Introduction to Basic Toric Geometry

Gottfried Barthel, Karl-Heinz Fieseler, Ludger Kaup

Contents

Introduction

The aim of these notes is to give a concise introduction to some fundamental notions of toric geometry, with applications to singularity theory in mind. Toric varieties and their singularities provide a lot of particularly interesting examples: Though belonging to a restricted class, they illustrate many central concepts for the general study of algebraic varieties and singularities. Quoting from the introduction of [Ful], one may say that "*toric varieties have provided a remarkably fertile testing ground for general theories*". Whereas a singular variety may not be "globally" toric, singularities often are "toroidal", i.e., locally analytically equivalent to toric ones, so toric geometry can help for a better understanding even of non-toric singular varieties. In addition to that, for studying certain classes of non-toroidal singularities, methods of toric geometry turn out to be most useful, e.g., for the resolution of "non-degenerate complete intersection singularities". As a key feature, toric varieties admit a surprisingly simple, yet elegant and powerful description that prominently uses objects from elementary convex and combinatorial geometry. These objects are "rational" convex polyhedral cones and compatible collections thereof, called "fans", in a real vector space of dimension equal to the complex dimension of the variety.

The attribute “toric” refers to the *algebraic torus* of algebraic group theory. In the complex setting we are dealing with exclusively, the complex algebraic n-torus is an n-fold product $\mathbb{T}_n := (\mathbb{C}^*)^n$, endowed both with its group structure and its structure as an affine algebraic variety. (It is the complexification of the familiar real n-torus $(\mathbb{S}^1)^n$ and includes the latter as an equivariant deformation retract.) A *toric variety* is an algebraic variety including $\mathbb{T}_n$ as an open dense subset such that the group structure extends to an action on the variety. It turns out that many familiar algebraic varieties actually are toric; basic singular ones are the quadric cones $V(\mathbb{C}^3; xy-z^2)$ and $V(\mathbb{C}^4; xy-zw)$.

In these notes, we focus on fundamental parts of the theory that are indispensable if one wants to apply toric methods as a tool for singularity theory. The picture presented here is by no means complete since important applications to singularity theory, let alone to other parts of mathematics, had to be left out. As examples, we just mention the role of toric geometry in studying non-degenerate complete intersection singularities or in the general resolution of singularities.

We assume that the reader is familiar with elementary concepts of algebraic geometry. Affine complex algebraic varieties and their morphisms are in one-to-one contravariant correspondence to finitely generated reduced $\mathbb{C}$-algebras and their homomorphisms: The elements of the algebra yield the regular functions on the variety, and the points of the variety correspond to the maximal ideals of the algebra. Ideals determine closed subvarieties; conversely, to any closed subvariety corresponds its vanishing ideal. General varieties are obtained from affine ones by a natural gluing procedure that respects the separation condition. All varieties to be considered here are of finite type, i.e., they admit a finite covering by open affine subspaces. Moreover, we exclusively deal with (connected) *normal* varieties, i.e., the “coordinate algebras” corresponding to their affine open subsets are integral domains and integrally closed in their field of fractions.

Besides these fundamental notions of algebraic geometry, we use some basic concepts of group actions like orbits, invariant subsets, isotropy subgroups, and fixed points.

1 Fundamental Notions

1.1 Group embeddings

Let G be a **complex algebraic group**, which means that G is both, a group and a complex algebraic variety, and these structures are compatible: The group multiplication

$$G \times G \longrightarrow G\,,\ (g,h) \longmapsto gh$$

and the inversion

$$G \longrightarrow G\,,\ g \longmapsto g^{-1}$$

are morphisms of algebraic varieties. A (homo-)morphism between algebraic groups is a homomorphism of groups which at the same time is a morphism of varieties.

Standard examples are the general linear group $\mathrm{GL}_n(\mathbb{C})$ and its closed subgroups like $\mathrm{SL}_n(\mathbb{C})$, regular upper and lower triangular matrices, and regular diagonal matrices. The latter family of commutative connected complex algebraic groups plays the key role in these notes:

1.1.1 Definition. *A (complex)* **algebraic (n-) torus**, *usually denoted by $\mathbb{T}$ or $\mathbb{T}_n$, is an algebraic group isomorphic to the n-fold cartesian product of the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers:*

$$\mathbb{T} = \mathbb{T}_n \cong (\mathbb{C}^*)^n.$$

With such a torus, we usually associate a fixed isomorphism $\mathbb{T}_n \cong (\mathbb{C}^*)^n$. – The toric varieties, to be considered in the sequel, are *embeddings* of algebraic tori. We first define that notion for an arbitrary algebraic group G:

1.1.2 Definition. *A* G-**embedding** *is an algebraic variety X together with*

(1) an algebraic action

$$G \times X \longrightarrow X\,,\ (g, x) \longmapsto g{\cdot}x = gx$$

of the group G on X, i.e., that mapping is both, a G-action and a morphism of algebraic varieties;

(2) an open embedding $j\colon G \to X$ with dense image such that $j(gh) = g{\cdot}j(h)$ holds for arbitrary elements $g, h \in G$, i.e., the action of G on X extends the G-action on $G \cong j(G)$ by left translation.

An immediate example for the group $G = \mathrm{GL}_n(\mathbb{C})$ is its embedding into the vector space $\mathbb{C}^{n\times n}$ of square matrices.

1.1.3 Remark. Condition (2) may be equivalently restated as follows: *There is a "big" (i.e., open and dense) G-orbit $O = G{\cdot}x_0$ in X such that the isotropy subgroup G_{x_0} is trivial.* The embedding j then is just the orbit map $g \mapsto g{\cdot}x_0$, where x_0 is the j-image of the unit element of G. – The point x_0 is often called the *base point.*

We usually identify G with its image $j(G)$ in X. – Next, we consider morphisms of group embeddings:

1.1.4 Definition. *Given a homomorphism $q\colon G \to H$ of algebraic groups, a morphism $\varphi\colon X \to Y$ from a G-embedding X to an H-embedding Y is called*

a) a q-**extension** *if $\varphi \circ j_X = j_Y \circ q$;*

b) q-**equivariant** *if $\varphi(gx) = q(g)\varphi(x)$ for arbitrary $g \in G$ and $x \in X$.*

Every q-extension φ is q-equivariant, since the equality $\varphi(g{\cdot}x) = q(g){\cdot}\varphi(x)$ holds on the dense open subset G of X and thus on all of X. Conversely, for an abelian group G, if $\varphi\colon X \to Y$ is q-equivariant and the image $\varphi(x_0)$ of the base point x_0 of X lies in the big orbit $H \cong H{\cdot}y_0$ of Y – say $\varphi(x_0) = h_0{\cdot}y_0$ –, then $\psi := h_0^{-1}{\cdot}\varphi$ is a q-extension. So a q-equivariant morphism $\varphi\colon X \to Y$ is a q-extension if (and only if) it maps the base point of X to the base point of Y.

Referring to the subsequent remarks for the notion of a "normal" variety, we introduce the main object of the present course:

1.1.5 Definition. *For a torus $\mathbb{T}$, a $\mathbb{T}$-embedding into a normal*[1] *algebraic variety X is called a* **($\mathbb{T}$-) toric variety**.

Sometimes it is useful to use a more precise notation for a toric variety, writing a pair $(X, \mathbb{T})$ or even a triplet $(X, \mathbb{T}, x_o)$ instead of X.

Normal varieties

We briefly recall that an algebraic variety X is called *normal* if all its local rings $\mathcal{O}_{X,x}$ are normal integral domains, i.e., they are integrally closed in their respective field of fractions $Q(\mathcal{O}_{X,x})$. A connected normal variety is irreducible.

If X is affine, then normality is equivalent to the fact that the restriction of functions

$$\mathcal{O}(X) \longrightarrow \mathcal{O}(X_{\mathrm{reg}})$$

from all of X to the regular locus $X_{\mathrm{reg}} := X \setminus S(X)$ is an isomorphism of rings. (This is a strong "Riemann removable singularity" property.) If furthermore X is irreducible, then X is normal if (and only if) the ring $\mathcal{O}(X)$ of globally regular functions is integrally closed in its field of fractions $Q\big(\mathcal{O}(X)\big) = \mathbb{C}(X)$, the function field of X.

Smooth varieties are normal, since their local rings are factorial, and normal varieties are "not too singular": If an n-dimensional irreducible variety X is normal, then its singular locus satisfies $\dim S(X) \leqq n-2$. For hypersurfaces, the converse holds.

A standard example of an irreducible variety that is *not normal* is provided by "Neil's parabola" $X = V(\mathbb{C}^2; y^2 - x^3)$: The rational function $h = y/x \in \mathbb{C}(X)$ satisfies the integral equation $h^3 = y$, but it is not regular. (The $\mathbb{T}_1$-action $t{\cdot}(x, y) := (t^2x, t^3y)$ with base point $(1, 1)$ actually would make this singular curve a non-normal $\mathbb{T}_1$-embedding.)

1.2 Toric varieties: Basic examples

After these preparations, we proceed to discuss our main object of interest, namely, the toric varieties. According to Remark 1.1.3, their definition sums up to the following: A normal algebraic $\mathbb{T}$-variety X is toric if and only if X has a base point x_0 with trivial isotropy and dense orbit. The embedding then is provided by the orbit map

$$j = j_X : \mathbb{T} \xrightarrow{\cong} \mathbb{T}{\cdot}x_0 \Subset X\,,\ t \mapsto t \cdot x_0\,,$$

where "$\Subset$" means an open inclusion.

We present a few fundamental examples, most of which will be considered repeatedly in these notes.

1.2.1 Example. The following varieties, endowed with the torus action and base point as indicated, are toric:

[1] For some problems in algebraic geometry, the normality condition is unnecessarily restrictive. Since those problems lie outside the scope of these notes, we stick here to the "classical" definition.

(0) The torus $\mathbb{T}$, acting on itself by translation, with the natural base point $(1,\dots,1)$.

(1) The linear space $\mathbb{C}^n$, with $\mathbb{T}_n = (\mathbb{C}^*)^n \subsetneq \mathbb{C}^n$ acting by componentwise multiplication, and the natural base point $(1,\dots,1)$.

(2) The (singular) two-dimensional affine quadric cone $Y := V(\mathbb{C}^3;\, xz-y^2)$ with the $\mathbb{T}_2$-action $(s,t)\cdot(x,y,z) := (sx, sty, st^2z)$ and base point $(1,1,1)$. – The $\mathbb{T}_2$-action $(s,t)\cdot(x,y,z) := (sx, ty, s^{-1}t^2z)$ on that variety yields another toric structure, denoted Y' for distinction, which is q-isomorphic to the first one (for which q?), but *not* isomorphic in the sense of Def. 1.2.4. – Normality is assured by the fact that Y is a hypersurface with an isolated singularity.

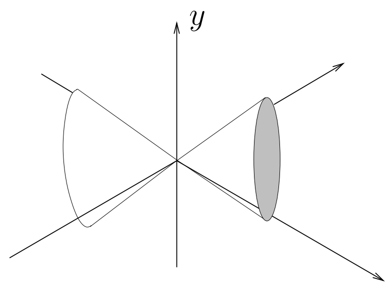

Figure 1: The set of real points of Y

With regard to Figure 1, the reader should keep in mind that it does not faithfully reflect the situation in the complex case: Whereas the punctured real part $Y_{\mathbb{R}} \setminus \{0\}$ is disconnected, its complex counterpart $Y \setminus \{0\}$ is connected.

(3) The (singular) three-dimensional "determinantal variety" $Z \hookrightarrow \mathbb{C}^{2\times 2}$ consisting of all singular 2×2-matrices

$$Z := \left\{ A := \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbb{C}^{2\times 2} \;;\; \det A = 0 \right\}$$

with the $\mathbb{T}_3$-action

$$(s,t,u)\cdot\begin{pmatrix} x & y \\ z & w \end{pmatrix} := \begin{pmatrix} sx & ty \\ suz & tuw \end{pmatrix}$$

and base point $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Normality is seen as above. – This variety Z will also be interpreted as the three-dimensional "Segre cone": The obvious identification $\mathbb{C}^{2\times 2} \cong \mathbb{C}^4$ yields $Z \cong V(\mathbb{C}^4;\, xw-yz)$, the affine cone over the smooth projective quadric surface in $\mathbb{P}_3$ that is the image of the Segre embedding of $\mathbb{P}_1 \times \mathbb{P}_1$.

(4) The projective n-space $\mathbb{P}_n$ with $\mathbb{T}_n$-action $t\cdot[x] := [x_0, t_1x_1, \dots, t_nx_n]$ and base point $[1,\dots,1]$. This is the most basic example of a compact toric variety.

We remark that compact toric surfaces are always projective, whereas higher-dimensional compact toric varieties in general are not.

The two-dimensional quadric cone Y actually is the lowest-degree member in an infinite family of singular two-dimensional hypersurfaces in $\mathbb{C}^3$ that are toric varieties, playing an important role in the theory of surface singularities:

1.2.2 Remark. For every integer $k \geqq 2$, the variety $Y_k := V(\mathbb{C}^3; xz - y^k)$, with the $\mathbb{T}_2$-action $(s,t)\cdot(x,y,z) := (sx, sty, s^{k-1}t^k z)$ and the natural base point $(1,1,1)$, is toric. In the literature, the singularity at the origin of Y_k is called a "*rational double point of type* A_{k-1}".

With respect to the projection onto the (x,z) plane, the surface Y_k is a finite covering branched along the coordinate axes – such coverings occur during the resolution of arbitrary singular surfaces. We note that Y_k is a "*cyclic quotient singularity*": The cyclic group C_k of k-th roots of unity acts on the plane $\mathbb{C}^2$ as a subgroup of $\mathrm{SL}_2(\mathbb{C})$ via $\zeta\cdot(u,v) := (\zeta u, \zeta^{k-1}v)$. The quotient variety $\mathbb{C}^2/C_k$ is a normal surface. The map $\mathbb{C}^2 \to \mathbb{C}^3$, $(u,v) \mapsto (u^k, uv, v^k)$ given by invariant polynomials induces an isomorphism $\mathbb{C}^2/C_k \xrightarrow{\cong} Y_k$ (see also Example 3.1.10). – The restriction of this map to $\mathbb{R}^2$ yields a parametrization of $(Y_k)_{\mathbb{R}} = Y_k \cap \mathbb{R}^3$ if k is odd.

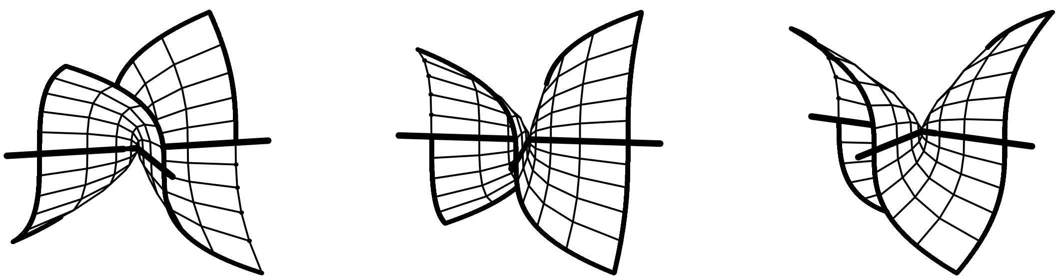

Figure 2: Three views of the set $(Y_3)_{\mathbb{R}}$ with the real A_2 surface singularity

1.2.3 Remark. There are some natural ways of constructing new toric varieties from given ones:

(1) Every nonempty open $\mathbb{T}$-invariant subset of a toric variety is itself toric.

(2) A finite product of toric varieties is again toric (with respect to the direct product of the involved tori).

(3) Let X be a $\mathbb{T}$-toric variety and G, a closed subgroup of the torus $\mathbb{T}$. The residue class group $\mathbb{T}/G$ is again a torus of dimension $n - \dim G$, see Remark 1.3.4 (4). This quotient torus acts on the topological orbit space X/G of the induced G-action on X. The embedding $\mathbb{T} \Subset X$ induces an open $\mathbb{T}/G$-equivariant inclusion $\mathbb{T}/G \Subset X/G$.

If G is finite, then X/G has a natural $\mathbb{T}/G$-toric structure that makes the projection $X \twoheadrightarrow X/G$ a toric morphism. This is discussed at the end of subsection 2.3 when X is affine; using Sumihiro's theorem 1.2.5, the general case then follows by a natural guing procedure.

If G is not finite, then the G-orbit $G\cdot t$ of a point $t \in \mathbb{T}$ is closed in $\mathbb{T}$, but it may fail to be closed in X; see Example 2.3.12 for three typical subgroups $G \cong \mathbb{C}^*$ of $\mathbb{T}_2$ acting on $X = \mathbb{C}^2$. In that case, the topological orbit space is not separated; in particular, it is not an algebraic variety! To obtain a "categorical quotient", a more involved approach is needed, since such a quotient morphism identifies different orbits if their closures intersect. For the affine case, we discuss some aspects of the "algebraic quotient" in the paragraph on quotients at the end of subsection 2.3. The general situation lies outside the scope of these notes.

Morphisms of toric varieties are defined as in 1.1.4:

1.2.4 Definition. *Let $q\colon \mathbb{T} \to \mathbb{T}'$ be a homomorphism of algebraic tori, and $(X, \mathbb{T}, x_0)$ and $(X', \mathbb{T}', x_0')$ be toric varieties. Then a base point preserving q-equivariant morphism (i.e., a q-extension) is called a* q-toric morphism. *– In the case $\mathbb{T} = \mathbb{T}'$ and $q = \mathrm{id}_{\mathbb{T}}$, we simply speak of a toric morphism.*

The theory of toric varieties heavily relies on the following result:

1.2.5 Theorem (Sumihiro). *Every point in a (normal) toric variety admits an affine open $\mathbb{T}$-invariant neighbourhood.*

Thus, in order to analyse *arbitrary* $\mathbb{T}$-toric varieties, it suffices to consider *affine* $\mathbb{T}$-toric varieties, what we shall do in section 2, and then to study how they can be patched together, see section 3.2.

Without assuming normality, the conclusion of Sumihoro's theorem is no longer valid:

1.2.6 Example. The binary cubic forms $\big(-4tu(t+u), -4tu(t-u), (t+u)^3\big)$ define a morphism $\mathbb{P}_1 \twoheadrightarrow C \hookrightarrow \mathbb{P}_2$ onto the projective nodal cubic curve $C = V\big(\mathbb{P}_2; y^2z - x^2(x+z)\big)$. The map is injective except for identifying the points $0 := [0,1]$ and $\infty := [1,0]$, and it induces an isomorphism $\mathbb{P}_1/(0 \sim \infty) \cong C$. The $\mathbb{C}^*$-action $s \cdot [t,u] := [st, u]$ on $\mathbb{P}_1$ thus defines a almost transitive algebraic action of the 1-torus on that (non-normal) projective curve with one big orbit and the single fixed point $[0,0,1]$, so the fixed point does not have an invariant affine neighbourhood.

1.3 Characters and one-parameter subgroups of tori

In the study of toric varieties, algebraic group homomorphisms between the acting torus and $\mathbb{C}^*$ play a key role. This starts with the following easy but crucial fact: *Every algebraic group endomorphism of the algebraic 1-torus $\mathbb{T}_1 = \mathbb{C}^*$ is of the form $s \mapsto s^k$ with a unique integer $k \in \mathbb{Z}$.* The resulting *canonical group isomorphism* $\mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{Z}$ *sending* $\mathrm{id}_{\mathbb{C}^*}$ *to* 1 can be generalized in two ways:

1.3.1 Definition. *Let $\mathbb{T}_n \cong (\mathbb{C}^*)^n$ be an algebraic n-torus. A homomorphism of algebraic groups $\chi\colon \mathbb{T}_n \to \mathbb{C}^*$ is called a* character *of $\mathbb{T}_n$, and a homomorphism $\lambda\colon \mathbb{C}^* \to \mathbb{T}_n$ is called a* one-parameter subgroup.

With respect to the argumentwise multiplication, the sets

$$(1.3.1.1) \qquad \mathbb{X}(\mathbb{T}_n) := \mathrm{Hom}(\mathbb{T}_n, \mathbb{C}^*) \quad \text{and} \quad \mathbb{Y}(\mathbb{T}_n) := \mathrm{Hom}(\mathbb{C}^*, \mathbb{T}_n)$$

are abelian groups. The canonical group isomorphism $\mathbb{X}(\mathbb{C}^*) = \mathbb{Y}(\mathbb{C}^*) \cong \mathbb{Z}$ is generalized as follows:

1.3.2 Remark. The set $\mathbb{X}(\mathbb{T}_n)$ is a *lattice* (i.e., a free abelian group) of rank n: In fact, the mapping

$$M := (\mathbb{Z}^n, +) \longrightarrow \big(\mathbb{X}(\mathbb{T}_n), \cdot\big)\ ,\ \mu \longmapsto \Big(\chi^\mu \colon t \mapsto t^\mu := \prod_{i=1}^n t_i^{\mu_i}\Big)$$

(with coordinates $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ and $t = (t_1, \dots, t_n) \in (\mathbb{C}^*)^n$) is an isomorphism of abelian groups. Hence, every character of $\mathbb{T}$ is a Laurent monomial in the coordinate functions (i.e., the basis characters) $t_1, \dots, t_n$ on $\mathbb{T}_n$. The *Laurent algebra* generated by these monomials is the coordinate ring of the torus as an affine algebraic variety, i.e.,

$$\mathcal{O}(\mathbb{T}_n) \;=\; \mathbb{C}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}] \;=\; \bigoplus_{\chi \in \mathbb{X}(\mathbb{T}_n)} \mathbb{C}{\cdot}\chi\,. \tag{1.3.2.1}$$

Dually, the set $\mathbb{Y}(\mathbb{T}_n)$ is a lattice of rank n, too: There is an isomorphism

$$N := \mathbb{Z}^n \longrightarrow \mathbb{Y}(\mathbb{T}_n)\ ,\ \ \nu = (\nu_1, \dots, \nu_n) \longmapsto \big(\lambda_\nu \colon s \mapsto (s^{\nu_1}, \dots, s^{\nu_n})\big)\,.$$

By a slight abuse of terminology, we occasionally call M and N the lattice of characters and of one-parameter subgroups, respectively. – Each of these lattices M and N determines the torus $\mathbb{T}$: Using the canonical $\mathbb{Z}$-module structure on the abelian group $\mathbb{C}^*$, there are functorial isomorphisms

$$\mathbb{T} \;\cong\; \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \quad \text{and} \quad \mathbb{T} \;\cong\; N \otimes_{\mathbb{Z}} \mathbb{C}^* \;=:\; \mathbb{T}_N\,. \tag{1.3.2.2}$$

1.3.3 Remark. Via the isomorphisms $M = \mathbb{Z}^n \xrightarrow{\cong} \mathbb{X}(\mathbb{T})$ and $N = \mathbb{Z}^n \xrightarrow{\cong} \mathbb{Y}(\mathbb{T})$ provided by the fixed identification $\mathbb{T}_n \cong (\mathbb{C}^*)^n$, the composition pairing

$$\mathbb{X}(\mathbb{T}) \times \mathbb{Y}(\mathbb{T}) \longrightarrow \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*)\,,\ \ (\chi^\mu, \lambda_\nu) \longmapsto \langle \chi^\mu, \lambda_\nu\rangle := \chi^\mu \circ \lambda_\nu \tag{1.3.3.1}$$

corresponds to the usual inner product

$$\langle\_\,,\_\rangle : M \times N \longrightarrow \mathbb{Z}\,,\ \ (\mu, \nu) \longmapsto \langle \mu, \nu\rangle := \sum_{i=1}^n \mu_i \nu_i\ , \tag{1.3.3.2}$$

i.e.,

$$(\chi^\mu \circ \lambda_\nu)(s) = s^{\langle \mu, \nu\rangle} \quad \text{holds for every} \ \ s \in \mathbb{C}^*\,. \tag{1.3.3.3}$$

We use the same symbol for the extended dual pairing

$$\langle\_\,,\_\rangle \colon M_{\mathbb{R}} \times N_{\mathbb{R}} \longrightarrow \mathbb{R}$$

of real vector spaces, where for a lattice $L \cong \mathbb{Z}^n$, we set

$$L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n\,. \tag{1.3.3.4}$$

Moreover, for the dual pair of standard lattice bases in M and in N and thus, dual vector space bases of $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$, we shall use this notation:

(1.3.3.5) $$(e_1, \dots, e_n) \quad \text{in } M, \quad \text{and} \quad (f_1, \dots, f_n) \quad \text{in } N.$$

To better understand the mutual relations between the torus $\mathbb{T}$ and the lattice N expressed in formulae (1.3.1.1) and (1.3.2.2) as well as the structure of closed subgroups of the torus, it is helpful to use an intermediate "analytic" object.[2]

1.3.4 Remark. (1) We use the exact "exponential sequence" $0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to 1$ (where $\exp(z) := e^{2\pi i z}$). "Tensoring" with the lattice N yields a new exact sequence

(1.3.4.1) $$0 \longrightarrow \underbrace{N}_{\cong \mathbb{Z}^n} \longrightarrow \underbrace{N_{\mathbb{C}}}_{:= N \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^n} \xrightarrow{\exp} \underbrace{\mathbb{T}_N}_{\cong (\mathbb{C}^*)^n} \longrightarrow \mathbf{1}_{\mathbb{T}}\ ,$$

where the function exp is applied componentwise, and $\mathbf{1}_{\mathbb{T}}$ is the unit element of $\mathbb{T} := \mathbb{T}_N$. Occasionally, we interpret $N_{\mathbb{C}}$ as tangent space $T_{\mathbf{1}}(\mathbb{T})$ of the torus at the neutral element. The exact sequence immediately provides identifications

$$N = \ker(\exp\colon N_{\mathbb{C}} \twoheadrightarrow \mathbb{T}) \quad \text{and} \quad \mathbb{T} \cong N_{\mathbb{C}}/N\,.$$

In this setting, the isomorphism $N \cong \mathbb{Y}(\mathbb{T})$ can be seen as follows: There is a canonical identification $N \cong \mathrm{Hom}(\mathbb{Z}, N)$, with $\nu \in N$ corresponding to the map $\mathbb{Z} \to N$, $1_{\mathbb{Z}} \mapsto \nu$ and conversely. Scalar extension of lattice homomorphisms then yields a natural identification of N with the group $\mathrm{Hom}\big((\mathbb{C}, \mathbb{Z}), (N_{\mathbb{C}}, N)\big)$ of vector space homomorphisms respecting the given lattices. Passing to the quotient modulo these sublattices, such a linear homomorphim then uniquely "descends" to a homomorphism of tori $\mathbb{C}^* \to \mathbb{T}$. The inverse homomorphism $\mathbb{Y}(\mathbb{T}) \to N$ is obtained by "lifting" a one-parameter subgroup $\lambda \in \mathbb{Y}(\mathbb{T})$ to such a vector space homomorphism $\widetilde{\lambda} = d\lambda\colon (\mathbb{C}, \mathbb{Z}) \to (N_{\mathbb{C}}, N)$.

(2) Any homomorphism $q\colon \mathbb{T}' \to \mathbb{T}$ of tori lifts to a vector space homomorphism $\widetilde{q} =: dq\colon N'_{\mathbb{C}} \to N_{\mathbb{C}}$ that respects the lattices and hence induces a lattice homomorphism $\widetilde{q}\colon N' \to N$. We thus have a commutative "ladder"

(1.3.4.2) $$\begin{tikzcd} 0 \arrow[r] & N' \arrow[r] \arrow[d, "\widetilde{q}|_{N'}"'] & N'_{\mathbb{C}} \arrow[r, "\exp"] \arrow[d, "\widetilde{q}"', "=dq"] & \mathbb{T}' \arrow[r] \arrow[d, "q"'] & 1 \\ 0 \arrow[r] & N \arrow[r] & N_{\mathbb{C}} \arrow[r, "\exp"'] & \mathbb{T} \arrow[r] & 1\,. \end{tikzcd}$$

In the "differential" interpretation, the map dq actually is the derivative of q at $\mathbf{1}_{\mathbb{T}'}$.

Conversely, to any lattice homomorphism $\varphi\colon N' \to N$ corresponds a homomorphism $\mathbb{T}(\varphi) := \varphi \otimes \mathrm{id}_{\mathbb{C}^*}\colon \mathbb{T}_{N'} \to \mathbb{T}_N$ of tori. After a choice of bases, the map φ is explicitly represented by a matrix $A = (a_{ij}) \in \mathbb{Z}^{n \times m}$. In the corresponding coordinates for the tori, the homomorphism $\mathbb{T}(\varphi) := \varphi \otimes \mathrm{id}_{\mathbb{C}^*}$ is given by

$$\mathbb{T}(\varphi)\colon (\mathbb{C}^*)^m \longrightarrow (\mathbb{C}^*)^n\,, \quad (t_1, \dots, t_m) \longmapsto \left(\prod_{j=1}^{m} t_j^{a_{ij}}\right)_{i=1,\dots,n}.$$

[2] From the "categorical" point of view, this is quite natural since a lattice of positive rank, being an infinite discrete group, is not an algebraic group, but rather an analytic group.

(3) The pair of covariant functors $N \rightsquigarrow \mathbb{T}_N$ and $\mathbb{T} \rightsquigarrow \mathbb{Y}(\mathbb{T}) \cong N$ actually establishes an equivalence of categories. One should note, however, that the behaviour of morphisms is not quite straightforward since the functor $N \rightsquigarrow \mathbb{T}_N$ is right-exact only: For the inclusion $\iota\colon N' \hookrightarrow N$ of a sublattice, the resulting homomorphism $\mathbb{T}(\iota)\colon \mathbb{T}_{N'} \to \mathbb{T}_N$ is injective if and only if N' is a *saturated sublattice* (i.e., if N/N' is torsion-free and thus, free: In that case, there is a one-sided inverse to ι that, upon tensoring with $\mathbb{C}^*$, provides a one-sided inverse on the level of tori). In particular, if $\operatorname{rank}(N') = \operatorname{rank}(N)$, then the homomorphism $\mathbb{T}(\iota)$ is surjective with the finite abelian group N/N' as kernel. Analogously, the inverse $\mathbb{T} \rightsquigarrow N$ is left-exact only.

Readers with an interest in "categorical" aspects might wonder why this behaviour does not contradict the equivalence property. Looking closer, one notes a subtle difference between the two categories: both are additive, neither of them is abelian, but they fail "on different sides". In fact, in the category of finitely generated lattices, morphisms do not always have cokernels, whereas in the category of tori, morphisms do not always have kernels.

(4) Any closed subgroup G of $\mathbb{T}$ can be *diagonalised*: There is an isomorphism of tori transforming the pair $(\mathbb{T}, G)$ into a product $\prod_{i=1}^n (\mathbb{C}^*, G_i) := \big((\mathbb{C}^*)^n, \prod_{i=1}^n G_i\big)$, where each G_i, a closed subgroup of the one-torus $\mathbb{C}^*$, is either $\mathbb{C}^*$ or a finite (cyclic) group of roots of unity. To obtain such a diagonalization, consider the inverse image $\exp^{-1}(G)$ in $N_\mathbb{C}$. It splits (non-canonically) into a direct sum $V \oplus L$ of the vector subspace $V := \exp^{-1}(G^\circ) \cong T_\mathbf{1} G$ and a "transversal" lattice L, where G° denotes the connected component of the identity in G. The lattice L spans a vector subspace $L_\mathbb{C}$ of $N_\mathbb{C}$ that is complementary to V, and $L' := N \cap L_\mathbb{C}$ is included in L as a sublattice of finite index. Splitting $N \cap V$ into sublattices of rank one, applying the structure theorem for subgroups of finitely generated free abelian groups to the pair of lattices (L, L'), and then passing to the image in $\mathbb{T}$ then yields a diagonalization. This immediately implies that the residue class group $\mathbb{T}/G$ is a torus, as stated in Remark 1.2.3 (3).

If G is finite, then its inverse image $\exp^{-1}(G)$ is discrete, so it consists only of the lattice L. The latter includes N, and the exponential mapping induces an isomorphism $L/N \cong G$. Conversely, given the inclusion $\iota\colon N' \hookrightarrow N$ of a sublattice of finite index, the corresponding surjective homomorphism $\mathbb{T}(\iota)\colon \mathbb{T}_{N'} \to \mathbb{T}_N$ of tori identifies $\mathbb{T}_N$ with the quotient $\mathbb{T}_N/G$ by a finite subgroup $G \cong N/N'$ (see also (2) and (3) above).

For the study of quotient structures, it is useful to have an alternative approach to the closed subgroups of a torus $\mathbb{T}$: Associating to G the sublattice $K := \{\mu \in M\,;\ \chi^\mu|_G = 1\}$, and to such a sublattice $K \subseteq M$ the subgroup $G := \bigcap_{\mu \in K} \ker(\chi^\mu)$, establishes a one-to-one correspondence.

We illustrate the correspondence $N \longleftrightarrow \mathbb{T}_N$ on the level of morphisms with an example. The idea will be used again several times (see 2.2.13, 2.2.17, and 2.2.19) since it is essential for one of our standard examples:

1.3.5 Example. We consider the standard lattice $N = \mathbb{Z}^2$ with the standard basis f_1, f_2, and the sublattice $\widetilde{N}$ spanned by $v_1 = 2f_1 - f_2$ and $v_2 = f_2$. With respect to this basis, the inclusion $\iota\colon \widetilde{N} \hookrightarrow N$ is given by the matrix $\left(\begin{smallmatrix} 2 & 0 \\ -1 & 1 \end{smallmatrix}\right)$. Under the basis isomorphism $\widetilde{N} \cong \mathbb{Z}^2$, the torus

$\mathbb{T}_{\widetilde{N}} = \widetilde{N} \otimes_{\mathbb{Z}} \mathbb{C}^*$ is identified with $(\mathbb{C}^*)^2$ by sending $v_1 \otimes s$ to $(s, 1)$, and $v_2 \otimes t$ to $(1, t)$. Thus, with the standard identification $\mathbb{T}_N \cong (\mathbb{C}^*)^2$, the associated morphism of tori takes the form $\mathbb{T}(\iota)\colon (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$, $(s,t) \mapsto (s^2, t/s)$, with kernel $\pm(1,1)$.

2 Affine Toric Varieties

2.1 Algebraic description: The coordinate ring

An affine variety X is completely determined by its ring $\mathcal{O}(X)$ of regular functions. If X is toric, then the restriction of global regular functions to the (open dense) embedded torus $\mathbb{T}$ provides an injective algebra homomorphism from $\mathcal{O}(X)$ into $\mathcal{O}(\mathbb{T})$, the Laurent monomial algebra of (1.3.2.1). We may thus identify $\mathcal{O}(X)$ with a subalgebra of the latter:

(2.1.0.1)
$$\mathcal{O}(X) \;\cong\; \mathcal{O}(X)|_{\mathbb{T}} \;\subset\; \mathcal{O}(\mathbb{T}) \;=\; \bigoplus_{\chi \in \mathbb{X}(\mathbb{T})} \mathbb{C}{\cdot}\chi\,.$$

The characters of $\mathbb{T}$ that extend to a regular function on X – and thus, are elements of $\mathcal{O}(X)$ – play a key role in the study of the coordinate ring. Evidently, the set

(2.1.0.2)
$$S = S_X := \mathcal{O}(X) \cap \mathbb{X}(\mathbb{T})$$

of these characters is a (multiplicative) submonoid of the character group. We first study its role for the vector space structure of $\mathcal{O}(X)$:

2.1.1 Lemma. *The set $S = S_X$ provides a vector space basis of the coordinate ring:*

(2.1.1.1)
$$\mathcal{O}(X) = \bigoplus_{\chi \in S} \mathbb{C}{\cdot}\chi\,.$$

Proof. The torus action on X induces an action on the coordinate ring:

$$\mathbb{T} \times \mathcal{O}(X) \longrightarrow \mathcal{O}(X)\,,\ (t, f) \longmapsto f^t \quad \text{where} \quad f^t(x) := f(t{\cdot}x).$$

By (2.1.0.1), each non-zero function $f \in \mathcal{O}(X)$ can uniquely be written as $f = \sum_{i=1}^r \lambda_i \chi_i$ with distinct characters $\chi_i \in \mathbb{X}(\mathbb{T})$ and non-zero complex coefficients λ_i. Applying the torus action yields

$$f^t = \sum_{i=1}^r \lambda_i{\cdot}\chi_i(t){\cdot}\chi_i \;\in\; \mathcal{O}(X) \quad \text{for every } t \in \mathbb{T}\,.$$

Since the characters $\chi_1, \dots, \chi_r \in \mathcal{O}(\mathbb{T})$ are linearly independent, we find points $u_1, \dots, u_r$ in $\mathbb{T}$ such that the matrix $\big(\chi_i(u_j)\big)$ is nonsingular. Hence, each $\lambda_i \chi_i$ and thus, each χ_i lies in the span of the functions f^{u_j}; consequently, all χ_i belong to $\mathcal{O}(X)$. $\square$

We now study an additive description of $\mathcal{O}(X)$ using the corresponding subset of "exponents"

(2.1.1.2)
$$E = E_X := \{\mu \in M\,;\ \chi^\mu \in S_X\}$$

in $M = \mathbb{Z}^n$. Since S_X is a monoid, this set E_X is a sub-semigroup of M, so we can describe $\mathcal{O}(X)$ as the semigroup algebra:

$$\mathbb{C}[E_X] := \bigoplus_{\mu\in E} \mathbb{C}\cdot\chi^\mu = \mathcal{O}(X)\,. \tag{2.1.1.3}$$

We list some essential properties:

2.1.2 Remark. (1) $0 \in E$;

(2) E is finitely generated;

(3) E generates M as a group;

(4) E is a saturated sub-semigroup of M: If $k\mu \in E$ holds for some $k \in \mathbb{N}_{>1}$ and $\mu \in M$, then $\mu \in E$.

Proof. (1) holds, since $1 = \chi^0 \in \mathcal{O}(X)$.

(2) This assertion is true since $\mathcal{O}(X)$ is a finitely generated $\mathbb{C}$-algebra that is spanned by characters. Hence, we find elements $\mu^1, \dots, \mu^r \in M$ with[3] $E = \sum_{i=1}^r \mathbb{N}\cdot\mu^i$.

(3) We have to verify that $E + (-E) = M$. To that end, let $\chi := \chi^\mu$ with $\mu := \sum_{i=1}^r \mu^i$ as in the proof of (2). Since none of the generators $\chi^{\mu^1}, \dots, \chi^{\mu^r}$ of $S_X = \mathbb{X}(\mathbb{T}) \cap \mathcal{O}(X)$ has a zero on the principal open subset $U := X_\chi$ of X, every character in S_X even yields an (invertible) function on U. In particular, this implies that the character $\chi^{-1} = (\chi^{\mu^1} \cdots \chi^{\mu^r})^{-1}$ belongs to $\mathcal{O}(U)^\times$, and since $\mathcal{O}(U) = \mathcal{O}(X)[\chi^{-1}]$, this yields $S_U := \mathcal{O}(U) \cap \mathbb{X}(\mathbb{T}) \subset \mathcal{O}(U)^\times$. It now suffices to show that U is just the embedded torus $\mathbb{T}$ of X, since then $\mathcal{O}(U) = \mathbb{C}\,[E + \mathbb{N}\cdot(-\mu)]$ agrees with $\mathcal{O}(\mathbb{T}) = \mathbb{C}\,[M]$, so

$$M \;=\; E + \mathbb{N}\cdot(-\mu) \;\subseteq\; E + (-E) \subseteq M\,.$$

The (open) inclusion $\mathbb{T} \subset U$ being obvious, we have to verify that the complementary subset $Z := U \setminus \mathbb{T}$ of U is empty. This complement being a proper closed $\mathbb{T}$-invariant subset, its vanishing ideal $I(Z)$ is non-zero and spanned by characters in S_U. We have seen that such characters are invertible functions on U. Hence, the ideal $I(Z)$ is the unit ideal in $\mathcal{O}(U)$, and thus Z is empty.

(4) If some power $(\chi^\mu)^k = \chi^{k\mu}$ of a character $\chi^\mu \in \mathbb{X}(\mathbb{T}) \subset Q\big(\mathcal{O}(X)\big)$ lies in $\mathcal{O}(X)$, then χ^μ is integral over this normal ring and thus lies in it. □

For any affine variety X, the generators of its coordinate ring $\mathcal{O}(X)$ are just the components of a closed embedding $X \hookrightarrow \mathbb{C}^r$ and vice versa. In the toric case, we may thus characterize the generators of the semigroup of exponents:

[3] We adopt the convention that $\mathbb{N} := \mathbb{N}_{\geqq 0} = \mathbb{Z}_{\geqq 0}$.

2.1.3 Remark. Let X be an affine toric variety, and fix $\mu^1, \dots, \mu^r \in E_X$ with corresponding character functions $\chi_i := \chi^{\mu^i} \in S_X$. Then

$$E_X = \sum_{i=1}^{r} \mathbb{N}\cdot\mu^i \,,$$

i.e., $\mu^1, \dots, \mu^r$ generate the semigroup E_X, if and only if the morphism

$$X \longrightarrow \mathbb{C}^r\,,\; x \longmapsto \big(\chi_1(x), \dots, \chi_r(x)\big) \tag{2.1.3.1}$$

is a closed embedding (see also Lemma 2.3.1). In that case, the $\mathbb{T}$-action extends to the ambient space $\mathbb{C}^r$ in the form

$$t\cdot(z_1, \dots, z_r) = \big(\chi_1(t)\cdot z_1, \dots, \chi_r(t)\cdot z_r\big)\,. \tag{2.1.3.2}$$

On the other hand, if the equivariant morphism (2.1.3.1) given by the vectors $\mu^1, \dots, \mu^r$ is a closed embedding, then these vectors generate E_X as a semigroup. □

We apply this remark to our standard examples (see also Remark 2.2.6):

2.1.4 Example. Using the numbering of 1.2.1, we obtain:

(1) The semigroup E_X for $X = \mathbb{C}^n$ is generated by $e_1, \dots, e_n$.

(2) The semigroup E_Y for the quadric cone Y is generated by e_1, $e_1{+}e_2$, $e_1{+}2e_2$.

More generally, for the two-dimensional toric hypersurfaces Y_k (with $k \geqq 2$) discussed in Remark 1.2.2, the semigroup E_{Y_k} is generated by e_1, $e_1{+}e_2$, $(k{-}1)e_1{+}ke_2$.

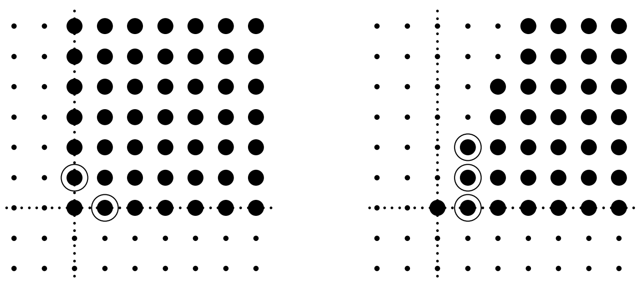

Figure 3: Generators and points of $E_{\mathbb{C}^2}$ and of E_Y

(3) The semigroup E_Z for the Segre cone Z is generated by e_1, e_2, e_1+e_3, e_2+e_3. (See Figure 7 in Example 2.2.7: these generators are the four points on the first cross-section.)

2.2 Geometric description: Polyhedral lattice cones

We now prepare for the geometric description of toric varieties in terms of cones and fans as announced in the introduction.

(I) Recollection: Polyhedral cones

The semigroup of exponents E is the set of lattice points $E = \gamma \cap M$ in a suitable n-dimensional "lattice cone" γ in $M_{\mathbb{R}}$. We briefly recall the general notion of a (lattice) cone:

2.2.1 Definition. *For a lattice $L \cong \mathbb{Z}^n$, let γ be a subset of $V := L_{\mathbb{R}}$.*

(1) The set γ is called a **polyhedral cone** *if there are finitely many vectors $v_1, \dots, v_r$ in V such that*

$$\gamma = \sum_{i=1}^{r} \mathbb{R}_{\geq 0} \cdot v_i =: \operatorname{cone}(v_1, \dots, v_r) .$$

These vectors are called spanning vectors *or* generators *of the cone. A cone spanned by a single non-zero vector v is called a* ray, *denoted by*

$$\operatorname{ray}(v) := \operatorname{cone}(v) .$$

(2) A polyhedral cone γ is called an **L-cone** *(or* lattice cone*) if its spanning vectors $v_1, \dots, v_r$ can be chosen in the lattice L.*

(3) A polyhedral cone γ is called **strongly convex** *(or* pointed*) if it does not include a line through the origin.*

(4) A subset δ of a polyhedral cone γ is called a **face**, *denoted $\delta \preceq \gamma$, if it is of the form*

$$\delta = \gamma \cap \{v^* = 0\} \quad \textit{for some linear form } v^* \in V^* \textit{ with } v^*|_\gamma \geq 0 .$$

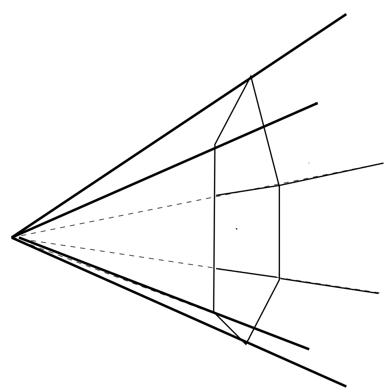

Figure 4: A strongly convex polyhedral cone and its faces

Since $v^* = 0$ is admissible, each cone is a face of itself. We use the notation $\delta \precneqq \gamma$ if we want to emphasize that a face δ is *proper*. – For future use, we introduce some conventions:

2.2.2 Convention. All cones to be considered in the sequel will be polyhedral, and usually, they are assumed to be lattice cones. A system of spanning vectors $v_1, \dots, v_r$ for a cone γ is usually assumed to be irredundant. Moreover, if γ is a lattice cone, then each such vector v_i is usually assumed to be a *primitive lattice vector*, i.e., not a non-trivial positive integer multiple of another lattice vector.

We add a few remarks and introduce some additional notions and notations:

2.2.3 Remark. A polyhedral cone γ as in 2.2.1 (1) can be equivalently described as finite intersection of closed half spaces

$$\gamma = \bigcap_{j=1}^{s} \{v_j^* \geq 0\} \quad \text{with suitable linear forms } v_j^* \in V^* .$$

The boundary hyperplanes $\{v_j^* = 0\}$ with $v_j^* \neq 0$ are called *supporting hyperplanes* of γ. – In the case of a lattice cone, the linear forms v_j^* can be chosen as vectors of the dual lattice $\mathrm{Hom}(L, \mathbb{Z})$. Hence, each *face of a lattice cone* is again a lattice cone.

The intersection $\gamma \cap (-\gamma)$ is the largest linear subspace included in γ. In particular, $\gamma \cap (-\gamma)$ is the *zero cone*

$$o := \{0\}$$

if and only if γ is strongly convex.

A *proper* face δ of γ is cut out by a supporting hyperplane. The *relative interior* γ° is the set of all points in γ not included in a proper face. – The *dimension* of γ is defined as

$$\dim \gamma := \dim \mathrm{lin}(\gamma) \ , \quad \text{where } \ \mathrm{lin}(\gamma) = \gamma + (-\gamma)$$

is the linear subspace of V spanned by *gamma.* A cone of dimension d is usually called a d-cone; a $(d-1)$-face of γ is called a *facet*, and and a 1-face, an *edge.* As a notational convention, we mostly use symbols like σ, τ, ϱ for N-cones, and γ, δ etc. for M-cones.

(II) "Contravariant" description

For the next results, we need some more notation:

2.2.4 Notation. For a ring R, we denote by $\mathrm{Sp}(R)$ its maximal spectrum. Furthermore, we denote by

$\mathfrak{ATV}_{\mathbb{T}}$ the category of affine $\mathbb{T}$-toric varieties with ($\mathrm{id}_{\mathbb{T}}$-)toric morphisms,

$\mathfrak{C}_L$ for a lattice L, the category of L-cones (in $L_{\mathbb{R}}$) and their inclusions,

$\mathfrak{C}_{L,d}$ the full subcategory of d-cones,

$\mathfrak{SC}_L$ and $\mathfrak{SC}_{L,d}$ the respective full subcategories of strongly convex cones.

We may now start the construction of the bridge between the algebraic geometry of toric varieties and the elementary real convex geometry of cones that will be provided by the Equivalence Theorem 2.2.11.

2.2.5 Anti-Equivalence Theorem. *For an n-torus $\mathbb{T}$ with corresponding group M of exponents for $\mathbb{X}(\mathbb{T})$, the assignment*

$$\mathfrak{C}_{M,n} \longrightarrow \mathfrak{ATV}_{\mathbb{T}} \ , \ \gamma \mapsto X^\gamma := \mathrm{Sp}\big(\mathbb{C}[\gamma \cap M]\big)$$

is an anti-equivalence of categories.

Indication of proof. On the one hand, for an n-cone $\gamma = \operatorname{cone}(\mu^1, \ldots, \mu^r)$ with primitive generators μ^i, one verifies that $E := \gamma \cap M$ shares the properties of Remark 2.1.2 and that this fact guarantees that X^γ actually is a toric variety: The semigroup E is generated by the finite set $P \cap M$, where $P := \left\{\sum_{i=1}^r t_i\mu^i\,;\ 0 \leqq t_i \leqq 1 \text{ for } i = 1, \ldots, r\right\}$ is the "fundamental polytope" of the cone γ (see the left-hand side of Figure 5 in 2.2.6).

Furthermore, $E + (-E) = M$ (this guarantees that $\mathbb{T} \to X^\gamma$ is an open embedding), since for every sufficiently "long" vector $\mu \in M \cap \gamma^\circ$ we have $\mu + e_i \in E$ for $i = 1, \ldots, n$. Finally, we may write $\gamma = \bigcap_{j=1}^s H_j$, an intersection of closed half spaces $H_j = \{\nu^j \geq 0\}$ in $M_\mathbb{R}$ with suitable $\nu^j \in N \cong M^*$. To this corresponds a description

$$\mathbb{C}\,[\gamma \cap M] \;=\; \bigcap_{j=1}^{s} \mathbb{C}\,[H_j \cap M]\;.$$

Since the subrings $\mathbb{C}\,[H_j \cap M] \cong \mathcal{O}\big(\mathbb{C} \times (\mathbb{C}^*)^{n-1}\big)$ of the Laurent algebra $\mathbb{C}[M]$ are normal, so is their intersection.

On the other hand, to an affine toric variety X corresponds a semigroup of exponents E_X as in (2.1.1.2) with finitely many generators, say $\mu^1, \ldots, \mu^r$, in M. Then, for the cone $\gamma := \operatorname{cone}(\mu^1, \ldots, \mu^r)$, the variety X is isomorphic to X^γ. For more details, see Lemma 2.3.1. □

Whereas the above elements $\mu^1, \ldots, \mu^r$ of E also generate the corresponding cone γ, the converse need not be true, even if the generating vectors of the cone are primitive; cf. example 2.2.7 (2).

2.2.6 Remark. For the semigroup $E_\gamma := \gamma \cap M$ cut out by a strongly convex cone γ, there is a canonical minimal system of generators, sometimes called a *Hilbert basis*: It is the set $\dot{E} \setminus (\dot{E} + \dot{E})$ of *indecomposable* elements in E, with $\dot{E} := E \setminus \{0\}$.

In the two-dimensional case, it consists of those primitive lattice vectors μ^i which lie on the boundary of the (unbounded) "polyhedron" $K := \operatorname{conv}(\dot{E})$. This can be seen as follows: According to the regularity criterion in Corollary 3.1.6, two neighbouring points μ^i, μ^{i+1} generate a regular cone, so each lattice point in this cone is a linear combination of μ^i and μ^{i+1}.

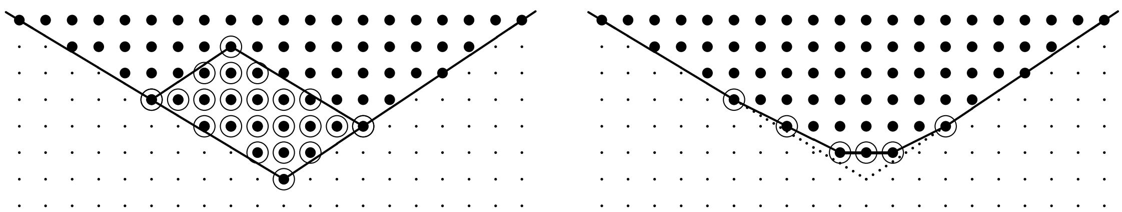

Figure 5: Fundamental cell P_γ and polyhedron K_γ for $\gamma = \operatorname{cone}(-5e_1+3e_2, 3e_1+2e_2)$

We illustrate the correspondence $X = X^\gamma \longleftrightarrow \gamma$ between affine toric varieties and M-cones with our basic examples from 2.1.4:

2.2.7 Example. (1) The linear space $X = \mathbb{C}^n$ corresponds to $\gamma = \operatorname{cone}(e_1, \ldots, e_n)$.

(2) The affine quadric cone Y corresponds to $\gamma = \text{cone}(e_1, e_1{+}2e_2)$.

More generally, the toric surface Y_k (with $k \geqq 2$) discussed in Remark 1.2.2 corresponds to $\gamma = \text{cone}\big(e_1, (k{-}1)e_1{+}ke_2\big)$.

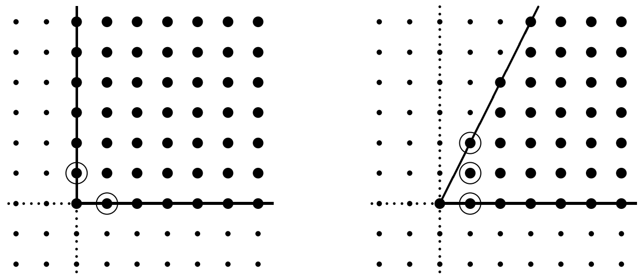

Figure 6: The M-cones cutting out $E_{\mathbb{C}^2}$ and E_Y (cf. Figure 3)

In Figure 6, the encircled dots represent a Hilbert basis of the pertinent semigroups $E_{\mathbb{C}^2}$ and E_Y.

(3) The Segre cone Z corresponds to $\gamma = \text{cone}(e_1, e_2, e_1{+}e_3, e_2{+}e_3)$ with generators forming a Hilbert basis (see Figure 7).

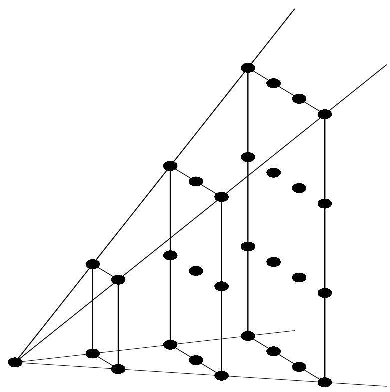

Figure 7: The M-cone $\gamma = \text{cone}(e_1, e_2, e_1{+}e_3, e_2{+}e_3)$ cutting out E_Z

(III) Dual cones and the "covariant" description

For a truly elegant and powerful geometric description of toric varieties, it is essential to complement the correspondence of the Anti-Equivalence Theorem 2.2.5 with a "covariant" version, which is obtained by dualization of cones.

For a lattice L, its *dual lattice* is defined as $L^* := \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$:

2.2.8 Definition. *For an L-cone γ, its* **dual** *is the cone*

$$\gamma^\vee := \big\{ u \in (L^*)_{\mathbb{R}} \ ; \ \langle u, \gamma \rangle \geqq 0 \big\} ,$$

where $\langle u, \gamma \rangle \geq 0$ means that $\langle u, v \rangle \geq 0$ holds for every $v \in \gamma$.

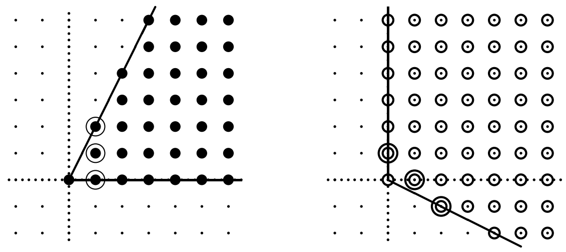

Figure 8: A pair of dual cones $\gamma, \gamma^\vee$ for dual lattices L, L^*

2.2.9 Remark. The dualization of cones enjoys the following properties:

(1) The dual of an L-cone is an L^*-cone, so dualization defines a map $\mathfrak{C}_L \to \mathfrak{C}_{L^*}$.

(2) Dualization is inclusion-reversing, i.e., $\delta \subseteq \gamma$ implies $\delta^\vee \supseteq \gamma^\vee$.

(3) Dualization is involutive, i.e., $(\gamma^\vee)^\vee = \gamma$.

(4) If γ is a linear subspace, then $\gamma^\vee = \gamma^\perp$ holds.

(5) $(\delta+\gamma)^\vee = \delta^\vee \cap \gamma^\vee$ and $(\delta \cap \gamma)^\vee = \delta^\vee + \gamma^\vee$.

(6) $\dim \sigma^\vee = n - \dim\bigl(\sigma \cap (-\sigma)\bigr)$. □

In particular, according to properties (3) and (6), there is a one-to-one correspondence between the objects of the category $\mathfrak{S}\mathfrak{C}_L$ of strongly convex L-cones, and of the category $\mathfrak{C}_{L^*,n}$ of full-dimensional L^*-cones. Applying this observation to the dual lattices M and N, we obtain:

2.2.10 Corollary. *The dualization of cones*

$$\mathfrak{S}\mathfrak{C}_N \longleftrightarrow \mathfrak{C}_{M,n}\ ,\ \sigma \longleftrightarrow \sigma^\vee$$

is an anti-equivalence of categories.

Combining this result with the Anti-Equivalence Theorem 2.2.5, we achieve the construction of the bridge, which is fundamental for the toric geometry:

2.2.11 Corollary (Equivalence Theorem). *The functor*

$$\mathfrak{S}\mathfrak{C}_N \longrightarrow \mathfrak{A}\mathfrak{T}\mathfrak{V}_{\mathbb{T}}\ ,\ \sigma \longmapsto X_\sigma := X^{\sigma^\vee} = \mathrm{Sp}\bigl(\mathbb{C}[\sigma^\vee \cap M]\bigr)$$

is a (covariant) equivalence of categories,

2.2.12 Convention. In view of the above correspondence, *all N-cones considered in the sequel will usually be assumed to be* ***strongly convex***. The symbol σ will denote such a cone.

We illustrate the correspondence $X = X_\sigma \longleftrightarrow \sigma$ between affine toric varieties and N-cones again with our basic examples of 2.1.4.

2.2.13 Example. (1) The linear space $X = \mathbb{C}^n$ corresponds to $\sigma = \operatorname{cone}(f_1, \dots, f_n)$.

(2) The affine quadric cone Y corresponds to $\sigma = \operatorname{cone}(2f_1 - f_2, f_2)$ (see Figure 9).

More generally, the toric surface Y_k (with $k \geqq 2$) discussed in Remark 1.2.2 corresponds to $\sigma = \operatorname{cone}\big(kf_1 - (k-1)f_2, f_2\big)$.

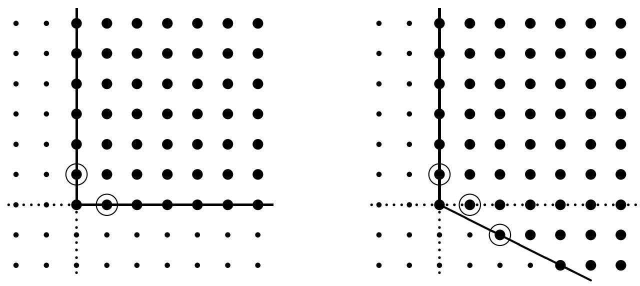

Figure 9: The N-cones for $\mathbb{C}^2$ and Y

(3) The Segre cone Z corresponds to $\sigma = \operatorname{cone}(f_1, f_2, f_3, f_1 + f_2 - f_3)$.

Let us also give an example for the correspondence of morphisms in Corollary 2.2.11 that at the same time exhibits the important role played by faces:

2.2.14 Example. To the face relation $\tau \preceq \sigma$ of N-cones corresponds an open embedding $X_\tau \to X_\sigma$ of affine toric varieties.

Proof. According to Remark 2.2.9, the "face equality" $\tau = \sigma \cap \mu^\perp$ (with some $\mu \in \sigma^\vee \cap M$) translates into $\tau^\vee = \sigma^\vee + \mathbb{R} \cdot \mu = \sigma^\vee + \operatorname{ray}(-\mu)$. This in turn implies that $\mathcal{O}(X_\tau) = \mathcal{O}(X_\sigma)[\chi^{-1}]$, where $\chi := \chi^\mu$. As a consequence, the morphism $X_\tau \to X_\sigma$ decomposes into an isomorphism $X_\tau \cong (X_\sigma)_\chi$ and the open inclusion of the principal open subset $(X_\sigma)_\chi$ into its ambient variety X_σ. $\square$

So far, we mainly have been interested in full-dimensional N-cones. Fortunately, their investigation is essentially sufficient for the general theory. This fact, to be applied time and again, is a consequence of the following result, where we use this notation:

(2.2.15.0) $$N_\sigma := N \cap \operatorname{lin}\sigma \hookrightarrow N \quad \text{and} \quad \mathbb{T}_\sigma := N_\sigma \otimes_{\mathbb{Z}} \mathbb{C}^* \hookrightarrow \mathbb{T} = N \otimes_{\mathbb{Z}} \mathbb{C}^* .$$

2.2.15 Proposition (Product decomposition). *For a d-dimensional N-cone σ, there exists a complementary subtorus $\mathbb{T}_{n-d}$ of $\mathbb{T}_\sigma$ in $\mathbb{T}_n$ and a $\mathbb{T}_\sigma$-toric variety Z_σ, such that*

(2.2.15.1) $$\mathbb{T} \;\cong\; \mathbb{T}_\sigma \times \mathbb{T}_{n-d} \quad \textit{and} \quad X_\sigma \;\cong\; Z_\sigma \times \mathbb{T}_{n-d}\,,$$

endowed with the product action, where $\mathbb{T}_{n-d}$ acts on itself by translation.

Proof. Since σ is an N-cone, the lattice N_σ is a saturated sublattice of rank d in N. Hence, the residue class group N/N_σ is free of rank $n-d$, and hence, the sublattice N_σ admits a complement N' in N. We then choose $\mathbb{T}_{n-d} := N' \otimes_{\mathbb{Z}} \mathbb{C}^*$. Eventually, we define Z_σ as the $\mathbb{T}_\sigma$-toric variety associated to the cone σ, considered as a (full-dimensional) cone in $(N_\sigma)_{\mathbb{R}}$. □

The factors Z_σ and $\mathbb{T}_{n-d}$ actually are naturally embedded in X_σ as closed subvarieties; see Remark 2.3.8 for details.

A word of warning may be in order concerning the role of the lattice in the correspondence between cones and affine toric varieties, a role that is not apparent from the usual notation:

2.2.16 Remark. Given a lattice $N \cong \mathbb{Z}^n$, then every finite collection $v_1, \ldots, v_r$ in the rational vector space $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ that includes a vector space basis generates a new lattice $\widetilde{N} := \sum_{i=1}^{r} \mathbb{Z}{\cdot}v_i$ (in general, this sum is not direct). The two lattices intersect in a sublattice of finite index (such lattices are called *commensurable* in the associated real vector space $N_{\mathbb{R}}$).

An N-cone σ then also is an $\widetilde{N}$-cone, here denoted by $\widetilde{\sigma}$ for distinction. Disregarding the primitivity condition, one may even choose the same set of generators for σ and $\widetilde{\sigma}$. Obviously, the resulting affine toric varieties X and $\widetilde{X}$ are not isomorphic if $N \neq \widetilde{N}$, since then $\mathbb{T} \neq \widetilde{\mathbb{T}}$. Isomorphy may even fail when disregarding the torus action and only considering the underlying structures of affine algebraic varieties:

2.2.17 Example. Let $N = \mathbb{Z}^2$ be the standard lattice in $\mathbb{R}^2$ and σ, the cone spanned by $v_1 = 2f_1 - f_2$ and $v_2 = f_2$ as in Example 2.2.13 (2). Hence, $X_{(N,\sigma)}$ is the singular two-dimensional quadric cone Y. With respect to the sublattice $\widetilde{N} := \mathbb{Z}{\cdot}v_1 \oplus \mathbb{Z}{\cdot}v_2$, however, the cone is spanned by a lattice basis, and the resulting variety $X_{(\widetilde{N},\widetilde{\sigma})}$ is the affine plane $\mathbb{C}^2$.

These two surfaces are even topologically distinct since the local fundamental group of Y at the origin is the cyclic group C_2. □

This example fits into an important generalization of the equivalence of categories described in Remark 1.3.4:

2.2.18 Remark. Given two lattices N_1, N_2 and N_i-cones σ_i, a *morphism of lattice cones* $(N_1, \sigma_1) \to (N_2, \sigma_2)$ is a lattice homomorphism $\varphi\colon N_1 \to N_2$ such that $\varphi_{\mathbb{R}}(\sigma_1) \subsetneqq \sigma_2$ holds. Then the homomorphism of tori $\mathbb{T}(\varphi)\colon \mathbb{T}_{N_1} \to \mathbb{T}_{N_2}$ extends to a toric morphism $X(\varphi)\colon X_{(N_1,\sigma_1)} \to X_{(N_2,\sigma_2)}$. The correspondence

(2.2.18.1) $$X\colon (N, \sigma) \mapsto X_{(N,\sigma)}\,,\ \varphi \mapsto X(\varphi)$$

is an *equivalence* beween the categories of (strictly convex) lattice cones and of affine toric varieties.

We come back to the last example to illustrate this correspondence:

2.2.19 Example (2.2.17 continued). The inclusion of lattices $\iota\colon \widetilde{N} \hookrightarrow N$ defines a morphism of lattice cones $(\widetilde{N}, \widetilde{\sigma}) \to (N, \sigma)$. To describe the corresponding morphism of toric varieties, we identify $\mathbb{T}_N$ and $\mathbb{T}_{\widetilde{N}}$ with $(\mathbb{C}^*)^2$ by the choice of the above bases. According to Example 1.3.5, the morphism of tori $\mathbb{T}(\iota)$ takes the form $(s, t) \mapsto (s^2, t/s)$. Composed with the embedding of $\mathbb{T}_N$

into $X_{(N,\sigma)} = Y$ given by the orbit map $(u,v) \cdot (1,1,1) = (u, uv, uv^2)$ of Example 1.2.1 (2), we obtain the map $(\mathbb{C}^*)^2 \hookrightarrow Y$, $(s,t) \mapsto (s^2, st, t^2)$ that clearly extends to a map from $\mathbb{C}^2 = X_{(\widetilde{N},\widetilde{\sigma})}$ to $X_{(N,\sigma)}$.

2.3 Cones and orbit structure

We now analyse how the orbit structure of an affine toric variety X_σ can be read off from the (strongly convex) N-cone σ. The following facts turn out to be easy, but crucial:

2.3.1 Lemma. *Let σ be an N-cone, moreover, let $\nu \in N$ and $\mu \in M$. Then:*

(1) $\mu \in \sigma^\vee \iff \chi^\mu$ extends to a regular function on X_σ.

(2) $\nu \in \sigma \iff \lambda_\nu \colon \mathbb{C}^* \longrightarrow \mathbb{T} \Subset X_\sigma$ extends to a morphism $\mathbb{C} \longrightarrow X_\sigma$.

Proof. (1) is true by definition.

(2) Using (1), a system of semigroup generators $\mu^i \in \sigma^\vee \cap M$, $i = 1, \ldots, r$, with corresponding characters $\chi_i := \chi^{\mu^i}$ defines a closed embedding

$$X \hookrightarrow \mathbb{C}^r\,,\ x \mapsto \big(\chi_1(x), \ldots, \chi_r(x)\big)\,, \tag{2.3.1.1}$$

which is equivariant with respect to the torus action $tz = \big(\chi_1(t)z_1, \ldots, \chi_r(t)z_r\big)$ of (2.1.3.2) on $\mathbb{C}^r$. The proof now follows from the equivalence of these five properties:

(a) The one-parameter subgroup λ_ν extends to a morphism $\mathbb{C} \to X_\sigma$;

(b) The group homomorphisms

$$\chi^{\mu^i} \circ \lambda_\nu : \mathbb{C}^* \longrightarrow \mathbb{C}^* \Subset \mathbb{C}\,,\ s \mapsto s^{\langle \mu^i, \nu \rangle}$$

extend to regular functions $\mathbb{C} \to \mathbb{C}$,

(c) $\langle \mu^i, \nu \rangle \geq 0$ for $i = 1, \ldots, r$; (d) $\langle \sigma^\vee, \nu \rangle \geq 0$; (e) $\nu \in \sigma = (\sigma^\vee)^\vee$. ☐

By the above equivalence (2), given an affine toric variety, we may recover the set of lattice points in the defining cone and thus, recover the cone:

2.3.2 Remark. For an affine toric variety X, the set of lattice points in its defining N-cone σ_X is explicitly given as follows:

$$\sigma_X \cap N \;=\; \{\nu \in N\ ;\ \lambda_\nu(0) \text{ exists in } X\}\,.$$

In the preceding formula, we have adopted the following convenient notation:

2.3.3 Convention. Let X be a (not necessarily affine) toric variety and $\lambda \in \mathbb{Y}(\mathbb{T})$, a one-parameter subgroup. If λ extends to a morphism from $\mathbb{C}$ to X, then the limit

$$\lambda(0) := \lim_{s \to 0} \lambda(s) \tag{2.3.3.1}$$

exists in X. We only use the symbol $\lambda(0)$ in this situation.

In Example 2.2.14, we have seen that the inclusion $\tau \preceq \sigma$ of a face induces an open embedding $X_\tau \Subset X_\sigma$ of affine toric varieties. We are now ready to prove the converse, a result that plays an essential role for the gluing of affine toric varieties to general ones:

2.3.4 Proposition. *Let $\sigma' \subseteq \sigma$ be an inclusion of N-cones. Then the induced morphism $X_{\sigma'} \to X_\sigma$ is an open embedding (if and) only if $\sigma' \preceq \sigma$.*

Proof. We may interpret $U := X_{\sigma'}$ as an open subset of $X := X_\sigma$. Since $X \setminus U$ is a closed $\mathbb{T}$-invariant subvariety, its vanishing ideal in $\mathcal{O}(X)$ is generated by finitely many characters $\chi_i = \chi^{\mu^i} \in S_X$. As a consequence, U is the union of the principal open toric subvarieties $X_{\chi_i} = X_{\tau_i}$ corresponding to the faces $\tau_i = \sigma \cap (\mu^i)^\perp$ (cf. Example 2.2.14). According to Remark 2.3.2, the analoguous relation $\sigma' = \bigcup \tau_i$ holds. Hence, there is a face $\tau_i \preceq \sigma$ satisfying $\dim \tau_i = \dim \sigma'$. Together with $\tau_i \subseteq \sigma' \subseteq \sigma$, this implies that σ' is included in $\mathrm{lin}(\tau_i) \cap \sigma = \tau_i$, thus proving $\sigma' = \tau_i$. $\square$

The one-to-one correspondence between faces and affine open toric subvarieties thus established actually is only one aspect of a larger picture: The combinatorial face structure of a cone also corresponds to the orbit structure and the structure of invariant irreducible closed subvarieties.

2.3.5 Theorem. *Let σ be an N-cone.*

(1) There is a one-to-one correspondence between the following sets:

(a) The set $\Delta(\sigma) := \{\tau \subset N_\mathbb{R}\,;\, \tau \preceq \sigma\}$ of faces of σ,

(b) the set $X_\sigma/\mathbb{T} := \{\mathbb{T}{\cdot}x\,;\, x \in X_\sigma\}$ of $\mathbb{T}$-orbits in X_σ,

(c) the set of non-empty closed irreducible $\mathbb{T}$-invariant subvarieties,

(d) the set of open affine toric subvarieties.

(2) This correspondence is explicitly given as follows: To a face $\tau \preceq \sigma$, we first associate a "base point"

$$x_\tau := \lambda_\nu(0) \in X_\sigma\,,$$

where ν is an arbitrary lattice vector in the relative interior τ°. Then the orbit, the closed irreducible $\mathbb{T}$-invariant subvariety, and the affine open toric subvariety corresponding to τ are

$$\mathbb{O}_\tau := \mathbb{T}{\cdot}x_\tau\,, \quad \mathbb{V}_\tau := \overline{\mathbb{O}}_\tau\,, \ \text{and} \quad X_\tau\,,$$

respectively. Here the closure of $\mathbb{O}_\tau$ is taken with respect to X_σ, and X_τ is identified with the image of the open embedding into X_σ.

(3) The correspondence $\tau \longleftrightarrow \mathbb{V}_\tau$ is inclusion-reversing. Each orbit $\mathbb{O}_\tau$ is a locally closed subvariety in X_σ, and it is isomorphic to a torus of the complementary dimension

$$\dim \mathbb{O}_\tau = \dim \mathbb{V}_\tau = \dim \sigma - \dim \tau\,.$$

Moreover, this orbit is the unique closed orbit in X_τ; on the other hand, it is open in its closure

$$\mathbb{V}_\tau \;=\; \bigcup_{\tau \preceq \tau' \preceq \sigma} \mathbb{O}_{\tau'}\,,$$

the union of $\mathbb{O}_\tau$ and finitely many orbits of strictly smaller dimension.

(4) Dually, the correspondence $\tau \longleftrightarrow X_\tau$ is inclusion-preserving, and the affine open toric subvariety

$$X_\tau \;=\; \bigcup_{\tau' \preceq \tau} \mathbb{O}_{\tau'}$$

is the union of $\mathbb{O}_\tau$ and finitely many orbits of strictly larger dimension.

The reader might wish to keep in mind that the notation $\mathbb{V}_\tau$ has no "absolute" meaning, since the closure depends on the ambient toric variety X_σ. – Before proving the theorem, we first add a few comments: The zero cone o corresponds to the "big" orbit $\mathbb{O}_o = \mathbb{T}$ (the embedded torus), whereas the orbit $\mathbb{O}_\sigma$ corresponding to a full-dimensional cone σ consists of precisely one point x_σ, which is then the unique fixed point of the affine toric variety X_σ.

Next, we look at a basic example (see Figure 10 for $n = 2$):

2.3.6 Example. Assume that $\sigma = \operatorname{cone}(\nu^1, \ldots, \nu^n)$ is spanned by a lattice basis of N. Denote by $\mu^1, \ldots, \mu^n$ the corresponding dual basis of M. Then X_σ is isomorphic to the linear space $\mathbb{C}^n$, endowed with the action (2.1.3.2). The base point is $(1, \ldots, 1)$. The faces of σ are of the form $\sigma_J := \operatorname{cone}(\nu^j\,;\, j \in J)$ given by the subsets J of $\{1, \ldots, n\}$. To σ_J corresponds the orbit

$$O_{\sigma_J} = \big\{(z_1, \ldots, z_n)\,;\; z_i = 0 \iff i \in J\big\}\,,$$

the closure of which is a coordinate subspace. The pertinent orbit base point x_{σ_J} is the unique point of O_{σ_J} with $z_i = 1$ for $i \notin J$. $\square$

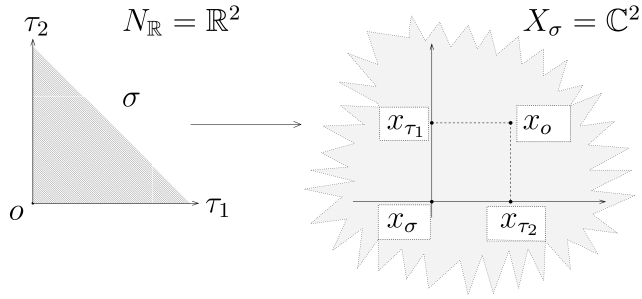

Figure 10: Faces of $\operatorname{cone}(f_1, f_2)$ and corresponding orbit base points in $\mathbb{C}^2$

If σ is a general d-cone, then the orbit O_σ is isomorphic to a torus of the complementary dimension $n - d$:

2.3.7 Corollary. *In the product decomposition* (2.2.15.1)*, the variety Z_σ has a (unique) fixed point z_σ, and the orbit O_σ corresponds to $\{z_\sigma\} \times \mathbb{T}_{n-d}$ (with isotropy group $\mathbb{T}_\sigma$ at each point).* □

In order to reduce the proof of Theorem 2.3.5 to the case of full-dimensional cones, we add the following complement to the product decomposition of Proposition 2.2.15:

2.3.8 Remark. In the affine toric variety X_σ defined by a d-cone σ, let x_o and x_σ denote the base points of the orbits $O_o \cong \mathbb{T}$ and O_σ, respectively. In the product decomposition $(X_\sigma, \mathbb{T}) \equiv (Z_\sigma, \mathbb{T}_\sigma) \times \mathbb{T}_{n-d}$ of (2.2.15.1), the two factors admit a natural closed embedding into X_σ as follows:

$$Z_\sigma \cong \overline{\mathbb{T}_\sigma \cdot x_0} \quad \text{and} \quad \mathbb{T}_{n-d} \cong \mathbb{T}\cdot x_\sigma = O_\sigma \,. \qquad \square$$

Proof of Theorem 2.3.5. We proceed by induction on $d := \dim \sigma$. For $d = 0$, there is nothing to prove since then $\sigma = o$, so $X_o = \mathbb{O}_o = \mathbb{V}_o = \mathbb{T}$ because of $o^\vee = M_\mathbb{R}$. For $d > 0$, we may assume that $d = n$: By the product decomposition $X_\sigma \cong Z_\sigma \times \mathbb{T}_{n-d}$ of Proposition 2.2.15 together with Remark 2.3.8, $\mathbb{T}$-objects like orbits, invariant irreducible closed subvarieties, and affine open invariant subvarieties in X_σ correspond to the respective $\mathbb{T}_\sigma$-objects in Z_σ via $Y \mapsto Y \cap Z_\sigma$ and vice versa.

For convenience, we fix a closed equivariant embedding $X \hookrightarrow \mathbb{C}^r$ given by the character functions $\chi_i = \chi^{\mu^i} \in S_X$ corresponding to a system of non-zero generators $\mu^1, \ldots, \mu^r$ for E_X as in Remark 2.1.3. These characters also describe the torus action on the ambient space $\mathbb{C}^r$. Since they are non-trivial, the origin is the only fixed point on $\mathbb{C}^r$. We have to show that $0 \in X_\sigma$.

By induction hypothesis, the theorem holds if we replace σ with any proper face τ. From Example 2.2.14, we know that each natural morphism $X_\tau \to X_\sigma$ is an open $\mathbb{T}$-equivariant embedding. We now consider the open "quasi-affine" toric subvariety $X_{\partial\sigma} := \bigcup_{\tau \precneqq \sigma} X_\tau$ of X_σ and its complement, the closed invariant subvariety $F := X_\sigma \backslash X_{\partial\sigma}$. The proof of the theorem will essentially be deduced from the following three properties:

(i) $F = \{0\}$, which thus is the unique fixed point on X_σ,

(ii) F is included in each orbit closure $\overline{\mathbb{O}}_\tau$ (taken with respect to X_σ),

(iii) the equality $\lambda_\nu(0) = 0$ holds for any $\nu \in \sigma^\circ$.

In fact, properties (iii) and (i) imply that the base point $x_\sigma = 0$ is well defined and that its orbit $\mathbb{O}_\sigma = \mathbb{T}\cdot 0 = F$ is closed, so $\mathbb{O}_\sigma$ equals $\mathbb{V}_\sigma$, and it has the asserted dimension $n - \dim \sigma = 0$. This proves part (2) for $\tau = \sigma$.

To establish the correspondence between faces and orbits, we fix an arbitrary non-zero orbit $\mathbb{O} = \mathbb{T}\cdot x$ through some point $x \in X \setminus \{0\} = X_{\partial\sigma}$. We thus have $x \in X_\tau$ for a suitable proper face τ of σ, so by induction hypothesis, there exists a face $\tau_0 \preceq \tau$ with $\mathbb{O} = \mathbb{O}_{\tau_0} = \mathbb{T}\cdot x_{\tau_0}$. We may thus replace x with the orbit base point x_{τ_0}. By hypothesis, the unicity of τ_0 is valid in every affine open subvariety $X_{\tau'}$ that includes $\mathbb{O}$, so it is valid in X_σ.

For the asserted correspondence between faces and invariant irreducible closed subvarieties, we consider such subvariety $A \hookrightarrow X_\sigma$ with $\dot{A} := A \backslash F \neq \emptyset$. Then $\dot{A} = \bigcup_{\tau \in \partial\sigma}(A \cap X_\tau)$ implies the equality $\dim(A \cap X_\tau) = \dim A$ for some face $\tau \in \partial\sigma$. Again by induction hypothesis, there is a face $\tau_0 \preceq \tau$ with $A \cap X_\tau = \mathbb{V}(\tau_0) = \overline{\mathbb{O}}(\tau_0)$, the closure being taken in X_τ; moreover, $\mathbb{O}(\tau_0)$ is open in $A \cap X_\tau$ and hence, has the same dimension. Since a $\mathbb{T}$-orbit is irreducible, this readily implies that A is the closure of $\mathbb{O}(\tau_0)$ in X_σ. The unicity of τ_0 is seen as above.

We still have to prove the properties (i)–(iii): The coordinate functions $\chi_i \in S_X$ generate $\mathcal{O}(X_\sigma)$ and thus have at most one common zero. On the other hand, for any lattice vector $\nu \in N \cap \sigma^\circ$ and for each μ^i, the inner product satisfies $\langle \mu^i, \nu \rangle > 0$, and this implies that the limit $\lambda_\nu(0) = 0$ exists in X_σ. □

From the proof of Theorem 2.3.5 and Proposition 2.2.15, we deduce the following consequence:

2.3.9 Corollary. *In an affine toric variety X_σ, the unique closed orbit O_σ is a $\mathbb{T}$-equivariant deformation retract of X_σ.*

Proof. It clearly suffices to consider the case of a full-dimensional cone σ, or, in other words, the case where X_σ has a (unique) fixed point x_σ. We consider an equivariant embedding $X_\sigma \hookrightarrow \mathbb{C}^r$, $x \mapsto \big(\chi_1(x), \ldots, \chi_r(x)\big)$ as in (2.3.1.1) with $\chi_i := \chi^{\mu^i}$. To each fixed lattice vector $\nu \in N$ corresponds a set of exponents $k_i := \langle \mu^i, \nu \rangle$ with $\chi_i\big(\lambda_\nu(s)\big) = s^{k_i}$ and thus, an induced $\mathbb{C}^*$-action

$$\mathbb{C}^* \times \mathbb{C}^r \longrightarrow \mathbb{C}^r\,,\ (s,z) \longmapsto \big(\chi_i\big(\lambda_\nu(s)\big)\cdot z_i\big)_{i=1,\ldots,r} = (s^{k_1} z_1, \ldots, s^{k_r} z_r)$$

on $\mathbb{C}^r$ that respects X_σ. For a lattice vector ν in the relative interior σ° (which here coincides with the topological interior), these exponents satisfy the strict inequality $k_i > 0$. It follows that the corresponding $\mathbb{C}^*$-action extends to $s = 0$. Restricting to scalars $s \in [0,1]$ yields a $\mathbb{T}$-equivariant (why?) homotopy, providing an equivariant retraction by deformation to the fixed point. □

To study the geometry of orbit closures in an affine toric variety, let τ be a face of a cone σ. We recall from Corollary 2.3.7 that the subtorus $\mathbb{T}_\tau$ of $\mathbb{T}$ is the isotropy group at each point of O_τ. Hence, restricting the $\mathbb{T}$-action to O_τ provides an action of the quotient torus $\mathbb{T}/\mathbb{T}_\tau$ on O_τ.

2.3.10 Proposition. *Every orbit closure $\overline{O}_\tau$ in X_σ is an affine $\mathbb{T}/\mathbb{T}_\tau$-toric variety:*

$$\overline{O}_\tau \cong X_{\sigma/\tau},$$

where $\sigma/\tau := \pi(\sigma)$ denotes the image cone for the quotient map $\pi\colon N_\mathbb{R} \to (N/N_\tau)_\mathbb{R}$.

Proof. We may regroup the "weight subspaces" $\mathbb{C}\cdot\chi^\mu$ occuring in formula (2.1.1.3) according to the decomposition $E = F \mathbin{\dot\cup} J$ of the exponent semigroup $E := M \cap \sigma^\vee$ into the sub-semigroups

$$F := E \cap \tau^\perp \quad \text{and} \quad J := E \setminus F$$

(where J actually is an "ideal", i.e., $J + E \subsetneqq J$). We claim that this regrouping yields a direct sum decomposition

$$\mathcal{O}(X_\sigma) \;=\; \mathcal{O}(X_\sigma)^{\mathbb{T}_\tau} \oplus I(\overline{O}_\tau) \tag{2.3.10.1}$$

of the coordinate ring into the subalgebra of all $\mathbb{T}_\tau$-invariant functions

$$\mathcal{O}(X_\sigma)^{\mathbb{T}_\tau} \;=\; \bigoplus_{\mu\in F} \mathbb{C}\cdot\chi^\mu$$

and the ideal of $\overline{O}_\tau$ in X_σ. To that end, it suffices to verify that

$$I(\overline{O}_\tau) \;=\; \bigoplus_{\mu\in J} \mathbb{C}\cdot\chi^\mu\,. \tag{2.3.10.2}$$

We choose an arbitrary lattice vector $\nu \in \tau^\circ \cap N$. For a character $\chi = \chi^\mu \in \mathcal{O}(X_\sigma) \cap \mathbb{X}(\mathbb{T})$ (i.e., with exponent $\mu \in E$), we have the following chain of equivalences $\big($since $x_\tau = \lambda_\nu(0)\big)$:

$$\chi \in I(\overline{O}_\tau) \iff \chi(x_\tau) = 0 \iff \langle \mu, \nu\rangle > 0 \iff \mu \notin \tau^\perp\ .$$

As a consequence of formulae (2.3.10.1) and (2.3.10.2), factoring out the ideal yields an isomorphism

$$\mathcal{O}(\overline{O}_\tau) \;\cong\; \mathcal{O}(X_\sigma)^{\mathbb{T}_\tau} \;=\; \bigoplus_{\mu\in F} \mathbb{C}\cdot\chi^\mu\,.$$

Under the natural dual pairing of M and N, an arbitrary exponent vector $\mu \in F$ vanishes on the sublattice N_τ. It thus corresponds to a unique linear form $\overline{\mu}$ on the quotient lattice N/N_τ. Such a linear form is the exponent of a character for the quotient torus $\mathbb{T}/\mathbb{T}_\tau$, and the condition $\mu \in \sigma^\vee$ then is equivalent to $\overline{\mu}$ belonging to the dual cone of the image $\sigma/\tau = \pi(\sigma)$. □

Quotients of affine toric varieties

To finish the discussion of affine toric varieties, we briefly come back to quotients, continuing Remark 1.2.3 (3). We use this notation: Let G be a closed subgroup of a torus $\mathbb{T}$, and let $q\colon \mathbb{T} \twoheadrightarrow \mathbb{T}' := \mathbb{T}/G$ be the quotient map. We denote by $N' \cong \mathrm{Hom}(\mathbb{C}^*, \mathbb{T}')$ the lattice of one-parameter subgroups of $\mathbb{T}'$, and by $\widetilde{q} = dq\colon N \to N'$, the lattice homomorphism corresponding to q. Its kernel $N_0 := \ker(dq)$ is a saturated sublattice of N. Finally, we consider the sublattice $K = N_0^\perp \subset M$ corresponding to G introduced in Remark 1.3.4 (4). We note that for each $\mu \in K$, the character χ^μ passes to the quotient torus $\mathbb{T}'$; in fact, the lattice K is naturally identified with the lattice $M' = (N')^*$ of characters of $\mathbb{T}'$.

2.3.11 Remark. Let σ be a full-dimensional (strongly convex) N-cone with dual M-cone $\sigma^\vee$. The image $\sigma' := q_{\mathbb{R}}(\sigma)$ is an N'-cone, but it may fail to be *strongly* convex (this is illustrated by the "elliptic" case of the following example). The strong convexity of σ' is equivalent to the condition that N_0 does not intersect the relative interior σ°.

If σ' is strongly convex, then dq induces a map of cones $(N, \sigma) \to (N', \sigma')$, so the surjective homomorphism $q\colon \mathbb{T} \twoheadrightarrow \mathbb{T}'$ extends to a q-toric map

$$\varphi\colon X := X_{(N,\sigma)} \longrightarrow X_{(N',\sigma')} =: X' .$$

This map is surjective, since each orbit base point $x_{\tau'}$ of X' lies in the image: For a face $\tau' \preceq \sigma'$, there is a face $\tau \preceq \sigma$ and a vector $\nu \in \tau^\circ$ such that $dq(\nu)$ lies in the relative interior of τ'. Then φ maps $x_\tau = \lambda_\nu(0)$ to $\lambda_{dq(\nu)}(0) = x_{\tau'}$.

The comorphism $\varphi^*\colon \mathcal{O}(X') \to \mathcal{O}(X)$ induces an isomorphism between $\mathcal{O}(X')$ and the ring

$$\mathcal{O}(X)^G := \{f \in \mathcal{O}(X)\,;\ \forall\, t \in G : f^t = f\}$$

of G-invariant functions on X. This provides an identification

$$X' \;\cong\; X/\!/G := \mathrm{Sp}\big(\mathcal{O}(X)^G\big) \tag{2.3.11.1}$$

with the *algebraic quotient* of X by G. The quotient morphism $X \twoheadrightarrow X/\!/G$ admits a factorisation

$$X \twoheadrightarrow X/G \twoheadrightarrow X/\!/G$$

through the topological orbit space.

As stated in Remark 1.2.3 (3), this orbit space need not be separated if G is not finite:

2.3.12 Example. We consider three closed embeddings $\iota_*\colon \mathbb{C}^* \hookrightarrow \mathbb{T}^2$ with the index $* = e, h, p$ indicating "elliptic, hyperbolic, parabolic", given by $\iota_h\colon s \mapsto (s, s^{-1})$, $\iota_p\colon s \mapsto (s, 1)$, and $\iota_e\colon s \mapsto (s, s)$. We denote by $G = G_h := \iota_h(\mathbb{C}^*)$, G_p, and G_e the respective image group. The quotient map $q\colon \mathbb{T}_2 \twoheadrightarrow \mathbb{T}_2/G \cong \mathbb{C}^*$ is then given by a character $\chi^{\pm 1}$, namely $\chi_h(t,u) = tu$, $\chi_p(t,u) = u$, and $\chi_e(t,u) = tu^{-1}$.

The map of lattices $dq\colon \mathbb{Z}^2 \to \mathbb{Z}$ is represented by the (2×1)-matrices $(1,1)$ for G_h, $(0,1)$ for G_p, and $(1,-1)$ for G_e. The respective sublattices $\ker(dq)$ of one-parameter subgroups are $N_h = \mathbb{Z}{\cdot}(1,-1)$, $N_p := \mathbb{Z}{\cdot}(1,0)$, and $N_e = \mathbb{Z}{\cdot}(1,1)$.

Let $\sigma = \mathrm{cone}(f_1, f_2)$ be the standard lattice cone defining $X := \mathbb{C}^2$. The respective image cone $dq(\sigma)$ in $N'_{\mathbb{R}} \cong \mathbb{R}$ is $\sigma' = \mathrm{ray}(1)$ for G_h and G_p. For G_e, we obtain $\sigma' = \mathrm{cone}(1,-1) = N'_{\mathbb{R}}$, so it is not strongly convex; accordingly, the lattice point $(1,1) \in N_e$ lies in the relative interior of σ.

The $\mathbb{C}^*$-actions on X given by these groups are as follows:

(1) For G_h, we have the *hyperbolic* action $s{\cdot}(x,y) = (sx, s^{-1}y)$. Each "generic" orbit is a fibre of the quotient map $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2/\!/G_h \cong \mathbb{C}$, $(x,y) \mapsto xy$, so it is closed. The fibre $(xy = 0)$ consists of three orbits: the fixed point $(0,0)$ and the punctured coordinate axes $\mathbb{C}^* \times 0$ and $0 \times \mathbb{C}^*$. In the topological orbit space, these punctured coordinate axes are non-closed points having the fixed point in their closure. The map $\mathbb{C}^2/G_h \twoheadrightarrow \mathbb{C}^2/\!/G_h$ identifies these three points.

(2) For G_p, we get the *parabolic* action $s{\cdot}(x,y) = (sx, y)$. The quotient map $\varphi\colon \mathbb{C}^2 \twoheadrightarrow \mathbb{C}^2/\!/G_p \cong \mathbb{C}$ is given by $(x,y) \mapsto y$. Each fibre $\varphi^{-1}(y) = \mathbb{C} \times y$ consists of two orbits, namely the (non-closed) "pointed" horizontal line $\mathbb{C}^* \times \{y\}$, and the (closed) fixed point $(0,y)$. The topological orbit space consists of two copies of $\mathbb{C}$, say $0 \times \mathbb{C}$ and $1 \times \mathbb{C}$. The points in the first copy are closed, whereas the closure of any point $(1,y)$ in the second copy consists of the two points $(1,y), (0,y)$. The map $\mathbb{C}^2/G_p \twoheadrightarrow \mathbb{C}^2/\!/G_p$ identifies these two points.

(3) For G_e, we obtain the *elliptic* action $s\cdot(x,y) = (sx, sy)$. The fixed point $(0,0)$ is the only closed orbit; any other orbit is a "pointed" line. The topological orbit space consists of the (closed) unique fixed point and a projective line, with the closure of any point consisting of that point and the fixed point. The only invariant functions are the constants, so the algebraic quotient $\mathbb{C}^2/\!/G_e$ consists of a single point, with the obvious quotient map.

To end this discussion of quotients, we add a few further remarks that might help for a better understanding of the situation. Using the notation introduced above, we again assume that σ' is strongly convex.

2.3.13 Remark (2.3.11 continued). For each $\mathbb{T}'$-orbit $\mathbb{O}(\tau')$ in X', given by a face $\tau' \preceq \sigma'$, there is a unique face $\tau \preceq \sigma$ such that the $\mathbb{T}$-orbit $\mathbb{O}(\tau)$ in X is relatively closed in the preimage $\varphi^{-1}\big(\mathbb{O}(\tau')\big)$: We write $\tau' = \sigma' \cap (\mu')^{\perp}$ with a suitably chosen lattice vector $\mu' \in M' \cap (\sigma')^{\vee}$. The map $\chi^{\mu'} \circ q \colon \mathbb{T} \to \mathbb{C}^*$ is a character of $\mathbb{T}$; so it is of the form χ^{μ}, and its exponent $\mu = q^*(\mu')$ clearly lies in $\sigma^{\vee}$. It is not difficult to see that in fact, the $\mathbb{T}'$-orbit $\mathbb{O}(\tau')$ is the G-orbit space $\mathbb{O}(\tau)/G$.

If G is finite, then $\widetilde{q}\colon N \to N'$ is the inclusion of a sublattice of finite index $|G|$ (see item (4) in Remark 1.3.4). Any (strongly convex) N-cone also is an N'-cone; the face structure is of course independent of the lattice, so there is a one-to-one correspondence between $\mathbb{T}$-orbits and $\mathbb{T}'$-orbits. In that case, the induced map $X/G \twoheadrightarrow X/\!/G$ is a homeomorphism, so the algebraic quotient is just the topological orbit space. In such a situation, one calls $X/\!/G$ a *geometric quotient.*

We finally note that the algebraic quotient $X/\!/G := \mathrm{Sp}\big(\mathcal{O}(X)^G\big)$ is defined for any closed subgroup G of $\mathbb{T}$, and it always admits a natural $\mathbb{T}/G$-action. The strong convexity condition for the image cone $\sigma' = \widetilde{q}(\sigma)$ implies that the canonical morphism $\mathbb{T}/G \to X/\!/G$ is an open embedding, and conversely.

3 Toric Singularities

3.1 Local structure at a fixed point

The common theme of the notes collected in the present volume is the geometry of singularities. In our context, we have to study the singularities occuring on affine toric varieties. As a consequence of the product decomposition obtained in Proposition 2.2.15, it suffices to consider the variety defined by a full-dimensional N-cone. Here, the fixed point is the natural candidate for a singularity.

So first of all we have to find out under which conditions a fixed point is or is not singular:

3.1.1 Proposition. *Let σ be a full-dimensional (strongly convex) N-cone. Then the following statements are equivalent:*

(1) The affine variety X_σ is smooth.

(2) The unique fixed point x_σ of X_σ is a regular point.

(3) The M-cone $\sigma^\vee$ is spanned by a lattice basis $\mu^1, \dots, \mu^n$, i.e., $\sigma^\vee = \operatorname{cone}(\mu^1, \dots, \mu^n)$.

(4) The N-cone σ is spanned by a lattice basis $\nu^1, \dots, \nu^n$, i.e., $\sigma = \operatorname{cone}(\nu^1, \dots, \nu^n)$.

(5) $X_\sigma \cong \mathbb{C}^n$ with the base point $(1, \dots, 1)$ and the $\mathbb{T}$-action

$$t \cdot z = (\chi_1(t)z_1, \dots, \chi_n(t)z_n),$$

where $\chi_i = \chi^{\mu^i}$ for a lattice basis $\mu^1, \dots, \mu^n$ of M.

A cone satisfying these properties is called **regular** (see Definition 3.1.2 for the case of not full-dimensional cones).

Proof. Since the implication "(4) $\implies$ (5)" has been discussed in Example 2.2.13, the only non-trivial implication is "(2) $\implies$ (3)": The maximal ideal of all regular functions vanishing at x_σ is of the form

$$\mathfrak{m} \;=\; \bigoplus_{\mu \in \dot{E}} \mathbb{C}\cdot\chi^\mu$$

with $\dot{E} := E \setminus \{0\}$ for $E = \sigma^\vee \cap M$, so $\mathfrak{m}^2 = \bigoplus_{\mu \in \dot{E}+\dot{E}} \mathbb{C}\cdot\chi^\mu$. Hence, the Zariski tangent space $T_{x_\sigma}X_\sigma = (\mathfrak{m}/\mathfrak{m}^2)^*$ has the dual

$$\mathfrak{m}/\mathfrak{m}^2 \;\cong\; \bigoplus_{\mu \in \mathfrak{B}} \mathbb{C}\cdot\chi^\mu\,,$$

where $\mathfrak{B} := \dot{E} \setminus (\dot{E}+\dot{E})$ denotes the Hilbert basis of the semigroup E (cf. Remark 2.2.6). Now

$$\big|\mathfrak{B}\big| = \dim_{\mathbb{C}}(\mathfrak{m}/\mathfrak{m}^2) = \dim X_\sigma = n$$

since x_σ is a regular point. As a semigroup, E is generated by $\mathfrak{B}$ (why?); furthermore, we know that $M = E + (-E)$ by Remark 2.1.2 (2). Hence, the elements in $\mathfrak{B}$ generate M as a lattice and thus form a lattice basis, since there are only n of them. ☐

Somewhat more general are the cones spanned by linearly independent lattice vectors:

3.1.2 Definition. *A d-dimensional L-cone is called* **simplicial** *if it has exactly d edges. It is called* **regular** *if it is simplicial and if the primitive spanning vectors of the edges are part of a lattice basis of L.*

The name "simplicial" is to indicate that transversal sections of such a cone are $(d$–$1)$-simplices. (In [Ew], they are called "simplex cones", whereas "simplicial cone" there means a cone that has simplicial polytopes as cross-sections.) Two-cones are always simplicial, whereas cones of dimension $d \geqq 3$ in general are not. Our standard example is provided by $\sigma = \operatorname{cone}(f_1, f_2, f_3, f_1{+}f_2{-}f_3)$ defining the Segre cone Z of 2.2.13 (3) (see also 2.2.7 (3) and Figure 7 for its dual).

To a simplicial d-cone $\sigma = \mathrm{cone}(v_1, \dots, v_d)$, we associate the sublattice

$$\Gamma_\sigma \ := \ \sum_{i=1}^{d} L_i \subseteqq L_\sigma \quad \text{with} \quad L_i := L \cap \mathrm{lin}(v_i)$$

(recalling $L_\sigma := L \cap \mathrm{lin}(\sigma)$, cf. Proposition 2.2.15), generated by the lattice points on the edges (see Figure 11 for an example with $d = 2$). Then the regularity of σ can be

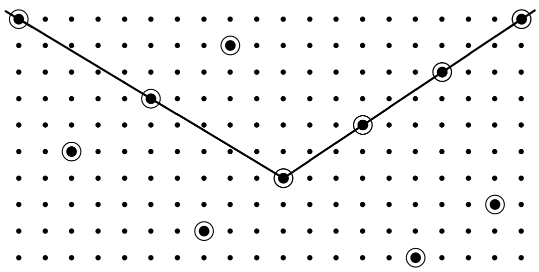

Figure 11: Sublattice Γ_σ for $\sigma = \mathrm{cone}(-5f_1+3f_2, 3f_1+2f_2)$

characterized by the equality $\Gamma_\sigma = L_\sigma$. More precisely, the deviation of a simplicial L-cone from regularity is measured by the index of this sublattice:

3.1.3 Definition (Multiplicity of a simplicial cone). *For a d-dimensional simplicial L-cone σ, the positive integer*

$$\mathrm{mult}(\sigma) := m_\sigma := [L_\sigma : \Gamma_\sigma] = |L_\sigma / \Gamma_\sigma|$$

is called its multiplicity.

It follows from the structure theorem for finitely generated abelian groups that the multiplicity of a full-dimensional simplicial L-cone is readily computed in terms of a primitive generating system:

3.1.4 Remark. If the generators $v_1, \dots, v_n \in L$ of a simplicial n-cone are primitive and $n = \mathrm{rank}(L)$, then, after fixing an isomorphism $L \cong \mathbb{Z}^n$,

$$\mathrm{mult}\big(\mathrm{cone}(v_1, \dots, v_n)\big) = \big|\det(v_1, \dots, v_n)\big| \, .$$

A topological interpretation of the multiplicity for full-dimensional cones is given in Corollary 3.1.9.

The next observation follows immediately from the definition of the multiplicity.

3.1.5 Remark (Geometric interpretation of the multiplicity). For a simplicial d-cone σ spanned by *primitive* lattice vectors $v_1, \dots, v_d \in L$, let

$$P = P(v_1, \dots, v_d) = \left\{ \sum_{i=1}^{d} t_i v_i \ ; \ 0 \leqq t_i < 1 \right\}$$

be the "half open" fundamental parallelotope of the sublattice Γ_σ. Then the multiplicity is given by the number of N-lattice points in P:

$$\operatorname{mult}(\sigma) \;=\; \#(P \cap L)\,. \qquad \square$$

As a useful consequence, one may check for regularity by counting lattice points:

3.1.6 Corollary (Geometric regularity criterion). *A simplicial d-cone σ as above is regular if its fundamental parallelotope P intersects the lattice L_σ only at the origin.*

In dimension $d = 2$, the same conclusion holds if we replace P with the d-simplex $\operatorname{conv}(0, v_1, \ldots, v_d)$.

Toric varieties given by a simplicial N-cone have a remarkably simple structure that we are now going to describe. In the two-dimensional case (where cones are always simplicial), we have already seen that the affine toric surface Y_k (with $k \geqq 2$) introduced in Remark 1.2.2 is a quotient $\mathbb{C}^2/C_k$ of the affine plane by a suitable action of a finite group, namely the cyclic group of order k. We want to show that an analoguous result, except for the cyclic structure of the group, holds for arbitrary affine toric varieties in the "simplicial class".

Using the product decomposition of Proposition 2.2.15, we again may assume that $\dim \sigma = n$. We then consider on $\mathbb{C}^n$ the action of $\mathbb{T} \cong (\mathbb{C}^*)^n$ by componentwise multiplication, thus identifying the elements in $\mathbb{T}$ with diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$:

3.1.7 Proposition. *For a full-dimensional simplicial N-cone σ, the toric variety X_σ is the algebraic quotient $\mathbb{C}^n/G := \operatorname{Sp}\bigl(\mathcal{O}(\mathbb{C}^n)^G\bigr)$ of $\mathbb{C}^n$ by the induced action of a finite subgroup $G \cong N/\Gamma_\sigma$ of the torus $\mathbb{T}$, so X_σ is just the usual orbit space of the action of G on $\mathbb{C}^n$.*

In the literature, varieties that locally are orbit spaces of a smooth manifold with respect to an action of a finite group often are called "orbifolds".

Proof. With respect to the sublattice $N' := \Gamma_\sigma$ of the lattice N, the cone $\sigma =: \sigma'$ is regular. By Proposition 3.1.1, the associated toric variety $X_{(N',\sigma')}$ is the affine space $\mathbb{C}^n$. The assertion now follows from our discussion of quotients in Remark 1.3.4 and 2.3.13. $\square$

The group $G \cong N/\Gamma_\sigma$ actually can be recovered from the toric variety by topological means, namely as fundamental group of the open invariant subvariety of regular points:

3.1.8 Remark. There is an isomorphism

$$G \;\cong\; \pi_1\bigl((X_\sigma)_{\mathrm{reg}}\bigr)\,.$$

Indication of Proof. We let $\nu^1, \ldots, \nu^n \in N$ denote the primitive generators of σ, and $\iota\colon f_i' \mapsto \nu^i$, the inclusion of the sublattice $\mathbb{Z}^n \cong N' := \Gamma_\sigma$ into N. On the side of tori, we have the finite morphism

$$\mathbb{T}(\iota) =: q\colon \mathbb{T}' \twoheadrightarrow \mathbb{T} \cong \mathbb{T}'/G\,,$$

with $\mathbb{T}' := \mathbb{T}_{N'}$ and $\mathbb{T} := \mathbb{T}_N$ for ease of notation. Then q extends to the quotient morphism

$$\varphi = X(\iota)\colon X' = \mathbb{C}^n \twoheadrightarrow \mathbb{C}^n/G \cong X \ ,$$

with $X' := X_{(N',\sigma')}$ and $X = X_{(N,\sigma)}$. Furthermore, let $Z \hookrightarrow \mathbb{C}^n$ be the set of points where the G-action has non-trivial isotropy. It suffices to show thatφ induces an unramified covering $\mathbb{C}^n \setminus Z \to (X_\sigma)_{\mathrm{reg}}$ which is the universal covering.

First of all, we verify that $\mathbb{C}^n \setminus Z$ is simply connected: We use the fact that a lattice vector $\nu \in N$ is primitive if and only if the corresponding one-parameter subgroup $\lambda_\nu\colon \mathbb{C}^* \to \mathbb{T}$ is an injective homomorphism. Since $\iota(f_i') = \nu^i$, this implies for each coordinate subtorus $\mathbb{T}_i := \lambda_{f_i'}(\mathbb{C}^*)$ of $\mathbb{T}_{N'}$ that the group $G \cap \mathbb{T}_i$ is trivial. Restated in other words: If a matrix in $G \subset \mathbb{T} \subset \mathrm{GL}_n(\mathbb{C})$ has the eigenvalue 1 with multiplicity at least $n-1$, then it is the identity; such a group G is called a *small* subgroup of $\mathrm{GL}_n(\mathbb{C})$. So the subset $Z \hookrightarrow \mathbb{C}^n$ is a union of coordinate subspaces of dimension at most $n-2$. Using the fact that spheres of dimension at least 2 are simply connected, the complement $\mathbb{C}^n \setminus Z$ can be seen to be simply connected as well.

Eventually, we have to show that the preimage of the singular locus $S(X_\sigma)$ coincides with Z. Since $Z = \varphi^{-1}\big(\varphi(Z)\big)$, this follows from the equality $S(X_\sigma) = \varphi(Z)$. The inclusion "$\subseteq$" is clear, while "$\supseteq$" follows from the fact that the set of points where a dominant morphism between equidimensional smooth varieties is not a local analytic isomorphism has codimension one. $\square$

3.1.9 Corollary. *The multiplicity of a full-dimensional simplicial N-cone σ satisfies*

$$\mathrm{mult}(\sigma) \ = \ \big|\pi_1\big((X_\sigma)_{\mathrm{reg}}\big)\big| \ . \qquad \square$$

3.1.10 Example. (1) Let $n = 2$. For coprime integers k, ℓ with $\ell > 0$, we consider $\sigma = \mathrm{cone}(\ell f_1 - k f_2, f_2)$ in $\mathbb{R}^2$. Then $q\colon \mathbb{T} \to \mathbb{T}$ of (3.1.7.1) satisfies $q(t_1, t_2) = (t_1^\ell, t_1^{-k} t_2)$; thus G is the cyclic subgroup of order ℓ in $\mathrm{GL}_2(\mathbb{C})$ generated by (ζ, ζ^k) with $\zeta := e^{2\pi i/\ell}$. The singularity $\mathbb{C}^2/G$ thus obtained is a *cyclic quotient singularity*, and $\mathrm{mult}(\sigma) = |G| = \ell$.

The group G lies in $\mathrm{SL}_2(\mathbb{C})$ if and only if $k = \ell - 1$. In that case, for $\ell \geqq 2$, the semigroup $\sigma^\vee \cap M$ is generated by $e_1, e_1 + e_2, (\ell-1)e_1 + \ell e_2$. The closed equivariant embedding of (2.3.1.1) shows that X_σ is a toric hypersurface in $\mathbb{C}^3$, namely the surface Y_ℓ of Remark 1.2.2. In particular, for $\ell = 2$, we thus obtain one of our standard examples, the quadric cone Y of Example 1.2.1 (2).

(2) Under the primitivity hypothesis of Remark 3.1.8, for any dimension n, the group G has a system of at most $n-1$ generators. Non-cyclic groups actually occur already in dimension $n = 3$. An example is furnished by $\sigma = \mathrm{cone}(v_1, v_2, v_3)$ with $v_1 = f_1$, $v_2 = f_1 + 2f_2$, and $v_3 = f_1 + 2f_3$.

To close this subsection, we have to add at least some remarks on the non-simplicial case. As usual, we may restrict to full-dimensional cones.

3.1.11 Remark. The structure of general toric singularities is considerably more complicated. There is a close relation with polytopes spanned by lattice vectors (called lattice polytopes for short): Intersecting a given n-cone σ with the affine hyperplanes $H_\mu := \{v \in N_\mathbb{R}\,;\ \langle v, \mu\rangle = l\}$ for any $\mu \in (\sigma^\vee)^\circ \cap M$ (on a suitable integral "level" $l > 0$) associates to σ a family of $(n-1)$-dimensional lattice polytopes with fixed combinatorial type. Conversely, any lattice polytope in $\mathbb{R}^{n-1}$, placed in the affine hyperplanes $(x_n = l)$ on different levels $l \in \mathbb{N}_{>0}$, spans a family of n-cones.

For $n = 3$, the associated polytopes are plane polygons. Their combinatorial type is just given by the number of vertices (or edges). For any fixed number k, however, there are countably many non-equivalent realizations of a k-gon as a lattice polygon. For $n = 4$, we have to look at three-dimensional polytopes (classically also called "polyhedra"). Here the situation gets much worse already on the side of combinatorial types: The enumeration results known so far show that the number of such types, considered as a function of the number f_i of i-faces (vertices, edges, or facets for $i = 0$, 1, or 2), grows rather rapidly. (An empirical formula gives $(k-6)^{(k-8)/3}$ as approximate number of types for $f_1 = k$.)

For special types of toric singularities, there are satisfactory classification results: Toric complete intersection singularities, for example, correspond to the so-called Nakajima polytopes placed on the level $l = 1$. Starting from points and line segments, these polytopes are inductively constructed as follows: One takes a sufficiently high prism over a Nakajima base polytope and then makes a "skew" truncation by a linear height function that is strictly positive on the relative interior of the base and integer-valued at lattice points. – For a survey of results about toric singularities, we refer to section 2 in [Cox]. A more detailed exposition is outside the scope of the present notes.

3.2 General toric varieties and fans

Except for the general definition, the example of the projective space (see 1.2.1), and Sumihiro's Theorem 1.2.5, we so far only have studied affine toric varieties. This is sufficient for the local investigation of toric singularities, but it does not allow to deal with problems like resolution of such singularities. Since we intend to address this topic in the final subsection, we have to provide the necessary tools.

As a consequence of Sumihiro's Theorem, every toric variety, say X, can be covered by open affine toric subvarieties. From Remark 2.3.2, we know that to each affine toric variety, say U, corresponds a unique N-cone $\sigma = \sigma_U$ such that $U = X_\sigma$.

To the general toric variety X, we may thus associate the following collection of N-cones:

$$\Delta := \Delta(X) := \{\sigma = \sigma_U \in \mathrm{Ob}(\mathfrak{S}\mathfrak{C}_N)\,;\ U \Subset X\}\,,$$

where U runs through the affine open toric subvarieties of X. For any two cones $\sigma, \sigma' \in \Delta$, the intersection $X_\sigma \cap X_{\sigma'}$ is a $\mathbb{T}$-invariant affine open subspace of X, and thus $X_\sigma \cap X_{\sigma'} = X_\tau$ with a cone $\tau \in \Delta$. Since X is separated, a one-parameter subgroup $\lambda \in \mathbb{Y}(\mathbb{T})$ has at most one limit $\lambda(0) \in X$. Hence, according to Remark 2.3.2 (2), the following holds:

$$\begin{aligned} \tau \cap N &= \{\nu \in N \,;\, \lambda_\nu(0) \in X_\sigma \cap X_{\sigma'}\} \\ &= \{\nu \in N \,;\, \lambda_\nu(0) \in X_\sigma\} \cap \{\nu \in N \,;\, \lambda_\nu(0) \in X_{\sigma'}\} \\ &= (\sigma \cap N) \cap (\sigma' \cap N)\ . \end{aligned}$$

This readily implies that $\tau = \sigma \cap \sigma'$. Furthermore, it follows from Proposition 2.3.4 that $\sigma \cap \sigma'$ is a common face of both, σ and σ'. That leads to the following notion:

3.2.1 Definition. *An (N-lattice)* **fan** *in $N_\mathbb{R}$ is a finite non-empty set Δ of (strongly convex) N-cones satisfying*

(1) $\tau \preceq \sigma \in \Delta \implies \tau \in \Delta$ *;*

(2) $\sigma, \sigma' \in \Delta \implies \sigma \cap \sigma' \preceq \sigma, \sigma'$*; in particular,* $\sigma \cap \sigma' \in \Delta$.

A fan is called simplicial or regular, respectively, if each of its cones has that property.

There are two fans that naturally correspond to a cone:

3.2.2 Remark. A single (strongly convex) N-cone σ generates its *full face fan*

$$\Delta(\sigma) := \{\tau \,;\, \tau \preceq \sigma\}\,,$$

also called an *affine* fan. If $\sigma \neq o$, then

$$\partial\sigma := \{\tau \,;\, \tau \precneqq \sigma\}$$

is a subfan, called the *boundary fan* of σ.

We have just seen that a toric variety determines a fan. On the other hand, given a fan, there is a unique corresponding toric variety:

3.2.3 Proposition. *To every N-fan Δ, one associates the toric variety*

$$X_\Delta \ := \ \bigcup_{\sigma \in \Delta} X_\sigma$$

by gluing the family of affine toric varieties $(X_\sigma)_{\sigma\in\Delta}$, where X_σ and $X_{\widetilde\sigma}$ are glued along the common open affine invariant subvariety $X_{\sigma\cap\widetilde\sigma}$.

The fact that the prevariety X_Δ is separated and thus, a variety, can be seen as follows: For cones σ and $\widetilde\sigma$ in Δ, the cone $\tau := \sigma \cap \widetilde\sigma$ lies in Δ, too. Hence, the intersection $X_\sigma \cap X_{\widetilde\sigma} \cong X_\tau$ again is affine. We have to verify that the corresponding comorphism

$$d^*\colon \mathcal{O}(X_\sigma \times X_{\widetilde\sigma}) = \mathbb{C}[M \cap \sigma^\vee] \otimes_\mathbb{C} \mathbb{C}[M \cap \widetilde\sigma^\vee] \longrightarrow \mathcal{O}(X_\tau) = \mathbb{C}[M \cap \tau^\vee]$$

of affine algebras, given by $\chi^\mu \otimes \chi^{\widetilde\mu} \mapsto \chi^{\mu+\widetilde\mu}$, is surjective. This is an immediate consequence of the equality $\tau^\vee = \sigma^\vee + \widetilde\sigma^\vee$, see Remark 2.2.9 (5). □

Obviously, Proposition 3.1.1 implies that the variety X_Δ is smooth if and only if the fan Δ is regular.

3.2.4 Example. In $N_{\mathbb{R}} \cong \mathbb{R}^n$, we set $f_0 := -\sum_{i=1}^{n} f_i$ and

$$\Delta := \left\{\sigma_J := \operatorname{cone}(f_i\,;\ i \in J)\ ;\ \ J \subsetneqq \{0, \ldots, n\}\right\}. \tag{3.2.4.1}$$

Then X_Δ is isomorphic to the projective space $\mathbb{P}_n$ with the toric structure of Example 1.2.1 (4). – In Figure 12, we depict the fan Δ in the two-dimensional case.

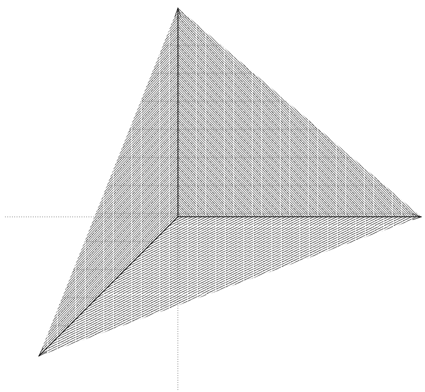

Figure 12: The fan for the projective plane $\mathbb{P}_2$

Proof. We write N_r in order to indicate the rank of the lattice under consideration; thus Δ is a fan in $(N_n)_{\mathbb{R}}$. In N_{n+1}, we exceptionally denote the standard lattice basis by $(g_0, \ldots, g_n)$, and consider the regular cone $\sigma := \operatorname{cone}(g_0, \ldots, g_n)$. The homomorphism

$$p\colon \mathbb{T}_{n+1} \longrightarrow \mathbb{T}_n\,,\ \ u := (u_0, \ldots, u_n) \longmapsto (u_1 u_0^{-1}, \ldots, u_n u_0^{-1})$$

induces the linear map

$$dp\colon (N_{n+1})_{\mathbb{R}} \longrightarrow (N_n)_{\mathbb{R}}\,,\ g_0 \longmapsto -\sum_{i=1}^{n} f_i\,,\ g_i \longmapsto f_i \text{ for } i = 1, \ldots, n\,,$$

which maps the cones in $\partial\sigma$ onto the cones in Δ. Hence, p extends to a morphism

$$\mathbb{C}^{n+1}\backslash\{0\} \ = \ X_{\partial\sigma} \longrightarrow X_\Delta\,.$$

The kernel of p is the diagonal $D := \left\{(s, \ldots, s) \in (\mathbb{C}^*)^{n+1}\right\}$. As a consequence, the map p is D-invariant and factors through $\mathbb{P}_n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/D$. In order to establish the isomorphism $\mathbb{P}_n \cong X_\Delta$, we consider a cone $\sigma_J \in \Delta$ and denote by τ_J the unique cone in $\partial\sigma$ with $dp(\tau_J) = \sigma_J$. Then

$$p^{-1}(X_{\sigma_J}) = X_{\tau_J} \ \cong\ Z_{\tau_J} \times D \ \longrightarrow\ X_{\sigma_J} \cong Z_{\tau_J}$$

is the projection onto the first factor. That proves the claim. $\square$

Many of the general remarks already made in the affine case remain valid in this more general setting. Firstly, the dependence on the lattice N has to be kept in mind (cf. Remark 2.2.16): If N and $\widetilde{N}$ are commensurable lattices, then any N-fan Δ can be considered as an $\widetilde{N}$-fan $\widetilde{\Delta}$. The resulting toric varieties need not be isomorphic as abstract varieties:

3.2.5 Example (Weighted projective spaces). For an $(n+1)$-vector $\mathbf{a} := (a_0, \ldots, a_n)$ of integers $a_i \geqq 1$ with $\gcd(a_0, \ldots, a_n) = 1$, we consider the fan Δ of (3.2.4.1), but replace the standard lattice $N \cong \mathbb{Z}^n$ with the finer lattice $\widetilde{N}$ generated by the rational vectors $(1/a_i)f_i$ for $i = 0, \ldots, n$. Then Δ of course remains simplicial, but in general, it is no longer regular. The resulting toric variety is called the *weighted projective space* $\mathbb{P}(\mathbf{a})$.

The open affine "charts" given by the n-cones σ_J (for $J \subsetneqq \{0, \ldots, n\}$ as above) are cyclic quotients $\mathbb{C}^n/G_J$. There is an isomorphism $\mathbb{P}(\mathbf{a}) \cong \mathbb{P}_n/G(\mathbf{a})$ with $G(\mathbf{a}) := \prod_{i=0}^n C_{a_i}$ acting coordinatewise on $\mathbb{P}_n$, so in particular, $\mathbb{P}(1, \ldots, 1) \cong \mathbb{P}_n$. Moreover, the description $\mathbb{P}_n \cong (\mathbb{C}^{n+1})/D$ given above generalizes to the weighted projective space if one replaces the diagonal 1-subtorus $D \subset (\mathbb{C}^*)^{n+1}$ with $D(\mathbf{a}) := \{t^{\mathbf{a}} = (t^{a_0}, \ldots, t^{a_n})\,;\ t \in \mathbb{C}^*\}$. $\square$

Secondly, there is a similar equivalence of suitable categories (cf. Remark 2.2.18); see (3) below. Thirdly, the results pertaining to orbits and orbit closures carry over to the general case:

3.2.6 Remark. (1) For each cone $\sigma \in \Delta$, there exists an associated orbit

$$O_\sigma \ := \ \mathbb{T}\cdot\lambda_\nu(0)\,,$$

where $\nu \in \sigma^\circ$. Again, Theorem 2.3.5 and formula (2.3.10) hold. In fact, the orbit O_σ is the unique closed orbit in the open subvariety X_σ of X_Δ, and the $\mathbb{T}/\mathbb{T}_\sigma$-toric variety $\overline{O}_\sigma \hookrightarrow X_\Delta$ satisfies

$$\overline{O}_\sigma \cong X_{\Delta/\sigma}$$

with the quotient fan $\Delta/\sigma := \{\tau/\sigma\,;\ \tau \in \Delta, \sigma \preceq \tau\}$ in $(N/N_\sigma)_\mathbb{R}$.

In Figure 13, we indicate schematically the correspondence between cones and orbits.

(2) A toric variety X_Δ is **complete** if and only if the fan Δ is *complete* (that is, its *support* $|\Delta| := \bigcup_{\sigma\in\Delta} \sigma$ is the entire linear space $N_\mathbb{R}$). – In the strong topology of complex varieties, "complete" means compact.

(3) Let Δ, Λ be fans in $N_\mathbb{R}$. Then there is a toric morphism $X_\Delta \to X_\Lambda$ if and only if each cone of Δ is included in some cone of Λ. – More generally, the equivalence of categories stated for the affine case as in (2.2.18.1) easily carries over to lattice fans and general toric varieties. (For a lattice homomorphism $\varphi\colon N_1 \to N_2$ and N_i-lattice fans Δ_i, the condition for a morphism of lattice fans is that the $\varphi_\mathbb{R}$-image of each cone of Δ_1 be included in some cone of Δ_2.)

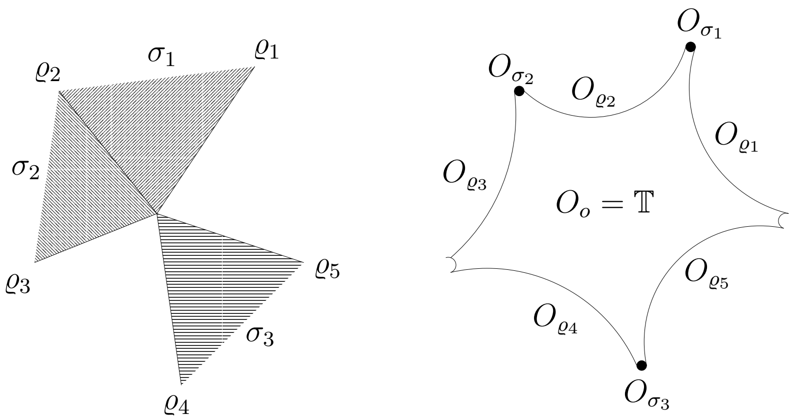

Figure 13: Correspondence between fan cones and orbits

(4) For fans Δ and Λ in $N_{\mathbb{R}}$, we assume that each cone of Δ is included in some cone of Λ. The resulting morphism $X_\Delta \to X_\Lambda$ is **proper** if and only if $|\Lambda| = |\Delta|$; and in that case Δ is called a **subdivision** or **refinement** of Λ. – In the strong topology of complex varieties, "proper" means that the inverse image of a compact subset is again compact.

For a proof of (2) and (4), we refer to the standard literature.

3.2.7 Exercise. In a two-dimensional fan Δ, let a ray ϱ be the common edge of two 2-cones. Describe Δ/ϱ and show that $\overline{O}_\varrho$ is a projective line.

3.2.8 Exercise. Given the lattice vectors $v_1 = f_2$, $v_2 = 2f_1 - f_2$, and $v_3 = -f_1$, the 2-cones $\sigma_{i,j} := \operatorname{cone}(v_i, v_j)$ determine a fan Δ, indicated in Figure 14. Show that two of the cones are regular and one is singular, defining the affine quadric cone of Example 1.2.1 (2).

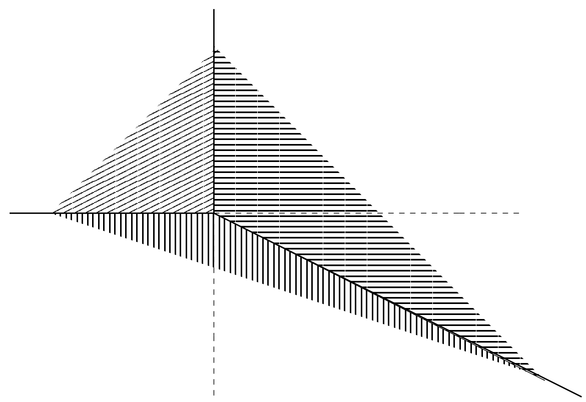

Figure 14: The fan for the projective quadric cone

The fan Δ is complete, and X_Δ is the projective quadric cone. – Figure 15 shows the set $(X_\Delta)_{\mathbb{R}}$ of real points, also called the "pinched torus" (tore pincé). The *caveat* about interpreting "real" pictures of complex varieties stated in Example 1.2.1 (2) also applies here.

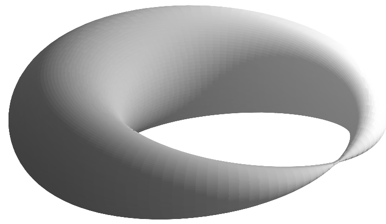

Figure 15: The set of real points of the projective quadric cone

The projective quadric cone is the closure in $\mathbb{P}_3$ of the affine quadric cone Y in $\mathbb{C}^3$. In general, the projective closure of an affine toric variety needs not be normal:

3.2.9 Example. Among the singular two-dimensional toric hypersurfaces $Y_k = V(\mathbb{C}^3; xz - y^k)$ considered in Remark 1.2.2, only this cone $Y = Y_2$ and the cubic Y_3 with the A_2 singularity have projective closures in $\mathbb{P}_3$ with isolated (and thus, normal) singularities. Since the torus action on the affine part clearly extends, the closure then is a (normal) toric variety.

We add a few remarks on the closure of Y_3 in $\mathbb{P}_3$: In homogeneous coordinates $[x, y, z, w]$, it is given by the homogenized equation $wxz = y^3$. Hence, there are two additional A_2 singularities at infinity, namely at the origins of the affine charts $(x = 1)$ and $(z = 1)$, respectively. The affine quotient representation $Y_3 \cong \mathbb{C}^2/C_3$ explained in Remark 1.2.2 also carries over to the closure: In homogeneous coordinates $[r, s, t]$ for $\mathbb{P}_2$, the action takes the form $\zeta \cdot [r, s, t] := [r, \zeta s, \zeta^2 t]$, and the quotient $\mathbb{P}_2/C_3$ is embedded in $\mathbb{P}_3$ via $[r, s, t] \mapsto [r^3, s^3, rst, t^3]$. – The defining fan is spanned by $v_2 = f_2$, $v_1 = 3f_1 - 2f_2$, and $v_0 = -3f_1 + f_2$, so it consists of three 2-cones σ_i with $\mathrm{mult}(\sigma_i) = 3$.

Moreover, the projective quadric cone is isomorphic to the weighted projective plane $\mathbb{P}(1, 1, 2)$. This is a particular case of the following exercise (whereas the cubic is not of this form):

3.2.10 Exercise. Let $\Delta(v_0, v_1, v_2)$ be the complete two-dimensional lattice fan given by primitive lattice vectors v_i, and let m_i be the multiplicity of $\mathrm{cone}(v_j, v_k)$. Then the toric surface X_Δ is a weighted projective plane – and then isomorphic to $\mathbb{P}(m_0, m_1, m_2)$ – if and only if $\gcd(m_0, m_1, m_2) = 1$. That condition in turn is equivalent to the fact that N coincides with its sublattice $\sum_{i=0}^{2} \mathbb{Z} \cdot v_i$.

Toric divisors

For applications in Shihoko Ishii's course, we introduce the concept of *toric divisors.* First, we briefly recall the general notion: In order to study the zeros and poles of a rational function f on a normal variety Z , one first notes that the locus of zeros and poles of f has only finitely many irreducible components, all of codimension one. Hence, one associates to every 1-codimensional irreducible subvariety $A \hookrightarrow Z$ a "multiplicity" $v_A(f) \in \mathbb{Z}$, the *vanishing order* of f along V. Now the pertinent information may be encoded in the "divisor" (f) of f: By definition, a *divisor* on Z is a formal sum

$$D = \sum_{A \hookrightarrow Z} v_A \cdot A,$$

over all one-codimensional irreducible subvarieties, where only *finitely many* of the integer coefficients v_A are allowed to be non-zero. Then the *divisor of the rational function* f is

$$(f) \;:=\; \sum_{A \hookrightarrow Z} v_A(f) \cdot A\,.$$

Returning to the toric situation, the $\mathbb{T}$-invariant irreducible subvarieties of codimension one in a toric variety X_Δ are just the orbit closures $D_{\varrho_i} := \overline{O}_{\varrho_i}$, where the ϱ_i denote the rays of Δ. The characters of the torus are regular functions on $\mathbb{T}$ without zeros. Considering them as rational functions on X_Δ, it follows that non-trivial multiplicities only can occur on the "boundary" $X_\Delta \setminus \mathbb{T} = \bigcup_{i=1}^k D_{\varrho_i}$. – Let $\nu^i \in N$ be the primitive generator of $\varrho_i = \mathrm{ray}(\nu^i)$.

3.2.11 Remark. The divisor in X_Δ of the character χ is

$$(\chi) \;=\; \sum_{i=1}^{k} \langle \chi, \lambda_{\nu^i} \rangle \; D_{\varrho_i}\,. \tag{3.2.11.1}$$

Proof. For a fixed index i, we set $\varrho := \varrho_i$ and $\lambda = \lambda_{\nu^i}$. Then λ maps $\mathbb{C}^*$ isomorphically onto the subtorus $\mathbb{T}_\varrho$ of $\mathbb{T}$. As in 2.2.15 and 2.3.7, we write $X_\varrho = Z_\varrho \times \mathbb{T}_{n-1} = \mathbb{C} \times \mathbb{T}_{n-1}$ with $O_\varrho = \{0\} \times \mathbb{T}_{n-1}$ and closure $Y := \overline{O}_\varrho$. For $f \in \mathbb{C}(X_\varrho)$, the multiplicity $v_Y(f)$ in the divisor (f) is just the multiplicity of the function $s \mapsto f(s,t)$ at $s = 0$, for generic $t \in \mathbb{T}_{n-1}$. Applying this to the special case $f = \chi$, the equation $\chi(s, \mathbf{1}) = \chi\big(\lambda(s)\big) = s^{\langle \chi, \lambda \rangle}$ implies that $\chi(s,t) = \chi(1,t) \cdot s^{\langle \chi, \lambda \rangle}$ has multiplicity $\langle \chi, \lambda \rangle$ at $s = 0$ for all $t \in \mathbb{T}_{n-1}$. $\square$

A divisor on X_Δ of the form

$$D \;=\; \sum_{i=1}^{k} n_i D_{\varrho_i}$$

is called a *toric divisor*. The special case where all coefficients $n_i = 1$ is of particular interest. Its negative,

$$K_X := -\sum_{i=1}^{k} D_{\varrho_i}\,,$$

is the famous *canonical divisor*.

3.3 Resolution of toric singularities

In general, smooth (i.e., non-singular) varieties are much better understood and usually enjoy much nicer formal properties than singular ones. In studying singular varieties, it is thus a natural attempt to "resolve" the singularities. This means to find a non-singular "model" $\widetilde{X}$ of the given singular variety X, i.e., a smooth variety $\widetilde{X}$ together with a proper morphism $\widetilde{X} \to X$ that is an isomorphism over the regular locus X_{reg}. The general resolution of singularities is rather involved. For complex varieties, it has been achieved by a celebrated result of HIRONAKA.

In the toric case, resolution of singularities is much more accessible: We recall that such a variety X_Δ is smooth if and only if the defining fan Δ is regular. According to Remark 3.2.6, a subdivision Δ' of a (general) fan Δ corresponds to a proper toric morphism $X_{\Delta'} \to X_\Delta$ that induces an isomorphism on the common open invariant subvariety $X_{\Delta\cap\Delta'}$. Hence, an equivariant resolution of singularities is given by a *regular* subdivision Δ' of Δ, such that the subdividing fan Δ' contains Δ_{reg} as a subfan. In that case we call Δ' a *resolution of* Δ $\big($or of the cone σ if $\Delta = \Delta(\sigma)\big)$.

3.3.1 Theorem (Equivariant resolution of toric singularities). *For every toric variety, there exists a resolution of the defining fan and thus, an equivariant resolution of singularities.*

Since a one-dimensional normal variety is smooth, the case of (normal!) toric curves is without relevance for singularities: In fact, the only such curves are $\mathbb{C}^*$, $\mathbb{C}$, and $\mathbb{P}_1$. Thus, we first discuss the resolution of singular toric surfaces.

(I) The surface case

This situation can be dealt with most explicitly: A two-dimensional fan necessarily is simplicial, so the singularities are of the nice "quotient" type discussed in Proposition 3.1.7, and they are necessarily isolated.

3.3.2 Theorem. *For every toric surface, there exists a unique minimal resolution of the defining fan and thus, a canonical equivariant resolution of singularities.*

It obviously suffices to prove this statement in the affine case:

3.3.3 Lemma. *A two-dimensional N-cone admits a unique minimal resolution; in particular, every resolution is a refinement of the minimal one.*

Proof. We may assume that $N = N_\sigma$. To construct a minimal resolution, we consider the convex hull K of the set $N \cap \sigma \setminus \{0\}$ as in Remark 2.2.6. The boundary of this polyhedron consists of two unbounded half-lines, each one included in an edge of σ, and finitely many bounded line segments. It thus contains only finitely many primitive lattice points, say $\nu^0, \ldots, \nu^{r+1} \in N$ in clockwise order. We set $\varrho_i := \operatorname{ray}(\nu^i)$, $\sigma_i := \varrho_i + \varrho_{i+1}$, and

$$\Delta := \{\tau\ ;\ \tau \preceq \sigma_i \text{ for some } i\,,\ 0 \le i \le r\}$$

(see Figure 16). According to Remark 3.1.6, each cone σ_i is regular, since the triangle with vertices 0, ν^i, ν^{i+1} contains no further lattice vector.

Now let Δ' be an arbitrary regular subdivision of σ. To verify that Δ' is a refinement of Δ – thus also proving minimality and unicity of Δ –, it suffices to show that each ray ϱ_i of Δ is a ray of Δ': The ray ϱ_i is included in a 2-cone $\tau := \operatorname{cone}(b_1, b_2)$ of Δ'. Since τ is regular, we may assume that (b_1, b_2) is a basis of N. Hence, by the regularity criterion of Remark 3.1.6, there is no further lattice point in the triangle spanned by 0,

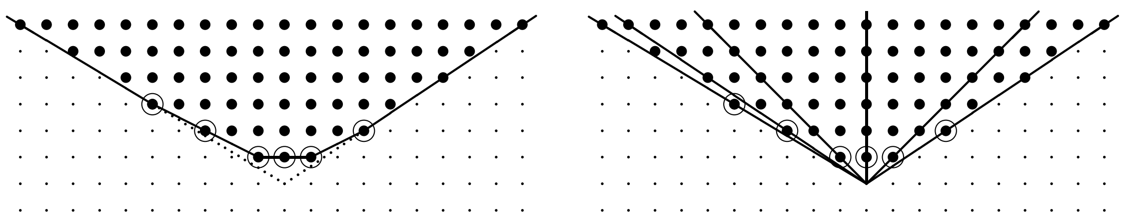

Figure 16: Polyhedron K_σ and minimal resolution for $\sigma = \text{cone}(-5f_1{+}3f_2, 3f_1{+}2f_2)$

b_1, b_2. Moreover, the lattice points b_1, b_2, ν^i lie in K, so the line segments $[b_1, b_2]$ and $[0, \nu^i]$ intersect in a point of K. Since ν^i is an element of ∂K, it lies on the line segment $[b_1, b_2]$, so it is one of the endpoints.

We indicate how to construct the first subdividing vector ν^1; iterating that step then enables a recursive computation of all vectors ν^i: For the primitive spanning vectors $\nu^0, \nu \in N$ of σ, there is a lattice basis b_1, b_2 of $N \cong \mathbb{Z}^2$ such that

$$(3.3.3.1) \qquad \nu^0 = b_2 \qquad \text{and} \qquad \nu = m_\sigma b_1 - k b_2 \quad \text{with an integer} \quad 1 \leqq k < m_\sigma$$

(see the following exercise). Then ν^1 is the vector b_1. □

3.3.4 Exercise. (1) Given two primitive lattice vectors $v_1, v_2 \in \mathbb{Z}^2$ with $\det(v_1, v_2) = m > 1$, prove that there exists a (positively oriented) lattice basis (b_1, b_2) and an integer k with $1 \leqq k < m$, $\gcd(m, k) = 1$ such that $v_1 = mb_1 - kb_2$ and $v_2 = b_2$.

(2) In the proof of Lemma 3.3.3, show that the first subdivision yields a resolution if and only if $k = 1$.

(3) Show that the maximal number of necessary subdivisions equals $m_\sigma{-}1$, and characterize the case when this occurs.

The theory of toric surface singularities and their resolution is particularly rich and fascinating. Within the scope of these notes, we have to content ourselves to indicating some key results:

3.3.5 Remark. In the situation of Lemma 3.3.3, let $X_\Delta \to X_\sigma$ be the minimal resolution of the singular affine surface X_σ , and $\pi\colon X_{\Delta'} \to X_\sigma$, an arbitrary resolution. We denote by $\varrho_0, \ldots, \varrho_{r+1}$ the rays of Δ' in clockwise order.

(1) The "exceptional fiber" $E := \pi^{-1}(x_\sigma)$ over the singular point of X_σ consists of the "new" orbit closures $E_i := \overline{O}_{\varrho_i} \cong \mathbb{P}_1$ (cf. Exercise 3.2.7) corresponding to the subdividing rays $\varrho_1, \ldots, \varrho_r$. They form a "chain" $E_1, \ldots, E_r$ of curves as depicted in Figure 17: Each E_i only intersects its neighbours; the intersection is *transverse* and consists of one point (transversality means that the curves meet like coordinate axes).

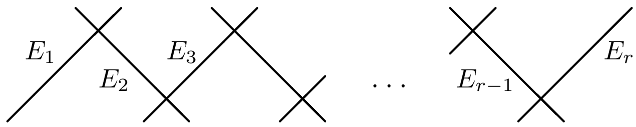

Figure 17: System of exceptional curves

(2) To curves E_i and E_j, one attaches their "intersection number" $E_i{\cdot}E_j$. For $i \neq j$, this is just the number of intersection points since a non-empty intersection is transverse.

For $i = j$, the "self intersection number" $E_i{\cdot}E_i = E_i^2$ expresses the "twisting" of the ambient smooth surface along the curve. In the present case, this number equals $-a_i$, where $a_i \geqq 1$ is the multiplicity of the cone $\varrho_{i-1}+\varrho_{i+1}$ spanned by the two neighbouring rays of ϱ_i. For a minimal resolution, we determine the a_i in (3).

The resulting "intersection matrix" $\big(E_i \cdot E_j\big)$ is an integral symmetric tridiagonal $(r{\times}r)$-matrix. It is negative definite with $\big|\det\big(E_i{\cdot}E_j\big)\big| = \operatorname{mult}(\sigma)$.

Instead of schematically indicating the chain of curves as in Figure 17, it is customary to depict its *dual graph* (see Figure 18): The dual graph has one node for each of

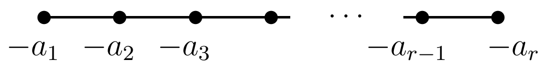

Figure 18: Weighted dual graph of exceptional curves

the curves; two nodes are joined by a simple edge if the corresponding curves meet. Moreover, the nodes are weighted by their self-intersection numbers.

(3) Since Δ' is a refinement of Δ, there is a factorization $X_{\Delta'} \to X_\Delta \to X_\sigma$ of π. The first morphism $X_{\Delta'} \to X_\Delta$ is a finite composition of "blow ups" in (regular) fixed points: Such a blow up of X_Δ in the fixed point x_τ corresponding to a 2-cone $\tau = \operatorname{cone}(b_1, b_2) \in \Delta$ spanned by a lattice basis, corresponds to subdividing τ with the diagonal $\varrho := \operatorname{ray}(b_1{+}b_2)$.

In the exceptional fibre, the blow up introduces a new chain-link $E_\tau \cong \mathbb{P}_1$ of self-intersection -1. (Such a curve that may be "contracted" to a smooth point is called an "exceptional curve of the first kind"). The self-intersection number of each "old" curve E_i passing through x_τ drops by 1. In particular, if all exceptional curves E_i of the resolution have self-intersection number $-a_i \leqq -2$, then the resolution is minimal.

(4) For the minimal resolution Δ of σ, each multiplicity a_i is at least two. Furthermore, in the situation of (3.3.3.1), the numbers a_i are the integers occurring in the following

"Hirzebruch-Jung continued fraction"

$$\frac{m_\sigma}{k} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_{r-1} - \frac{1}{a_r}}}}. \tag{3.3.5.1}$$

Proof. (3) It suffices to verify the description of blow ups for the "affine" fan Δ generated by the regular 2-cone $\tau := \mathrm{cone}(f_1, f_2)$. We have to show that the diagonal ray ϱ spanned by $f := f_1+f_2$ belongs to every non-trivial regular subdivision Δ' of τ, or, equivalently, that a regular subdivision Δ' of τ not containing ϱ as an edge coincides with Δ.

In fact, assuming $\varrho \notin \Delta'$, then the primitive lattice vector f lies in the interior of a regular cone $\tau' := \mathrm{cone}(b_1, b_2) \subsetneqq \tau$ spanned by a lattice basis. We thus may write $f = n_1b_1+n_2b_2$ with integers $n_1, n_2 > 0$. Since both, b_1 and b_2, are non-zero, non-negative integral linear combinations of f_1, f_2 and the decomposition $f = f_1 + f_2$ is unique, we necessarily have $\tau' = \mathrm{cone}(b_1, b_2) = \mathrm{cone}(f_1, f_2) = \tau$.

(4) We verify formula (3.3.5.1) by induction on r. For $r = 1$, we find $k = 1$ and $a_1 = m_\sigma$. In the induction step, we apply the induction hypothesis to the cone $\sigma' := \mathrm{cone}(b_1, m_\sigma b_1 - kb_2)$. It has multiplicity $m' = k$. We write $m_\sigma = qm' - k' = qk - k'$ with $0 \le k' < k$ and consider the lattice basis $b_2' := b_1, b_1' := qb_1 - b_2$. Then $a_1 = q$ and

$$\frac{m'}{k'} = a_2 - \cfrac{1}{a_3 - \cfrac{1}{\ddots - \cfrac{1}{a_{r-1} - \frac{1}{a_r}}}}$$

by induction hypothesis. Formula (3.3.5.1) now is an immediate consequence. $\square$

We describe the blowing up of the origin in local coordinates:

3.3.6 Example. Let Δ be the regular fan obtained by subdividing $\sigma = \mathrm{cone}(f_1, f_2)$ with $\varrho = \mathrm{ray}(v)$ for $v := f_1+f_2$. The basic characters e_1, e_2 yield coordinates (x, y) on $X_\sigma = \mathbb{C}^2$. For $\sigma_1 = \mathrm{cone}(f_1, v)$ and $\sigma_2 = \mathrm{cone}(v, f_2)$, the dual cones are $\sigma_1^\vee = \mathrm{cone}(e_1-e_2, e_2)$ and $\sigma_2^\vee = \mathrm{cone}(e_2-e_1, e_1)$, thus providing coordinates (u_i, v_i) for $X_{\sigma_i} \cong \mathbb{C}^2$. The inclusion $\sigma_i \hookrightarrow \sigma$ then corresponds to $(u_1, v_1) \mapsto (x, y) = (u_1v_1, v_1)$ and $(u_2, v_2) \mapsto (x, y) = (v_2, u_2v_2)$. Hence, the u_i-axes $v_i = 0$ get collapsed to the origin. The two coordinate charts for X_Δ are glued along $\mathbb{C}^* \times \mathbb{C}$ by the transition functions $u_2 = 1/u_1$ – thus gluing together the two u_i-axes to the exceptional fibre of the blow-up map $X_\Delta \to \mathbb{C}^2$ over the origin, a projective line – and $v_2 = u_1v_1 = v_1/u_2$. Hence, the constant function $v_1 = 1$ is transformed into the rational function $v_2 = 1/u_2$ with a simple pole at the origin. This vanishing order -1 is the self-intersection number of the exceptional curve.

(II) The general case

The basic idea for the toric resolution in higher dimensions is to proceed in two steps: Firstly, the fan is made simplicial, secondly, it is regularized. Both steps rely on the process of "stellar subdivision". In the following definition, we do not assume that the ray ϱ is an edge of the cone σ; we even allow ϱ not to be included as a subset.

3.3.7 Definition (Stellar subdivision). *Let σ be a cone and ϱ, an arbitrary ray. Then the union of face fans*

$$\Sigma_\varrho(\sigma) \ := \ \begin{cases} \bigcup\limits_{\tau\in\partial\sigma\,,\,\varrho\not\subset\tau} \Delta(\varrho+\tau) & \text{if } \varrho \subset \sigma \\ \Delta(\sigma) & \text{if } \varrho \not\subset \sigma \end{cases}$$

is a fan subdividing σ, called the stellar subdivision of σ with center ϱ.

If Δ is a fan and ϱ an arbitrary ray included in the support of Δ, then the stellar subdivision of Δ with center ϱ is the fan

$$\Sigma_\varrho(\Delta) \ := \ \bigcup_{\sigma\in\Delta} \Sigma_\varrho(\sigma)\,. \tag{3.3.7.1}$$

3.3.8 Remark. (1) If the ray ϱ even is an element of Δ, then no simplicial cone of Δ gets subdivided.

(2) The resolution of toric surface singularities described in the proof of Lemma 3.3.3 is obtained by an iterated stellar subdivision. Similarly, the blowing up of regular fixed points in a toric surface discussed in Remark 3.3.5 (4) is nothing but the stellar subdivision of $\operatorname{cone}(b_1, b_2)$ with respect to $\operatorname{ray}(b_1 + b_2)$. This procedure generalizes from $n = 2$ to regular cones of arbitrary dimension $n \geqq 3$, thus describing the higher-dimensional "blowing up" of regular fixed points.

(3) For a reader familiar with the notion of "blowing up an ideal", we add the following: On the level of toric varieties, a stellar subdivision $\Sigma_\varrho(\sigma)$ of a cone σ with respect to $\varrho := \operatorname{ray}(\nu)$ corresponds to the blow up of a $\mathbb{T}$-invariant ideal I in $\mathcal{O}(X_\sigma)$: There is a positive integer k such that the affine hyperplane $\langle \dots, \nu\rangle = k$ in $M_\mathbb{R}$ intersects each edge of the dual cone $\sigma^\vee$ not contained in $\varrho^\perp$ in a lattice point. Then I may be chosen as the ideal generated by the characters corresponding to those lattice points.

We now apply the stellar subdivision to achieve the first step of a resolution.

3.3.9 Lemma ("Simplicialization"). *Every fan admits a simplicial subdivision.*

Proof. We introduce the following terminology: An edge ϱ of a cone σ is said to *split* σ if there is a "complementary" facet τ of σ, i.e., such that $\sigma = \tau+\varrho$; a cone is called *stout* if it has no splitting edges. A cone is simplicial if (and only if) it does not include any stout face. If a ray ϱ is included in a cone of a fan Δ, then $\Sigma_\varrho(\Delta) \setminus \Delta$ obviously does not contain any stout cone. As a consequence, a stellar subdivision with center included in a stout cone lowers the number of such cones. Hence, after finitely many subdivisions with centers in stout cones, one arrives at a simplicial fan. ☐

3.3.10 Remark. For such an iterated stellar subdivision Δ' of Δ, there exist two extreme possibilities for the choice of the centers:

(1) We call the subdivision Δ' *thin* if each center ϱ is an edge in Δ; in other words, there are no "new" rays. We denote by E the "exceptional locus" of the associated toric morphism $X_{\Delta'} \to X_\Delta$, that is, the union of all infinite fibres. Then each irreducible component E_i of E is of the form $\mathbb{V}(\tau_i')$ where τ_i' is a minimal new cone of Δ', so E_i has codimension at least 2.

(2) The subdivision Δ' is called *fat* if each center ϱ is generated by a lattice vector in the relative interior of a stout cone. Then each E_i is of codimension 1.

This follows from the fact that the irreducible components of E are the orbit closures corresponding to cones $\sigma \in \Delta'$ with boundary $\partial\sigma$ included in Δ.

As an example, we discuss the Segre cone:

3.3.11 Example. For the three-dimensional toric variety $X_\sigma = N(\mathbb{C}^4;\ z_1z_4 - z_2z_3)$ of example 3.1.10 (2), one can show that there are exactly two different thin simplicial subdivisions Δ_1, Δ_2 of the cone σ. The fans Δ_i actually are regular, and in both cases, the corresponding exceptional set $E := \pi^{-1}(x_\sigma)$ is a projective line. – We indicate such a dividing cone τ_i in the next figure.

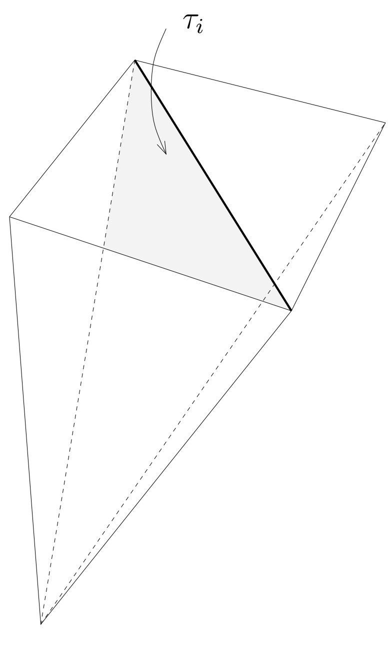

Figure 19: A thin simplicial subdivision

A fat simplicial subdivision for the pertinent cone σ is provided by the stellar subdivision $\Delta' := \Sigma_\varrho(\sigma)$ with center $\varrho := \mathrm{cone}(f_1+f_2)$; the exceptional fiber E is isomorphic to the surface $\mathbb{P}_1 \times \mathbb{P}_1$. We remark that $X_{\Delta'} \to X_\sigma$ factors through the two thin simplicial resolutions Δ_i of the preceding exercise: The fan Δ' consists of all cones which are the intersection of a cone in Δ_1 with a cone in Δ_2. In fact, Δ' also is a resolution.

Finally we show

3.3.12 Lemma ("Regularization"). *Every simplicial fan admits a regular subdivision.*

Proof. We first remark that a stellar subdivision $\Sigma_\varrho(\Delta)$ of a simplicial fan again is simplicial, and that a ray ϱ not included in a regular cone of the fan Δ provides an inclusion

$$\Delta_{\mathrm{reg}} \subset \Sigma_\varrho(\Delta)_{\mathrm{reg}} .$$

Obviously a fan Δ is regular if all its maximal cones are regular.

One now successively lowers the multiplicities of maximal cones: We fix such a cone σ of maximal multiplicity $m_\sigma = m > 1$ and a minimal face $\tau \preceq \sigma$ of multiplicity $m_\tau > 1$. It is of the form $\tau = \mathrm{cone}(b_1, \dots, b_d)$ with linearly independent primitive lattice vectors $b_1, \dots, b_d$. According to the regularity criterion in Remark 3.1.6, there is a (w.l.o.g. primitive) lattice vector $\nu \in N$ in the parallelotope spanned by $b_1, \dots, b_d$ which is not a vertex. Then necessarily $\nu = \sum_{i=1}^d \alpha_i b_i$ with rational coefficients α_i strictly between 0 and 1. For $\varrho := \mathrm{ray}(\nu)$, we now consider the stellar subdivision $\Sigma_\varrho(\Delta)$ and show that the multiplicity of each new maximal cone $\hat\sigma$ in $\Sigma_\varrho(\Delta)$ is strictly less than m_σ. Such a cone $\hat\sigma$ includes a face of the form $\hat\tau = \gamma + \varrho$ with a facet γ of τ, say $\gamma = \mathrm{cone}(b_1, \dots, b_{d-1})$. Now an easy computation, using the fact that $b_1, \dots, b_{d-1}$ are part of a basis of N_τ ($\partial\tau$ being a regular fan) shows that $m_{\hat\tau} = \alpha_d m_\tau < m_\tau$. Finally there are exact sequences

$$0 \longrightarrow N_\tau/\Gamma_\tau \longrightarrow N_\sigma/\Gamma_\sigma \longrightarrow N_\sigma\big/(\Gamma_\sigma + N_\tau) \longrightarrow 0$$

and

$$0 \longrightarrow N_{\hat\tau}/\Gamma_{\hat\tau} \longrightarrow N_{\hat\sigma}/\Gamma_{\hat\sigma} \longrightarrow N_{\hat\sigma}\big/(\Gamma_{\hat\sigma} + N_{\hat\tau}) \longrightarrow 0$$

of finite abelian groups. Since the third terms are isomorphic, counting elements yields that $m_{\hat\sigma} = \alpha_d \cdot m_\sigma < m_\sigma$.

Since each maximal cone of Δ that includes τ may take the role of Σ, the fan $\Sigma_\varrho(\Delta)$ has less maximal cones of multiplicity m than the original fan Δ. □

3.3.13 Remark. If the simplicialization step only consists of fat stellar subdivisions, the subsequent regularization yields a resolution where the exceptional set

$$E = \pi^{-1}(S(X)) \hookrightarrow X'$$

is a divisor, as it only has irreducible components of codimension 1. Being a smooth toric variety, X' can be covered by invariant coordinate patches of the type $\mathbb{C}^d \times (\mathbb{C}^*)^{n-d}$. Hence, the invariant subvariety E intersects them in unions of coordinate hyperplanes. Such a divisor E is called a "divisor with normal crossings".

3.3.14 Exercise. Consider the singular simplicial cone $\sigma = \mathrm{cone}(f_1, f_2, \sum_{j=1}^{3} j{\cdot}f_j)$ of dimension three.

(1) Prove that its boundary subfan $\partial\sigma$ is *regular*.

(2) Prove that $\mathrm{ray}\big(f_1{+}2(f_2{+}f_3)\big)$ passes through the relative interior of σ and that the stellar subdivision with respect to it yields cones of smaller multiplicity.

References

[Bry] J.-L. Brylinski, *Eventails et variétés toriques*, in: *Séminaire sur les Singularités des Surfaces*, Lecture Notes in Mathematics **777**, Springer-Verlag, 1980.

[Cox] D. Cox, *Update on toric geometry*, pp. 1–41 in *Geometry of Toric Varieties*, L. Bonavero, M. Brion (ed.), Séminaires et Congrès **6**, Société Math. de France, 2002.

[Ew] G. Ewald, *Combinatorial Convexity and Algebraic Geometry*, Grad. Texts in Math. **168**, Springer-Verlag, 1996.

[FiKp] K.-H. Fieseler, L. Kaup, *Algebraische Geometrie – Grundlagen*, Berliner Studienreihe zur Mathematik **13**, Heldermann-Verlag, 2005.

[Ful] W. Fulton, *Introduction to Toric Varieties*, Annals of Math. Studies **131**, Princeton Univ. Press, 1993.

[Kp] L. Kaup, *Vorlesungen über torische Varietäten*, Konstanzer Schriften in Mathematik und Informatik 130, 2001, web address: `http://www.inf.uni-konstanz.de/Schriften`.

[Oda] T. Oda, *Convex Bodies and Algebraic Geometry*, Ergebnisse Math. Grenzgeb. (n.S.) **15**, Springer-Verlag, 1988.

[Oka] M. Oka, *Non-Degenerate Complete Intersection Singularity*, Actualités Mathématiques, Hermann, 1997.

Adresses

Gottfried Barthel, Ludger Kaup
Fachbereich Mathematik und Statistik
Universität Konstanz, Fach D 203
D-78 457 Konstanz
`Gottfried.Barthel@uni-konstanz.de`
`Ludger.Kaup@uni-konstanz.de`

Karl-Heinz Fieseler
Matematiska Institutionen
Box 480
SE-751 06 Uppsala
`Karl-Heinz.Fieseler@math.uu.se`