minimizing if*

$$\ddot{\gamma}(t) \in \mathbb{R}\mathbf{n}_{\gamma(t)} + \mathbb{R}\dot{\gamma}(t)$$

for all $t \in \mathcal{I}$*.*

Remark 1.3. If $||\dot{\gamma}(t)|| \equiv const$, then γ is locally minimizing iff $\ddot{\gamma}(t) \in \mathbb{R}\mathbf{n}_{\gamma(t)}$ for all $t \in \mathcal{I}$. Such a path is called a *geodesic*.

Program:

1. Consider not only surfaces M, but "submanifolds" $M \subset \mathbb{R}^{m+c}$ of arbitrary dimensions $m = \dim M$ and codimension c.
2. Replace $M \subset \mathbb{R}^{m+c}$ with *differentiable manifolds*.
3. Introduce the concept of a Riemannian metric g on a differentiable manifold.
4. Every $M \subset \mathbb{R}^{m+c}$ inherits a Riemannian metric g from $\mathbb{R}^{m+c}$.
5. Study pairs (M, g) (Riemanian manifolds) instead of $M \subset \mathbb{R}^{m+c}$.

We comment on 3): For a differentiable manifold M of dimension $\dim M = m$ one has the notion of

1. smooth paths $\gamma : \mathcal{I} \longrightarrow M$,
2. smooth functions $f : M \longrightarrow \mathbb{R}$,
3. for every $a \in M$ its tangent space $T_aM \cong \mathbb{R}^m$ at a, s.th. $\dot{\gamma}(t) \in T_{\gamma(t)}M$ is defined for smooth paths $\gamma : \mathcal{I} \longrightarrow M$,
4. of a Riemannian metric, a family $g = (g_a)_{a \in M}$ of inner products $g_a : T_aM \times T_aM \longrightarrow \mathbb{R}$ depending smoothly on $a \in M$

Definition 1.4. 1. Riemannian manifold = a pair (M, g) with a differentiable manifold M and a Riemannian metric g on M.

2. Length of smooth paths:

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$

In contrast to Jost's book we define geodesics by the differential equation they satisfy and then prove that they are locally minimizing. Further studies of the global behaviour of geodesics are planned.

2 Differentiable Manifolds

Definition 2.1. An m-dimensional topological manifold M is a Hausdorff topological space admitting an open cover

$$M = \bigcup_{i \in I} U_i$$

with open subset $U_i \subset M$ homeomorphic to open subsets $V_i \subset \mathbb{R}^m$.

Example 2.2. 1. Any open subset of $\mathbb{R}^m$ is an m-dimensional topological manifold.

2. Denote $||x|| = \sqrt{x_1^2 + ... + x_{n+1}^2}$ the euclidean norm of a vector $x \in \mathbb{R}^{n+1}$. Then the sphere

$$\mathbf{S}^n := \{x \in \mathbb{R}^{n+1}; ||x|| = 1\}$$

is an n-dimensional topological manifold: We have

$$\mathbf{S}^n = U_1 \cup U_2$$

with $U_1 := \mathbf{S}^n \setminus \{-e_{n+1}\}, U_2 := \mathbf{S}^n \setminus \{e_{n+1}\}$, where $U_i \cong \mathbb{R}^n$. For example the maps

$$\sigma_i : U_i \longrightarrow \mathbb{R}^n, x = (x', x_{n+1}) \mapsto \frac{x'}{1 - (-1)^i x_{n+1}}, i = 1, 2$$

are homeomorphisms: For $x \in U_i$ the point $(\sigma_i(x), 0)$ is the intersection of the line spanned by x and $-e_{n+1}$ (for $i = 1$) resp. e_{n+1} (for $i = 2$) with the hyperplane $\mathbb{R}^n \times 0$.

Now one could try to study a function $f : M \longrightarrow \mathbb{R}$ on a topological manifold M by considering what one gets by composing f with inverse homeomorphisms $\mathbb{R}^m \supset V \xrightarrow{\psi} U \subset M$ and then apply analysis to the composite $f \circ \psi : V \longrightarrow \mathbb{R}$. But then it will depend on the choice of the homeomorphism ψ, whether $f \circ \psi$ is differentiable or not. One can avoid that difficulty by restricting to a system, "atlas", of "mutually compatible" homeomorphisms, also called "charts":

Definition 2.3. Let M be an m-dimensional topological manifold.

1. A chart on M is a pair (U, φ), where $U \subset M$ is open and $\varphi : U \longrightarrow V$ is a homeomorphism between U and an open subset $V \subset \mathbb{R}^m$. The component functions $\varphi^1, ..., \varphi^m$ then are also called (local) coordinates for M on $U \subset M$.

2. Two charts $(U_i, \varphi_i), i = 1, 2$, on a topological manifold M are called (C^∞-)compatible if either $U_{12} := U_1 \cap U_2$ is empty or the transition map ("coordinate change")
$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_{12}) \longrightarrow \varphi_2(U_{12})$$
is a diffeomorphism between the open sets $\varphi_1(U_{12}) \subset V_1 \subset \mathbb{R}^m$ and $\varphi_2(U_{12}) \subset V_2 \subset \mathbb{R}^m$.

3. A differentiable atlas $\mathcal{A}$ on a topological manifold M is a system
$$\mathcal{A} = \{(U_i, \varphi_i); i \in I\}$$
of mutually (C^∞-)compatible charts, such that $M = \bigcup_{i \in I} U_i$.

Example 2.4. 1. The charts $(U_i, \sigma_i), i = 1, 2$ on $\mathbf{S}^n$, cf. 2.2.3, constitute a differentiable atlas: Again we have $\mathbf{S}^n = U_1 \cup U_2$ and the transition map
$$\sigma_2 \circ \sigma_1^{-1} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\},\ x \mapsto \frac{x}{||x||^2}.$$

2. Let $W \subset \mathbb{R}^n$ be an open subset and
$$F : W \longrightarrow \mathbb{R}^{n-m}$$
a differentiable map, such that for all $a \in M := F^{-1}(0) \subset W$ the Jacobian map
$$DF(a) : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-m}$$

is surjective. For every point $a \in M$ we shall construct a local chart (U_a, φ_a). We may assume that $\frac{\partial F}{\partial(x_{m+1},...,x_n)}(a) \neq 0$. Then the map $\Phi : (x_1, ..., x_m, F_1, ..., F_{n-m}) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ induces, according to the inverse function theorem, a diffeomorphism $\tilde{U} \longrightarrow \tilde{V}$ between an open neighborhood $\tilde{U} \subset W$ of $a \in \mathbb{R}^n$ and an open neighborhood $\tilde{V}$ of $(a_1, ..., a_m, 0) \in \mathbb{R}^n$. As a consequence the map

$$\varphi_a : U_a := \tilde{U} \cap M \longrightarrow V_a := \{y \in \mathbb{R}^m; (y, 0) \in \tilde{V} \subset \mathbb{R}^m \times \mathbb{R}^{n-m}\},$$

$$x \mapsto (x_1, ..., x_m)$$

is a homeomorphism. Then the collection $\mathcal{A} := \{(U_a, \varphi_a); a \in M\}$ constitutes a differentiable atlas on M. Note that all local coordinates are obtained by choosing m suitable restrictions $x_{i_1}|_M, ..., x_{i_m}|_M$ of the coordinate functions $x_1, ..., x_n$, i.e., with the choice of the set $\{i_1, ..., i_m\}$ depending on the point $a \in M$.

The last example shows that a differentiable atlas may depend on a lot of choices and can be unnecessarily big as well. So we need to say when two atlases are "equivalent":

Definition 2.5. 1. *Two atlases $\mathcal{A}$ and $\tilde{\mathcal{A}}$ on an m-dimensional topological manifold M are called equivalent if any chart in $\mathcal{A}$ is compatible with any chart in $\tilde{\mathcal{A}}$.*

2. *A differentiable structure on a topological manifold is an equivalence class of differentiable atlases.*

3. *A differentiable manifold M is a topological manifold together with a differentiable structure. We say that a differentiable atlas $\mathcal{A}$ is an atlas for the differentiable manifold M, if $\mathcal{A}$ defines (or belongs to) the differentiable structure of M.*

If M is a differentiable manifold, then a "chart (U, φ) on M" means always a chart compatible with all the charts of a (resp. all) atlases defining the differentiable structure of M.

We leave the details of the following remark to the reader:

Remark 2.6. 1. Any open subset $U \subset M$ of a differentiable manifold inherits a natural differentiable structure.

2. The cartesian product $M \times N$ of differentiable manifolds M, N carries a natural differentiable structure.

Definition 2.7. *Let M, N be differentiable manifolds of dimension m, n respectively.*

1. *A function $f : M \longrightarrow \mathbb{R}$ is differentiable if the functions $f \circ \varphi^{-1} : V \longrightarrow \mathbb{R}$ are differentiable for all charts $(U, \varphi : U \longrightarrow V) \in \mathcal{A}$ in a differentiable atlas for M. The same definition applies for maps $F : M \longrightarrow \mathbb{R}^n$. We denote*

$$C^\infty(M) := \{f : M \longrightarrow \mathbb{R} \text{ differentiable}\}$$

   the set of all differentiable functions, indeed a real vector space which is even closed with respect to the multiplication of functions.

2. *A continuous map $F : M \longrightarrow N$ is called differentiable if all the maps $\psi \circ (F|_{F^{-1}(W)}) : F^{-1}(W) \longrightarrow \mathbb{R}^n$ are differentiable, where $(W, \psi) \in \mathcal{B}$ is any chart in an atlas $\mathcal{B}$ defining the differentiable structure of N.*

3. *A diffeomorphism $F : M \longrightarrow N$ between two differentiable manifolds M and N is a bijective differentiable map, such that its inverse $F^{-1} : N \longrightarrow M$ is differentiable as well.*

4. *We say that M is diffeomorphic to N and write $M \cong N$ if there is a diffeomorphism $F : M \longrightarrow N$.*

Note that differentiable functions are continuous, and that the definition of differentiability is independent from the choice of the differentiable atlases for M and N.

Remark 2.8. Given a topological manifold M there are a lot of distinct differentiable structures, but the corresponding differentiable manifolds may be diffeomorphic nevertheless: By a smooth type on M we mean the diffeomorphism class of the differentiable manifold defined by some differentiable structure on M.

1. For $\dim M \leq 3$ there is exactly one smooth type.

2. For $\dim M \geq 4$ it may happen that there is no differentiable structure on M at all.

3. A compact topological manifold of dimension at least 5 admits only finitely many smooth types. E.g., the sphere $\mathbf{S}^n$ has the standard smooth type as described above, but there may be other ones: For $n = 5, ..., 20$ we obtain $1, 1, 28, 2, 8, 6, 992, 1, 3, 2, 16256,\ 2, 16, 16, 523264, 24$ smooth types respectively. For $n = 4$ it is not known whether there are exotic smooth types, i.e. smooth types different from the standard smooth type.

4. For $n \neq 4$ there is only the standard smooth type on $\mathbb{R}^n$, while on $\mathbb{R}^4$ there are uncountably many different smooth types; some of them are obtained as follows: One considers an open subset $U \subset \mathbb{R}^4$ with the standard smooth type; if there is a homeomorphism $U \cong \mathbb{R}^4$ one gets an induced differentiable structure on $\mathbb{R}^4$.

Definition 2.9. Let M be an m-dimensional differentiable manifold. A subset $L \subset M$ is called a submanifold of codimension k iff for every point $a \in L$ there is a chart (U, φ), such that $\varphi(U \cap L) = \{x = (x_1, ..., x_m) \in V := \varphi(U); x_{m-k+1} = ... = x_m = 0\}$.

Note that a submanifold $L \subset M$ inherits from M a unique differentiable structure, such that the inclusion $L \hookrightarrow M$ is differentiable.

3 Vector Fields

Imagine $M = F^{-1}(0)$ with $F : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-m}$, where $DF(a) \in \mathbb{R}^{n-m,n}$ has rank $n - m$ for all $a \in M = F^{-1}(0)$. Then

$$T_aM := \ker(DF(a)) \hookrightarrow \mathbb{R}^n.$$

and the affine subspace $a + T_aM$ is the intuitive idea behind the tangent space. Here we describe the abstract notion:

Definition 3.1. *A tangent vector X_a (or derivation) at a point $a \in M$ of a differentiable manifold M is a linear map $X_a : C^\infty(M) \longrightarrow \mathbb{R}$ satisfying the following Leibniz rule:*

$$X_a(fg) = f(a)X_a(g) + X_a(f)g(a).$$

The set of all tangent vectors of M at $a \in M$ forms a vector space T_aM, called the tangent space *of M at $a \in M$.*

Remark 3.2. (1) For $\mathbb{R} \subset C^\infty(M)$, the constant functions, we have

$X_a(\mathbb{R}) = 0$ for every tangent vector $X_a \in T_aM$, since $X_a(1) = X_a(1^2) = X_a(1) + X_a(1)$.

(2) Take a chart $\varphi : U \to V \subset \mathbb{R}^n$ with $a \in U$ and $\varphi(a) = 0$. Then the maps

$$\partial_i^a := \partial_i^{\varphi,a} : f \mapsto \frac{\partial f \circ \varphi^{-1}}{\partial x_i}(0),\ i = 1, ..., n,$$

are tangent vectors at a.

(3) Another, may be more geometric, construction that avoids the choice of charts is the following: To any curve, i.e. differentiable map, $\gamma : \mathcal{I} \to M$ defined on an open interval $\mathcal{I} \subset \mathbb{R}$ with $\gamma(t_0) = a$ for some $t_0 \in \mathcal{I}$ we can associate the tangent vector $\dot{\gamma}(t_0) \in T_aM$ defined by

$$\dot{\gamma}(t_0) : f \mapsto (f \circ \gamma)'(t_0)\ .$$

The vector $\dot{\gamma}(t_0)$ is called the tangent vector of the curve $\gamma : \mathcal{I} \to M$ at $t_0 \in \mathcal{I}$.

Theorem 3.3. *Using the notation of Remark 3.2.2 we have*

$$T_aM = \bigoplus_{i=1}^{n} \mathbb{R} \cdot \partial_i^a,$$

i.e. the tangent vectors $\partial_i^a := \partial_i^{\varphi,a}$ *form a basis of the tangent space* T_aM.

For the proof we need

Lemma 3.4. *1. If* $f \in C^\infty(M)$ *vanishes in a neighborhood of* $a \in M$*, then* $X_a(f) = 0$ *for all tangent vectors* $X_a \in T_aM$.

2. Let $U \subset M$ *be open. Denote* $\varrho : C^\infty(M) \longrightarrow C^\infty(U), f \mapsto f|_U$ *the restriction from* M *to* U*. Then the map*

$$T_aU \longrightarrow T_aM, X_a \mapsto X_a \circ \varrho,$$

is an isomorphism.

Proof. *1.)* If f vanishes near a, take a function $g \in C^\infty(M)$ with $g = 1$ near a and $fg = 0$. Then $0 = X_a(fg) = g(a)X_a(f) + f(a)X_a(g) = X_a(f)$.
2.) Injectivity: Assume $X_a \circ \varrho = 0$. Take any function $f \in C^\infty(U)$. Choose $\tilde{f} \in C^\infty(M)$ with $\tilde{f} = f$ near a. Then, according to the first part, we have $X_a(f) = X_a(\tilde{f}|_U) = 0$. Now the function $f \in C^\infty(U)$ being arbitrary, we obtain $X_a = 0$.
Surjectivity: For $Y_a \in T_aM$ define $X_a \in T_aU$ by its value on $f \in C^\infty(U)$ as follows

$$X_a(f) := Y_a(\tilde{f}),$$

where again $\tilde{f} \in C^\infty(M)$ with $\tilde{f} = f$ near a. Then $X_a(f)$ is well defined as a consequence of the first part and obviously $X_a \circ \varrho = Y_a$. □

Proof of 3.3. As a consequence of 3.4 we may assume, with the notation of Rem.3.2.2, $M = U = V \subset \mathbb{R}^n$ and show that the tangent vectors $\partial_i^0 \in T_0V$ with $\partial_i^0(f) := \frac{\partial f}{\partial x_i}(0)$ form a basis of T_0V. Since $\partial_i^0(x_j) = \delta_{ij}$ they are linearly independent. On the other hand, for any $X_0 \in T_0V$ we have

$$X_0 = \sum_{i=1}^n X_0(x_i)\partial_i^0.$$

Take $f \in C^\infty(V)$. After, may be, a shrinking of V we may, according to the below lemma 3.5, assume $f = f(0) + \sum_{i=1}^n x_i f_i$ with $f_i \in C^\infty(V)$ and then obtain $X_0(f) = \sum_{i=1}^n X_0(x_i)f_i(0) = \sum_{i=1}^n X_0(x_i)\partial_i^0(f)$. □

Lemma 3.5. *Let $f \in C^\infty((-\varepsilon, \varepsilon)^m)$. Then we may write*

$$f = f(0) + \sum_{i=1}^n x_i f_i$$

with functions $f_1, ..., f_m \in C^\infty((-\varepsilon, \varepsilon)^m)$.

Proof. We have

$$f(x_1, ..., x_m) = \sum_{i=0}^{m-1} (f(x_1, ..., x_{m-i}, 0, .., 0) - f(x_1, ..., x_{m-i-1}, 0, .., 0)) + f(0, ..., 0)$$

$$= \sum_{i=1}^m x_{m-i} \left(\int_0^1 \frac{\partial f}{\partial x_{m-i}}(x_1, ..., x_{m-i-1}, tx_{m-i}, 0, .., 0)dt \right) + f(0, ..., 0).$$

□

Remark 3.6. 1. Any finite dimensional vector space V is a differentiable manifold: Take an atlas containing one chart (V, φ), where $\varphi : V \longrightarrow \mathbb{R}^n$ is a linear isomorphism. The resulting manifold does not depend on the choice of φ.

2. For any $a \in V$ there is a natural isomorphism

$$V \xrightarrow{\cong} T_a V, v \mapsto \dot{\gamma}_{a,v}(0)$$

with $\gamma_{a,v}(t) = a + tv$.

Differentiable maps induce linear maps between tangent spaces:

Definition 3.7. Given a differentiable map $F : M \to N$ between the differentiable manifolds M and N, there is an induced homomorphism of tangent spaces:

$$T_a F : T_a M \to T_{F(a)} N$$

defined by

$$T_a F(X_a) : C^\infty(N) \to \mathbb{R}, f \mapsto X_a(f \circ F) \ .$$

It is called the *tangent map* of F at $a \in M$.

Obviously we have for a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = a$ that

$$T_a F(\dot{\gamma}(0)) = \dot{\delta}(0), \text{ where } \delta := F \circ \gamma \ .$$

For explicit computations we note that, if $F = (F_1, ..., F_m) : U \to W$ is a differentiable map between the open sets $U \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$, and $b = F(a)$ for $a \in U$, then with respect to the bases $\partial_1^a, ..., \partial_n^a$ of $T_a U$ and $\partial_1^b, ..., \partial_m^b$ of $T_b W$ the linear map $T_a F$ has the matrix:

$$DF(a) = \left(\frac{\partial F_i}{\partial x_j}(a) \right)_{1 \leq i \leq m, 1 \leq j \leq n} \in \mathbb{R}^{m,n} \ ,$$

the Jacobi matrix of F at $a \in U$.

Furthermore it is immediate from the definition, that the tangent map behaves functorially, i.e. if $F : M_1 \to M_2$ and $G : M_2 \to M_3$ are differentiable maps, then $G \circ F : M_1 \to M_3$ is again differentiable and the chain rule

$$T_a(G \circ F) = T_{F(a)} G \circ T_a F$$

holds.

All the tangent vectors at points in a differentiable m-manifold M form a differentiable m^2-manifold $TM = \bigcup_{a\in M} T_aM$, the *tangent bundle* of M. For example, for $M = F^{-1}(0) \hookrightarrow \mathbb{R}^n$ as in the introduction we take

$$TM := \{(a,\xi) \in \mathbb{R}^n \times \mathbb{R}^n; F(a) = 0, DF(a)\xi = 0\}$$

$$= \bigcup_{a\in M} \{a\} \times T_aM \hookrightarrow \mathbb{R}^n \times \mathbb{R}^n.$$

We introduce first the notion of a vector bundle, a smooth family of vector spaces, over M.

Definition 3.8. A vector bundle of rank n over M is a triple (E,π,M), where E is a differentiable manifold and $\pi : E \longrightarrow M$ a differentiable map, such that

1. the fibers $E_a := \pi^{-1}(a)$ carry the structure of an n-dimensional vector space,
2. there is an open covering $M = \bigcup_{i\in I} U_i$ together with diffeomorphisms

   $$\tau_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n,$$

   such that $\pi = \mathrm{pr}_{U_i} \circ \tau_i$, and inducing vector space isomorphisms

   $$\mathrm{pr}_{\mathbb{R}^n} \circ \tau_i : E_a \longrightarrow \mathbb{R}^n \times \{a\} \cong \mathbb{R}^n.$$

   The τ_i are called *trivializations*.

Remark 3.9. If E is a vector bundle, the comparison of two trivializations τ_i, τ_j results over $U_{ij} := U_i \cap U_j$ in the following *transition function*

$$\tau_j \circ \tau_i^{-1} : U_{ij} \times \mathbb{R}^n \longrightarrow U_{ij} \times \mathbb{R}^n, (\mathbf{x},\mathbf{y}) \longrightarrow (\mathbf{x}, A_{ij}(\mathbf{x})\mathbf{y})$$

with a smooth function $A_{ij} : U_{ij} \longrightarrow GL_n(\mathbb{R})$.

Example 3.10.
1. $E := M \times \mathbb{R}^n$, the trivial vector bundle of rank n.
2. Let M be a differentiable n-manifold. The *tangent bundle* TM is, as a set, the disjoint union

   $$TM := \bigcup_{a\in M} T_aM$$

of all tangent spaces at points $a \in M$. Take a differentiable atlas $\mathcal{A} = \{(U_i, \varphi_i); i \in I\}$. Denote $\pi : TM \to M$ the map, which associates to a tangent vector $X_a \in T_aM$ its "base point" $a \in M$. We pick the following trivialization

$$\tau_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^m, X_a = \sum_{j=1}^{m} \lambda_j \partial_j^{\varphi_i,a} \mapsto (a, \lambda_1, ..., \lambda_m).$$

We endow TM with a topology: A set $W \subset TM$ is open if $\tau_i(W \cap U_i) \subset U_i \times \mathbb{R}^m$ is open for all charts $i \in I$. in an atlas $\mathcal{A}$ for M. We obtain a topological manifold TM with differentiable atlas $\{(\pi^{-1}(U_i), \tau_i)$. Indeed the corresponding transition functions are

$$\tau_j \circ \tau_i^{-1} : U_{ij} \times \mathbb{R}^n \longrightarrow U_{ij} \times \mathbb{R}^n, (\mathbf{x}, \mathbf{y}) \longrightarrow (\mathbf{x}, DF_{ij}(\mathbf{x})\mathbf{y})$$

with $F_{ij} := \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij})$.

Now we can generalize Definition 3.7: given a differentiable map $F : M \to N$ the pointwise tangent maps $T_aF : T_aM \to T_{F(a)}N$ combine to a differentiable map $TF : TM \to TN$, i.e.

$$TF|_{T_aM} := T_aF : T_aM \to T_{F(a)}N.$$

Indeed, the map TF fits into a commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{TF} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{F} & N \end{array},$$

i.e. $\pi_N \circ TF = F \circ \pi_M$ holds with the projections $\pi_M : TM \longrightarrow M$ and $\pi_N : TN \to N$ of the respective tangent bundles.

Definition 3.11. The restriction of a vector bundle (E, π, M) to an open subset $U \subset M$ is

$$E|_U := (\pi^{-1}(U), \pi, U).$$

Definition 3.12. Let E be a vector bundle over M.

1. A section of E is a differentiable map $\sigma : M \longrightarrow E$ with $\pi \circ \sigma = \mathrm{id}_M$. We denote
$$\Gamma(E) := \{\sigma : M \longrightarrow E, \text{section of } E\},$$
a $C^\infty(M)$-module with the scalar multiplication $(f\sigma)(a) := f(a)\sigma(a)$.

2. Let $U \subset M$ be open. A vector field on U is a section
$$X \in \Gamma(TM|_U), a \mapsto X_a \in T_aM.$$
We write as well
$$\Theta(U) := \Gamma(TM|_U).$$

Remark 3.13. 1. Let (U, φ) be a chart. Then
$$\partial_i := \partial_i^\varphi : U \longrightarrow TM, a \mapsto \partial_i^a \quad \text{for } i = 1, ..., m,$$
with
$$\partial_i^a(f) := \frac{\partial f \circ \varphi^{-1}}{\partial x_i}(\varphi(a))$$
are vector fields on U, the "coordinate vector fields" associated to the local chart (or local coordinates) $x_i = \varphi_i(a), i = 1, ..., m$. Since $T_aM = \bigoplus_{i=1}^m \mathbb{R}\partial_i^a$, any section $X : U \longrightarrow TM$ of $\pi : TM \longrightarrow M$ can be written
$$X = \sum_{i=1}^m g_i \partial_i,$$
with unique functions $g_i \in C^\infty(U)$.

2. Note that on an arbitrary differentiable manifold M it is in general not possible to find vector fields $X_1, ..., X_m \in \Theta(M)$, such that $(X_1)_a, ..., (X_m)_a$ is a *frame* at a, i.e., a basis of T_aM, for all $a \in M$. If such vector fields exist, the manifold M is called *parallelizable*. In a more algebraic way: M is parallelizable iff $\Theta(M) \cong C^\infty(M)^m$.

3. The vector fields on M can be identified with derivations $D : C^\infty(M) \to C^\infty(M)$, i.e. linear maps satisfying the Leibniz rule $D(fg) = D(f)g + fD(g)$ for all $f, g \in C^\infty(M)$. Given a vector field $X \in \Theta(M)$ the corresponding derivation $X : C^\infty(M) \to C^\infty(M), f \mapsto X(f)$ is defined by $(X(f))(a) := X_a(f)$. In fact, every derivation $D : C^\infty(M) \to C^\infty(M)$ is obtained from a vector field: Take $X \in \Theta(M)$ with
$$X_a : C^\infty(M) \to \mathbb{R}, f \mapsto D(f)(a) \ .$$

4. For an open subset $U \subset M$ the tangent bundle TU is identified, in a natural way, with the open subset $\pi^{-1}(U) \subset TM$.

5. Let $F : M \to N$ be a differentiable map. Given a vector field $X \in \Theta(M)$, we can consider $TF \circ X : M \to TN$, but that map does not in general factor through N, e.g. if F is not injective. But it does if $F : M \to N$ is a diffeomorphism: then we may define a map
$$F_* : \Theta(M) \to \Theta(N), X \mapsto F_*(X) := TF \circ X \circ F^{-1} \in \Theta(N),$$
the push forward of vector fields with respect to a diffeomorphism.

The vector space $\Theta(M)$ carries further algebraic structure: Though the compositions XY and YX of two derivations (vector fields) $X, Y : C^\infty(M) \to C^\infty(M)$ are no longer derivations, their commutator is:

$$\begin{aligned}
(XY - YX)fg &= XY(fg) - YX(fg) \\
&= X(fYg + gYf) - Y(fXg + gXf) \\
&= fXYg + (Xf)Yg + gXYf + (Xg)Yf \\
&- fYXg - (Yf)(Xg) - gYXf - (Yg)(Xf) \\
&= fXYg - fYXg + gXYf - gYXf \\
&= f(XY - YX)g + g(XY - YX)f.
\end{aligned}$$

Definition 3.14. The *Lie bracket* $[X, Y] \in \Theta(M)$ of two vector fields $X, Y \in \Theta(M)$ is the commutator of the derivations $X, Y : C^\infty(M) \to C^\infty(M)$, i.e.
$$[X, Y] := XY - YX,$$
or, in other words, the vector field $[X, Y]$ satisfying
$$[X, Y]_a(f) := X_a(Y(f)) - Y_a(X(f))$$
for all differentiable functions $f \in C^\infty(M)$ at every point $a \in M$.

Note that the tangent vector $[X, Y]_a$ is not a function of the values $X_a, Y_a \in T_aM$ only, since the local behavior of the vector fields X, Y near $a \in M$ also enters in the computation rule. If $x_1, ..., x_n$ are local coordinates on $U \subset M$, and $X, Y \in \Theta(U)$ have representations
$$X = \sum_{i=1}^{n} f_i\, \partial_i,\ Y = \sum_{i=1}^{n} g_i\, \partial_i,$$

then
$$[X,Y] = \sum_{i=1}^{n} (X(g_i) - Y(f_i))\, \partial_i \ .$$
So, in particular, $[\partial_i, \partial_j] = 0$ for coordinate vector fields. On the other hand we mention:

Theorem 3.15. (Frobenius Theorem) *Let $X_1, ..., X_n \in \Theta(M)$ be pairwise commuting vector fields, i.e. $[X_i, X_j] = 0$ for $1 \leq i, j \leq n$. Then every point $a \in M$, such that $(X_1)_a, ..., (X_n)_a$ is a frame at a (i.e. a basis of the tangent space T_aM) admits a neighborhood $U \subset M$ with local coordinates $x_1, ..., x_n \in C^\infty(U)$ such that*
$$X_i|_U = \partial_i.$$

For the proof one needs the notion of an integral curve of a vector field:

Definition 3.16. Let $X \in \Theta(M)$ be a vector field. A smooth curve $\gamma : \mathcal{I} \to M$ is called an *integral curve* of the vector field X, if $\dot\gamma(t) = X_{\gamma(t)}$ holds for all $t \in \mathcal{I}$.

Theorem 3.17. *Let $X \in \Theta(M)$ be a vector field.*

1. *Given a relatively compact open set $U \subset M$ there is an $\varepsilon > 0$ and a smooth map*
$$\gamma : U \times (-\varepsilon, \varepsilon) \longrightarrow M,$$
such that for all $a \in U$ the path $t \mapsto \gamma_a(t) := \gamma(a,t)$ is an integral curve of the vector field X with $\gamma_a(0) = a$. The path γ_a is unique.

2. *If $X_a \neq 0$, there is a local chart (U, φ) around a, such that $X|_U = \partial_1$.*

3. *The set*
$$\mathbb{D}(X) := \{(a,t) \in M \times \mathbb{R}; \exists\, X\text{-integral curve } \gamma_a : \mathcal{I} \longrightarrow M; 0, t \in \mathcal{I}, \gamma_a(0) = a\}$$
is open and the map
$$\gamma : \mathbb{D}(X) \longrightarrow M, (a,t) \mapsto \gamma_a(t),$$
is smooth.

4. *If M is compact, we have $\mathbb{D}(X) = M \times \mathbb{R}$.*

Proof. The first statement follows from the fundamental theorem of ODE's including the fact, that solutions depend smoothly on initial conditions. For the second one we may assume $a = 0 \in U \subset \mathbb{R}^m, X_0 = \partial_1^0$. Take an open subset $V \subset \mathbb{R}^{m-1}$, such that $\{0\} \times V$ is relatively compact in U and note that the map $V \times \mathbb{R} \longrightarrow U, (\mathbf{x}, t) \mapsto \gamma_{(\mathbf{x},0)}(t)$ is a diffeomorphism near the origin. For the third one we refer to Lang, Differential and Riemannian manfolds, Th.2.6. Finally, if M is compact, choose $\varepsilon > 0$ as in the first point, with $U = M$. Then, if $\gamma : (t_0, t_1) \longrightarrow M$ is an integral curve of X, it can be extended to $(t_0 - \varepsilon, t_1 + \varepsilon)$, since the corresponding statement holds for $[t_2, t_3]$ with $t_0 < t_2 < t_3 < t_1$. $\square$

Remark 3.18.

1. An integral curve $\gamma : (0, t_0) \longrightarrow M$ has a set of limit points in M, the points of the form $\lim_{n\to\infty} \gamma(t_n)$ with some sequence $t_n \to t_0$. Then either γ can be extended over t_0 or the limit point set is empty.

2. If M is compact, $t \mapsto \mu_t$ generates a one parameter group of diffeomorphisms $\mu_t : M \longrightarrow M$, since $\mu_0 = \mathrm{id}_M$ and $\mu_s \circ \mu_t = \mu_{s+t}$, called the flow of X.

3. The proof of Th.3.15 relies on the fact that given the flows (μ_t) and $(\tilde{\mu}_t)$ of commuting vector fields $X, \tilde{X} \in \Theta(M)$, one has $\mu_s \circ \tilde{\mu}_t = \tilde{\mu}_t \circ \mu_s$ for all sufficiently small $s, t \in \mathbb{R}$.

4 Vector space constructions

The tangent space T_aM at a point $a \in M$ of the manifold M has been defined without referring to local coordinates, in particular there is no distinguished basis. For the definition of an infinitesimal metric on M, which allows us to measure the length of piecewise smooth curves and the study of the corresponding geometry we need Linear Algebra constructions applying to an n-dimensional vector space V without a distinguished basis. For explicit calculations one needs of course a basis of V (resp. local coordinates near $a \in M$ in the case $V = T_aM$), so it is important to understand how the induced bases behave under a change of basis of V (resp. a coordinate change). This is most easily remembered when taking into account the "naturality" (or "functoriality") of our constructions, and one need not know by heart a lot of complicated formulae with many indices.

We shall discuss

1. the space of (k, ℓ)-tensors $T^{k,\ell}(V)$,
2. the tensor algebra $T^*(V)$,
3. the exterior algebra $\Lambda^* V$.

The **tensor product** $V \otimes W$ of two vector spaces V, W is characterized by the following features:

1. There is a bilinear map
$$\tau : V \times W \longrightarrow V \otimes W,$$
where one traditionally writes
$$\mathbf{v} \otimes \mathbf{w} := \tau(\mathbf{v}, \mathbf{w}),$$
2. $V \otimes W = \mathrm{span}\{\mathbf{v} \otimes \mathbf{w}; \mathbf{v} \in V, \mathbf{w} \in W\}$, so any element in $V \otimes W$ is of the form $\mathbf{v}_1 \otimes \mathbf{u}_1 + ... + \mathbf{v}_r \otimes \mathbf{u}_r$,
3. $\dim V \otimes W = n \cdot m$.

Remark 4.1.

1. If $\mathbf{e}_1, ..., \mathbf{e}_n$ is a basis of V and $\mathbf{f}_1, ..., \mathbf{f}_m$ a basis of W, then the $\mathbf{e}_i \otimes \mathbf{f}_j$ form a basis of $V \otimes W$.
2. The universal mapping property (UMP): Given a bilinear map $\varphi : V \times W \longrightarrow U$ there is a unique linear map $\hat{\varphi} : V \otimes W \longrightarrow U$, s.th. the diagram
$$\begin{tikzcd} V \otimes W \arrow[r, "\hat{\varphi}"] & A \\ V \times W \arrow[u, "\tau"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$
is commutative. Indeed, define
$$\hat{\varphi}(\mathbf{e}_i \otimes \mathbf{f}_j) := \varphi(\mathbf{e}_i, \mathbf{f}_j).$$
3. Given $f : V \longrightarrow V', g : W \longrightarrow W'$ there is an induced linear map
$$f \otimes g : V \otimes W \longrightarrow V' \otimes W', \mathbf{v} \otimes \mathbf{w} \mapsto f(\mathbf{v}) \otimes g(\mathbf{w}).$$

Example 4.2. 1. There is a natural isomorphism

$$\hat{\varphi} : V^* \otimes W \longrightarrow \mathrm{Hom}(V, W)$$

induced by

$$\varphi : V^* \times W \longrightarrow \mathrm{Hom}(V, W), (\mathbf{v}^*, \mathbf{w}) \mapsto \mathbf{v}^*(..)\mathbf{w}.$$

2. The space of k-tensors: We define $T^k(V), k \in \mathbb{N}$, by

$$T^0(V) := \mathbb{R}, T^{k+1}(V) := T^k(V) \otimes V$$

and obtain a k-linear map

$$V^k \longrightarrow T^k(V), (\mathbf{v}_1, ..., \mathbf{v}_k) \mapsto \mathbf{v}_1 \otimes ... \otimes \mathbf{v}_k,$$

such that the UMP for k-linear (instead of bilinear) maps is satisfied.

3. A k-tensor is an expression of the form

$$\alpha = \sum_{(i_1,...,i_k)} \lambda_{i_1,...,i_k} \mathbf{v}_{i_1} \otimes ... \otimes \mathbf{v}_{i_k}.$$

4. Denote $\mathrm{Mult}^k(V)$ the vector space of k-linear forms. There is a natural isomorphism

$$T^k(V^*) \xrightarrow{\cong} \mathrm{Mult}^k(V)$$

induced by the k-linear map

$$(V^*)^k \longrightarrow \mathrm{Mult}^k(V), (\mathbf{v}_1^*, ..., \mathbf{v}_k^*) \mapsto \mathbf{v}_1^*(..) \cdot ..... \cdot \mathbf{v}_k^*(..).$$

In the sequel we identify elements in $T^k(V^*)$ with k-multilinear forms, such that

$$\mathbf{v}_1^* \otimes ... \otimes \mathbf{v}_k^*(\mathbf{v}_1, ..., \mathbf{v}_k) = \mathbf{v}_1^*(\mathbf{v}_1) \cdot ..... \cdot \mathbf{v}_k^*(\mathbf{v}_k).$$

5. There is a natural isomorphism

$$T^k(V) \otimes T^\ell(V) \cong T^{k+\ell}(V),$$

the "concatenation":

$$(\mathbf{v}_1 \otimes ... \otimes \mathbf{v}_k) \otimes (\mathbf{u}_1 \otimes ... \otimes \mathbf{u}_\ell) = \mathbf{v}_1 \otimes ... \otimes \mathbf{v}_k \otimes \mathbf{u}_1 \otimes ... \otimes \mathbf{u}_\ell.$$

6. In particular
$$T^*(V) := \bigoplus_{k=0}^{\infty} T^k(V)$$
carries the structure of a graded algebra, the product being the tensor product. It is called the tensor algebra of V.

7. Any linear map $f : V \longrightarrow W$ induces a ring homomorphism $T^*f : T^*(V) \longrightarrow T^*(W)$.

8. If $\alpha \in \mathrm{Mult}^k(V) \cong T^k(V^*)$ and $\beta \in \mathrm{Mult}^\ell(V) \cong T^\ell(V^*)$, then $\alpha \otimes \beta \in \mathrm{Mult}^{k+\ell}(V)$ satisfies
$$\alpha \otimes \beta(\mathbf{v}_1, ..., \mathbf{v}_{k+\ell}) = \alpha(\mathbf{v}_1, ..., \mathbf{v}_k)\beta(\mathbf{v}_{k+1}, ..., \mathbf{v}_{k+\ell}).$$

Definition 4.3. A (k, ℓ) tensor α is an element in
$$T^{k,\ell}(V) := T^k(V) \otimes T^\ell(V^*),$$
it can be written:
$$\alpha := \sum_{(i_1,...,i_k,j_1,...,j_\ell)} \lambda_{i_1,...,i_k,j_1,...,j_\ell} \mathbf{v}_{i_1} \otimes ... \otimes \mathbf{v}_{i_k} \otimes \mathbf{v}^*_{j_1} \otimes ... \otimes \mathbf{v}^*_{j_\ell}.$$

Example 4.4. Here are some natural isomorphisms:

1. $T^{1,1}(V) \cong \mathrm{End}(V)$.

2. $T^{0,2}(V) \cong \mathrm{Mult}^2(V)$.

3. The tensors in
$$T^{1,2}V = V \otimes V^* \otimes V^* \cong \mathrm{Mult}^2(V) \otimes V$$
may be understood as bilinear maps $V \times V \longrightarrow V$.

4. The tensors in
$$T^{1,3}V = V \otimes V^* \otimes V^* \otimes V^* \cong \mathrm{Mult}^2(V) \otimes \mathrm{End}(V)$$
may be understood as bilinear maps $V \times V \longrightarrow \mathrm{End}(V)$.

Definition 4.5. The exterior algebra is the factor algebra

$$\Lambda^* V := T^*(V)/\mathfrak{a},$$

with the two-sided homogeneous ideal $\mathfrak{a} = \bigoplus_{k=2}^{\infty} \mathfrak{a}_k \subset T^*(V)$ generated by the elements $\mathbf{v} \otimes \mathbf{v} \in T^2(V)$. The induced product is denoted $\wedge$.

Remark 4.6. 1. We have

$$\mathbf{v} \wedge \mathbf{v} = 0, \mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}.$$

2. More generally, for $\alpha \in \Lambda^k V, \beta \in \Lambda^\ell V$, we have

$$\beta \wedge \alpha = (-1)^{k\ell} \alpha \wedge \beta.$$

3. If $f : V \longrightarrow W$ is a linear map, there is an induced homomorphism $\Lambda^k f : \Lambda^k V \longrightarrow \Lambda^k W$.

4. For an endomorphism $f : V \longrightarrow V$ the induced homomorphism is multiplication with the determinant:

$$\Lambda^n f = \mu_{\det f}.$$

Proposition 4.7. *Let $\Lambda^k V := T^k(V)/\mathfrak{a}_k$. We have*

$$\Lambda^k V = \{0\}, k > n.$$

In particular

$$\Lambda^* V = \bigoplus_{k=0}^{n} \Lambda^k V.$$

Furthermore

$$\dim \Lambda^k V = \binom{n}{k}$$

and the elements

$$\mathbf{e}_{i_1} \wedge .... \wedge \mathbf{e}_{i_k}, 1 \leq i_1 < ... < i_k \leq n,$$

form a basis of $\Lambda^k V$.

We may identify $\Lambda^* V$ with a subspace of $T^*(V)$. We start with:

Remark 4.8. There is a natural action of $\mathbf{S}_k$ on $T^k(V)$:

$$(\pi, \mathbf{v}_1 \otimes ... \otimes \mathbf{v}_k) \mapsto \pi_*(\mathbf{v}_1 \otimes ... \otimes \mathbf{v}_k) := \mathbf{v}_{\pi(1)} \otimes ... \otimes \mathbf{v}_{\pi(k)}.$$

Definition 4.9. A k-tensor $\alpha \in T^k(V)$ is called alternating, if

$$\pi_*(\alpha) = \mathrm{sign}(\pi)\alpha$$

holds for all permutations $\pi \in \mathbf{S}_k$.

Proposition 4.10. *The alternating k-tensors form a subspace*

$$A^k(V) \subset T^k(V),$$

such that the quotient map $T^k(V) \longrightarrow \Lambda^k V$ induces an isomorphism

$$A^k(V) \xrightarrow{\cong} \Lambda^k V.$$

Remark 4.11. 1. Though the direct sum

$$A^*(V) := \bigoplus_{k=0}^{n} A^k(V) \subset T^*(V)$$

is not a subring, we can describe the wedge product on $A^*(V)$ as follows: There is a projection operator:

$$\mathrm{Alt}^k : T^k(V) \longrightarrow T^k(V),$$

i.e. $\mathrm{Alt}^k(T^k(V)) = A^k(V), \mathrm{Alt}^k|_{A^k(V)} = \mathrm{id}_{A^k(V)}$, namely

$$\mathrm{Alt}^k(\alpha) := \frac{1}{k!} \sum_{\pi \in \mathbb{S}_k} \mathrm{sign}(\pi)\pi_*(\alpha).$$

If we define $\wedge$ on $A^k(V)$ as the pull back of the wedge product w.r.t. the isomorphism

$$A^*(V) \xrightarrow{\cong} \Lambda^* V,$$

we obtain

$$\alpha \wedge \beta = \frac{(k+\ell)!}{k!\ell!} \mathrm{Alt}^{k+\ell}(\alpha \otimes \beta).$$

for $\alpha \in A^k(V), \beta \in A^\ell(V)$.

2. Under the isomorphism $T^k(V^*) \cong \mathrm{Mult}^k(V)$ the alternating tensors in $A^k(V^*)$ correspond to alternating k-linear forms $\alpha \in \mathrm{Mult}^k(V)$, i.e. such that
$$\exists\, i \neq j : \mathbf{v}_i = \mathbf{v}_j \Longrightarrow \alpha(\mathbf{v}_1, ..., \mathbf{v}_k) = 0.$$

Euclidean vector spaces: Finally we consider the situation, where the vector space V is endowed with an inner product $g : V \times V \longrightarrow \mathbb{R}$. It induces an isomorphism
$$\psi : V \xrightarrow{\cong} V^*, \mathbf{v} \mapsto g(\mathbf{v}, ..),$$
such that, because of the symmetry of the inner product,
$$\psi = \psi^* \circ \beta : V \longrightarrow V^{**} \longrightarrow V^*.$$
Here $\beta : V \longrightarrow V^{**}$ is the biduality isomorphism. In the sequel we shall simply identify V with V^{**}, vectors $\mathbf{v} \in V$ acting on V^* by evaluation $\mathbf{v}(\mathbf{v}^*) := \mathbf{v}^*(\mathbf{v})$.

We want to show that there is an induced inner product on $T^{k,\ell}(V)$. First of all there is an isomorphism
$$T^{k,\ell}(V^*) \cong T^k(V^*) \otimes T^\ell(V) \cong T^{k,\ell}(V)^*,$$
induced by the multilinear map
$$(V^*)^k \times V^\ell \longrightarrow T^{k,\ell}(V)^*,$$
$$(\mathbf{v}_1^*, ..., \mathbf{v}_k^*, \mathbf{v}_1, ..., \mathbf{v}_\ell) \mapsto \mathbf{v}_1^* \otimes ... \otimes \mathbf{v}_k^* \otimes \mathbf{v}_1 \otimes .... \otimes \mathbf{v}_\ell,$$
the RHS being the tensor product of the linear maps $\mathbf{v}_i^* : V \longrightarrow \mathbb{R}$ and $\mathbf{v}_j : V^* \longrightarrow \mathbb{R}$.

Now the inner product on $T^{k,\ell}(V)$ correspond to the isomorphism
$$T^{k,\ell}(V) \longrightarrow T^{k,\ell}(V)^* \cong T^k(V^*) \otimes T^\ell(V)(\cong T^{\ell,k}(V))$$
$$\mathbf{v}_1 \otimes ... \otimes \mathbf{v}_k \otimes \mathbf{v}_1^* \otimes ... \otimes \mathbf{v}_\ell^* \mapsto$$
$$\psi(\mathbf{v}_1) \otimes ... \otimes \psi(\mathbf{v}_k) \otimes \psi^{-1}(\mathbf{v}_1^*) \otimes ... \otimes \psi^{-1}(\mathbf{v}_\ell^*).$$
Thus the inner product on $T^{k,\ell}(V)$ is defined without referring to a basis of V. For practical purposes we note that, given a g-orthonormal basis $\mathbf{e}_1, ..., \mathbf{e}_n$ of V, we have
$$\mathbf{e}_i^* = \psi(\mathbf{e}_i)$$

for the dual basis $\mathbf{e}_1^*, ..., \mathbf{e}_n^*$ of V^*. It follows that the (k,ℓ)-tensors

$$\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_k} \otimes \mathbf{e}_{j_1}^* \otimes .... \otimes \mathbf{e}_{j_\ell}^* \in T^{k,\ell}(V)$$

form a g-orthonormal basis.

A similar discussion applies to the exterior product $\Lambda^k V$.

5 Vector bundles and tensor fields

The above vector space constructions are of the following type: We have a map

$$\Phi : V \mapsto \Phi(V),$$

which associates to a vector space V another one, $\Phi(V)$, and with any isomorphism $f : V \longrightarrow W$ an isomorphism $\Phi(f) : \Phi(V) \longrightarrow \Phi(W)$, such that

1. $\Phi(g \circ f) = \Phi(g) \circ \Phi(f)$ and
2. $\Phi(\mathrm{id}_V) = \mathrm{id}_{\Phi(V)}$,
3. finally, if $f_a \in \mathrm{End}(V)$ smoothly depends on $a \in U \subset M$, so does $\Phi(f_a)$.

Example 5.1. 1. $\Phi(V) = V$ with $\Phi(f) = f$.

2. $\Phi(V) = V^*$ with $\Phi(f) = (f^{-1})^*$.
3. $\Phi(V) = T^{k,\ell}(V)$ with $\Phi(f) = T^k(f) \otimes T^\ell((f^{-1})^*)$.
4. $\Phi(V) = \Lambda^k(V)$ with $\Phi(f) := \Lambda^k(f)$.

Now, given a vector bundle $E \longrightarrow M$ we can form a vector bundle $\Phi(E)$ as follows: Take a covering

$$M = \bigcup_{i \in I} U_i$$

with trivializations

$$\tau_i : \pi^{-1}(U_i) \longrightarrow U_i \times V,$$

and transition functions:

$$\tau_j \circ \tau_i^{-1} : U_{ij} \times V \longrightarrow U_{ij} \times V, (\mathbf{x}, \mathbf{y}) \longrightarrow (\mathbf{x}, F_{ij}(\mathbf{x})\mathbf{y})$$

with smooth $F_{ij} := U_{ij} \longrightarrow \mathrm{End}(V)$.

Remark 5.2. Given smooth functions $F_{ij} := U_{ij} \longrightarrow \mathrm{End}(V)$, s.th. we have the following "cocycle relation"

$$F_{ik}(\mathbf{x}) = F_{jk}(\mathbf{x}) \circ F_{ij}(\mathbf{x}), \forall\ \mathbf{x} \in U_{ijk} := U_i \cap U_j \cap U_k$$

satisfied for all $i, j, k \in I$ as well as $F_{ii} \equiv \mathrm{id}_V$ (s.th. $F_{ij}(\mathbf{x}) \in GL(V)$), we consider on the disjoint union

$$\hat{E} := \bigcup_{i \in I} U_i \times V$$

the relation

$$U_i \times V \ni (\mathbf{x}, \mathbf{v}) \sim (\mathbf{x}, F_{ij}(\mathbf{x})\mathbf{v}) \in U_j \times V,$$

in particular that implies $\mathbf{x} \in U_{ij}$. Due to the cocycle relation $\sim$ it is an equivalence relation, and we can form the quotient space

$$E := \hat{E}/\sim,$$

which obviously is the total space of a vector bundle $\pi : E \longrightarrow M$.

Definition 5.3. Let E be a vector bundle over M with fiber V, obtained as in Rem.5.2 from an open cover $(U_i)_{i \in I}$ of M with $F_{ij} : U_{ij} \longrightarrow \mathrm{End}(V)$ satisfying the cocycle relation. Then we define

$$\Phi(E) := \bigcup_{i \in I} U_i \times \Phi(V)/\sim$$

with the equivalence relation

$$U_i \times \Phi(V) \ni (\mathbf{x}, \mathbf{v}) \sim (\mathbf{x}, \Phi(F_{ij}(\mathbf{x}))\mathbf{v}) \in U_j \times \Phi(V).$$

Example 5.4.

1. The dual bundle E^*. Indeed the sections in $\Gamma(E^*)$ may be regarded as smooth functions $E \longrightarrow \mathbb{R}$, which restrict to a linear form on each fiber E_a.
2. $T^{k,\ell}(E)$.
3. $\Lambda^k(E)$.

Definition 5.5. 1. The dual bundle of the tangent bundle

$$T^*M := (TM)^*.$$

is called the *cotangent bundle* of M. We use also the notation

$$\Omega(M) := \Gamma(T^*M)$$

and call sections $\omega \in \Omega(M)$ differential forms of degree 1 or simply 1-forms.

2. Given $f \in C^\infty(M)$ its *differential* $df \in \Omega(M)$ is defined as the 1-form given by

$$df_a : X_a \mapsto X_a f.$$

Remark 5.6. Indeed, if $x_1, ..., x_m$ are local coordinates on U, we have

$$\Omega(U) = \bigoplus_{i=1}^{m} C^\infty(U) dx_i$$

and

$$df|_U = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i} dx_i.$$

Note that on a manifold there is no notion of the gradient of a function!

Definition 5.7. The bundle of tensors of type (k, ℓ) is defined as

$$T^{k,\ell}(M) := T^{k,\ell}(TM).$$

A global section $\alpha \in \Gamma(T^{k,\ell}(M))$ is called a (k, ℓ)-tensor field.

Remark 5.8. If $f : M \longrightarrow N$ is differentiable, there is a pull back

$$F^* : \Gamma(T^{0,\ell}N) \longrightarrow \Gamma(T^{0,\ell}M)$$

given by

$$(F^*\sigma)_a((X_1)_a, ..., (X_\ell)_a) := \sigma_{F(a)}(T_aF((X_1)_a), ..., T_aF((X_\ell)_a)).$$

If $M = V \subset \mathbb{R}^n$ and $F = (F_1, ..., F_n)$ we have

$$F^*(dy_{i_1} \otimes ... \otimes dy_{i_\ell}) = dF_{i_1} \otimes ... \otimes dF_{i_\ell}.$$

Definition 5.9. 1. A pseudo-Riemannian metric on a differentiable manifold M is a section $g \in \Gamma(T^{0,2}M)$, such that the bilinear form

$$g_a : T_aM \times T_aM \longrightarrow \mathbb{R}$$

is symmetric and nondegenerate for all $a \in M$. It is called a Riemannian metric if the forms g_a are positive definite for all $a \in M$.

2. A Riemannian manifold is a pair (M, g) consisting of a differentiable manifold and a Riemannian metric $g \in \Gamma(T^{0,2}(M))$.

Remark 5.10. 1. On a local chart U with local coordinates $x_1, ..., x_m$ we have

$$g|_U = \sum_{i,j} g_{ij} dx_i \otimes dx_j.$$

with functions $g_{ij} \in C^\infty(M)$.

2. Let $M = F^{-1}(0) \subset W \subset \mathbb{R}^n$ be a submanifold. Then M is a Riemannian manifold with $g := \iota^*(\sum_{i=1}^n dx_i \otimes dx_i)$. Here $\iota : M \hookrightarrow W$ denotes the inclusion.

Definition 5.11. Differential forms of degree k are the sections of the bundle $\Lambda^k(T^*M)$; we set

$$\Omega^k(M) := \Gamma(\Lambda^k(T^*M)).$$

Note that $\Omega^0(M) = C^\infty(M), \Omega^1(M) = \Omega(M)$.

Remark 5.12. 1. Usually we regard a k-form ω as a map associating to each point $a \in M$ an alternating form $\omega_a \in A^k(T_a^*M) \cong \Lambda^k(T_a^*M)$.

2. If $x_1, ..., x_m$ are local coordinates on $U \subset M$, we have

$$\Omega^k(U) = \bigoplus_{1 \le i_1 < ... < i_k \le m} C^\infty(U) dx_{i_1} \wedge ... \wedge dx_{i_k}.$$

$$= \bigoplus_{I \subset \{1,...,m\}, |I| = k} C^\infty(U) d\mathbf{x}_I,$$

where

$$d\mathbf{x}_I := dx_{i_1} \wedge ... \wedge dx_{i_k},$$

if $I = \{i_1, ..., i_k\}, 1 \le i_1 < ... < i_k \le m$.

Definition 5.13. The exterior derivative

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

is defined as follows: Starting with the alternating k-$C^\infty(M)$-linear form

$$\omega : \Theta(M)^k \longrightarrow C^\infty(M),$$

we define the $C^\infty(M)$-$(k+1)$-linear form

$$d\omega : \Theta(M)^{k+1} \longrightarrow C^\infty(M)$$

by

$$d\omega(X_1, ..., X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i \omega(X_1, ..., \hat{X}_i, ..., X_{k+1})$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, ..., \hat{X}_i, .., \hat{X}_j, ..., X_{k+1}).$$

Remark 5.14.

1. In order to see that $d\omega \in \Omega^{k+1}(M)$, we have to check that
   (a) $d\omega$ is $C^\infty(M)$-multilinear
   (b) and alternating. For that we may assume that the vector fields X_i are coordinate vector fields (so the second sum vanishes) and then show that the exchange of two arguments results in a change of sign.
2. W.r.t.local coordinates we have
   $$d(f dx_{i_1} \wedge ... \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge ... \wedge dx_{i_k}.$$
3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$, where $\alpha \in \Omega^k(M)$.
4. $d(F^*\omega) = F^*(d\omega)$ for differentiable $F : M \longrightarrow N$.

Definition 5.15. A differemtial form $\omega \in \Omega^k(M)$ is called

1. *closed*, if $d\omega = 0$,
2. *exact*, if $\omega = d\eta$ with a form $\eta \in \Omega^{k-1}(M)$.

Example 5.16. 1. $\omega = f dx + g dy \in \Omega(V), V \subset \mathbb{R}^2$, is closed iff $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

2. Let $M := \mathbb{R}^2 \setminus \{0\}$ be the punctured plane. Then $\omega := -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ is closed, but not exact. Indeed, on $U := \mathbb{R}^2 \setminus \mathbb{R}_{\leq 0} \times \{0\}$ we havev $\omega = d\varphi$, but φ does not admit an extension to M.

Proposition 5.17. *1. Exact forms are closed, or equivalently $d \circ d = 0$.*

2. Lemma of Poincaré: On a starshaped open subset $U \subset \mathbb{R}^m$ every closed form is exact.

Proof. 1. W.l.o.g. $\omega = f dx_1 \wedge ... \wedge dx_k \in \Omega(V), V \subset \mathbb{R}^m$. Then we have

$$d\omega = df \wedge dx_1 \wedge ..... \wedge dx_k$$

and

$$dd\omega = ddf \wedge dx_1 \wedge ..... \wedge dx_k,$$

hence it suffices to show $ddf = 0$. Exercise!

2. We first show

Lemma 5.18. *Let $W \subset \mathbb{R} \times \mathbb{R}^m, U \subset \mathbb{R}^m$ be open with $[0,1] \times U \subset W$, furthermore*

$$\iota_0 : U \longrightarrow W, \mathbf{x} \mapsto (0, \mathbf{x}), \iota_1 : U \longrightarrow W, \mathbf{x} \mapsto (1, \mathbf{x}).$$

Let $\sigma \in \Omega^k(W)$ be a closed form. Then there is a form $\eta \in \Omega(U)$ with

$$(\iota_1)^*\sigma - (\iota_0)^*\sigma = d\eta.$$

Let us first prove the Poincaré lemma: We may assume that U is starshaped w.r.t. the origin and consider the map $\varphi : \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R}^m, (t, \mathbf{x}) \mapsto t\mathbf{x}$, and apply Lemma 5.18 with $W := \varphi^{-1}(U)$. We take $\sigma = \varphi^*(\omega)$ and obtain

$$d\eta = (\iota_1)^*\varphi^*(\omega) - (\iota_0)^*\varphi^*(\omega) = (\varphi \circ \iota_1)^*\omega - (\varphi \circ \iota_0)^*\omega = \omega.$$

Proof of 5.18. We write

$$\sigma = \sum_{|I|=k} f_I d\mathbf{x}_I + \sum_{|J|=k-1} g_J dt \wedge d\mathbf{x}_J$$

and then have

$$\iota_1^*\sigma = \sum_{|I|=k} f_I(1,\mathbf{x})d\mathbf{x}_I,\ \ \iota_0^*\sigma = \sum_{|I|=k} f_I(0,\mathbf{x})d\mathbf{x}_I.$$

We know

$$0 = d\sigma = \sum_I \left(\frac{\partial f_I}{\partial t} dt \wedge d\mathbf{x}_I + \sum_{i=1}^m \frac{\partial f_I}{\partial x_i} dx_i \wedge d\mathbf{x}_I \right)$$

$$- \sum_J \sum_{i=1}^m \frac{\partial g_J}{\partial x_i} dt \wedge dx_i \wedge d\mathbf{x}_J,$$

in particular

$$\sum_I \frac{\partial f_i}{\partial t} d\mathbf{x}_I = \sum_J \sum_{i=1}^m \frac{\partial g_J}{\partial x_i} dx_i \wedge d\mathbf{x}_J.$$

Now take

$$\eta := \sum_J \left(\int_0^1 g_J(t,\mathbf{x})dt \right) d\mathbf{x}_J.$$

Then we find

$$d\eta = \sum_J \sum_{i=1}^m \left(\int_0^1 \frac{\partial g_J}{\partial x_i}(t,\mathbf{x})dt \right) dx_i \wedge d\mathbf{x}_J.$$

$$\int_0^1 \left(\sum_J \sum_{i=1}^m \frac{\partial g_J}{\partial x_i}(t,\mathbf{x}) dx_i \wedge d\mathbf{x}_J \right) dt$$

$$= \int_0^1 \left(\sum_I \frac{\partial f_I}{\partial t}(t,\mathbf{x}) d\mathbf{x}_I \right) dt$$

$$= \sum_I (f_I(1,\mathbf{x}) - f_I(0,\mathbf{x}) d\mathbf{x}_I = (\iota_1)^*\sigma - (\iota_0)^*\sigma.$$

□

□

Definition 5.19. The factor vector spaces

$$H^q_{dR}(M) := \frac{\ker(d : \Omega^q(M) \longrightarrow \Omega^{q+1}(M))}{d\Omega^{q-1}(M)}$$

are called the de Rham cohomology groups of M, a topological invariant due to the lemma of Poincaré. The dimension

$$b_q(M) := \dim H^q_{dR}(M)$$

is called the q-th Betti number of M.

Remark 5.20. Integration on a Riemannian manifold: If $F : U \longrightarrow V$ is a differentiable map between open subsets $U, V \subset \mathbb{R}^m$, we have

$$F^*(f dy_1 \wedge ... \wedge dy_m) = f \circ F \cdot \det\left(\frac{\partial F_i}{\partial x_j}\right) dx_1 \wedge ... \wedge dx_m.$$

On the other hand: If F is even a diffeomorphism and $f : V \longrightarrow \mathbb{R}$ is continuous with compact support, then there is the following transformation formula for integrals

$$\int_V f dy_1...dy_m = \int_U f \circ F \cdot \left|\det\left(\frac{\partial F_i}{\partial x_j}\right)\right| dx_1....dx_m.$$

Now let us consider a Riemannian manifold M. On each connected $U \subset M$ with local coordinates $x_1, ..., x_m$ there are two normalized forms $\omega \in \Omega^m(M)$ as follows: Denote $X_1, ..., X_m$ the corresponding coordinate vector fields. Choose $A : U \longrightarrow GL_m(\mathbb{R})$, such that the components of $A\begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix}$ form an ON-basis of T_aM at all $a \in U$. (They are in general **not** coordinate vector fields!) Then

$$\pm \det(A(\mathbf{x}))dx_1 \wedge ... \wedge dx_m \in \Omega^m(U)$$

are the two normalized m-forms on U. Take $G : U \longrightarrow GL_m(\mathbb{R})$, such that $d\mathbf{x}^T \otimes G(\mathbf{x})d\mathbf{x}$ (with $d\mathbf{x} := \begin{pmatrix} dx_1 \\ \vdots \\ dx_m \end{pmatrix}$) is the metric tensor on U. It follows

$$A^T G A = E,$$

the unit matrix, hence

$$|\det A| = \frac{1}{\sqrt{|\det G|}}.$$

Thus

$$\pm\frac{dx_1 \wedge ... \wedge dx_m}{\sqrt{|\det(G)|}} \in \Omega^m(U)$$

are the two normalized m-forms over U. We may now integrate functions $f \in C_0(M)$. If $\mathrm{supp}(f) := \overline{f^{-1}(0)} \subset U$ with $U \cong V \subset \mathbb{R}^m$ as above, we set

$$\int_M f = \int_V \frac{f(\mathbf{x})}{\sqrt{|\det(G(\mathbf{x})|}} dx_1...dx_m.$$

Take $M = \bigcup_{i\in I} U_i$ with local charts $\varphi_i : U_i \longrightarrow V_i \subset \mathbb{R}^m$. Finally choose a partition of unity ψ_i subordinate to $(U_i)_{i\in I}$ and define

$$\int_M f := \sum_{i\in I}\int_M \psi_i f.$$

In order to see that this does not depend on the choices involved use the transformation formula for integrals.

In general there is no normalized m-form $\omega \in \Omega^m(M)$. The reason for that is as follows: Imagine we have two open subsets taken from the above open covering. Assuming the U_i to be connected we can take a normalized form $\omega_i \in \Omega^m(U_i)$, which is unique up to sign. But if $U_{ij} = W_1 \cup W_2$ is a disjoint union, we can not exclude that

$$\omega_j|_{W_1} = \omega_i|_{W_1}, \omega_j|_{W_2} = -\omega_i|_{W_2}.$$

So there is no normalized m-form on $U_i \cup U_j$!

Definition 5.21. 1. An m-dimensional differentiable manifold M is called orientable, if there is a nowhere vanishing m-form $\omega \in \Omega^m(M)$, or equivanlently, if $\Lambda^m T^*M \cong M \times \mathbb{R}$.

2. An orientation of M is given by such a form; two nonvanishing forms define the same orientation if they differ only by a positive function.

3. An oriented manifold is a manifold together with an orientation.

Remark 5.22. 1. M is orientable iff there is an atlas $\mathcal{A}$, such that the transition functions have a positive functional determinant.

2. On an oriented Riemannian manifold M we may define $\int_M \eta$ for m-forms η with compact support: We write $\eta = f\omega$, where ω is the normalized orientation form (volume form), and define

$$\int_M \eta := \int_M f.$$

6 Connections on Vector Bundles

If E is a vector bundle over M, there is no natural way to produce an isomorphism $E_a \cong E_b$ for different points $a, b \in M$. In this section we define an additional datum D on E, called a *connection* or *covariant derivative*, which can be used in order to define a derivative $D_{X_c}\mu \in E_c$ of a section $\mu \in \Gamma(E)$ w.r.t. a tangent vector $X_c \in T_cM$. Thus we are able to define what it means that $\mu \circ \gamma$ is constant, where $\gamma : \mathcal{I} \longrightarrow M$ is a curve from a to b: We want to have

$$E_{\gamma(t)} \ni D_{\dot{\gamma}(t)}\mu = 0$$

for all $t \in \mathcal{I}$. Then we may "transport" a vector $x \in E_a$ to a vector $y \in E_b$ along γ by requiring that there is a section $\mu \in \Gamma(E)$, which is constant along γ and satisfies $x = \mu(a), y = \mu(b)$.

Definition 6.1. An affine connection or covariant derivative on a vector bundle E over M is an $\mathbb{R}$-bilinear map

$$D : \Theta(M) \times \Gamma(E) \longrightarrow \Gamma(E), (X, \mu) \longrightarrow D_X\mu,$$

which is $C^\infty(M)$-linear in X and $\mathbb{R}$-linear in μ, such that the Leibniz rule

$$D_X(f\mu) = X(f)\mu + fD_X\mu$$

holds. A section $\mu \in \Gamma(E)$ is called (D-)flat if $D_X\mu = 0$ for all $X \in \Theta(M)$.

Example 6.2. 1. On $E = M \times \mathbb{R}^n$ we have the "trivial connection" D^0: We have $\Gamma(E) \cong C^\infty(M)^n$ and set

$$D^0_X(f_1, ..., f_n) := (X(f_1), ..., X(f_n)).$$

2. If $M = F^{-1}(0) \hookrightarrow \mathbb{R}^n$, we may consider the inclusion

$$TM \hookrightarrow T\mathbb{R}^n|_M = M \times \mathbb{R}^n$$

and the fiberwise orthogonal projection

$$\mathrm{pr} : M \times \mathbb{R}^n \longrightarrow TM,$$

i.e.

$$\mathbb{R}^n \cong \{a\} \times \mathbb{R}^n \longrightarrow T_a M$$

is the orthogonal projection onto $\mathrm{pr}_a : T_a M \hookrightarrow \mathbb{R}^n$. Then

$$D_X \mu := \mathrm{pr}(D^0_X(\mu))$$

defines a connection on $E = TM$. Here we regard $\mu \in \Gamma(TM)$ as section of $T\mathbb{R}^n|_M = M \times \mathbb{R}^n$, thus $\mu = (f_1, ..., f_n)$ and

$$D_X \mu = \mathrm{pr}(X(f_1), ..., X(f_n)).$$

Remark 6.3. 1. Let $U \subset M$ be an open subset. If $X|_U = 0$ or $\mu|_U = 0$, we have $D_X \mu|_U = 0$ as well. Given $a \in U$ take $f \in C^\infty(M)$ vanishing near a and $= 1$ in a neighourhood of the support of X resp. μ. We find $X = fX$ and thus

$$D_X \mu = D_{fX} \mu = f D_X \mu = 0$$

near a. On the other hand

$$D_X(\mu) = D_X(f\mu) = (Xf)\mu + f D_X \mu = 0$$

near a.

2. As a consequence a connection on E induces a connection of $E|_U$ for any open subset $U \subset M$.

3. Connections are determined by local data, the so called *Christoffel symbols*: Let $U \subset M$ be an open subset over which both TM and E are trivial:

$$\Theta(U) = \bigoplus_{i=1}^{m} C^\infty(U) X_i, \ \Gamma(E|_U) = \bigoplus_{j=1}^{n} C^\infty(U) \mu_j.$$

Then we may write

$$(D_{X_i}\mu_j)|_U = \sum_{k=1}^{n} \Gamma_{ij}^k \mu_k$$

with functions $\Gamma_{ij}^k \in C^\infty(U)$, the *Christoffel symbols*. We leave it to the reader to establish that connections on $E|_U$ are in one-to-one correspondence with systems of functions $\Gamma_{ij}^k \in C^\infty(U)$, furthermore that if $M = \bigcup_{i\in I} U_i$ is an open cover and D_i are connections on $E|_{U_i}$ that agree over $U_i \cap U_j$, they can be patched together to a connection on E. Details are left to the reader.

4. The value of $D_X\mu$ at $a \in M$ depends only on $X_a \in T_aM$ and the section μ: Given D we may define

$$T_aM \times \Gamma(E) \longrightarrow E_a, (X_a, \mu) \mapsto D_{X_a}\mu,$$

such that

$$(D_X\mu)_a = D_{X_a}\mu.$$

We have to show that $X_a = 0$ implies $(D_X\mu)_a = 0$ and may assume that TM is trivial. If $X = \sum_{i=1}^m f_iX_i$ with a frame $X_1..., X_m$, we have $f_i(a) = 0, i = 1, ..., m$, and obtain

$$(D_X\mu)_a = \sum_{i=1}^{m} f_i(a)(D_{X_i}\mu)_a = 0.$$

In general there are no non-zero flat sections over an open subset $U \subset M$. But over a curve in M there are! We need the following definition:

Definition 6.4. 1. Let $f : Q \longrightarrow M$ be a differentiable map. A section of E above or over f is a differentiable map $\mu : Q \longrightarrow E$ with $\mu_q \in E_{f(q)}$ for all $q \in Q$.

$$\begin{array}{ccc} & & E \\ \mu & \nearrow & \downarrow \\ Q & \xrightarrow{f} & M \end{array}$$

2. We denote $\Gamma_f(E)$ the vector space of all sections of E above f, a $C^\infty(Q)$-module.

3. For $E = TM$ we write $\Theta_f(M) := \Gamma_f(TM)$.

Remark 6.5. If D is a connection on E we may define $D_X\mu$, where $X \in \Theta(Q)$ is a vector field on Q and μ a section of E above f. Assume first there is a frame $\mu_1, ..., \mu_n \in \Gamma(E|_U)$ in an open neighbourhood U of $f(Q)$. Write then $\mu = \sum_{i=1}^n g_i(\mu_i \circ f)$ with functions $g_i \in C^\infty(Q)$ and define

$$D_X\mu = \sum_{i=1}^m Xg_i \cdot (\mu_i \circ f) + g_i(D_{Tf(X)}\mu_i) \circ f$$

with $D_{Tf(X)}\mu_i$ being defined pointwise as in Rem.6.3.4. This definition is independent from the chosen frame and thus can be used locally in order to patch together the definitions on the members of an open cover of $f(Q)$ in the general case. If $f = \gamma$ is a curve $t \mapsto \gamma(t)$ one writes also

$$\nabla_{\dot\gamma}\mu := \nabla_{\frac{d}{dt}}\mu.$$

Definition 6.6. A section $\mu \in \Gamma_f(E)$ over f is called D-flat if $D_X\mu = 0$ holds for all vector fields $X \in \Theta(Q)$.

Proposition 6.7. *Let $\gamma : \mathcal{I} \longrightarrow M$ be a smooth curve with start point $\gamma(0) = a \in M$ and end point $\gamma(1) = b \in M$.*

1. *For any $x \in E_a$ there is a unique flat section $\mu_x \in \Gamma_\gamma(E)$ with $\mu_x(0) = x$.*

2. *The parallel transport from a to b along γ is the map*

$$PT_\gamma : E_a \longrightarrow E_b, x \mapsto \mu_x(1).$$

3. *If $\tilde\gamma = \gamma \circ \tau$ with a reparametrization $\tau : \mathcal{J} \longrightarrow \mathcal{I}$ of γ, we have*

$$PT_{\tilde\gamma} = PT_\gamma.$$

Proof. We may subdivide

$$\mathcal{I} = \mathcal{I}_1 \cup ... \cup \mathcal{I}_r$$

into subintervals $\mathcal{I}_k$, s.th. $\gamma(\mathcal{I}_k)$ is contained in a local chart over which E is trivial and then prove the theorem separately for all the $\mathcal{I}_k, k = 1, ..., r$.

So w.l.o.g. there is a local frame $\mu_1, ..., \mu_n$ of E in a neighbouhood of $\gamma(\mathcal{I})$ and local coordionates $x_1, ..., x_m$. Writing $\mu = \sum_j f_j \mu_j$ with $f_j \in C^\infty(\mathcal{I})$ we want to have

$$0 = D_{\dot\gamma(t)} \left(\sum_{j=1}^n f_j \mu_j \right) = \sum_{j=1}^n \dot f_j \mu_j + f_j D_{\dot\gamma(t)} \mu_j.$$

$$= \sum_{j=1}^n \left(\dot f_j \mu_j + f_j \left(\sum_{i=1}^m \dot\gamma_i(t) D_{\partial_i} \mu_j \right) \right)$$

$$= \sum_{j=1}^n \dot f_j \mu_j + f_j \left(\sum_{i=1}^m \dot\gamma_i(t) \sum_{k=1}^n \Gamma_{ij}^k \mu_k \right)$$

$$= \sum_{j=1}^n \dot f_j \mu_j + \sum_{k=1}^n f_k \left(\sum_{i=1}^m \dot\gamma_i(t) \sum_{j=1}^n \Gamma_{ik}^j \mu_j \right),$$

whence we see that

$$0 = \dot f_j + \sum_{k=1}^n f_k \sum_{i=1}^m \dot\gamma_i(t) \Gamma_{ik}^j$$

should hold for $j = 1, ..., n$. As a linear differential equation it has a unique solution over the entire interval $\mathcal{I}$ for a given initial value. Finally a solution $f_1, ..., f_n$ induces the solution $\tilde f_j := f_j \circ \tau$ with $\tilde\gamma$ instead of γ, since passing from γ to $\tilde\gamma$ means for each term composition with τ and multiplication with $\tau'(s)$. □

Theorem 6.8. *If M is paracompact, there is a connection on E.*

Proof. Write $M = \bigcup_{i \in I} U_i$ with a locally finite cover $(U_i)_{i \in I}$, such that $E|_{U_i}$ is trivial over all $i \in I$. In particular there are connections D^i on $E|_{U_i}$. Choose a partition of unity $(\varphi_i)_{i \in I}$, subordinate to the cover $(U_i)_{i \in I}$. Then define our connection by the locally finite sum

$$D_X \mu := \sum_{i \in I} \varphi_i \cdot D_X^i(\mu),$$

where, by definition, $\varphi_i \cdot D_X^i(\mu) = 0$ outside U_i. □

Remark 6.9. If $D, \tilde D$ are connections, then their difference is a tensor:

$$D - \tilde D \in \mathrm{Hom}(\Theta(M), \mathrm{End}(E)) \cong \Gamma(T^*(M) \otimes \mathrm{End}(E)).$$

Definition 6.10. The curvature of D is the map

$$F_D : \Theta(M)^2 \times \Gamma(E) \longrightarrow \Gamma(E),$$

defined as

$$F_D(X,Y)\mu := D_X D_Y \mu - D_Y D_X \mu - D_{[X,Y]}\mu.$$

Remark 6.11. 1. We may also regard the curvature as a map

$$F_D : \Theta(M)^2 \longrightarrow \mathrm{End}_{C^\infty(M)}(\Gamma(E)).$$

Exercise: Check $C^\infty(M)$-linearity w.r.t. μ!

2. Since F_D is $C^\infty(M)$-linear in X, Y as well, it is a tensor field, can be regarded as follows

$$F_D \in \Gamma((T^{0,2}M) \otimes \mathrm{End}(E)),$$

indeed $F_D \in \Gamma(A^2(T^*M) \otimes \mathrm{End}(E))$, i.e. $F_D(Y,X) = -F_D(X,Y)$.

Theorem 6.12. *The following statements are equivalent:*

1. $F_D = 0$.

2. Every point $a \in M$ has a neighbourhood U, over which there is a frame of flat sections $\mu_1, ..., \mu_n \in \Gamma(E|_U)$.

Proof. "1) $\Longrightarrow$ 2)": We have

$$F_D(X,Y)\mu_i = 0, i = 1, .., n,$$

hence $F_D = 0$.
"1) $\Longrightarrow$ 2)": We may assume $U = (-1,1)^m$ and do induction on $m = \dim M$. Given a flat section $\mu \in \Gamma(E|_{(-1,1)^{n-1}\times\{0\}})$ we may extend it uniquely to a section $\hat{\mu} \in \Gamma(E|_U)$ by defining it for $x = (x', x_n)$ as

$$\hat{\mu}(x', x_n) := PT_{\gamma_x}(\mu(x', 0)),$$

where

$$\gamma_x(t) = (x', tx_n), 0 \le t \le 1.$$

We have to show that

$$D_{X_i}\hat{\mu} = 0, i = 1, ..., n,$$

holds for the coordinate vector fields $X_i = \frac{\partial}{\partial x_i}$. Since $[X_i, X_j] = 0$, we know that

$$D_{X_i} D_{X_j} \hat{\mu} = D_{X_j} D_{X_i} \hat{\mu}.$$

By construction we have

$$D_{X_n} \hat{\mu} = 0.$$

Now we show that

$$D_{X_i} \hat{\mu}(x', x_n), i = 1, ..., n-1,$$

is obtained from

$$D_{X_i} \mu(x', 0) = 0$$

by parallel transport along γ_x, hence it also vanishes. Indeed

$$D_{\dot{\gamma}} D_{X_i} \hat{\mu} = x_n D_{X_n} D_{X_i} \hat{\mu} \circ \gamma = x_n D_{X_i} D_{X_n} \hat{\mu} \circ \gamma = 0.$$

Thus, starting with a flat frame $\mu_1, .., \mu_n \in \Gamma(E|_{(-1,1)^{n-1} \times \{0\}})$ we obtain a flat frame $\hat{\mu}_1, .., \hat{\mu}_n \in \Gamma(E|_U)$ $\square$

Connections may also be understood in a geometric way: The corresponding geometric objects are here called *E-linear horizontal subbundles* of TE. In the literature there is also the term *linear Ehresmann connection*. We start with the definition of horizontal subbundles:

Definition 6.13. Let $V := \ker(T\pi) \subset TE$ (with the bundle projection $\pi : E \longrightarrow M$) be the "vertical subbundle".

1. A horizontal subbundle $H \hookrightarrow TE$ is a subbundle with $TE = V \oplus H$.
2. A curve $\varphi : I \longrightarrow E$ is called (H-)horizontal if $\dot{\varphi}(t) \subset H_{\varphi(t)}$ for all $t \in I$.
3. A section $\sigma : U \longrightarrow E$ is called (H-)horizontal, if $T_a\sigma(T_aM) \subset H_{\sigma(a)}$ for all $a \in U$.

Example 6.14. A horizontal bundle $H \subset TE$ induces a horizontal subbundle $H^{\oplus} \subset T(E \oplus E)$ as follows: For $x, y \in E_a$ we have

$$T_{(x,y)}(E \oplus E) = \{(X_x, Y_y) \in T_xE \oplus T_yE; T\pi(X_x) = T\pi(Y_y)\}$$

and define

$$H^{\oplus}_{(x,y)} := (H_x \oplus H_y) \cap T_{(x,y)}(E \oplus E).$$

Definition 6.15. A horizontal subbundle $H \subset TE$ is called E-linear if the scalar multiplication $\mu_\lambda : E \longrightarrow E$ for all $\lambda \in \mathbb{R}$ and the addition $\alpha : E \oplus E \longrightarrow E$ satisfy

$$T\alpha(H^{\oplus}) \subset H, T\mu_\lambda(H) \subset H.$$

Remark 6.16. Fix a horizontal subbundle $H \subset TE$.

1. Any vector field $X \in \Theta(M)$ has a unique lift to a vector field $\hat{X} \in \Gamma(H) \subset \Theta(E)$.

2. Horizontal subbundles are in one-to-one coorrespondence with right inverses
$$\pi^*(TM) \hookrightarrow TE$$
of
$$TE \xrightarrow{T\pi} \pi^*(TM).$$
In particular, given $a \in M$ and $x \in E_a$ we have a map
$$F_x : T_aM \cong \pi^*(TM)_x \longrightarrow T_xE \twoheadrightarrow E_a.$$
Here the projection $T_xE \twoheadrightarrow E_a$ depends on the choice of a trivialization $E|_U \cong U \times E_a$ on a neighbourhood of $U \ni a$, it is obtained from the projection $U \times E_a \longrightarrow E_a$ and the natural isomorphism $T_x(E_a) \cong E_a$.

3. A horizontal subbundle $H \subset TE$ is E-linear if and only if $E_a \longrightarrow \mathrm{Hom}(T_aM, E_a), x \mapsto F_x$, is a linear map for all $a \in M$.

4. For an E-linear horizontal subbundle $H \subset TE$ a parallel transport may be defined using horizontal liftings of curves: Given $\gamma : \mathcal{I} \longrightarrow M$ and $x \in E_a, a = \gamma(0)$, there exists a unique H-horizontal lifting $\tilde{\gamma}_x : \mathcal{I} \longrightarrow E$ of γ with $\tilde{\gamma}_x(0) = x$. Locally the lifting is the solution of a linear differential equation of first order.

Proposition 6.17. *There is a one-to-one correspondence between connections on a vector bundle E and E-linear horizontal subbundles $H \subset TE$.*

Proof. Let us start with $H \subset TE$. We want to define $D_{X_a}\sigma$ and pick a curve $\gamma : \mathcal{I} \longrightarrow M$ with start point $a \in M$ and $\dot{\gamma}(0) = X_a$. Denote $\tilde{\gamma}_1, ..., \tilde{\gamma}_n$

H-horizontal liftings of γ, such that $\tilde{\gamma}_1(0), ..., \tilde{\gamma}_n(0) \in E_a$ is a basis. Then $\tilde{\gamma}_1(t), ..., \tilde{\gamma}_n(t) \in E_{\gamma(t)}$ is a basis for all $t \in \mathcal{I}$. Now, given $\sigma \in \Gamma(E)$ write

$$\sigma \circ \gamma = \sum_{i=1}^{n} f_i \cdot \tilde{\gamma}_i$$

with functions $f_i : \mathcal{I} \longrightarrow \mathbb{R}$. Now define

$$D_{X_a}\sigma := \sum_{i=1}^{n} \dot{f}_i(0) \cdot \tilde{\gamma}_i(0).$$

On the other hand, given a connection D we are looking for the horizontal subspace $H_x \subset T_xE, x \in E$. Let $a := \pi(x)$. We define a linear injection $F_x : T_aM \hookrightarrow T_xE$ and take $H_x := F_x(T_aM)$. Let $X_a \in T_aM$. Take a curve $\gamma : \mathcal{I} \longrightarrow M$ with start point $a \in M$ and $\dot{\gamma}(0) = X_a$. Then

$$F_x(X_a) := \frac{d\hat{\gamma}_x}{dt}(0).$$

□

Definition 6.18.

1. A metric vector bundle $E \longrightarrow M$ is a vector bundle together with a fibre metric $\sigma \in \Gamma(T^{0,2}(E))$, i.e. σ_a is an inner product on E_a for all $a \in M$.

2. A metric connection on E is a connection D, such that $X\sigma(\mu, \nu) = \sigma(D_X\mu, \nu) + \sigma(\mu.D_X\nu)$. for $\mu, \nu \in \Gamma(E), X \in \Theta(M)$.

Definition 6.19. Let M be a metric bundle over the compact Riemannian manifold M. The map

$$D \mapsto YM(D) := \int_M ||F_D||^2$$

from the set of all metric connections on E to the reals is called the Yang-Mills functional of E.

An application of the Yang-Mills functional is discussed in the next section.

7 Four Manifolds: A Survey

For a four dimensional oriented simply connected compact Riemannian manifold M its (real) intersection form is the bilinear symmetric form

$$H^2_{dR}(M) \times H^2_{dR}(M) \longrightarrow \mathbb{R},$$

$$(\alpha + d\Omega^1(M), \beta + d\Omega^1(M)) \mapsto \int_M \alpha \wedge \beta.$$

Indeed, it may already be defined over the integers: Singular integral cohomology associates to a topological space M a $\mathbb{Z}$-module

$$H^2(M) \hookrightarrow H^2_{dR}(M),$$

the inclusion being nothing but $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$, where $n := b_2(X)$ is the second Betti number of X. In technical terms:

$$H^2_{dR}(M) \cong H^2(M) \otimes_{\mathbb{Z}} \mathbb{R}.$$

There is as well an integral intersection form

$$\sigma_M : H^2(M) \times H^2(M) \longrightarrow \mathbb{Z},$$

a unimodular symmetric bilinear form, the restriction of the dR-bilinear form, which thus in particular maps $\mathbb{Z}^n \times \mathbb{Z}^n \subset \mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{Z} \subset \mathbb{R}$. Fixing an isomorphism

$$H^2(M) \cong \mathbb{Z}^n,$$

we may write

$$\sigma_M(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T A \mathbf{v}$$

with a symmetric matrix $A \in GL_n(\mathbb{Z})$. In that case we write

$$\sigma_M \cong A.$$

In particular

$$A' \cong \sigma_M \cong A \Longleftrightarrow A' = S^T A S, S \in GL_n(\mathbb{Z}).$$

The classification of nondegenerate symmetric bilinear forms on $\mathbb{R}^n$ is given by $I_p \oplus -I_q$ with $p + q = n$. Here I_r denotes the unit matrix of size $r \times r$. But over the integers there are more refined invariants:

Definition 7.1. A symmetric bilinear form $\sigma : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$ is called

1. even if $\sigma(\mathbf{u}, \mathbf{u}) \in 2\mathbb{Z}$ holds for all $\mathbf{u} \in \mathbb{Z}^n$, or, equivalently, the diagonal entries of A are even,

2. odd otherwise.

Example 7.2. 1. $H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is even, while $I_1 \oplus -I_1$ is not. Both forms are indefinite and equivalent over the reals; both matrices satisfy $A \cong -A$.

2. $E_8 \in GL_8(\mathbb{Z})$, the Cartan matrix of the exceptional Lie algebra E_8, is even and definite.

3. Any odd indefinite form is diagonalizable, i.e. of the form $I_p \oplus -I_q$ with $p, q > 0$.

4. Any even indefinite form is of the type $pE_8 \oplus qH$ with unique $p \in \mathbb{Z}, q > 0$. The representation as sum of indecomposable forms is nevertheless not unique: Any indecomposable even positive definite form A gives rise to an indefinite form $A \oplus H$.

5. For definite A the decomposition into a direct sum of indecomposable forms is unique.

6. For given rank n there are only finitely many indecomposable forms.

Theorem 7.3 (Freedman)**.** *Every unimodular form σ on $\mathbb{Z}^n$ is realized as σ_M by some compact simply connected oriented topological four manifold M. Indeed*

1. *If σ_M is even, then M is determined up to homeomorphy by the isomorphy type of σ_M.*

2. *if σ_M is odd, then there are two non-homeomorphic possible M. For one of them, $M \times \mathbb{S}^1$ admits a differentiable structure, for the other one it does not.*

In particular the topological type of a differentiable manifold as above is uniquely determined by its intersection form σ_M.

So there are a lot of non-smoothable four manifolds. Th.7.6 gives further severe restriction for smoothability. For the sake of completeness we mention:

Theorem 7.4 (Quinn)**.** *Any non-compact four manifold admits a differentiable structure.*

Example 7.5. Here are the most basic building blocks for four manifolds:

1. $\mathbb{S}^4$ realizes the zero form.

2. $\mathbb{P}^2(\mathbb{C})$ realizes $\pm I_1$ depending on the choice of orientation.

3. $\mathbb{S}^2 \times \mathbb{S}^2$ realizes H.

4. We have
$$\sigma_M \oplus \sigma_N \cong \sigma_{M\#N}$$
with the connected sum $M\#N$ of M and N. Take closed disks $\overline{\mathbb{B}}^4(M) \subset M, \overline{\mathbb{B}}^4(N) \subset N$ with boundary spheres $\mathbb{S}(M), \mathbb{S}(N)$. Now glue as follows:
$$M \setminus \mathbb{B}^4(M) \supset \mathbb{S}(M) \xrightarrow{f} \mathbb{S}(N) \subset N \setminus \mathbb{B}^4(N),$$
where $f : \mathbb{S}(M) \longrightarrow \mathbb{S}(N)$ is an orientation reversing homeomorphism. That operation makes also sense in the category of differentiable manifolds (with an an orientation reversing diffeomorphism).

5. The Kummer surface (a complex(!) surface, hence a four manifold)
$$K := \{[z_0, .., z_3] \in \mathbb{P}_3(\mathbb{C}); z_0^4 + ... + z_3^4 = 0\}$$
has $\sigma_K \cong 2E_8 \oplus 3H$.

Theorem 7.6 (Donaldson)**.** *For a compact oriented simply connected differentiable four manifold M with definite intersection form σ_M we have*
$$\sigma_M \cong \pm I_n$$
with $n = b_2(M)$.

Theorem 7.7. *There is a differentiable structure on $\mathbb{R}^4$, such that there is no smooth embedding $j : \mathbb{S}^3 \hookrightarrow \mathbb{R}^4$, s.th. $\overline{\mathbb{B}}^4$ lies in the bounded component of $\mathbb{R}^4 \setminus j(\mathbb{S}^3)$.*

Proof. Write

$$K = M \# 3(\mathbb{S}^2 \times \mathbb{S}^2)$$

Then M does not admit a differentiable structure according to Th.7.6. The attempt to amputate $3(\mathbb{S}^2 \times \mathbb{S}^2)$ from K leads to an "exotic four plane". □

Sketch of the proof of Th.7.6. Let σ_M be definite. One considers a suitable metric rank 4 vector bundle $E \longrightarrow M$. We consider the space of metric connections

$$MC(E) = D + \Gamma(T^*(M) \otimes \mathrm{Ad}(E)).$$

Indeed, if $\tilde{D}, D$ are metric connections we have $\tilde{D} - D \in \Gamma(T^*(M) \otimes \mathrm{Ad}(E))$, where $\mathrm{Ad}(E) \subset \mathrm{End}(E)$ is the bundle of skew symmetric endomorphisms.

Definition 7.8. The group

$$\mathrm{Aut}(E) := \{f : E \longrightarrow E; \forall a \in M : f(E_a) \subset E_a, f|_{E_a} \text{ isometry}\}$$

is called the gauge group of E.

Remark 7.9.

1. $\mathrm{Aut}(E)$ acts on $MC(E)$ by conjugation:

$$f_*(D)_X(\mu) := f(D_X(f^{-1} \circ \mu)).$$

2. $F_{f_*(D)}(X, Y)\mu = f \circ F_D(X, Y) \circ (f^{-1} \circ \mu)$
3. $YM(f_*(D)) = YM(D)$.
4. The space $SD(E) \subset MC(E)$ of "self dual connections" (instantons) is contained in the set of critical points of YM, it is $\mathrm{Aut}(E)$-invariant.

One considers the "moduli space"

$$\mathfrak{M} := SD(E)/\mathrm{Aut}(E).$$

It is a topological space with the following properties:

1. There are points $a_1, ..., a_n$ with $n := b_2(M)$, s.th. the puncture $\mathfrak{M}^* := \mathfrak{M} \setminus \{a_1, ..., a_n\}$ is a five dimensional oriented manifold.

2. There are mutually disjoint open neighbourhoods U_i of the singular points a_i homeomorphic to a cone

$$U_i \cong C(\mathbb{P}_2(\mathbb{C})), i = 1, ..., b_2(M)$$

over the complex projective plane $\mathbb{P}_2(\mathbb{C})$. Here

$$C(X) := X \times [0,1)/\sim$$

with $(x,0) \sim (y,0)$ for all $x, y \in X$.

3. $\mathfrak{M}$ has one end diffeomorphic to $M \times \mathbb{R}$, i.e. there is a compact set $K \subset \mathfrak{M}$, s.th.
$$\mathfrak{M} \setminus K \cong M \times \mathbb{R}.$$

Then we may produce from $\mathfrak{M}$ a five dimensional manifold N with boundary

$$\partial N \cong M \cup \coprod_n \mathbb{P}_2(\mathbb{C}).$$

That implies

$$\sigma_M \cong \sigma_{\coprod_n \mathbb{P}_2(\mathbb{C})} \cong I_n.$$

□

8 Connections on TM

Denote

$$\nabla : \Theta(M) \times \Theta(M) \longrightarrow \Theta(M)$$

a connection on the tangent bundle of the differentiable manifold M.

Definition 8.1. The torsion tensor field $T_\nabla \in \Gamma(T^{1,2}M)$ of the connection ∇ is defined by

$$T_\nabla(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y].$$

We call ∇ torsion free if $T_\nabla = 0$.

Remark 8.2. If ∇ has Christoffel symbols Γ_{ij}^k with respect to the local coordinates $x_1, ..., x_m$, we have

$$T_\nabla = \sum_{i,j} (\Gamma_{ij}^k - \Gamma_{ji}^k)\, \partial_k \otimes dx_i \otimes dx_j.$$

Hence ∇ is torsion free if $\Gamma_{ji}^k = \Gamma_{ij}^k$ holds for $k = 1, ..., m$.

Theorem 8.3. *Let M be a pseudo-Riemannian manifold. Then there is a unique torsion free "metric" connection ∇ on TM, called the Levi-Civita-connection.*

Example 8.4.

1. $M = \mathbb{R}^m$ with $g = \sum_i dx_i \otimes dx_i$ has Levi-Civita connection with $\Gamma_{ij}^k = 0$ for all i, j, k. With other words

   $$\nabla_{\partial_i} \partial_j = 0, i, j = 0, ..., m$$

   resp.

   $$\nabla_X (\sum_{k=1}^m f_k \partial_k) = \sum_{k=1}^m X(f_k) \partial_k.$$

2. Let $M = F^{-1}(0) \hookrightarrow W \subset \mathbb{R}^n$ with $F : W \longrightarrow \mathbb{R}^{n-m}$. Denote $P : M \times \mathbb{R}^n \longrightarrow TM$ the orthogonal projection. Then $\nabla_X^M Y = P \circ \nabla_X Y$ is the Levi-Civita connection of M.

Proof. We show that on every coordinate patch U with coordinates $x_1, ..., x_m$ the Christoffel symbols are uniquely determined; hence the corresponding connections can be glued to a global connection.

Denote $g = \sum_{i,j} g_{ij} dx_i \otimes dx_j$ the metric tensor, $X_i := \partial_i$. Consider a triple in $\{1, .., m\}^3$. We want

$$X_k g(X_i, X_j) = g(D_{X_k} X_i, X_j) + g(X_i, D_{X_k} X_j)$$

and obtain three equations permuting for a given triple the indices. On the other hand

$$X_i g(X_j, X_k) - X_k g(X_i, X_j) + X_j g(X_i, X_k) = 2g(D_{X_i} X_j, X_k)$$

has to hold for a metric torsion free connection (use $\nabla_{X_i} X_j = \nabla_{X_j} X_i$). Indeed the two systems of three equations turn out to be equivalent for a torsion free connection ∇. Since

$$g(D_{X_i} X_j, X_k) = \sum_\ell \Gamma_{ij}^\ell \cdot g_{\ell k},$$

we obtain with the matrix $G = (g_{\ell k})$ the following equality

$$(g(D_{X_i}X_j, X_1), ..., g(D_{X_i}X_j, X_m) = (\Gamma^1_{ij}, ..., \Gamma^m_{ij}) \cdot G$$

respectively

$$(\Gamma^1_{ij}, ..., \Gamma^m_{ij}) = (g(D_{X_i}X_j, X_1), ..., g(D_{X_i}X_j, X_m)) \cdot G^{-1}.$$

Combining that with our previous result we arrive at an explicit formula for the Christoffel symbols. □

Definition 8.5. A smooth curve $\gamma : \mathcal{I} \longrightarrow M$ in a pseudo-Riemannian manifold M is called a geodesic, if its tangent field $\dot\gamma : \mathcal{I} \longrightarrow TM$ is (∇-)flat:

$$\nabla_{\dot\gamma}\dot\gamma = 0.$$

Proposition 8.6. *Given $a \in M$ and $X_a \in T_a(M)$ there is a unique geodesic $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ with $\gamma(0) = a, \dot\gamma(0) = X_a$.*

Proof. Assume that $x_1, ..., x_m$ are local coordinates on $U \subset M$. If then $\gamma = (\gamma_1, ..., \gamma_m)$ is a smooth path, and $\Gamma^k_{ij} \in C^\infty(M)$ the Christoffel symbols w.r.t. the coordinate vector fields $X_1, ..., X_m$, the condition for γ to be a geodesic is the following non-linear system of differential equations

$$\ddot\gamma_k + \sum_{i,j} \Gamma^k_{ij}\dot\gamma_i\dot\gamma_j, k = 1, ..., m.$$

Now apply the fundamental theorem of the theory of ODE. □

Remark 8.7.

1. If γ is the geodesic as in Prop.8.6, then $t \mapsto \gamma(\lambda t)$ is the geodesic starting at a with tangent vector λX_a.

2. Taking $\lambda = 0$ in the previous point we see that $\gamma \equiv a$ is a geodesic. Using that together with the above uniqueness result we obtain that a geodesic γ either is constant or $\dot\gamma(t) \neq 0$ for all t.

3. For a geodesic γ we have $g(\dot\gamma, \dot\gamma) \equiv c \in \mathbb{R}$.

In the theory of ODEs one reduces the order of a system of differential equations by introducing additional variables. Here is a geometric version of that process applied to the second order system for geodesics.

Definition 8.8. Given a smooth path γ in M denote $\dot{\gamma} : \mathcal{I} \longrightarrow TM$ and $\ddot{\gamma} : \mathcal{I} \longrightarrow TTM$ the first and second derivative of γ. The geodesic flow $Z \in \Theta(TM)$ is the vector field such that

$$Z_{X_a} := \ddot{\gamma}_{X_a}(0),$$

where $\gamma = \gamma_{X_a} : (-\varepsilon, \varepsilon) \longrightarrow M$ is the geodesic with $\gamma(0) = a, \dot{\gamma}(0) = X_a$.

Remark 8.9.

1. If $(x_1, ..., x_m, y_1, ..., y_m)$ are local coordinates on $\pi^{-1}(U) \subset TM$,, such that

   $$(x_1, ..., x_m, y_1, ..., y_m) \mapsto (x_1, ..., x_m, \sum_{i=1}^{m} y_i \partial_i^{\mathbf{x}})$$

   with $\mathbf{x} = (x_1, ..., x_m)$, we find

   $$Z = \sum_{k=1}^{n} \left(y_k \frac{\partial}{\partial x_k} - \left(\sum_{i,j} y_i y_j \Gamma_{ij}^k \right) \frac{\partial}{\partial y_k} \right),$$

   in particular $Z_{0_a} = 0$ holds for $0_a \in T_a M$.

2. Denote $\pi : TM \longrightarrow M$ the projection. For an integral curve $\sigma : \mathcal{I} \longrightarrow TM$ of Z through X_a the path $\gamma := \pi \circ \sigma : \mathcal{I} \longrightarrow M$ is the geodesic through a with tangent vector X_a.

For the next definition remember that $\mathbb{D}(Z) \subset TM \times \mathbb{R}$ consists of those points (X_a, t), such that Z has an integral curve defined on $[0, t]$ starting at X_a, see also Th.3.17.

Definition 8.10. Let

$$U := \{X_a \in TM; (X_a, 1) \in \mathbb{D}(Z) \subset TM \times \mathbb{R}\},$$

an open neighbourhood of the zero section in TM. The map

$$\exp : U \longrightarrow M, X_a \mapsto \gamma_{X_a}(1)$$

is called the exponential map for the pseudo-Riemannian manifold M. Here γ_{X_a} denotes the geodesic starting at a with tangent vector X_a. Set

$$\exp_a := \exp|_{U_a} : U_a \longrightarrow M.$$

with $U_a := T_a M \cap U$.

Lemma 8.11. *We have* $\exp(tX_a) = \gamma_{X_a}(t)$ *and*

$$T_0(\exp_a) = \mathrm{id}_{T_aM}$$

holds for the differential of the exponential

$$T_0(\exp_a) : T_0(U_a) = T_0(T_aM) \cong T_aM \longrightarrow T_aM$$

at $0 = 0_a \in T_aM$. *In particular* $\exp_a$ *induces a diffeomorphism from a neighbourhood ot* $0_a \in T_aM$ *onto a neighbourhood of* $a \in M$.

The next statement deals with the differential of the exponential outside the origin:

Proposition 8.12. *Let* $X_a \in U_a \subset T_aM$ *and* $\gamma(t) = \exp_a(tX_a)$ *be the geodesic with* $\dot\gamma(0) = X_a$. *Then the differential of the exponential map* $\exp_a : U_a \longrightarrow M$ *at* X_a*, the map*

$$T_{X_a}\exp_a : T_aM \cong T_{X_a}(T_aM) \longrightarrow T_{\gamma(1)}M$$

1. *induces an isometry* $\mathbb{R}X_a \longrightarrow \mathbb{R}\dot\gamma(1)$ *and*

2. *preserves the orthogonal complements:*

$$T_{X_a}\exp_a(X_a^\perp) \subset \dot\gamma(1)^\perp,$$

a statement also known as Gauß' lemma.

Proof. The first part of the statement follows from the fact that $||\dot\gamma(t)|| = ||X_a||$ for all t, in particular $||\dot\gamma(1)|| = ||X_a||$. For the second part take a vector $Y_a \perp X_a$ with $||Y_a|| = ||X_a||$. Now choose $\varepsilon > 0$, such that the map

$$f : \mathcal{I} \times (-\varepsilon, \varepsilon) \longrightarrow M, (t, w) \mapsto \exp_a(t(\cos(w)X_a + \sin(w)Y_a)),$$

is defined. Obviously all paths $t \mapsto f(t, w)$ are geodesics with tangent vectors of length $||X_a||$. We want to apply the below lemma with $T := Tf(\partial_t), W = Tf(\partial_w) \in \Theta_f(M)$. So we have to compute $g(W, T)$. We obtain

$$W_{(t,w)} = T\exp_a(t(-\sin(w)X_a + \cos(w)Y_a)).$$

In particular

$$W_{(t,0)} = T\exp_a(tY_a))$$

and thus, according to Rem. 8.13,

$$g(W_{(t,0)}, \dot\gamma(t)) = g(W_{(0,0)}, \dot\gamma(0)) = g(0, \dot\gamma(0)) = 0.$$

With $t = 1$ we obtain $T_{X_a}\exp_a(Y_a) \perp \dot\gamma(1)$. □

Lemma 8.13. *Assume $f : \mathcal{I} \times (-\varepsilon, \varepsilon) \longrightarrow M$ is a variation of the geodesic $\gamma(t) := f(t, 0)$, i.e. all curves $t \mapsto f(t, w)$ are geodesics. If in addition they have tangent vectors of the same length, then for $T := Tf(\partial_t), W := Tf(\partial_w) \in \Theta_f(M)$ the inner product $g(W, T) : \mathcal{I} \times (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ does not depend on t.*

Proof. The assumption yields

$$0 = \frac{\partial}{\partial w} g(T, T) = 2g(\nabla_{\partial_w} T, T),$$

whence

$$\frac{\partial}{\partial t} g(W, T) = g(\nabla_{\partial_t} W, T) + g(W, \nabla_{\partial_t} T) = g(\nabla_{\partial_w} T, T) = 0,$$

the connection ∇ being torsion free and $\nabla_{\partial_t} T = 0$. □

Theorem 8.14. *Let M be a pseudo-Riemannian manifold with metric tensor g, Levi-Civita connection ∇ and curvature $R = F_\nabla$. Then the following statements are equivalent:*

1. $R = 0$
2. *Every point $a \in M$ has a neighbourhood isomorphic to an open subset of $\mathbb{R}^m$ endowed with the standard metric $\sum_{i=1}^m \varepsilon_i dx_i \otimes dx_i$, where $\varepsilon_i = \pm 1$.*

Proof. According to Th. 6.12 there is a connected neighbourhood $U \ni a$ with flat vector fields $X_1, ..., X_m \in \Theta(U)$. We have

$$X_i g(X_j, X_k) = g(\nabla_{X_i} X_j, X_k) + g(X_j, \nabla_{X_i} X_k) = 0,$$

hence $g(X_i, X_j) \equiv g_{ij} \in \mathbb{R}$. On the other hand

$$[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0;$$

thus by Frobenius there are, after a shrinking of U at least, local coordinates $x_1, ..., x_m$ on U with $X_i = \frac{\partial}{\partial x_i}$. With other words

$$g|_U = \sum_{i,,j} g_{ij} dx_i \otimes dx_j.$$

Finally diagonalize. □

9 Length and distance

Definition 9.1. Given a Riemannian manifold M we define the metric $d = d_M$ by

$$d(p,q) = \inf\{L(\gamma); \gamma \text{ a broken smooth curve from } p \text{ to } q\}.$$

Proposition 9.2. *The manifold topology of M coincides with the topology of the metric space (M, d_M).*

Proof. Fix a point $p \in M$ and local coordinates

$$U \xrightarrow{\cong} \mathbb{B}_2 := \{\mathbf{x} \in \mathbb{R}^m; ||\mathbf{x}|| < 2\}, p \mapsto 0.$$

Here $||..||$ denotes the euclidean norm on $\mathbb{R}^n$. Consider

$$\hat{g} : \mathbb{B}_2 \times \mathbb{R}^m \longrightarrow \mathbb{R},$$

$$\hat{g}(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i,j} g_{ij}(\mathbf{x}) y_i y_j}$$

and

$$R := \sup\{\hat{g}(\mathbf{x}, \mathbf{y}); ||\mathbf{x}|| \leq 1, ||\mathbf{y}|| = 1\} < \infty,$$

and

$$r := \inf\{\hat{g}(\mathbf{x}, \mathbf{y}); ||\mathbf{x}|| \leq 1, ||\mathbf{y}|| = 1\} > 0.$$

We show for $\mathbf{x} \in \overline{\mathbb{B}}_1$ the estimates:

$$r \cdot ||\mathbf{x}|| \leq d(0, \mathbf{x}) \leq R \cdot ||\mathbf{x}||.$$

First of all looking at the path $t \mapsto t\mathbf{x}, 0 \leq t \leq 1$, we obtain

$$d(0, \mathbf{x}) \leq \int_0^1 \hat{g}(t\mathbf{x}, \mathbf{x}) dt \leq R \int_0^1 ||\mathbf{x}|| dt = R||\mathbf{x}||.$$

Second, take any path $\gamma : \mathcal{I} \longrightarrow M$ from 0 to $\mathbf{x}$. Denote $c \in \mathcal{I}$ the first point with $||\gamma(c)|| = ||\mathbf{x}||$. Then

$$L(\gamma) \geq L(\gamma|_{\mathcal{I}_0}) = \int_0^c \hat{g}(\gamma(t), \dot{\gamma}(t)) dt \geq r \int_0^c ||\dot{\gamma}(t)|| dt \geq$$

$$r \cdot || \int_0^c \dot{\gamma}(t) dt|| = r||\gamma(c)|| = r||\mathbf{x}||.$$

Since that holds for all γ, we are done. □

Theorem 9.3. *Assume that*

$$T_aM \supset B_\varepsilon(0_a) \xrightarrow{\exp_a} U \subset M$$

is a diffeomorphism. Then

1. $U = B_\varepsilon(a)$,
2. *for* $b = \exp_a(X_a), X_a \in B_\varepsilon(0_a)$, *we have* $d(a,b) = ||X_a||$, *and*
3. *any minimizing path from* a *to* b *has support* $\{\exp_a(tX_a), 0 \leq t \leq 1\}$.

Proof. Let $R := ||X_a||$. We show $L(\sigma) \geq R$ for any broken C^∞-path σ from a to b, furthermore that equality implies that σ is a reparametrization of the geodesic $\gamma_{X_a}|_{[0,1]}$.

1. We may assume $\sigma^{-1}(a) = \{0\}$ and then find a "subpath" $\sigma_0 = \exp \circ \tau$, where $\tau : \mathcal{I} = [0,c] \longrightarrow B_\varepsilon(0_a)$ with $\tau(0) = 0_a, ||\tau(c)|| = R$ and $0 < ||\tau(t)|| \leq R$.
2. We show $L(\sigma_0) \geq R$. We may assume
$$\tau(t) = r(t)\varphi(t)$$
with $r(t) > 0$ for $t > 0$ and $\varphi : \mathcal{I} \setminus \{0\} \longrightarrow \mathbb{S}(T_aM)$, i.e. one starts travelling at $t = 0$ and never returns to the origin $0_a \in T_aM$. For $\sigma_0 = \exp_a \circ \tau$ we have
$$\dot\sigma_0(t) = T_{\tau(t)} \exp_a(\dot r(t)\varphi(t) + r(t)\dot\varphi(t)),$$
where we use the isomorphism $T_{\tau(t)}T_aM \cong T_aM$. Since $\varphi(t) \perp \dot\varphi(t)$, we find with Gauß' lemma (Prop. 8.12) the estimate
$$||\dot\sigma_0(t)||^2 = ||T_{\tau(t)} \exp_a(\dot r(t)\varphi(t))||^2 + ||T_{\tau(t)} \exp_a(r(t)\dot\varphi(t))||^2 \geq \dot r(t)^2.$$
Indeed, if $\dot\varphi(t) \neq 0$ somewhere we obtain a strict inequality over some open subinterval $\mathcal{I}_0 \subset \mathcal{I}$, the map $T_{\tau(t)} \exp_a$ being an isomorphism, and thus
$$\int_0^c ||\dot\sigma_0(t)||dt > \int_0^c |\dot r(t)|dt \geq \int_0^R dr = R.$$
3. We thus know that $L(\sigma_0) = R$ implies $\dot\varphi \equiv 0$.

4. If σ_0 is broken, we may apply the above estimates to its smooth pieces. In particular we see, that for $L(\sigma_0) = R$ no breaks can occur, since that would mean that $\dot{\varphi} \not\equiv 0$ on some of its smooth pieces.

5. If $L(\sigma) = R$, it follows $\sigma = \sigma_0$ and because of $\dot{\varphi} \equiv 0$, we have

$$\tau(t) = r(t) R^{-1} X_a.$$

$\square$

Corollary 9.4. *Let γ be a piecewise smooth path from $a \in M$ to $b \in M$. Then if $L(\gamma) = d(a, b)$, it is a reparametrization of a geodesic, i.e. it is the composition of a nondecreasing surjective function $\mathcal{I} \longrightarrow \mathcal{J}$ and a geodesic $\mathcal{J} \longrightarrow M$.*

Proof. First of all any subpath of γ is minimizing as well. Pick a point $a \in |\gamma|$. Using $\exp_a : B_\varepsilon(0_a) \xrightarrow{\cong} B_\varepsilon(a)$ as in Prop. 9.3 we see that γ is immediately before and after a a reparametrized geodesic. It remains to exclude a break at a. For that we choose $\varepsilon > 0$, s.th. $\exp_c$ defines a diffeomorphism $B_\varepsilon(0_c) \longrightarrow B_\varepsilon(c)$ for all $c \in |\gamma|$. Take $c \in |\gamma|, c \neq a$, with $a \in B_\varepsilon(c)$ and apply Prop. 9.3 to $\exp_c : B_\varepsilon(0_c) \xrightarrow{\cong} B_\varepsilon(c)$ $\square$

10 Completeness

Theorem 10.1 (Hopf/Rinow)**.** *For a connected Riemannian manifold M denote $U \subset TM$ the domain of definition for the exponential* $\exp : U \longrightarrow M$. *Then the following statements are equivalent:*

1. *There is a point $a \in M$, such that $T_a M \subset U$.*

2. *The exponential is defined everywhere, i.e. $U = TM$.*

3. *(M, d) is a complete metric space.*

4. *The closed d-balls*

$$\overline{B}_r(a) := \{b \in M; d(a, b) \leq r\} \subset M$$

are compact.

Furthermore if M is complete, i.e. if one (or all) of the above conditions are satisafied, then for any two points $a, b \in M$ there is a (not necessarily unique) geodesic γ between a and b with $L(\gamma) = d(a.b)$.

Remark 10.2. Note that the exponential $\exp_a : T_aM \longrightarrow M$ for complete M in general neither is injective nor distance preserving; we only know

$$d(a, \exp_a(X_a)) \le ||X_a||$$

with equality for sufficiently short tangent vectors X_a.

Proof. "4) $\Longrightarrow$ 3)": Any d-Cauchy sequence is d-bounded, hence contained in a compact subset, thus has points of accumulation, indeed, exactly one, its limit.
"3) $\Longrightarrow$ 2)": Assume $\gamma : \mathcal{I} = [0, c) \longrightarrow M$ is a geodesic. Then if $c < \infty$ the completeness of M implies that the limit $\lim_{t \to c} \gamma(t) = b$ exists, and $K := \gamma(\mathcal{I}) \cup \{b\}$ is a compact set. Thus there is some $\varepsilon > 0$, s.th.

$$\{X_a \in TM; a \in K, ||X_a|| < \varepsilon\} \subset U.$$

It follows that γ can be extended to the interval $[0, d+\varepsilon)$ for all $d < c$, hence to $[0, c+\varepsilon)$. Thus, after all, γ can be extended to $[0, \infty)$.
"2) $\Longrightarrow$ 1)": Clear.
"1) $\Longrightarrow$ 4)": If we show that

$$E_r(a) := \exp_a(\overline{B}_r(0_a)) = \overline{B}_r(a)$$

holds for all $r > 0$, we are done: In that case, any $\overline{B}_r(a)$ is compact being the continuous image of a compact set. Furthermore, if $d(a, b) = r$, we have $b = \exp_a(X_a)$ with $||X_a|| = r$ and $t \mapsto \exp_a(tX_a), 0 \le t \le 1$, is a geodesic of minimal length joining a and b.

Now, for small $r > 0$ the equality $E_r(a) = \overline{B}_r(a)$ follows from Prop.9.3. We show then, that the set of all $r > 0$ with $E_\varrho(a) = \overline{B}_\varrho(a)$ for $\varrho \le r$ is both open and closed in $\mathbb{R}_{>0}$. In any case it is an interval.

1. It is closed: Assume that we have $E_\varrho(a) = \overline{B}_\varrho(a)$ for $\varrho < r$. Let $d(a, b) = r$. We claim that there is a sequence $b_n \to b$ with $d(a, b_n) < r$ for all n. If so, write $b_n = \exp_a(X_a(n))$. Since $||X_a(n)|| \le r$ for all n, we find a convergent subsequence resp. may assume that $X_a(n) \to X_a$. Then we have $b = \exp_a(X_a)$. To find the sequence b_n, choose paths γ_n

from a to b with $L(\gamma_n) \leq r + \frac{1}{n}$. Indeed, we take points $b_n \in |\gamma_n|$ with $r > d(a.b_n) > r - \frac{1}{n}$ and decompose γ_n into a path α_n from a to b_n and a second one, β_n, from b_n to b. Then we have $d(b_n, b) \leq L(\beta_n) \to 0$ because of

$$L(\alpha_n) + L(\beta_n) = L(\gamma_n) \to r$$

and $L(\alpha_n) \to r$ as well.

2. It is open: Assume $E_r(a) = \overline{B}_r(a)$. We show that $E_{r+\delta}(a) = \overline{B}_{r+\delta}(a)$ for sufficiently small $\delta > 0$. Choose $\varepsilon > 0$, such that for every point $c \in \mathbb{S}_r(a)$ (the set of all points in M at distance r from a, a closed subset of the compact set $\overline{B}_r(a)$, hence compact as well) the exponential map $\exp_c$ defines a diffeomorphism $B_\varepsilon(0_c) \longrightarrow B_\varepsilon(a)$. Let $\delta \leq \varepsilon$ and take a point $b \in \overline{B}_{r+\delta}(a)$. Choose $c \in \mathbb{S}_r(a)$ with minimal $d(b, c)$. We claim $d(a, b) = d(a, c) + d(c, b)$. We have to show $L(\gamma) \geq d(a, c) + d(c, b)$ for any path γ from a to b. We may cut γ into two subpaths at some point in $|\gamma| \cap \mathbb{S}_r(a)$. The first has length at least $r = d(a, c)$, the second one at least $d(c, b)$ because of the choice of c.

   Now write $c = \exp_a(X_a)$ and $b = \exp_c(Y_c)$. If our equality holds, then the geodesics $t \mapsto \exp_a(tX_a), 0 \leq t \leq 1$, and $t \mapsto \exp_c(tY_b), 0 \leq t \leq 1$, form together a minimizing path, hence, according to Cor 9.4, there is no break at c and they form a geodesic from a to c of length $d(a, b)$. Hence $b = \exp_a((r + d(c, b))X_a) \in E_{r+\delta}(a)$.

□

11 Jacobi fields

Remark 11.1. Let $f : Q \longrightarrow M$ be a differentiable map. Here M is a Riemannian manifold with Levi-Civita connection ∇, metric tensor g and curvature $R = R_\nabla$. If $X, Y \in \Theta_f(Q), W \in \Theta(Q)$, then

$$Wg(X, Y) = g(\nabla_W X, Y) + g(X, \nabla_W Y).$$

Definition 11.2. Let $\gamma : \mathcal{I} \longrightarrow M$ be a geodesic. A vector field $W : \mathcal{I} \longrightarrow TM$ above γ is called a Jacobi field, if

$$\nabla_{\dot\gamma}^2 W := \nabla_{\frac{d}{dt}}^2 W = R(\dot\gamma, W)\dot\gamma.$$

Proposition 11.3. *Let $\gamma : \mathcal{I} \longrightarrow M$ be a geodesic. Given a point $t_0 \in \mathcal{I}$ and tangent vectors $X_a, Y_a \in T_aM$ at $a = \gamma(t_0)$, there is a unique Jacobi field $W : \mathcal{I} \longrightarrow TM$ along γ with $W_{t_0} = X_a, \nabla_{\dot\gamma(t_0)}W = Y_a$.*

Proof. Denote $T_1, ..., T_m$ a basis of γ-parallel vector fields along γ. Then a vector field $W = \sum_{i=1}^m \varphi_i T_i$ with functions $\varphi_i \in C^\infty(\mathcal{I})$ is a Jacobi field iff

$$\nabla_{\dot\gamma}^2 W = \sum_{i=1}^m \ddot\varphi_i T_i = \sum_{i=1}^m \varphi_i R(\dot\gamma, T_i)\dot\gamma.$$

Now we may write

$$R(\dot\gamma, T_i)\dot\gamma = \sum_{j=1}^m a_{ji} T_j$$

with functions $a_{ji} \in C^\infty(\mathcal{I})$, so our equation is equivalent to the equation

$$\ddot\varphi = A\varphi$$

with $\varphi := \begin{pmatrix} \varphi_1 \\ \cdot \\ \cdot \\ \varphi_m \end{pmatrix}$ and the matrix $A = (a_{ji}) \in (C^\infty\mathcal{I})^{m,m}$. Now the solution theory for linear differential equations gives the result. □

Example 11.4.

1. A tangential vector field $W = f\dot\gamma, f \in C^\infty(\mathcal{I})$, is a Jacobi field, if and only if $\ddot f = 0$, i.e. f is an affine linear function: $f(t) = at + b$.

2. On $\mathbb{R}^n$ parallel vector fields are constant vector fields and Jacobi fields are of the form $tV + W$ with constant vector fields V, W.

3. On $\mathbb{S}^n$ geodesics γ are segments of intersections $\mathbb{S}^n \cap H$ with a two dimensional subspace $H \subset \mathbb{R}^{n+1}$, and the vector space of parallel vector fields is
$$\mathbb{R}\dot\gamma \oplus H^\perp,$$
where the elements in $H^\perp$ are regarded as restrictions to $\mathbb{S}^n$ of constant vector fields on $\mathbb{R}^{n+1}$. Futhermore
$$R(X, Y)Z = g(Y, Z)X - g(Z, X)Y,$$

see problem 7.3 a) with $L = \mathrm{id}_{T\mathbb{S}^n}$. So the equation for a Jacobi field W is

$$\nabla^2_{\frac{d}{dt}} W = g(W, \dot\gamma)\dot\gamma - g(\dot\gamma, \dot\gamma)W.$$

Now writing $W = f_1\dot\gamma + \sum_{i=2}^n f_i Y_i$ with a basis $Y_2, ..., Y_n$ of $H^\perp$ we obtain

$$\ddot f_1 \dot\gamma + \sum_{i=2}^n \ddot f_i Y_i = f_1\dot\gamma - f_1\dot\gamma - \sum_{i=2}^n f_i Y_i,$$

where we assume γ to be parametrized by arc length. Thus

$$\ddot f_1 = 0, \ddot f_i = -f_i, i = 2, ..., n,$$

and $f_1(t) = \lambda_1 t + \mu_1$, while $f_i = \lambda_i \cos(t) + \mu_i \sin(t), i = 2, ..., n$.

Proposition 11.5. *Assume $f : \mathcal{I} \times (-\varepsilon, \varepsilon) \longrightarrow M$ is a variation of the geodesic $\gamma(t) := f(t, 0)$, i.e. all curves $t \mapsto f(t, w)$ are geodesics. Then the vector field*

$$W := Tf \circ \frac{\partial}{\partial w} : \mathcal{I} \times (-\varepsilon, \varepsilon) \longrightarrow TM$$

restricts to a Jacobi field on $\mathcal{I} = \mathcal{I} \times \{0\}$. Indeed, every Jacobi field W along the geodesic γ is obtained in that way.

Proof. We leave it to the reader to check that W and

$$T := Tf \circ \frac{\partial}{\partial t} : \mathcal{I} \times (-\varepsilon, \varepsilon) \longrightarrow TM$$

satisfy

$$\nabla_{\frac{\partial}{\partial t}} W = \nabla_{\frac{\partial}{\partial w}} T$$

as well as

$$\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial w}} X - \nabla_{\frac{\partial}{\partial w}} \nabla_{\frac{\partial}{\partial t}} X = R(T, W)X.$$

Now

$$\nabla_{\frac{\partial}{\partial t}}(\nabla_{\frac{\partial}{\partial t}} W) = \nabla_{\frac{\partial}{\partial t}}(\nabla_{\frac{\partial}{\partial w}} T) = \nabla_{\frac{\partial}{\partial w}}(\nabla_{\frac{\partial}{\partial t}} T) + R(T, W)T = R(T, W)T.$$

For the proof of the second part we refer to the text book. $\square$

Example 11.6. Given a tangent vector $V_a \in T_aM$ and a geodesic $\gamma := \gamma_{X_a}$ with start point a we can explicitly write down the Jacobi field $W : \mathcal{I} \longrightarrow TM$ with $W_0 = 0, \nabla_{\dot\gamma(0)}W = V_a$. Denote $V \in \Theta(T_aM)$ the constant vector field $V \equiv V_a$ (remember $T_{Y_a}(T_aM) \cong T_aM$ naturally.).Then we have

$$W_t = T_{tX_a}\exp_a(tV_{tX_a}).$$

Indeed, it is obtained from the following variation of geodesics

$$f(t,w) := \exp_a(t(X_a + wV_a)).$$

Then we have

$$\begin{aligned}\nabla_{\frac{d}{dt}}W &= \nabla_{\frac{d}{dt}}(tT_{tX_a}\exp_a(V_{tX_a}))\\ &= T_{tX_a}\exp_a(V_{tX_a}) + t\nabla_{\frac{d}{dt}}(T_{tX_a}\exp_a(V_{tX_a})).\end{aligned}$$

Thus for $t = 0$ we obtain $(\nabla_{\frac{d}{dt}}W)_0 = V_a$. Note that if $V_a \perp X_a$, we have $g(W, \dot\gamma) \equiv 0$ as a consequence of Gauß' lemma.

Definition 11.7. A point $b \in M$ is called conjugate to $a \in M$ iff $b = \exp_a(X_a)$ and $\exp_a$ is not a diffeomorphism near X_a, i.e.

$$T_{X_a}\exp_a : T_aM \cong T_{X_a}(T_aM) \longrightarrow T_bM$$

is not an isomorphism.

Proposition 11.8. *The point $b = \exp_a(X_a)$ is conjugate to $a \in M$ iff there is a Jacobi field $W : \mathcal{I} \longrightarrow TM$ above $\gamma_{X_a} : \mathcal{I} \longrightarrow M$ with $W_0 = 0, W_1 = 0$.*

Proof. "$\Longrightarrow$": Denote $Z \in \Theta(T_aM)$ a nonzero constant vector field such that $Z_{X_a} \in \ker(T_{X_a}\exp_a)$ and take

$$W_t := T_{tX_a}\exp_a(tZ_{tX_a}).$$

"$\Longleftarrow$": Given $W : \mathcal{I} \longrightarrow TM$ we choose $Z \in \Theta(T_aM)$ as the constant vector field with $Z_0 = \nabla_{\dot\gamma(0)}W \neq 0$ (because of $W \not\equiv 0$). Then the Jacobi fields W and $Y_t = T_{tX_a}\exp_a(tZ_{tX_a})$ satisfy $W_0 = 0 = Y_0, \nabla_{\dot\gamma(0)}W = \nabla_{\dot\gamma(0)}Y$, hence coincide. In particular

$$T_{X_a}\exp_a(Z_{X_a}) = W_1 = 0.$$

$\square$

Corollary 11.9. *1. A point $b \in M$ is conjugate to the point $a \in M$ iff a is conjugate to b.*

2. If $\gamma : \mathcal{I} = [0,1] \longrightarrow M$ is a geodesic from $a \in M$ to $b \in M$ and b is not conjugate to a, then, given $X_a \in T_a M$ and $X_b \in T_b M$, there is a unique Jacobi field W along γ with $W_0 = X_a, W_1 = X_b$.

Theorem 11.10. First variation of arc length: *Let*

$$f : \mathcal{I} = [0,b] \times (-\varepsilon, \varepsilon) \longrightarrow M, (t,w) \mapsto f(t,w)$$

be a variation of paths and

$$L(w) := L(\gamma_w), \gamma_w(t) = f(t,w).$$

Furthermore $T := Tf(\partial_t), W := Tf(\partial_w)$. If γ_0 is parametrized by arc length, we have

$$L'(0) = g(W,T)\big|_{(0,0)}^{(0,b)} - \int_0^b g(W_{\gamma_0(t)}, (\nabla_{\partial_t} T)_{\gamma_0(t)})dt,$$

In particular $L'(0) = g(W,T)\big|_{(0,0)}^{(0,b)}$, if γ_0 is a geodesic.

Proof. Indeed

$$L'(w) = \int_0^b \frac{\partial}{\partial w}(\sqrt{g(T,T)})dt = \int_0^b \frac{g(\nabla_{\partial_w} T, T)}{\sqrt{g(T,T)}} dt.$$

For $w = 0$ we have $g(T,T) \equiv 1$ and

$$g(\nabla_{\partial_w} T, T) = g(\nabla_{\partial_t} W, T) = \frac{\partial}{\partial t} g(W,T) - g(W, \nabla_{\partial_t} T).$$

Finally integrate! $\square$

Corollary 11.11. *Let $p, q \in M$. For a smooth path $\sigma : [0,b] \longrightarrow M$ from p to q parametrized by arc length the following statements are equivalent:*

1. If $f : \mathcal{I} = [0,b] \times (-\varepsilon, \varepsilon) \longrightarrow M, (t,w) \mapsto f(t,w)$ is a variation of curves with fixed end points $f(0,w) \equiv p, f(b,w) \equiv q$ and "base curve" $\gamma_0 = \sigma$, then $L'(0) = 0$.

2. σ is a geodesic.

Proof. "2) $\Longrightarrow$ 1)": For a variation f with fixed end points we have $W_{(0,b)} = 0 = W_{(0,0)}$; hence Th.11.10 gives $L'(0) = 0$.
"1) $\Longrightarrow$ 2)": First of all note that any vector field $W : [0,b] \longrightarrow TM$ above σ with boundary values 0 can be realized by some variation of σ with fixed end points: Take

$$f(t,w) := \exp(wW_t).$$

Now assume $(\nabla_{\frac{d}{dt}}\sigma)_{t_0} = 0$ with $0 < t_0 < b$. Denote $f : [0,b] \longrightarrow \mathbb{R}_{\geq 0}$ a smooth function vanishing at the boundary points with $f(t_0) > 0$ and take $W := f\nabla_{\frac{d}{dt}}\sigma$. Then we find $L'(0) < 0$. □

Theorem 11.12. Second variation of arc length: *If in the situation of Th.11.11 the basic path γ_0 is a geodesic parametrized by arc length, we have*

$$L''(0) = g(\nabla_{\partial_w}W,T)\big|_{(0,0)}^{(b,0)}$$

$$+\int_0^b \left(g(R(W,T)W,T) + g(\nabla_{\partial_t}W, \nabla_{\partial_t}W) - (\frac{\partial}{\partial t}g(W,T))^2 \right) dt.$$

In particular, if $g(W,T)$ is constant along the base curve γ_0, then

$$L''(0) = g(\nabla_{\partial_w}W,T)\big|_{(0,0)}^{(b,0)} + \int_0^b (g(R(W,T)W,T) + g(\nabla_{\partial_t}W, \nabla_{\partial_t}W))\, dt.$$

Here the first term only depends on the restrictions $W|_{0\times(-\varepsilon,\varepsilon))}$ resp. $W|_{b\times(-\varepsilon,\varepsilon))}$. Finally, if furthermore W is a Jacobi field

$$L''(0) = \frac{\partial}{\partial w}g(T,W)\big|_{(0,0)}^{(b,0)}.$$

In particular

$$L''(0) = 0,$$

if all geodesics γ have the same start and end point

Proof. We compute

$$\frac{\partial^2}{\partial w^2}\sqrt{g(T,T)} = \frac{\partial}{\partial w}\left(\frac{g(\nabla_{\partial_w}T,T)}{\sqrt{g(T,T)}} \right)$$

$$= -\frac{g(\nabla_{\partial_w} T, T)^2}{\sqrt{g(T,T)}^3} + \frac{1}{\sqrt{g(T,T)}} \left(g(\nabla^2_{\partial_w} T, T) + g(\nabla_{\partial_w} T, \nabla_{\partial_w} T) \right).$$

We evaluate at $w = 0$ and obtain with $\nabla_{\partial_w} T = \nabla_{\partial_t} W$ the following

$$\frac{\partial^2}{\partial w^2} \sqrt{g(T,T)} = -g(\nabla_{\partial_t} W, T)^2 + g(\nabla_{\partial_w} \nabla_{\partial_t} W, T) + g(\nabla_{\partial_t} W, \nabla_{\partial_t} W)$$

$$= g(R(W,T)W + \nabla_{\partial_t} \nabla_{\partial_w} W, T) + g(\nabla_{\partial_t} W, \nabla_{\partial_t} W) - \left(\frac{\partial}{\partial t} g(W,T) \right)^2$$

$$= \frac{\partial}{\partial t} g(\nabla_{\partial_w} W, T) + g(R(W,T)W, T) + g(\nabla_{\partial_t} W, \nabla_{\partial_t} W) - \left(\frac{\partial}{\partial t} g(W,T) \right)^2.$$

Now integration gives the first formula. If W is Jacobi, we have along the bae curve

$$g(R(W,T)W,T) = -g(W, R(W,T)T) = g(R(T,W)T, W)$$

$$= g(\nabla^2_{\partial_t} W, T) = \frac{\partial}{\partial t} g(\nabla_{\partial_t} W, W) - g(\nabla_{\partial_t} W, \nabla_{\partial_t} W).$$

Finally use

$$\frac{\partial}{\partial w} g(T, W) = g(\nabla_{\partial_w} T, W) + g(T, \nabla_{\partial_w} W) = g(\nabla_{\partial_t} W, W) + g(T, \nabla_{\partial_w} W).$$

□

Here is a funny application of the second variation formula:

Theorem 11.13. *If $\gamma : [0, a] \longrightarrow M$ is a geodesic and there is a point $\gamma(b), 0 < b < a$, in between, which is conjugate to the start point $\gamma(0)$, then we have*

$$L(\gamma) > d(\gamma(0), \gamma(a)).$$

Remark 11.14. In the cylinder $M := \mathbb{S}^1 \times \mathbb{R} \hookrightarrow \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ there are no pairs of conjugate points. Nevertheless for the geodesic $\gamma(t) = (\cos(t), \sin(t), \lambda t), 0 \leq t \leq a$, we have $L(\gamma) > d(\gamma(0), \gamma(a))$ for $a > \pi$. Indeed the above theorem means that after a conjugate point there is a variation, such that the geodesic is not of minimal length within that family of curves. On the other hand such families do not exist before the first conjugate point.

For the proof of Th.11.13 we need

Proposition 11.15. *Let $\gamma : [0, b] \longrightarrow M$ be a geodesic. For a a continuous piecewise smooth vector field $W : [0, b] \longrightarrow TM$ along γ with $W_0 = 0$ we set*

$$F(W) := \int_0^b \Big(g(R(W,T)W,T) + g(\nabla_{\frac{d}{dt}} W, \nabla_{\frac{d}{dt}} W) \Big)\, dt.$$

Then, if $Z : [0, b] \longrightarrow TM$ denotes the Jacobi field with $Z_0 = 0, Z_b = W_b$ and no point $\gamma(t)$ is conjugate to the start point $\gamma(0)$, we have

$$F(W) \geq F(Z)$$

with equality iff $W = Z$.

We start with a useful observation:

Remark 11.16. If X, Y are Jacobi vector fields along the geodesic γ we have

$$g(\nabla_{\frac{d}{dt}} X, Y) - g(X, \nabla_{\frac{d}{dt}} Y) \equiv const.$$

Indeed

$$\frac{d}{dt}\Big(g(\nabla_{\frac{d}{dt}} X, Y) - g(X, \nabla_{\frac{d}{dt}} Y) \Big)$$
$$= g(R(\dot\gamma, X)\dot\gamma, Y) - g(X, R(\dot\gamma, Y)\dot\gamma)$$
$$= g(R(\dot\gamma, X)\dot\gamma, Y) - g(R(\dot\gamma, Y)\dot\gamma, X) = 0.$$

Proof of Th.11.15. Choose a basis of $T_{\gamma(0)}M$ and denote $V_1, ..., V_m \in \Theta(T_{\gamma(0)}M)$ the associated constant vector fields. Then $W_i := T\exp(V_i)$ is a frame of TM along γ, since there is no point conjugate to $\gamma(0)$, and the vector fields Z_i with $Z_{i,t} = tW_{i,t}$ are Jacobi fields. Write $W = \sum_{i=1}^m g_i W_i$ with continuous piecewise smooth functions $g_i : [0, b] \longrightarrow \mathbb{R}$, indeed $g_i(t) = tf_i(t)$ because of $W_0 = 0$ with continuous piecewise smooth functions $f_i : [0, b] \longrightarrow \mathbb{R}$. Thus $Z = \sum_{i=1}^m f_i(b)Z_i$, while $W = \sum_{i=1}^m f_i Z_i$.

We write

$$\nabla_{\frac{d}{dt}} W = A + B$$

with

$$A = \sum_{i=1}^m \dot f_i Z_i, B = \sum_{i=1}^m f_i \nabla_{\frac{d}{dt}} Z_i.$$

Note that the functions $\dot{f}_1, ..., \dot{f}_m$ are bounded and piecewise continuous – we need not care about their values at the breaks. We show that

$$F(W) = F(Z) + \int_0^b g(A_t, A_t)dt,$$

and obviously $\int_0^b g(A_t, A_t)dt = 0 \iff f_i \equiv f_i(b), i = 1, ..., m \iff W = Z$. We have

$$g(\nabla_{\frac{d}{dt}} W, \nabla_{\frac{d}{dt}} W) = g(A, A) + 2g(A, B) + g(B, B)$$

and

$$g(R(T, W)T, W) = \sum_{i=1}^m f_i g(R(T, Z_i)T, W) = \sum_{i=1}^m f_i g(\nabla^2_{\frac{d}{dt}} Z_i, W)$$

$$= \sum_{i=1}^m f_i \left(\frac{d}{dt} g(\nabla_{\frac{d}{dt}} Z_i, W) - g(\nabla_{\frac{d}{dt}} Z_i, \nabla_{\frac{d}{dt}} W) \right)$$

$$= \frac{d}{dt} g(B, W) - \sum_{i=1}^m \dot{f}_i g(\nabla_{\frac{d}{dt}} Z_i, W) - g(B, A) - g(B, B).$$

Hence

$$g(R(T, W)T, W) + g(\nabla_{\frac{d}{dt}} W, \nabla_{\frac{d}{dt}} W)$$

$$= \frac{d}{dt} g(B, W) + g(A, A) + g(B, A) - \sum_{i=1}^m \dot{f}_i g(\nabla_{\frac{d}{dt}} Z_i, W).$$

On the other hand

$$g(A, B) - \sum_{i=1}^m \dot{f}_i g(\nabla_{\frac{d}{dt}} Z_i, W) = \sum_{i,j} \dot{f}_i f_j \left(g(Z_i, \nabla_{\frac{d}{dt}} Z_j) - g(\nabla_{\frac{d}{dt}} Z_i, Z_j) \right) = 0$$

according to Lagrange's identity, since $Z_{i,0} = 0$ for $i = 1, ..., m$. Thus

$$F(W) = \int_0^b \left(g(R(W, T)W, T) + g(\nabla_{\frac{d}{dt}} W, \nabla_{\frac{d}{dt}} W) \right) dt = g(B_b, W_b) + \int_0^b g(A_t, A_t)dt,$$

since W is continuous and $W_0 = 0$. If we take $W = Z$ we have $W_b = Z_b$ and $A = 0$, while the respective vector fields B coincide at b. This gives the result. $\square$

Proof of Th.11.13. It suffices to show the inequality for $a > b$ close to b, since for $\tilde{a} > a$ we have

$$L(\gamma|_{[0,\tilde{a}]}) = L(\gamma|_{[0,a]}) + L(\gamma|_{[a,\tilde{a}]}) > d(0,a) + d(a,\tilde{a}) \geq d(0,\tilde{a}).$$

We choose a, such that $\gamma(b)$ is contained in the diffeomorphic image w.r.t. $\exp_{\gamma(a)}$ of a ball around $0_a \in T_aM$. Choose $c < b$ s.th. $\gamma(c)$ is contained in there as well.

If V is a vector field over $\gamma|_{\mathcal{I}}$ with a subinterval $\mathcal{I} \subset [0,a]$, we consider the variation

$$\mathcal{I} \times (-\varepsilon, \varepsilon) \longrightarrow M, (t,w) \mapsto \exp(wV_{\gamma(t)})$$

and denote $L_W(w)$ the length of the path $\mathcal{I} \longrightarrow M, t \mapsto \exp(wV_{\gamma(t)})$. Denote $Z : [0,b] \longrightarrow M$ a nontrivial Jacobi field vanishing at the end points. We have $g(Z,T) \equiv 0$ along γ, since $Z_t = T\exp(tU)$ with a constant vector field orthogonal to $\dot{\gamma}(0)$ (Gauß' lemma), and thus have

$$0 = L''_Z(0) = L''_{Z_0}(0) + L''_{Z_1}(0),$$

where $Z_0 = Z|_{[0,c]}, Z_1 = Z|_{[c,b]}$. Denote $Y : [c,a] \longrightarrow M$ the Jacobi field along γ with $Y_c = Z_c, Y_a = 0$ and define the broken vector field X by $X|_{[0,c]} = Z_0, X|_{[c,a]} = Y$. According to Prop.11.15 we have

$$L''_Y(0) < L''_W(0) = L''_{Z_1}(0),$$

where $W|_{[c,a]} = Z_1, W|_{[b,a]} = 0$. Thus

$$0 > L''_{Z_0}(0) + L''_Y(0) = L''_X(0),$$

and it follows that L_X has a local maximum at 0. □

Theorem 11.17. *Let $\gamma : \mathcal{I} \longrightarrow M$ be a geodesic emanating from $a \in M$. Then the set of points on γ conjugate to a is discrete.*

Proof. Assume $a = \gamma(0)$ and that $b = \gamma(c), c \in \mathcal{I}$, is conjugate to a, write $b = \exp_a(X_a)$. Take a basis of constant vector fields $Y_1, ..., Y_m \in \Theta(T_aM)$, such that $Y_{1,X_a}, ..., Y_{r,X_a}$ span the kernel of $T_{X_a}\exp_a$. Then consider the Jacobi vector fields $(Z_i)_t = T_{X_a}\exp_a(tY_i)$ along γ. We claim that the tangent vectors $(\nabla_{\frac{d}{dt}} Z_1)_c, ..., (\nabla_{\frac{d}{dt}} Z_r)_c, (Z_{r+1})_c, ..., (Z_m)_c$ form a basis of T_bM. If so, we obtain because of

$$(\nabla_{\frac{d}{dt}} Z_i)_c = \lim_{t\to c} \frac{(Z_i)_t}{t-b},$$

that for $t \neq c$ close to c, the vectors $\frac{(Z_1)_t}{t-b}, ..., \frac{(Z_r)_t}{t-b}, (Z_{r+1})_t, ..., (Z_m)_t$ are a basis for $T_{\gamma(t)}M$; in particular $\gamma(t)$ is not conjugate to $a = \gamma(0)$. First of all $(\nabla_{\frac{d}{dt}} Z_1)_c, ..., (\nabla_{\frac{d}{dt}} Z_r)_c$ are linearly independent: If

$$\lambda_1(\nabla_{\frac{d}{dt}} Z_1)_c + ... + \lambda_r(\nabla_{\frac{d}{dt}} Z_r)_c = 0,$$

then $X := \sum_{i=1}^r \lambda_i Z_i$ is be a Jacobi field with $X_c = 0 = (\nabla_{\frac{d}{dt}} X)_c$, hence $X = 0$. Now

$$0 = X_0 = \sum_{i=1}^r \lambda_i(\nabla_{\frac{d}{dt}} Z_i)_0 = \sum_{i=1}^r \lambda_i Y_{i,0}$$

implies $\lambda_1 = ... = \lambda_r = 0$. Now the claim follows from the fact that

$$(\nabla_{\frac{d}{dt}} Z_i)_c \perp Z_{j,c}$$

for all $t \in \mathcal{I}$, $i = 1, .., r, j = r+1, ..., m$: According to Rem. 11.16 we

$$g((\nabla_{\frac{d}{dt}} Z_i), Z_j) - g(Z_i, (\nabla_{\frac{d}{dt}} Z_j)) \equiv d,$$

while $t = 0$ gives $d = 0$, whence

$$g((\nabla_{\frac{d}{dt}} Z_i), Z_j) = g(Z_i, (\nabla_{\frac{d}{dt}} Z_j)).$$

The result now follows, since $Z_{i,c} = 0$ for $i = 1, ..., r$.

$\square$

12 Negative and Positive Curvature, Coverings

Theorem 12.1. *A complete simply connected Riemannian manifold M of constant sectional curvature K and dimension $m \geq 2$ is isomorphic to either*

1. *($K < 0$) m-diomensional hyperbolic space*

$$\mathbb{H}_m(K) := \{\mathbf{x} \in \mathbb{R}^{m+1}; x_1^2 + ... + x_m^2 - x_{m+1}^2 = \frac{1}{K}, x_{m+1} > 0\}$$

endowed with the restriction of the Lorentz metric

$$\sum_{i=1}^m dx_i \otimes dx_i - dx_{m+1} \otimes dx_{m+1},$$

2. *($K = 0$) euclidean m-space $\mathbb{R}^m$, or*

3. *($K > 0$) the m-dimensional sphere $\mathbb{S}^m(K) \hookrightarrow \mathbb{R}^{m+1}$ of radius $\frac{1}{\sqrt{K}}$.*

Proof. Given a complete M with constant curvature we construct a local isometry, indeed a covering, see Def.12.2,

$$\tilde{M} \longrightarrow M$$

with

$$\tilde{M} = \mathbb{H}_m(K), \mathbb{R}^m, \mathbb{S}^m(1/\sqrt{K}).$$

The fact that M is simply connected implies then that it is an isomorphism. Indeed, given a point $a \in M$ we study the exponential map $\exp_a : T_aM \longrightarrow M$ at some point $a \in M$ and find that the pair $(T_aM, \exp_a^*(g))$ does only depend on K. For $K \le 0$ it is a Riemannian manifold isomorphic to $\tilde{M}$, while for $K > 0$ we have to be slightly more careful. Indeed we obtain:

$$(\exp_a)^*(g)|_{T_{X_a}(T_aM)} = \hat{g}_a|_{\mathbb{R}X_a} \oplus h_K(||X_a||) \cdot \hat{g}_a|_{X_a^\perp}$$

with a function $h_K \in C^\infty([0,\infty))$ depending only on K. Here we denote $\hat{g}_a$ the metric tensor associated to the inner product g_a. For $K \le 0$ it has no zeros; thus $(\exp_a)^*(g)$ is a Riemannian metric.

The zeros of h_K for $K > 0$ are the integer multiples of $\pi/\sqrt{K}$. It follows, that $\exp_a$ maps the sphere of radius $\pi/\sqrt{K}$ to one point $b \in M$. Now apply the same arguments to $b \in M$ instead of $a \in M$. Since the two open balls with radius $\pi/\sqrt{K}$ in T_aM and T_bM (with the above metrics) can be patched together to a sphere $\mathbb{S}^m(K)$, we obtain a locally diffeomorphic map $\mathbb{S}^m(K) \longrightarrow M$. It is onto since its image is both closed and open in the connected manifold M.

Let us now establish the formula for the pull back of the metric g. First of all we may write the curvature tensor as follows

$$R(X,Y)Z = K(g(Y,Z)X - g(Z,X)Y).$$

We study Jacobi fields Z along a geodesic $\gamma(t) = \exp_a(tX_a), ||X_a|| = 1$, with initial value $Z_0 = 0$. (The above formula for the pull back of the metric is established for tX_a instead of the unnormalized X_a in the formula.) We have then $Z_t = T_{tX_a}\exp(tV_{tX_a})$ with a constant vector field $V \in \Theta(T_aM)$, in particular $\nabla_{\frac{d}{dt}} Z = V_0$. On the other hand look at the parallel vector field Y

over γ with $Y_0 = V_0$. We investigate whether fY with $f \in C^\infty([0,\infty))$ is a Jacobi field, assuming $V_0 \perp X_a$. We find

$$\ddot{f}Y = R(\dot\gamma, fY)\dot\gamma = Kf(g(Y,\dot\gamma)\dot\gamma - g(\dot\gamma,\dot\gamma)Y) = -KfY,$$

since $Y_t \perp \dot\gamma(t)$ for all t, that being true for $t = 0$. So fY is a Jacobi field if $\ddot{f} = -Kf$. The unique solution of that differential equation with $f(0) = 0, \dot{f}(0) = 1$ is

1. $f(t) = \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}t)$ for $K < 0$,
2. $f(t) = t$ for $K = 0$ and
3. $f(t) = \frac{1}{\sqrt{K}} \sin(\sqrt{K}t)$ for $K > 0$.

Finally, since they have the same value and the same (covariant) derivative at $t = 0$, we find $Z = fY$ and

$$T_{tX_a} \exp_a(tV_{tX_a}) = f(t)Y_t,$$

whence

$$T_{tX_a} \exp_a(V_{tX_a}) = h_K(t)Y_t$$

with $h_K(t) = f(t)t^{-1}$. □

Definition 12.2. *A surjective continuous map $\pi : X \longrightarrow Y$ between topological spaces X and Y is called a* covering *iff every point $b \in Y$ admits an open neighbourhood $V \subset Y$, such that its inverse image is the disjoint union*

$$\pi^{-1}(V) = \bigcup_{i \in I} U_i$$

of open subsets $U_i \subset X$ with $\pi|_{U_i} : U_i \longrightarrow V$ being a homeomorphism for every $i \in I$.

Remark 12.3. Note that a locally homeomorphic map need not be a covering. For example removing a point in X leads to a non-covering $X^* \longrightarrow Y$. On the other hand: If X is compact, then a local homeomorphism is even a covering. The fiber of $b \in Y$ is finite, say $a_1, ..., a_n$. Take neighbourhoods $U_i \ni a_i$, s.th. $\pi|_{U_i} : U_i \longrightarrow V_i$ is a homeomorphism. Finally $f(X \setminus U_1 \cap ... \cap U_n) \subset Y$ is compact, hence closed. Take V as its complement.

Example 12.4. 1. $\mathbb{S}^m \longrightarrow \mathbb{P}^m$.

2. If $f : M \longrightarrow N$ is a local isometry from a complete Riemannian manifold M to a connected one, then π is a covering and N is complete as well. Indeed, its image $f(M)$ is both open and closed, hence $f(M) = N$. Openness is trivial. If $b \in \overline{f(M)}$ take $\varepsilon > 0$, s.th. $\exp_b : B_\varepsilon(0_b) \xrightarrow{\cong} B_\varepsilon(b)$ is a diffeomorphism. In particular there is a geodesic from b to some point $c = f(a) \in B_\varepsilon(b)$. Now take the geodesic in M starting at a with tangent vector "opposite" to that one of the geodesic segment from b to c and follow it up to $t = d(b, c)$. (M is complete!) The end point d then satisfies $f(d) = b$. Finally if $f^{-1}(b) = \{a_i, i \in I\}$, we have
$$\pi^{-1}(B_\varepsilon(b)) = \bigcup_{i\in I} B_\varepsilon(a_i).$$

3. If M is complete and $a \in M$, s.th. there are no points conjugate to a, then $\exp_a : T_aM \longrightarrow M$ is a covering. Indeed, the pull back $\exp_a^*(g)$ is a Riemannian metric on T_aM and T_aM is complete w.r.t. it, the lines through the origin being geodesics.

In order to complete the proof of Th.12.1 we show that a covering $X \longrightarrow Y$ with simply connected Y and connected X is a homeomorphism. For that we need the following lifting theorem:

Theorem 12.5. *Let $y_0 \in Y$ and $x_0 \in X, z_0 \in Z$. Given a covering $\pi : X \longrightarrow Y$ with $\pi(x_0) = y_0$ and a continuous map $\varphi : Z \longrightarrow Y$ with $\varphi(z_0) = y_0$ with simply connected Z, there is a unique lifting $\hat\varphi : Z \longrightarrow X$ of φ, i.e. we have a commutative diagram:*

$$\begin{array}{ccc} & & X \\ & \nearrow & \downarrow \\ Z & \xrightarrow{\varphi} & Y \end{array}$$

s.th. $\hat\varphi(z_0) = x_0$.

Corollary 12.6. *A covering $\pi : X \longrightarrow Y$ with path connected X and simply connected Y is a homeomorphism.*

Proof. We apply Th.12.5 with $Z = Y, \varphi = \mathrm{id}_Y$, and suitable points $y_0 \in Y, x_0 \in X$. Then $\hat\varphi$ is a homeomorphism: Obviously it is injective and a local homeomorphism. Now take a point $x \in X$, denote $\gamma : \mathcal{I} \longrightarrow X$ a path from x_0 to x. Both γ and $\hat{\varphi} \circ \pi \circ \gamma$ are liftings of $\pi \circ \gamma$, hence coincide, in particular $x \in \varphi(Y)$. $\square$

Proof of Th.12.5. 1. The case $Z = [0,1], z_0 = 0$. Write $[0,1] = I_1 \cup ... \cup I_n$ with $I_k = [\frac{k-1}{n}, \frac{k}{n}]$. For sufficiently big $n \in \mathbb{N}$ every piece $\varphi(I_k) \subset \varphi(I)$ is contained in an open path connected set $V = V_k \subset Y$, such that $\pi^{-1}(V)$ is a disjoint union as in Def. 12.2. Now assume we have found a lift

$$\hat{\varphi}_k : \big[0, k/n\big] \longrightarrow X,$$

of $\varphi_k := \gamma|_{[0,\frac{k}{n}]}$. Choose $U \subset \pi^{-1}(V_{k+1})$ with $\pi|_U : U \longrightarrow V_{k+1}$ being homeomorphic and $\hat{\varphi}_k(\frac{k}{n}) \in U$. Now define $\hat{\varphi}_{k+1}$ by

$$\hat{\varphi}_{k+1}|_{[0,k/n]} := \hat{\varphi}_k \ , \quad \hat{\varphi}_{k+1}|_{I_{k+1}} := (\pi|_U)^{-1}|_{I_{k+1}}.$$

2. The case $Z = [0,1]^2, z_0 = (0,0)$. Fix $n \in \mathbb{N}$. We consider the subdivision of the unit square

$$[0,1]^2 = \bigcup_{1 \le i,j \le n} Q_{ij}$$

with

$$Q_{ij} := \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right], \quad 1 \le i,j \le n.$$

For sufficiently big $n \in \mathbb{N}$ every $\varphi(Q_{ij}) \subset Y$ is contained in an open connected set $V = V_{ij} \subset Y$, such that $\pi^{-1}(V)$ is a disjoint union as in Def. 12.2. Now assume we have found a lift

$$\hat{\varphi}_{ij} : B_{ij} := \bigcup_{(k,\ell) \prec (i,j)} Q_{k\ell} \longrightarrow X,$$

of $\varphi|_{B_{ij}}$, where $\prec$ is the lexicographic order on $\{1,...,n\}^2$. Since $B_{ij} \cap Q_{ij}$ is connected, we have $\hat{\varphi}_{ij}(B_{ij} \cap Q_{ij}) \subset U$ for one of the subsets $U \subset \pi^{-1}(V)$ with $\pi|_U : U \longrightarrow V$ being homeomorphic. Hence we may extend $\hat{\varphi}_{ij}$ to $B_{ij} \cup Q_{ij}$ defining it on Q_{ij} as $(\pi|_U)^{-1}$.

3. The general case: Given a point $z \in Z$, take a path $\beta_z : [0,1] \longrightarrow Z$ with $\beta_z(0) = z_0, \beta_z(1) = z$. Denote $\hat{\gamma}_z : [0,1] \longrightarrow X$ the lift of $\gamma_z := \varphi \circ \beta_z$ with $\hat{\gamma}_z(0) = x_0$ and take $\hat{\varphi}(z) := \hat{\gamma}_z(1)$. It remains to show that

different choices of $\beta_z : [0.1] \longrightarrow Z$ give the same value $\hat{\varphi}(z)$. This is done below.

□

We start with the notion of *homotopic paths*:

Definition 12.7. *Two paths $\alpha, \beta : [0,1] \longrightarrow Y$ with same start and end point are called homotopic: "$\alpha \sim \beta$", if there is a* homotopy *from α to β, i.e., a continuous map $F : [0,1] \times [0,1] \longrightarrow Y$ with the following properties:*

$$F(0,s) = \alpha(0) = \beta(0), F(1,s) = \alpha(1) = \beta(1)$$

and $F_s(t) := F(t,s)$ satisfies

$$F_0 = \alpha, \ F_1 = \beta.$$

Remark 12.8. 1. To be homotopic is an equivalence relation on the set of paths from a given point $x \in Y$ to another given point $y \in Y$. We denote $[\gamma]$ the equivalence class (homotopy class) of the path γ. If $\tau : [0,1] \longrightarrow [0,1]$ is a continuous map with $\tau(0) = 0, \tau(1) = 1$ (a "reparametrization"), then $\gamma \circ \tau \sim \gamma$.

2. Given paths $\alpha, \beta : [0,1] \longrightarrow Y$, such that $\beta(0) = \alpha(1)$, we define the concatenation $\alpha\beta : [0,1] \longrightarrow Y$ by

$$(\alpha\beta)(s) = \begin{cases} \alpha(2s) & , \quad \text{if } 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1) & , \quad \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

3. If $\alpha \sim \tilde{\alpha}, \beta \sim \tilde{\beta}$ and the end point of α is the starting point of β, then $\alpha\beta \sim \tilde{\alpha}\tilde{\beta}$, in particular we can concatenate homotopy classes. Note that in general $\alpha(\beta\gamma) \neq (\alpha\beta)\gamma$, but that $\alpha(\beta\gamma) \sim (\alpha\beta)\gamma$, i.e. on the level of homotopy classes concatenation becomes associative.

4. We can not only compose paths, but there is also the notion of an inverse path: Given $\alpha : [0,1] \longrightarrow Y$, we denote $\alpha^{-1} : [0,1] \longrightarrow Y$ the path $\alpha^{-1}(s) := \alpha(1-s)$. Note that $\alpha^{-1}\alpha \sim \alpha(0) \sim \alpha\alpha^{-1}$.

5. A path connected top. space X is simply connected if there is a point $x_0 \in X$ s.th. every closed curve with x_0 as starting and end point is homotopic to the constant path $\equiv x_0$. Indeed, if that condition is satisfied for one base point $x_0 \in X$ then it holds for all base points.

6. In a simply connected topological space X any two paths with the same start and the same end points are homotopic.

Example 12.9. 1. Obviously $\mathbb{R}^m$ is simply connected.

2. A path connected space $X = U \cup V$, which is the union of two open simply connected subsets $U, V \subset X$ with a path connected intersection $U \cap V$, is simply connected. In particular the spheres $\mathbf{S}^n, n \geq 2$, are simply connected. – To see this take a base point $x_0 \in U \cap V$ and consider a closed path $\gamma : [0,1] \longrightarrow X$. Then for sufficiently big $n \in \mathbb{N}$ every interval $I_k := [\frac{k-1}{n}, \frac{k}{n}]$ satisfies $\gamma(I_k) \subset U$ or $\gamma(I_k) \subset V$. Choose a path α_k from x_0 to $\gamma(\frac{k}{n})$ within U resp. V if $\gamma(\frac{k}{n}) \in U$ resp. $\gamma(\frac{k}{n}) \in V$. That is possible, since $U \cap V$ is connected. Then $\gamma \sim \beta_1 ... \beta_n := (..(\beta_1\beta_2)....\beta_n)$ with $\beta_1 := \gamma_1\alpha_1^{-1}, \beta_k := \alpha_{k-1}\gamma_k\alpha_k^{-1}, 2 \leq k < n$ and $\beta_n := \alpha_{n-1}\gamma_n$. Since both U and V are simply connected and $\beta_k([0,1]) \subset U$ or $\beta_k([0,1]) \subset V$, we get $\beta_k \sim x_0, 1 \leq k \leq n$, and thus $\gamma \sim x_0$.

Definition 12.10. *A covering $\pi : \hat{X} \longrightarrow X$ is called a universal covering of the path connected space X, if $\hat{X}$ is simply connected.*

Remark 12.11. 1. If $\hat{X} \longrightarrow X, \tilde{X} \longrightarrow X$ are universal coverings of X, then it follows from Th.12.5, that there is a homeomorphism $\hat{X} \longrightarrow \tilde{X}$ making

$$\begin{array}{ccc} \hat{X} & \longrightarrow & \tilde{X} \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

a commutative diagram. But it is not unique, since it depends on the choice of base points.

2. If $\hat{X} \longrightarrow X$ is a universal covering and $\hat{x}_0 \in \hat{X}, x_0 \in X$ base points with $\pi(\hat{x}_0) = x_0$, then

$$[\gamma] \mapsto \hat{\gamma}(1),$$

where $\gamma : [0,1] \longrightarrow X$ is a path with $\gamma(0) = x_0$ and $\hat{\gamma} : [0,1] \longrightarrow \hat{X}$ its lifting with $\hat{\gamma}(0) = \hat{x}_0$, defines a bijection between the set of homotopy equivalence classes of paths in X starting at x_0 and the points $\hat{x} \in \hat{X}$. Here γ, β are called homotopic if $\gamma(1) = \beta(1)$ and $\gamma \sim \beta$.

As next we want to construct, given a topological space X, its universal covering $\hat{X} \longrightarrow X$. The second part of the last remark suggests how to do that.

Call a topological space locally simply connected if every point has an open simply connected neighbourhood.

Theorem 12.12. *Every path connected and locally simply connected topological space* X *admits a covering* $\pi : \hat{X} \longrightarrow X$ *with a simply connected* $\hat{X}$, *called* the universal covering of X.

Proof. We choose a base point $x_0 \in X$ and define $\hat{X}$ to be the set of all homotopy classes of paths with the base point x_0 as start point. The map $\pi : \hat{X} \longrightarrow X$ then is defined as $\pi([\gamma]) := \gamma(1)$. The topology on $\hat{X}$ is defined as follows: Given a point $\hat{x} := [\gamma]$ with $x := \pi(\hat{x})$ and a simply connected neighbourhood U of x, we set

$$U(\hat{x}) := \{[\gamma\delta]; \delta : [0,1] \longrightarrow U, \delta(0) = x\}\ ,\ \hat{x} = [\gamma].$$

Then the sets $U(\hat{x})$ with an open neighbourhood $U \subset X$ of $x \in X$ constitute a basis for the topology of $\hat{X}$. We claim, that for simply connected U we have $U(a) \cap U(b) = \emptyset$ for $a, b \in \pi^{-1}(x), a \neq b$. Let $a = [\alpha], b = [\beta]$. Assume that $[\alpha\delta] = [\beta\delta']$ with paths $\delta, \delta' : [0,1] \longrightarrow U$. Take a path $\gamma : [0.1] \longrightarrow U$ from $\delta(1) = \delta'(1)$ to $\alpha(1) = \beta(1)$. Now, U being simply connected, we have $\delta\gamma \sim 0 \sim \delta'\gamma$ and thus

$$\alpha\delta \sim \beta\delta' \Longrightarrow \alpha\delta\gamma \sim \beta\delta'\gamma \Longrightarrow \alpha \sim \beta,$$

a contradiction. It follows easily that $\hat{X}$ is Hausdorff and $\pi : \hat{X} \longrightarrow X$ a covering.

Finally we show that $\hat{X}$ is simply connected: Let $\hat{\alpha} : [0,1] \longrightarrow \hat{X}$ be a closed path with start end point $\hat{x}_0 := [x_0]$ (where x_0 is regarded as the constant path). Consider the path $\alpha := \pi \circ \hat{\alpha}$. We have $\hat{\alpha}(1) = [\alpha]$, since $\hat{\alpha}$ is a lift of α with starting point $\hat{x}_0$ as well as the path $t \mapsto [\alpha_t]$ with the path $\alpha_t : [0,1] \longrightarrow \hat{X}, s \mapsto \alpha(ts)$. So because of the unique lifting property we obtain $\hat{\alpha}(1) = [\alpha_1] = [\alpha]$. But $\hat{\alpha}$ was a closed path, i.e. $[\alpha] = \hat{\alpha}(1) = \hat{\alpha}(0) = x_0$, i.e. $\alpha \sim x_0$. By Proposition **??** we obtain $\hat{\alpha} \sim \hat{x}_0$. □

Remark 12.13. Fundamental Group: The construction of the universal covering $\pi : \hat{X} \longrightarrow X$ can be used to associate to any connected and locally simply connected space a group, namely the set

$$\mathrm{Deck}(X) := \{f : \hat{X} \longrightarrow \hat{X} \text{ homeomorphism}; \pi \circ f = \pi\}$$

of all π-fiber preserving homeomorphisms of $\hat{X}$ ("deck transformations") with the composition of maps as group law. From our above reasoning it follows that, given a base point $x_0 \in X$, the restriction $\mathrm{Deck}(X) \longrightarrow \mathbb{S}(\pi^{-1}(x_0)), f \mapsto f|_{\pi^{-1}(x_0)}$ is injective. On the other hand, given points $a, b \in \pi^{-1}(x_0)$ there is exactly one $f \in \mathrm{Deck}(X)$ with $f(a) = b$. If one wants to avoid the universal covering $\pi : \hat{X} \longrightarrow X$ in the definition of $\mathrm{Deck}(X)$, one can construct an isomorphic group as follows: Take again a base point $x_0 \in X$ and define the *fundamental group* of X as the set

$$\pi_1(X, x_0) := \{[\gamma];\ \gamma \text{ path in } X, \gamma(0) = x_0 = \gamma(1)\}$$

of homotopy classes of closed paths in X with start and end point x_0, the group law being the concatenation of paths representing homotopy classes:

$$[\alpha][\beta] := [\alpha\beta].$$

Then there is a natural isomorphism

$$\pi_1(X, x_0) \cong \mathrm{Deck}(X)$$

as follows: Given $[\gamma]$ take any lifting $\hat{\gamma}$ of γ, then the unique $f \in \mathrm{Deck}(X)$ with $f(\hat{\gamma}(1)) = \hat{\gamma}(0)$ is the image of $[\gamma]$.

Furthermore note that a path α from $x_0 \in X$ to $x_1 \in X$ induces an isomorphism

$$\pi_1(X.x_0) \xrightarrow{\cong} \pi_1(X, x_1), [\gamma] \mapsto [\alpha^{-1}\gamma\alpha].$$

Theorem 12.14. *Denote M a connected complete Riemannian manifold. Then we have:*

1. *If M has everywhere nonpositive sectional curvature then there are no pairs of conjugate points in M. In particular for any $a \in M$ the exponential map $\exp_a : T_aM \longrightarrow M$ is the universal covering of M.*

2. *If M has sectional curvature $\geq K > 0$, then*

$$d(a, b) \leq \frac{\pi}{\sqrt{K}}$$

holds for $a, b \in M$. In particular M is compact and the fundamental group of M is finite, the universal covering of M being of the same type.

Example 12.15. If M has finite fundamental group it is the quotient

$$M = \hat{M}/\Gamma$$

of a simply connected compact manifold by the free action of a finite group Γ: Take $\hat{M}$ as the universal covering of M and Γ its deck transformation group. E.g. take $\hat{M} = \mathbb{S}^{2n-1} \subset \mathbb{C}^n$ and $\Gamma = C_q$, the group of q-th roots of unity, (lens spaces).

We mention without proof the theorem of Synge:

Theorem 12.16. *If M is a compact manifold with positive sectional curvature, then $\pi_1(M)$ is trivial or $\mathbb{Z}_2$, if $m = \dim M$ is even, and M is orientable if m is odd.*

Remark 12.17.

1. There is no positive curvature metric on $\mathbb{P}^2 \times \mathbb{P}^2$.
2. Conjecture (Hopf): There is no such metric on $\mathbb{S}^2 \times \mathbb{S}^2$.

Proof. 1.) We consider a geodesic $\gamma : \mathbb{R} \longrightarrow M$ parametrized by arc length, write $a := \gamma(0)$. Denote $X \in \Theta(M)$ a constant vector field with $X_0 \perp \dot{\gamma}(0), ||X_0|| = 1$, furthermore Z the vector field along γ with

$$Z_t := T_{t\dot{\gamma}(0)} \exp(X_{t\dot{\gamma}(0)}).$$

For the function

$$F(t) := ||tZ_t|| - t.$$

we have $F(0) = 0$ as well as $F'(0) = 0$ and show $F''(t) \geq 0$. It follows that F is a convex function and $F(t) \geq 1$ for $t \geq 0$, whence $||Z_t|| \geq 1$ for $t \geq 0$. Since that holds for all vector fields Z with the above properties it follows that the exponential is a local diffeomorphism along γ, the kernel of $T_{t\dot{\gamma}(0)} \exp$ being orthogonal to $\dot{\gamma}(0)$ according to Gauß' lemma. Now $Y = tZ$ is a Jacobi field and

$$F''(t) = \frac{d^2}{dt^2}\sqrt{g(Y,Y)}.$$

On the other hand we have $Y_t \neq 0$ for small $t > 0$ and thus

$$\frac{d}{dt}\sqrt{g(Y,Y)} = \frac{g(\nabla_{\frac{d}{dt}} Y, Y)}{||Y||}$$

and thus

$$\frac{d^2}{dt^2}\sqrt{g(Y,Y)} = \frac{1}{||Y||}\left(g(\nabla^2_{\frac{d}{dt}}Y,Y) + g(\nabla_{\frac{d}{dt}}Y,\nabla_{\frac{d}{dt}}Y) - \frac{g(\nabla_{\frac{d}{dt}}Y,Y)^2}{||Y||^2}\right)$$

$$= \frac{1}{||Y||^3}\left(g(\nabla^2_{\frac{d}{dt}}Y,Y)\cdot||Y||^2 + g(\nabla_{\frac{d}{dt}}Y,\nabla_{\frac{d}{dt}}Y)\cdot||Y||^2 - g(\nabla_{\frac{d}{dt}}Y,Y)^2\right)$$

$$= \frac{1}{||Y||^3}\left(g(R(\dot\gamma,Y)\dot\gamma,Y)\cdot||Y||^2 + ||\nabla_{\frac{d}{dt}}Y||^2\cdot||Y||^2 - g(\nabla_{\frac{d}{dt}}Y,Y)^2\right)$$

$$\geq -K(\mathbb{R}Y_t + \mathbb{R}\dot\gamma(t))\cdot||Y_t|| \geq 0$$

with Cauchy-Schwartz and the fact that the curvature tensor is skew symmetric. It follows that the function $t \mapsto ||Y_t||$ is convex with nonzero derivative at $t = 0$, hence increasing and without zeros $t > 0$; in particular the above argument applies to all $t > 0$.

2.) We show that on every geodesic $\gamma : [0, \frac{\pi}{\sqrt{K}}] \longrightarrow M$ parametrized by arc length there is a point conjugate to the starting point $\gamma(0)$. Assume the contrary. Denote E parallel unit vector field orthogonal to $\dot\gamma$. We consider the vector field W along γ given by

$$W_t = \sin\left(\sqrt{K}\cdot t\right)E_t.$$

It vanishes at the end points of $[0, \frac{\pi}{\sqrt{K}}]$ and $Z = 0$ is the unique Jacobi vector field which agrees with W at the boundary points. We thus have

$$L''_W(0) > L''_Z(0) = 0.$$

On the other hand with $b = \frac{\pi}{\sqrt{K}}$ and $K(t) := K(\mathbb{R}\dot\gamma(t) + \mathbb{R}E_t)$ we have

$$L''_W(0) = \int_0^b \left(g(R(W,\dot\gamma)W,\dot\gamma) + ||\nabla_{\frac{d}{dt}}W||^2\right)dt$$

$$= \int_0^b \left(-K(t)\sin^2(\sqrt{K}\cdot t) + K\cos^2(\sqrt{K}\cdot t)\right)dt$$

$$\leq K\int_0^b \left(-\sin^2(\sqrt{K}\cdot t) + \cos^2(\sqrt{K}\cdot t)\right)dt = 0.$$

□

Example 12.18. 1. For $m \leq 3$ the only simply connected differentiable manifolds underlying a Riemannian manifold of curvature $\geq K > 0$ are the spheres $\mathbb{S}^m$.

2. The metric product of Riemanian manifolds M, N of curvature $\geq K > 0$ has sectional curvature ≥ 0, not more, since the "sections" $U \subset T_{(a,b)}(M \times N) \cong T_aM \oplus T_bN$ spanned by vectors $(X_a, 0), (0, Y_b)$ have curvature 0.

3. For $m = 2n \geq 4$ there are Riemannian manifolds of curvature $\geq K > 0$, which are not spheres, e.g. the complex projective spaces $\mathbb{P}^n(\mathbb{C})$ to be discussed in the next section.

13 Complex projective space

In this section we study complex projective space

$$\mathbb{P}^n(\mathbb{C}) := \{L = \mathbb{C} \cdot \mathbf{z}; \mathbf{z} \in \mathbb{C}^{n+1} \setminus \{0\}\},$$

the set of all complex lines (one dimensional subspaces) in $\mathbb{C}^{n+1}$, from the point of view of differential geometry. Write

$$[\mathbf{z}] := [z_0, ..., z_n] := \mathbb{C}\mathbf{z}.$$

We consider $\mathbb{C}^{n+1}$ with the standard hermitian metric and identify the tangent space at a point $z \in \mathbb{C}^{n+1}$ with $\mathbb{C}^{n+1}$. The first thing we note is that the tangent map of the quotient map

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n(\mathbb{C}), \mathbf{z} \mapsto [\mathbf{z}]$$

induces an isomorphism

$$\mathbf{z}^\perp \longrightarrow T_{[\mathbf{z}]}\mathbb{P}^n(\mathbb{C}).$$

Hence $T_{[\mathbf{z}]}\mathbb{P}^n(\mathbb{C})$ is even in a natural way a complex vector space and $T\mathbb{P}^n(\mathbb{C})$ a complex vector bundle. Furthermore assuming $||z|| = 1$ the tangent space inherits a hermitian metric giving rise to a hermitian metric

$$g + i\omega$$

on $T\mathbb{P}^n(\mathbb{C})$, the so called Fubini-Study metric; its real part g is a Riemannian metric, while $\omega \in \Omega^2(\mathbb{P}^n(\mathbb{C}))$.

Geodesics: A non-constant geodesic on $\mathbb{P}^1(\mathbb{C})$ is given by

$$\{[\mathbf{z}]; \mathbf{z}^T A\mathbf{z} = 0\}$$

with a nondegenerate nondefinite selfadjoint matrix $A \in \mathbb{C}^{2,2}$. Under the stereographic projection $\mathbb{S}^2 \longrightarrow \mathbb{P}^1(\mathbb{C})$ they correspond to great circles, while when interpreting the projective line as extended complex plane $\mathbb{C}\cup\{\infty\}$ they are the generalized circles (circles or lines together with the point at infinity) meeting the unit circle in (at least) two antipodal points. If $\mathbb{P}(U) \subset \mathbb{P}^n(\mathbb{C})$ with a two dimensional subspace U, then the geodesics in $\mathbb{P}(U)$ are obtained from $\mathbb{P}(U) \cong \mathbb{P}^1(\mathbb{C})$ induced by any isometric isomorphism $U \cong \mathbb{C}^2$. Every geodesic is obtained in that way.
Curvature: Projective spaces are homogeneous, but don't have constant sectional curvature - for $n > 1$ they are not spheres. We have

$$K(\mathbb{R}X_a + \mathbb{R}Y_a) = 1 + 3g(iX_a, Y_a)^2,$$

where X_a, Y_a are orthonormal. As a consequence of the Cauchy-Schwartz inequality we find $1 \le K(U) \le 4$, where $K(U) = 4$ means that $U \subset T_a\mathbb{P}^n(\mathbb{C})$ is a complex subspace.

Remark 13.1. Complex projective spaces are among the most basic examples of complex manifolds; we conclude our notes with some remarks about that subject: Replace $\mathbb{R}$ with $\mathbb{C}$ and "differentiable" with "holomorphic" in order to obtain the definition of a complex (analytic) manifold M. In particular complex manifolds are smooth manifolds, hence one should understand how real and complex notions are related. First of all we need germs of complex valued smooth functions: Every smooth function $f : U \longrightarrow \mathbb{C}$, where U is an open neighbourhood of a given point $a \in M$, gives rise to its germ at a: We have $f_a = g_a$, if the representing functions coincide in some sufficiently small neighbourhood $W \ni a$. Germs can be added and multiplied, they form a $\mathbb{C}$-algebra $\mathcal{E}_a$ with $\mathcal{O}_a$, consisting of the germs of holomorphic functions, as subalgebra. Now a real tangent vector may be regarded as a derivation

$$X_a : \mathcal{E}_a \longrightarrow \mathbb{C}$$

mapping real valued functions to $\mathbb{R}$, s.th. we obtain an isomorphism

$$T_aM \hookrightarrow \mathrm{Der}_a(\mathcal{E}_a, \mathbb{C}) \twoheadrightarrow \mathrm{Der}_a(\mathcal{O}_a, \mathbb{C})$$

of real vector spaces, composed of an injection and a surjection, the latter being the restriction of a derivation to $\mathcal{O}_a \subset \mathcal{E}_a$. Thus the left hand side, T_aM, inherits the structure of a complex vector space from the right hand $\mathrm{Der}_a(\mathcal{O}_a, \mathbb{C})$. We remark that the kernel of the right arrow is generated by the operators

$$\frac{\partial}{\partial \overline{z}_\nu} = \frac{1}{2}\left(\frac{\partial}{\partial x_\nu} + i\frac{\partial}{\partial y_\nu}\right), \nu = 1, ..., n$$

and thus we see that $i \in \mathbb{C}$ acts on T_aM as follows:

$$i\frac{\partial}{\partial x_\nu} = \frac{\partial}{\partial y_\nu},\ i\frac{\partial}{\partial y_\nu} = -\frac{\partial}{\partial x_\nu}.$$

Definition 13.2. A hermitian metric $g + i\omega$ on the tangent bundle TM of a complex manifold is called a Kähler metric if $\omega \in \Omega^2(M)$ is closed: $d\omega = 0$.