Counting the faces of a polytope

1 Basic Definitions

Definition 1.1. 1. By a closed half space $H \subset V$ of a (finite dimensional) real vector space V we mean a set of the form

$$H := \{v \in V; \alpha(v) \le c\},$$

where $\alpha : V \longrightarrow \mathbb{R}$ is a nonzero linear form and $c \in \mathbb{R}$.

2. A polytope $P \subset V$ is any non-empty compact subset which is the intersection
$$P = \bigcap_{i=1}^{r} H_i$$
of finitely many closed half spaces $H_i \subset V$.

Remark 1.2. 1. A polytope $P \subset V$ is convex as intersection of convex sets.

2. The dimension $\dim P$ of a polytope $P \subset V$ is defined as the dimension of the subspace generated by the differences $v - w$ of vectors $v, w \in P$.

Definition 1.3. A polytope Q is a called a *face* of P, if either $Q = P$ or we can write
$$Q = P \cap \partial H$$
with a closed half space $H \supset P$. In that case we write $Q \preceq P$. If $\dim Q = i$, we simply call Q an i-face, and the notation $Q \preceq_k P$ means that $Q \preceq P$ is a face of codimension $k = n - \dim Q$.

In particular, if $n = \dim P$, its

1. $(n-1)$-faces are called *facets*,

2. 1-faces are called *edges*, and

3. its 0-faces are called *vertices*.

There is an alternative description of polytopes: They are exactly the sets

$$P = \mathrm{CH}(v_1, ..., v_r) := \{t_1 v_1 + ... + t_r v_r; t_1, ..., t_r \geq 0, t_1 + ... + t_r = 1\}$$

obtained as the convex hull of finitely many points $v_1, ..., v_r \in V$. The minimal choice of vectors $v_1, ..., v_r$ for a fixed polytope P is given by the vertices of P.

Remark 1.4. Given an n-polytope $P \subset V$ $(\dim V = n)$ the set

$$P^* := \{v^* \in V^*; v^*|_P \leq 1\}$$

is a polytope in the dual vector space V^*, called the polar polytope of P. Note that there is a bijection $P \succeq F \mapsto F^* \preceq P^*$ between the set of faces of P and that of faces of P^*, which is inclusion reversing with $\dim F^* + \dim F = n$.

Definition 1.5. The f**-vector** of the n-polytope P is the sequence

$$(f_0(P), ..., f_n(P)),$$

$$\text{where } \; f_i(P) := \text{number of } i\text{-faces of } \; \text{the polytope } P.$$

In these notes we consider the following

Question: Which sequences $(f_0, ..., f_n)$ can be realized as f-vectors of an n-polytope?

For $n = 2$ a necessary and sufficient condition obviously is $f_2 = 1, f_1 = f_0$, while for $n = 3$ there is the theorem of Steinitz:

Theorem 1.6 (Steinitz). *The quadruple* (f_0, f_1, f_2, f_3) *is the* f*-vector of a* 3*-polytope* P *iff the following conditions are satisfied:*

1. $f_3 = 1$ *and* $f_0 - f_1 + f_2 = 2$ *(Euler's relation).*
2. $4 \leq f_0 \leq 2f_2 - 4$.
3. $4 \leq f_2 \leq 2f_0 - 4$.

The above conditions are of a purely topological nature. Even better, the following result holds:

Theorem 1.7 (Steinitz). *Given any cell decomposition of the* 2*-sphere* $\mathbb{S}^2$ *(where we require that the intersection of two closed cells is again a closed cell), there is a* 3*-polytope* P*, such that the face decomposition of* $\partial P \cong \mathbb{S}^2$ *is combinatorially equivalent to the given cell decomposition of the* 2*-sphere* $\mathbb{S}^2$.

Indeed, the statement fails to hold for n-polytopes in dimensions $n > 3$.

Remark 1.8. For an arbitrary dimension n we have apart from $f_n = 1$ Euler's relation:

$$\sum_{i=0}^{n} (-1)^i f_i = 1.$$

Furthermore, an n-polytpe has at least $n + 1$ vertices:

$$f_0 \geq n + 1,$$

and since at a vertex at least n edges meet (see below), we obtain

$$2f_1 \geq nf_0$$

as a necessary condition.

But up to now there is no complete answer to our question for dimensions $n > 3$. We have to restrict to "simple polytopes". Note first that a vertex $v \in P$ of an n-polytope P is contained in at least n edges: If one takes a half space $H_0 \subset V$ with $H_0 \cap P = \{v\}$ and denotes $H \subset V$ a nearby half space with $H \supsetneq H_0$ (so the boundaries ∂H and ∂H_0 are parallel affine hyperplanes), then the "stub" $H \cap P$ is a pyramid over the $(n-1)$-polytope $\partial H \cap P$, and an $(n-1)$-polytope has at least n vertices.

Definition 1.9. An n-polytope P is called **simple** if each vertex is contained in exactly n edges (and hence $2f_1(P) = nf_0(P)$).

So for a vertex $v \in P$ of a simple polytope P the basis $\partial H \cap P$ of the pyramid $H \cap P$ is an $(n-1)$-simplex. In particular, it follows that a vertex of a simple n-polytope belongs to exactly n facets.

Remark 1.10. A polytope $P \subset V$ is simple iff its polar polytope $P^* \subset V^*$ is "simplicial", i.e. all its faces are simplices.

2 Algebraic Constructions

The Stanley Reisner ring: In order to attack our problem we attach to a simple polytope P a graded ring $SR(P)$, its *Stanley Reisner ring*: Denote $F_1, ..., F_r \prec_1 P$ the facets of P. Then

$$SR(P) := \mathbb{R}[F_1, ..., F_r]/\mathfrak{a},$$

where the facets $F_i \prec_1 P$ are considered as indeterminates and the (homogeneous) ideal $\mathfrak{a} \subset \mathbb{R}[F_1, ..., F_r]$ is generated by the monomials

$$\prod_{i \in A} F_i, \quad \text{where } A \subset \{1, ..., r\}, \ \bigcap_{i \in A} F_i = \emptyset.$$

The graded ring $SR(P)$ has another quite useful realization as an algebra of certain real valued functions on the dual space V^*:

Piecewise Polynomials: The polytope P gives rise to a fan[1], also called the "outer normal fan" of the polytope P,

$$\Delta := \Delta(P) := \{\sigma_Q; Q \prec P\}$$

in the dual vector space V^*: For a facet $F \prec_1 P$ denote $\mathbf{n}_F \in V^*$ a "normal vector" for P at F, i.e. such that $\mathbf{n}_F|_F \equiv c \geq \mathbf{n}_F|_P$. For any face $Q \prec P$ now set

$$\sigma_Q := \sum_{Q \preceq F \prec_1 P} \mathbb{R}_{\geq 0} \mathbf{n}_F.$$

[1]A collection Δ of strictly convex polyhedral cones $\sigma = \mathbb{R}w_1 + ... + \mathbb{R}w_r \subset W$ in a vector space W (a strictly convex cone σ does not contain a line) is called a fan, if with a cone all its faces belong to Δ and the intersection of two cones in Δ is a face of both.

Note that the map $Q \mapsto \sigma_Q$ is inclusion reversing and that

$$\dim \sigma_Q + \dim Q = n,$$

where $\dim \sigma_Q := \dim(\sigma_Q + (-\sigma_Q))$. The fan $\Delta(P)$ is complete, i.e. its cones cover the entire dual space:

$$V^* = \bigcup_{Q \prec P} \sigma_Q = \bigcup_{Q \prec_n P} \sigma_Q.$$

Now for a cone $\sigma \in \Delta$ we set

$$A^*_\sigma := \{\text{all polynomial functions } f : \sigma \longrightarrow \mathbb{R}\},$$

so A^*_σ is a polynomial ring in $\dim \sigma$ indeterminates, the symmetric algebra $S^*(V/\text{span}(\sigma)^\perp)$. Then the algebra of all Δ-piecewise polynomial functions

$$A^*_\Delta := \{f : V^* \longrightarrow \mathbb{R}; f|_\sigma \in A^*_\sigma,\ \forall \sigma \in \Delta\}$$

is, for $\Delta = \Delta(P)$, isomorphic to the Stanley-Reisner ring $SR(P)$: Since every vertex of P is contained in exactly n facets, the n-dimensional cones $\sigma \in \Delta(P)$ are simplicial, i.e., spanned by bases of V^*. Hence, in order to determine a piecewise linear function $\chi \in A^1_\Delta$, we can prescribe its values on the rays $\rho_i := \mathbb{R}_{\geq 0}\mathbf{n}_i, i = 1, ..., r$. Now $\chi_i \in A^1_\Delta$ is taken to be a function positive on $\rho_i \setminus \{0\}$ and vanishing on all remaining rays $\rho_j, j \neq i$, and our isomorphism takes the form

$$\mathbb{R}[F_1, ..., F_r]/\mathfrak{a} \longrightarrow A^*_{\Delta(P)}, \quad F_i + \mathfrak{a} \mapsto \chi_i.$$

The algebra A^*_Δ is in a natural way a (graded) module over the (graded) algebra of "global polynomials"

$$A^* := S(V)$$

on V^*, (using biduality $(V^*)^* \cong V$). Denote $\mathfrak{m} \subset A^*$ the maximal ideal $\mathfrak{m} := A^{>0}$ of all polynomials vanishing at the origin. Now we define the "algebraic object" $H^*(P)$, by means of which we can encode the f-vector in a more accessible way. We define it as

$$H^*(P) := A/\mathfrak{m} \otimes_{A^*} A^*_{\Delta(P)},$$

a graded commutative $\mathbb{R}$-algebra ($\mathbb{R} \cong A^*/\mathfrak{m}$).

Remark 2.1 (Toric Varieties). Assume that $\Lambda \subset V$ is a lattice of maximal rank and the vertices of P are lattice points (We say: "P is a lattice polytope"). Identify $\Lambda \cong \mathbb{Z}^n$ with the (complex) character group of an n-dimensional torus $\mathbb{T} \cong (\mathbb{S}^1)^n$. Now let us construct a topological space $X = X(P) := (P \times \mathbb{T})/\sim$ with the following equivalence relation $\sim$: We associate to any face $F \preceq P$ a closed subtorus $\mathbb{T}(F) \subset \mathbb{T}$, the common kernel of the characters in $\Lambda \cap V(F)$, where $V(F) \subset V$ denotes the subspace parallel to the affine span of $F \subset V$. Then two points in $F \times \mathbb{T} \subset P \times \mathbb{T}$ are equivalent if their second components differ only by a factor in $\mathbb{T}(F)$. The torus action on $P \times \mathbb{T}$ induces an action on $X(P)$ and we have

$$H^*(P) \cong H^{2*}(X(P); \mathbb{R})$$

as well as

$$SR(P) \cong H^{2*}_{\mathbb{T}}(X(P); \mathbb{R}),$$

while $H^{odd}(X(P); \mathbb{R}) = 0 = H^{odd}_{\mathbb{T}}(X(P); \mathbb{R})$. Indeed, it turns out that $X(P)$ even is "almost" a complex projective manifold (a complex projective variety with quite mild singularities, a rational homology manifold), and that fact gives us interesting information about the commutative $\mathbb{R}$-algebra $H^*(P)$, as for example Poincaré duality and the Hard Lefschetz theorem. But we shall try below to indicate how the corresponding properties can be proved completely in the framework of convex geometry.

Definition 2.2. The h**-vector** of the n-polytope P is the sequence

$$(h_0(P), ..., h_n(P)),$$

where $h_i(P) := \dim H^i(P)$.

Indeed, the f-vector of a simple n-polytope determines its h-vector completely and vice versa:

Proposition 2.3. *For a simple n-polytope $P \subset V$ its h-polynomial*

$$h_P(T) := \sum_{i=0}^{n} h_i(P) \cdot T^i$$

and its face polynomial

$$f_P(T) := \sum_{i=0}^{n} f_i(P) \cdot T^i$$

are related as follows

$$h_P(T) = f_P(T-1),$$

in particular $h_P(T)$ is a polynomial of degree n.

As a consequence we may reformulate our original problem in terms of the h-vector:

Problem: Which sequences $(h_0, ..., h_n)$ can be realized as h-vectors of a simple n-polytope?

Before we formulate the answer, we indicate the proof of Propositon 2.3. It is based on the following result (see also [BBFK$_2$]):

Theorem 2.4. *Let $\Delta = \Delta(P)$ be the outer normal fan of the polytope P. The A^*-module A^*_Δ is a finitely generated free A^*-module*

$$A^*_\Delta \cong \bigoplus_{i=0}^{n} (A^*[-i])^{h_i(P)},$$

where $A^*[-i]$ denotes the graded module A^* shifted i steps upward, and there is an exact sequence

$$0 \longrightarrow A^*_\Delta \longrightarrow C_n(\Delta) \longrightarrow ... \longrightarrow C_0(\Delta) \longrightarrow 0$$

of graded A^*-modules with

$$C_j(\Delta) := \bigoplus_{\sigma \in \Delta^j} A^*_\sigma$$

with $\Delta^j \subset \Delta$ being the subset of all j-dimensional cones in Δ. The differential $C_j(\Delta) \longrightarrow C_{j-1}(\Delta)$ is the natural one obtained after having chosen orientations for V^* and all the subspaces span(σ) for lower dimensional cones $\sigma \in \Delta$.

Proof of 2.3. For a graded real vector space W^* with finite dimensional weight subspaces $W^i, i \geq 0$, denote

$$P_{W^*}(T) := \sum_{i=0}^{\infty} \dim W^i \cdot T^i \in \mathbb{Z}[[T]]$$

its Poincaré series, e.g.

$$P_{A^*}(T) = (1-T)^{-n},\ P_{A^*_\sigma}(T) = (1-T)^{-\dim \sigma}.$$

Furthermore for a graded A^*-module M^* set

$$\overline{M}^* := M^*/\mathfrak{m}M^*.$$

Then the above theorem 2.4 implies

$$P_{A^*_\Delta}(T) = \sum_{j=0}^{n} (-1)^{n-j} P_{C_j(\Delta)}(T) = \sum_{j=0}^{n} (-1)^{n-j} f_{n-j} \cdot (1-T)^{-j}.$$

Finally, with

$$P_{A^*_\Delta}(T) = P_{\overline{A}^*_\Delta}(T)(1-T)^{-n}$$

we obtain

$$h_P(T) = \sum_{j=0}^{n} f_j \cdot (T-1)^j.$$

$\square$

3 The Main Theorem

Theorem 3.1 (Conjectured by McMullen 1971, proved 1980/81 by Billera, Lee and Stanley)**.** *The sequence* $(h_0, ..., h_n)$ *of natural numbers is the* h*-vector of a simple* n*-polytope if and only if the following conditions are satisfied*

1. *Eulers relation:* $h_0 = 1$
2. *The "Dehn-Sommerville equation"* $h_{n-i} = h_i$ *holds for all* i*.*
3. $h_i \geq h_{i-1}$ *för* $1 \leq i \leq n/2$*.*

4. *$h_i - h_{i-1} \le (h_{i-1} - h_{i-2})^{\langle i-1 \rangle}$ for $2 \le i \le n/2$, where $a^{\langle i \rangle}$ denotes the i-th pseudopower of $a \in \mathbb{N}$, see Def. 3.2 below.*

Definition 3.2. The pseudopower map

$$\mathbb{N} \times \mathbb{N}_{>0} \longrightarrow \mathbb{N}, (a, i) \mapsto a^{<i>}$$

is defined as follows: First of all

$$0^{<i>} := 0,$$

while for $a > 0$ writing

$$a = \sum_{k=j}^{i} \binom{n_k}{k}$$

with (unique!) natural numbers $n_i > n_{i-1} > ... > n_j \ge j \ge 1$ one sets

$$a^{<i>} := \sum_{k=j}^{i} \binom{n_k + 1}{k+1}.$$

4 The Proof of the Main Theorem

Dehn-Sommerville equations: There is a natural nondegenerate pairing

$$A^*_\Delta \times A^*_\Delta \longrightarrow A^*[-n]$$

of graded free A^*-modules, obtained as the composite of the multiplication of functions

$$A^*_\Delta \times A^*_\Delta \longrightarrow A^*_\Delta$$

and an evaluation map

$$\varepsilon : A^*_\Delta \longrightarrow A^*[-n],$$

see Def. 4.1. It descends to a dual pairing, called the "*intersection pairing*"

$$H^*(P) \times H^*(P) \longrightarrow \mathbb{R}[-n], (a, b) \mapsto a \cap b$$

resp. to dual pairings

$$H^i(P) \times H^{n-i}(P) \longrightarrow \mathbb{R}.$$

So in particular $h_i(P) = h_{n-i}(P)$.

Definition 4.1. The "evaluation map"

$$\varepsilon : A^*_\Delta \longrightarrow A^*[-n]$$

is defined as follows: Fix an orientation $\omega \in \bigwedge^n V$. For each n-cone σ, we denote $g_\sigma \in A^n_\sigma = A^n$ the unique non-trivial function ≥ 0 vanishing on its boundary $\partial\sigma$, which is the product of linear forms in $A^1 \cong V$, whose wedge product agrees, up to sign, with ω. Then the map ε is the composite

$$A^*_\Delta \subset \bigoplus_{\sigma\in\Delta^n} A_\sigma \;\longrightarrow\; A^* \;,\quad f = (f_\sigma)_{\sigma\in\Delta^n} \longmapsto \sum_{\sigma\in\Delta^n} \frac{f_\sigma}{g_\sigma}\;,$$

the sum lying not only in $Q(A^*)$, but even in A^*, since the singularities along the facets of n-cones cancel.

The function $i \mapsto h_i(P)$ is nondecreasing in the range $0 \leq i \leq \frac{n}{2}$:
The vertices $v_1, ..., v_s \in V$ of our polytope P combine to a function $\psi \in A^1_\Delta$ as follows: If $\sigma_i = \sigma_{v_i}$ denotes the n-cone associated to the vertex v_i, then $\psi|_{\sigma_i} = v_i \in V \cong (V^*)^* = A^1$. The multiplication

$$A^*_\Delta \longrightarrow A^*_\Delta[2], f \mapsto \psi f$$

induces a degree 2 map, the *Lefschetz operator*,

$$L : H^*(P) \longrightarrow H^*(P).$$

Theorem 4.2 (Hard Lefschetz)**.** *The iterated Lefschetz operator L^{n-2i} induces an isomorphism*

$$L^{n-2i} : H^i(P) \longrightarrow H^{n-i}(P)$$

for $i < n/2$.

Since L^{n-2i} factorizes over $H^{i+1}(P)$, it follows that $h_{i+1}(P) \geq h_i(P)$ for $i < n/2$.

Growth estimates for the "Betti numbers" $h_i(P)$:
We apply the following result to the algebra $B^* := H^*(P)/(\overline{\psi})$, where $\overline{\psi} \in H^1(P)$ denotes the residue class of $\psi \in A^1_\Delta$.

Theorem 4.3 (Macaulay). *A sequence* $(d_0, d_1, d_2, ...)$ *of natural numbers is the Hilbert sequence of a commutative graded* $\mathbb{R}$*-algebra* $B^* = \bigoplus_{i=0}^{\infty} B^i$ *generated by the elements of degree* 1 *(*$B^* = \mathbb{R}[B^1]$*), i.e. we have* $d_i = \dim B^i$*, if and only if*

1. $d_0 = 1$,
2. $d_{i+1} \leq (d_i)^{<i>}$ *for all* $i > 0$.

Hard Lefschetz: Nearby polytopes of a simple polytope P (i.e. nearby, when P is represented as intersection of half spaces) are simple and of the same combinatorial type as P, so we may assume that P is a lattice polytope for some lattice $\Lambda \subset V$ and then apply the Hard Lefschetz Theorem for projective algebraic varieties. This is the argument of Stanley in [St$_1$]. A convex geometry proof of Th.4.4 is due to McMullen (1993), see [Mc].

It is by far the most sophisticated part of the proof, and here we can only indicate a very rough outline. First of all there is a more detailed version of Th. 4.2, the "*Hodge Riemann relations*":

Theorem 4.4. *For* $i \leq n/2$ *the pairing*

$$\sigma_i : H^i(P) \times H^i(P) \longrightarrow \mathbb{R}, (a, b) \mapsto a \cap L^{n-2i} b$$

restricts to a $(-1)^i$*-definite pairing*

$$K^i(P) \times K^i(P) \longrightarrow \mathbb{R}$$

on the i*-th "primitive subspace"*

$$K^i(P) := \ker(L^{n-2i+1} : H^i(P) \longrightarrow H^{n-i+1}(P)).$$

Remark 4.5. The Hodge Riemann relations imply the Hard Lefschetz theorem: Indeed, using the fact that L is self-adjoint for the intersection pairing, we obtain a σ_i-orthogonal decomposition

$$H^i(P) = \bigoplus_{0 \leq q \leq i} L^q(K^{i-q}(P)).$$

To begin with, for $P = S_n$, an n-simplex, we have (with $\deg(T) = 1$)

$$H^*(S_n) = \mathbb{R}[\overline{\psi}] \cong \mathbb{R}[T]/(T^{n+1}),$$

and one checks easily that $\sigma_0(\overline{1}, \overline{1}) > 0$ holds for $P = S_n$.

Now consider a simple $(n+1)$-polytope $Q \subset V \times \mathbb{R}$, such that the projection $f := \mathrm{pr}_{\mathbb{R}} : V \times \mathbb{R} \longrightarrow \mathbb{R}$ separates the vertices of Q. We study the behaviour of the level polytopes

$$Q_s := \mathrm{pr}_V(Q \cap f^{-1}(s)) \subset V.$$

A value $s \in \mathbb{R}$ is then called a regular value for $f|_Q$ if Q_s does not contain a vertex of Q. Then any simple n-polytope $P \subset V$ can be realized as a level set Q_s of a regular value s for a suitable choice of the simple $(n+1)$-polytope $Q \subset V \times \mathbb{R}$. For regular s a little bit above $\min f|_Q$ the polytope Q_s is a simplex, so Th. 4.4 holds; hence we have to check what happens if s passes through a critical value of $f|_Q$, i.e. a vertex $v \in Q$:

The index $I_v(f)$ of $f|_Q$ in such a critical point v then is defined to be the number of edges of Q approaching v from below (with respect to the height function $f|_Q : Q \longrightarrow \mathbb{R}$), so $0 \leq I_v(f) \leq n+1$. Then the fan $\Delta(Q_s)$ as well as $A^*_{\Delta(Q_s)}$ and $H^*(Q_s)$ are independent from $s \in [a.b]$ for any interval $[a.b]$ containing only regular values of $f|_Q$, but the corresponding function $\psi_s \in A^1_\Delta$ (with, say, $\Delta := \Delta(P_a)$) is not.

Let us now investigate the situation that there is exactly one critical value $c = f(v)$ (with the vertex $v \in Q$) between the regular values a and b, denote $d := I_v(f)$ its index: The passage from Q_a to Q_b is called a *d-flip*: Denote $F_s \prec Q_s$ the face spanned by the vertices of Q_s lying on the edges of Q meeting at v: For $s < c$ the face F_s is an $(d-1)$-simplex and for $s > c$ an $(n-d)$-simplex, while $F_c \prec Q_c$ is a vertex. In order to compare the f- resp. h-polynomials we may, for symmetry reasons, assume $d \leq \frac{n+1}{2}$.

During a 1-flip a new facet is created, while for $d > 1$ there is a bijection between the faces of Q_a not contained in F_a and the faces of Q_b not contained in F_b. Denoting f^s resp. h^s the f- resp. h-polynomial of Q_s we thus obtain

$$f^b(T) = f^a(T) + T^{-1}((T+1)^{n-d+1} - (T+1)^d) \ ,$$

the f-polynomial of an m-simplex S_m being

$$f_{S_m}(T) = T^{-1}((T+1)^{m+1} - 1).$$

Replacing T with $T-1$ yields then

$$h^b(T) = h^a(T) + T^d + ... + T^{n-d} \ .$$

So $\dim K^d(Q_b) = \dim K^d(Q_a) + 1$. At the same time we have reproved the Dehn-Sommerville equations. But to show that the Hodge Riemann relations are preserved when passing from Q_a to Q_b as well as in the opposite direction requires a much more detailed and sophisticated argument; here we have to refer to [Mc] and [Ti].

5 General Polytopes

For nonsimple polytopes one can of course again define an h-vector as the transformed f-vector, but it turns out that it does not have the same nice properties as in the case of simple polytopes. Another idea is to look for lattice polytopes P only, and then to define

$$h_i(P) := \dim H^{2i}(X(P); \mathbb{R}),$$

but then, $X(P)$ being singular, the cohomology algebra has as well "pathological" features, e.g. it is not any longer a combinatorial invariant of the face lattice nor does it live only in even degrees.

On the other hand there is a cohomology theory specially adapted to singular varieties, the intersection cohomology $IH^*(..)$ of M. Goresky and R. MacPherson. So one could think of setting

$$h_i(P) := \dim IH^{2i}(X(P); \mathbb{R}).$$

It turns out that the intersection cohomology Betti numbers are again combinatorial invariants and can be computed recursively. Indeed, Stanley used the computation algorithm in order to define his generalized h-vector, see [St_2]. Intersection cohomology satisfies both Poincaré duality and Hard Lefschetz, but does this imply that $h_{n-i}(P) = h_i(P)$ and $h_{i+1}(P) \geq h_i(P)$ for $i < n/2$ holds even for non-lattice polytopes? For a simple polytope there is always a lattice polytope of the same combinatorial type, but for general polytopes that is definitely false. So in order to answer our question it was necessary to construct a "combinatorial intersection cohomology" in terms of polytopes or rather fans, extending the approach via piecewise polynomials, see [BBFK_2] and [BreLu_1]. The proof of the combinatorial hard Lefschetz theorem was - in accordance with the name - the most difficult part: Its proof was given in [Ka], and there are now even some simplified versions available, see [BreLu_2] and [BBFK_4]. In any case, one derives it from the Hard Lefschetz theorem for simple polytopes via desingularisation or rather "simplification" of a general polytope.

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