ON SELFADJOINT FUNCTORS SATISFYING POLYNOMIAL RELATIONS

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ABSTRACT. We study selfadjoint functors acting on categories of finite dimensional modules over finite dimensional algebras with an emphasis on functors satisfying some polynomial relations. Selfadjoint functors satisfying several easy relations, in particular, idempotents and square roots of a sum of identity functors, are classified. We also describe various natural constructions for new actions using external direct sums, external tensor products, Serre subcategories, quotients and centralizer subalgebras.

1. INTRODUCTION

The main motivation for the present paper stems from the recent activities on categorification of representations of various algebras, see, in particular, [CR, FKS, HS, HK, KMS1, MS1, MS3, MS4, R], the review [KMS2] and references therein. In these articles one could find several results of the following kind: given a field $\Bbbk$, an associative $\Bbbk$-algebra Λ with a fixed generating set $\{a_i\}$, and a Λ-module M, one constructs a *categorification* of M, that is an abelian category $\mathcal{C}$ and exact endofunctors $\{\mathrm{F}_i\}$ of $\mathcal{C}$ such that the following holds: The Grothendieck group $[\mathcal{C}]$ of $\mathcal{C}$ (with scalars extended to an appropriate field) is isomorphic to M as a vector space and the functor F_i induces on $[\mathcal{C}]$ the action of a_i on M. Typical examples of algebras, for which categorifications of certain modules are constructed, include group algebras of Weyl groups, Hecke algebras, Schur algebras and enveloping algebras of some Lie algebras. There are special reasons why such algebras and modules are of importance, for example, because of applications to link invariants (see [St1]) or Broué's abelian defect group conjecture (see [CR]). Introducing some extra conditions one could even establish some uniqueness results, see [CR, R].

In this paper we would like to look at this problem from a different perspective. The natural question, which motivates us, is the following: Given Λ and $\{a_i\}$ can one classify *all* possible categorifications of *all* Λ-modules up to some natural equivalence? Of course in the full generality the problem is hopeless, as even the problem of classifying all Λ-modules seems hopeless for wild algebras. So, to start with, in this paper we make the main emphasis on the most basic example,

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that is the case when the algebra Λ is generated by one element, say a. If Λ is finite-dimensional, then we necessarily have $f(a) = 0$ for some nonzero $f(x) \in \Bbbk[x]$. To make the classification problem more concrete, it is natural to look for a finite-dimensional $\Bbbk$-algebra A and an exact endofunctor F of A-mod, which should satisfy some sensible analogue of the relation $f(\mathrm{F}) = 0$. Assume that all coefficients of $f(x)$ are integral and rewrite $f(a) = 0$ as $g(a) = h(a)$, where both g and h have nonnegative coefficients. Setting

$$\mathtt{k}\mathrm{F} := \begin{cases} \underbrace{\mathrm{F} \oplus \mathrm{F} \oplus \cdots \oplus \mathrm{F}}_{\mathtt{k}\text{ times}}, & \mathtt{k} \in \{1, 2, \dots\}; \\ \mathbf{0}, & \mathtt{k} = 0; \end{cases} \tag{1}$$

and interpreting $+$ as $\oplus$, it makes sense to require $g(\mathrm{F}) \cong h(\mathrm{F})$ for our functor F.

To simplify our problem further we make another observation about the examples of categorification available from the literature mentioned above. All algebras appearing in this literature are equipped with an involution, which in the categorification picture is interpreted as "taking the adjoint functor" (both left- and right-adjoint). We again take the simplest case of the trivial involution, and can now formulate our main problem as follows:

Problem 1. *Given a finite dimensional $\Bbbk$-algebra A and two polynomials $g(x)$ and $h(x)$ with nonnegative integral coefficients, classify, up to isomorphisms, all selfadjoint endofunctors* F *on A*-mod *which satisfy $g(\mathrm{F}) \cong h(\mathrm{F})$.*

Using Morita equivalence, in what follows we may assume that A is basic (i.e. has one dimensional simple modules). In this paper we obtain an answer to Problem 1 for relations $x^2 = x$ (Section 5), $x^2 = k$, $k \in \mathbb{Z}_+$, (Sections 2, 3 and 6) and $x^k = x^m$ (Section 6). For semisimple algebras Problem 1 reduces to solving certain matrix equations over matrices with nonnegative integer coefficients (Section 7).

Another natural and important general question, which we address in this paper, is how to produce new functorial actions by selfadjoint functors (e.g. new solutions to Problem 1) from already known actions (known solutions to Problem 1). In particular, in Section 4 we describe the natural operations of external direct sums and external tensor products. In Section 8 we study how functorial actions by selfadjoint functors can be restricted to centralizer subalgebras. In the special case of the algebra A having the double centralizer property for a projective-injective module X, we show that there is a full and faithful functor from the category of selfadjoint functors on A-mod to the category of selfadjoint functors on $\mathrm{End}_A(X)^{\mathrm{op}}$-mod. We also present an example for which this functor is not dense (essentially surjective).

Finally, in Section 9 we study restriction of selfadjoint functors to invariant Serre subcategories and induced actions on quotient categories, which we show are realized via induced actions on centralizer subalgebras.

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2. Group actions on module categories

Let $\Bbbk$ be an algebraically closed field, A a basic finite dimensional unital $\Bbbk$-algebra, and $Z(A)$ the center of A. All functors we consider are assumed to be additive and $\Bbbk$-linear. We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{Z}_+$ the set of nonnegative integers. Let $\{L_1, L_2, \dots, L_n\}$ be a complete list of pairwise nonisomorphic simple A-modules. Let P_i, $i = 1, \dots, n$, denote the indecomposable projective cover of L_i. We denote by ID the identity functor and by $\mathbf{0}$ the zero functor.

To start with we consider the easiest possible nontrivial equation

$$\mathrm{F} \circ \mathrm{F} \cong \mathrm{ID}, \tag{2}$$

which just means that F is a (covariant) involution on A-mod. The answer to Problem 1 for relation (2) reduces to the following fairly well-known result (for which we did not manage to find a reasonably explicit reference though):

Proposition 2. *(i) For an algebra automorphism $\varphi : A \to A$ let ${}_{\varphi}A$ denote the bimodule A in which the left action is twisted by φ (i.e. $a \cdot x \cdot b = \varphi(a)xb$). Then $\mathrm{F}_{\varphi} := {}_{\varphi}A \otimes_A {}_-$ is an autoequivalence of* A-mod.

(ii) We have $\mathrm{F}_{\varphi} \circ \mathrm{F}_{\psi} \cong \mathrm{F}_{\varphi\circ\psi}$ for any automorphisms φ and ψ of A.

(iii) Every autoequivalence of A-mod *is isomorphic to F_{φ} for some algebra automorphism $\varphi : A \to A$.*

(iv) $\mathrm{F}_{\varphi} \cong \mathrm{ID}$ if and only if φ is an inner automorphism.

(v) F_{φ} is selfadjoint if and only if φ^2 is an inner automorphism.

Proof. The functor F_{φ} just twists the action of A by φ. This implies claim (ii) and, in particular, that $\mathrm{F}_{\varphi^{-1}}$ is an inverse to F_{φ}, which yields claim (i).

Let $\mathrm{F} : A\text{-mod} \to A\text{-mod}$ be an autoequivalence. Then F maps indecomposable projectives to indecomposable projectives, in particular, $\mathrm{F}_A A \cong {}_A A$ and we can identify these modules fixing some isomorphism, say $\alpha : {}_A A \xrightarrow{\sim} \mathrm{F}_A A$. Let $\beta : A^{\mathrm{op}} \to \mathrm{End}_A({}_A A)$ be the natural isomorphism sending a to the right multiplication with a, which we denote by r_a. Using the following sequence:

$$ {}_AA \xrightarrow{\alpha} \mathrm{F}\,{}_AA \xrightarrow{F(r_a)} \mathrm{F}\,{}_AA \xleftarrow{\alpha} {}_AA $$

we can define $\varphi(a) := \beta^{-1}(\alpha^{-1}F(r_a)\alpha)$. Then φ is an automorphism of A. It is straightforward to verify that $\mathrm{F} \cong \mathrm{F}_{\varphi^{-1}}$. This proves claim (iii).

If $\varphi : A \to A$ is inner, say $\varphi(a) = sas^{-1}$ for some invertible $s \in A$, it is straightforward to check that the map $a \mapsto sa$ is a bimodule isomorphism from A to ${}_\varphi A$. This means that $\mathrm{F}_\varphi \cong \mathrm{ID}$ in this case. Conversely, if $\mathrm{F}_\varphi \cong \mathrm{ID}$, then there is a bimodule isomorphism $f : A \to {}_\varphi A$. Let $s = f(1)$. Then s is invertible as $1 \in f(A) = f(1) \cdot A = sA$ and $1 \in f(A) = A \cdot f(1) = \varphi(A)s$. Also $\varphi(a)s = a \cdot f(1) = f(a) = f(1) \cdot a = sa$, which yields $\varphi(a) = sas^{-1}$. This proves claim (iv).

As F_φ is an autoequivalence by claim (i), the adjoint of F_φ is $\mathrm{F}_{\varphi^{-1}}$. Therefore claim (v) follows from claim (iv) □

We note that the bimodule ${}_\varphi A$ occurring in Proposition 2 is sometimes called a *twisted bimodule*, see for example [EH].

Let G be a group. A *weak* (resp. *strong*) action of G on A-mod is a collection $\{\mathrm{F}_g : g \in G\}$ of endofunctors of A-mod such that $\mathrm{F}_g \circ \mathrm{F}_h \cong \mathrm{F}_{gh}$ (resp. $\mathrm{F}_g \circ \mathrm{F}_h = \mathrm{F}_{gh}$) for all $g, h \in G$, and $\mathrm{F}_1 \cong \mathrm{ID}$ (resp. $\mathrm{F}_1 = \mathrm{ID}$). Two weak actions $\{\mathrm{F}_g : g \in G\}$ and $\{\mathrm{F}'_g : g \in G\}$ are called *equivalent* provided that $\mathrm{F}_g \cong \mathrm{F}'_g$ for all $g \in G$. Let $\mathrm{Aut}(A)$ denote the group of all automorphisms of A and $\mathrm{Inn}(A)$ denote the normal subgroup of $\mathrm{Aut}(A)$ consisting of all inner automorphisms. Set $\mathrm{Out}(A) := \mathrm{Aut}(A)/\mathrm{Inn}(A)$. From Proposition 2 we have:

Corollary 3. *Equivalence classes of weak actions of a group G on A-*mod *are in one-to-one correspondence with group homomorphisms from G to* $\mathrm{Out}(A)$.

Proof. Let $\{\mathrm{F}_g : g \in G\}$ be a weak action of G on A-mod. Then for any $g \in G$ the functor F_g is an autoequivalence of A-mod and hence is isomorphic to the functor F_{φ_g} for some automorphism φ_g of A (Proposition 2(iii)). By Proposition 2(iv), the automorphism φ_g is defined up to a factor from $\mathrm{Inn}(A)$, hence, by Proposition 2(ii), the map $g \mapsto \varphi_g \mathrm{Inn}(A)$ is a homomorphism from G to $\mathrm{Out}(A)$. From the definitions it follows that equivalent actions produce the same homomorphism and nonequivalent actions produce different homomorphisms. The claim follows. □

Corollary 4. *If* $\mathrm{Inn}(A)$ *is trivial, then every weak action of a group G on A-*mod *is equivalent to a strong action.*

Proof. If $\mathrm{Inn}(A)$ is trivial, the automorphism φ_g from the proof of Corollary 3 is uniquely defined, so the action $\{\mathrm{F}_g : g \in G\}$ is equivalent to the strong action $\{\mathrm{H}_g : g \in G\}$, where H_g denotes the functor of twisting the A-action by φ_g. The claim follows. □

Corollary 5. *Isomorphism classes of selfadjoint functors* F *satisfying* (2) *are in one-to-one correspondence with group homomorphisms from* $\mathbb{Z}_2$ *to* $\mathrm{Out}(A)$*. The correspondence is given by:*

$$\begin{array}{c} \mathrm{F} \mapsto f, \text{ where } f : \mathbb{Z}_2 \to \mathrm{Out}(A) \text{ is such that} \\ \mathrm{F} \cong \mathrm{F}_\varphi \text{ for any } \varphi \in f(1). \end{array} \tag{3}$$

Proof. Note that any autoequivalence of A-mod satisfying (2) is selfadjoint (as $\mathrm{F} \cong \mathrm{F}^{-1}$ by (2)). Therefore the claim follows from Corollary 3 and its proof. □

Corollary 6. *(i) Let* $n \in \{2, 3, 4, \dots\}$*. Then isomorphism classes of endofunctors* F *of* A-mod *satisfying*

$$\mathrm{F}^n := \underbrace{\mathrm{F} \circ \mathrm{F} \circ \cdots \circ \mathrm{F}}_{n \text{ times}} \cong \mathrm{ID} \tag{4}$$

are in one-to-one correspondence with group homomorphisms from $\mathbb{Z}_n$ *to* $\mathrm{Out}(A)$ *(the correspondence is given by* (3)*, where* $\mathbb{Z}_2$ *is substituted by* $\mathbb{Z}_n$*).*

(ii) The endofunctor F *from* (i) *is selfadjoint if and only if* $\mathrm{F}^2 \cong \mathrm{ID}$.

Proof. Claim (i) is proved similarly to Corollary 5. Claim (ii) is obvious. □

3. Selfadjoint functorial square roots

In this section we consider the generalization

$$\mathrm{F} \circ \mathrm{F} \cong \mathtt{k}\mathrm{ID}, \quad \mathtt{k} \in \{2, 3, 4, \dots\}, \tag{5}$$

of the equation (2) (see (1) for notation). Our main result here is the following:

Theorem 7. *(i) A selfadjoint endofunctor* F *of* A-mod *satisfying* (5) *exists if and only if* $\mathtt{k} = \mathtt{m}^2$ *for some* $\mathtt{m} \in \{2, 3, 4, \dots\}$*.*

(ii) If $\mathtt{k} = \mathtt{m}^2$ *for some* $\mathtt{m} \in \{2, 3, 4, \dots\}$*, then isomorphism classes of selfadjoint endofunctors* F *on* A-mod *satisfying* (5) *are in one-to-one correspondence with isomorphism classes of selfadjoint endofunctors* F' *on* A-mod *satisfying* (2)*. The correspondence is given by:* $\mathtt{m}\mathrm{F}' \mapsto \mathrm{F}'$*.*

Let $[A]$ denote the Grothendieck group of A-mod. For $M \in A$-mod we denote by $[M]$ the image of M in $[A]$. The group $[A]$ is a free abelian group with basis $\mathbf{l} := ([L_1], [L_2], \dots, [L_n])$. Every exact endofunctor G on A-mod defines a group endomorphism $[\mathrm{G}]$ of $[A]$. We denote by M_{G} the matrix of $[\mathrm{G}]$ in the basis $\mathbf{l}$. Obviously, $M_{\mathrm{G}} \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_+)$.

If G is selfadjoint, it is exact and maps projective modules to projective modules (and injective modules to injective modules, see for example [Ma, Corollary 5.21]). Then $\mathrm{G}P_j = \oplus_{i=1}^{n} \mathtt{x}_{ij} P_i$. Define $N_{\mathrm{G}} = (\mathtt{x}_{ij})_{i,j=1,\dots,n}$.

Lemma 8. *We have* $N_{\mathrm{G}} = M_{\mathrm{G}}^t$*, where* $\cdot^t$ *denotes the* transposed *matrix.*

Proof. Let $M_{\mathrm{G}} = (\mathtt{y}_{ij})_{i,j=1,\dots,n}$. The claim follows from the selfadjointness of G as follows:

$$\mathtt{x}_{ij} = \dim \mathrm{Hom}_A(\mathrm{G}P_j, L_i) = \dim \mathrm{Hom}_A(P_j, \mathrm{G}L_i) = \mathtt{y}_{ji}. \qquad \square$$

To prove Theorem 7 we will need to understand M_{F} for selfadjoint functors F satisfying (5). Let $\mathbf{1}_n$ denote the identity matrix in $\mathrm{Mat}_{n\times n}(\mathbb{Z}_+)$. Then from (5) we obtain $M_{\mathrm{F}}^2 = \mathtt{k}\cdot\mathbf{1}_n$. The canonical form for such M_{F} is given by the following lemma:

Lemma 9. *Let* $M \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_+)$ *be such that* $M^2 = \mathtt{k}\cdot\mathbf{1}_n$*. Then there exists a permutation matrix* S *such that* SMS^{-1} *is a direct sum of matrices of the form*

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \quad a, b \in \mathbb{Z}_+, ab = \mathtt{k}; \qquad \textit{and} \qquad (a), \quad a \in \mathbb{Z}_+, a^2 = \mathtt{k}.$$

Proof. We proceed by induction on n. If $n = 1$ the claim is obvious. Let $M = (m_{ij})_{i,j=1,\dots,n}$ and $M^2 = (k_{ij})_{i,j=1,\dots,n}$. If $m_{11} \neq 0$, then $m_{1j} = 0$, $j = 2,\dots,n$, for otherwise $k_{1j} \neq 0$ (as all our entries are in $\mathbb{Z}_+$). Similarly $m_{j1} = 0$, $j = 2,\dots,n$. This means that M is a direct sum of the block (m_{11}), where $m_{11}^2 = \mathtt{k}$, and a matrix $\hat{M}$ of size $n-1\times n-1$ satisfying $\hat{M}^2 = \mathtt{k}\cdot\mathbf{1}_{n-1}$. The claim now follows from the inductive assumption.

If $m_{11} = 0$ then, since $k_{11} = \mathtt{k}$, there exists $j \in \{2, 3, \dots, n\}$ such that $m_{1j} \neq 0$ and $m_{j1} \neq 0$. Substituting M by SMS^{-1}, where S is the transposition of j and 2, we may assume $j = 2$. Then from $k_{1j} = k_{j1} = 0$ for all $j \neq 1$, and $k_{2j} = k_{j2} = 0$ for all $j \neq 2$, it follows that $m_{1j} = m_{j1} = 0$ for all $j \neq 2$ and $m_{2j} = m_{j2} = 0$ for all $j \neq 1$. This means that M is a direct sum of the block $\begin{pmatrix} 0 & m_{1,2} \\ m_{2,1} & 0 \end{pmatrix}$, where $m_{12}m_{21} = \mathtt{k}$, and a matrix $\hat{M}$ of size $n-2\times n-2$ satisfying $\hat{M}^2 = \mathtt{k}\cdot\mathbf{1}_{n-2}$. Again, the claim now follows from the inductive assumption. This completes the proof. $\square$

Lemma 10. *Assume that* F *is an endofunctor on* A-mod *satisfying* (5)*. Then* F *preserves the full subcategory* $\mathcal{S}$ *of* A-mod*, which consists of semisimple* A*-modules.*

Proof. As F is additive, to prove the claim we have to show that F sends simple modules to semisimple modules. Since F satisfies (5), the matrix M_{F} satisfies $M_{\mathrm{F}}^2 = \mathtt{k}\cdot\mathbf{1}_n$ and hence is described by Lemma 9. From the latter lemma it follows that for any $i \in \{1, 2, \dots, n\}$ we have $[\mathrm{F}L_i] = a[L_j]$ for some $j \in \{1, 2, \dots, n\}$ and $a \in \mathbb{N}$, and, moreover, $[\mathrm{F}L_j] = b[L_i]$ for some $b \in \mathbb{N}$ such that $ab = \mathtt{k}$. Applying F to any inclusion $L_i \hookrightarrow \mathrm{F}L_j$ we get $\mathrm{F}L_i \hookrightarrow \mathrm{FF}L_j$. However, $\mathrm{FF}L_j \overset{(5)}{\cong} \mathtt{k}L_j$ is a semisimple module. Therefore $\mathrm{F}L_i$, being a submodule of a semisimple module, is semisimple itself. $\square$

Proof of Theorem 7(i). If $\mathtt{k} = \mathtt{m}^2$ for some $\mathtt{m} \in \{2, 3, 4, \dots\}$, then $\mathrm{F} = \mathtt{m}\mathrm{ID}$ is a selfadjoint functor satisfying (5). Hence to prove Theorem 7(i) we have to show that in the case $\mathtt{k} \neq \mathtt{m}^2$ for any $\mathtt{m} \in \{2, 3, 4, \dots\}$ no selfadjoint F satisfies (5).

In the latter case let us assume that F is a selfadjoint endofunctor on A-mod satisfying (5). From Lemma 9 we have that, after a reordering of simple modules, the matrix M_{F} becomes a direct sum of matrices of the form $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$, where $ab = \mathtt{k}$ and $a \neq b$. In particular, we have $[\mathrm{F}L_1] = \mathtt{b}[L_2]$ and $[\mathrm{F}L_2] = \mathtt{a}[L_1]$ for some $\mathtt{a}, \mathtt{b} \in \mathbb{N}$ such that $\mathtt{ab} = \mathtt{k}$ and $\mathtt{a} \neq \mathtt{b}$. By Lemma 10, we even have $\mathrm{F}L_1 \cong \mathtt{b}L_2$ and $\mathrm{F}L_2 \cong \mathtt{a}L_1$. Using this and the selfadjointness of F, we have:

$$\begin{aligned} \mathtt{b} &= \dim \mathrm{Hom}_A(\mathtt{b}L_2, L_2) \\ &= \dim \mathrm{Hom}_A(\mathrm{F}L_1, L_2) \\ &= \dim \mathrm{Hom}_A(L_1, \mathrm{F}L_2) \\ &= \dim \mathrm{Hom}_A(L_1, \mathtt{a}L_1) \\ &= \mathtt{a}, \end{aligned}$$

a contradiction. The claim of Theorem 7(i) follows. $\square$

Proof of Theorem 7(ii). This proof is inspired by [LS]. Assume that $\mathtt{k} = \mathtt{m}^2$ for some $\mathtt{m} \in \{2, 3, 4, \dots\}$. If F' is a selfadjoint endofunctor on A-mod satisfying (2), then $\mathtt{m}\mathrm{F}'$ is a selfadjoint endofunctor on A-mod satisfying (5). Hence to prove Theorem 7(ii) we have to establish the converse statement.

Let F be some selfadjoint endofunctor on A-mod satisfying (5). Our strategy of the proof is as follows: we would like to show that the functor F decomposes into a direct sum of $\mathtt{m}$ nontrivial functors and then use the results from Section 2 to get that these functors have the required form. To prove decomposability of F we produce $\mathtt{m}$ orthogonal idempotents in the endomorphism ring of F. For this we first show that the necessary idempotents exist in the case of a semisimple algebra, and then use lifting of idempotents modulo the radical. All the above requires some preparation and technical work.

From Lemma 9 and the above proof of Theorem 7(i) it follows that, re-indexing, if necessary, simple A-modules, the matrix M_{F} reduces to a direct sum of the blocks

$$\begin{pmatrix} 0 & \mathtt{m} \\ \mathtt{m} & 0 \end{pmatrix} \qquad \text{and/or} \qquad (\mathtt{m})\,. \tag{6}$$

Lemma 11. *The claim of Theorem 7*(ii) *is true in the case of a semisimple algebra A.*

Proof. Assume first that A is semisimple (and basic). Then $A \cong \oplus_{i=1}^{n}\mathbb{k}$ and A-mod $\cong \oplus_{i=1}^{n}\mathbb{k}$-mod. The only (up to isomorphism) indecomposable nonzero functor from $\mathbb{k}$-mod to $\mathbb{k}$-mod is the identity functor (as $\mathbb{k}\otimes\mathbb{k}\cong\mathbb{k}$). Therefore from (6) we get that F is isomorphic to a direct sum of functors of the form

$$\mathbb{k}\text{-mod} \underset{\mathtt{m}\mathrm{ID}}{\overset{\mathtt{m}\mathrm{ID}}{\rightleftarrows}} \mathbb{k}\text{-mod} \qquad \text{and/or} \qquad \mathbb{k}\text{-mod} \circlearrowright^{\mathtt{m}\mathrm{ID}}$$

(corresponding to the blocks from (6)). Define F' as the corresponding direct sum of functors of the form

$$\mathbb{k}\text{-mod} \underset{\mathrm{ID}}{\overset{\mathrm{ID}}{\rightleftarrows}} \mathbb{k}\text{-mod} \qquad \text{and/or} \qquad \mathbb{k}\text{-mod}. \circlearrowright^{\mathrm{ID}}$$

Then F' is selfadjoint and obviously satisfies (2), moreover, $\mathrm{F}\cong\mathtt{m}\mathrm{F}'$. This proves Theorem 7(ii) in the case of a semisimple algebra A. □

Let V be an A-A-bimodule such that $\mathrm{F}\cong V\otimes_A {}_-$. The right adjoint of F is F itself, in particular, this right adjoint is an exact functor and hence is given by tensoring with the bimodule $\mathrm{Hom}_{A-}(V,A)$ (that is $V\cong\mathrm{Hom}_{A-}(V,A)$). The bimodule V is projective both as a right A-module (as F is exact) and as a left A-module (as F sends projective modules to projective modules). Hence we have an isomorphism of A-A-bimodules as follows:

$$\begin{array}{ccc} \mathrm{Hom}_{A-}(V,A)\otimes_A V & \cong & \mathrm{Hom}_{A-}(V,V) \\ f\otimes v & \mapsto & (w\mapsto f(w)v). \end{array}$$

This gives us the following isomorphism of A-A-bimodules:

$$V\otimes_A V\cong\mathrm{Hom}_{A-}(V,A)\otimes_A V\cong\mathrm{Hom}_{A-}(V,V).$$

Note that the functor $\mathrm{F}\circ\mathrm{F}$ is given by tensoring with the bimodule $V\otimes_A V\cong\mathrm{Hom}_{A-}(V,V)$. From (5) we thus get an isomorphism

$$\mathrm{Hom}_{A-}(V,V)\cong\mathtt{k}A \tag{7}$$

of A-bimodules. Taking on both sides of the latter isomorphism elements on which the left and the right actions of A coincide, we get an isomorphism

$$\mathrm{Hom}_{A-A}(V,V)\cong\mathtt{k}Z(A) \tag{8}$$

of $Z(A)$-bimodules.

Let R_l and R_r denote the radical of V, considered as a left and as a right A-module, respectively. As $V\otimes_A {}_-$ sends simple modules to semisimple modules (Lemma 10), it follows that $R_l=R_r=:R$.

From (6) we have that the matrix M_{F} is symmetric. Hence $N_{\mathrm{F}}=M_{\mathrm{F}}$ (Lemma 8). Therefore from (6) it follows that each indecomposable projective module occurs in $\mathrm{F}_A A$ with multiplicity $\mathtt{m}$, that is $\mathrm{F}_A A\cong$

$\mathtt{m}\,_AA$. Using decomposition (7) we thus can choose a basis $\{b_i : i = 1, \dots, \mathtt{k}\}$ of $\mathrm{Hom}_{A-}(V,V)$ as a free left A-module such that the left and the right actions of A on the elements of this basis coincide. Then all b_i's belong to $\mathrm{Hom}_{A-A}(V,V)$ and form there a basis as a free $Z(A)$-module (both left and right).

Lemma 12. *Let $L = \oplus_{i=1}^{n} L_i$ and b be a nontrivial $\Bbbk$-linear combination of b_i's. Then b is a natural transformation from F to F and the induced endomorphism b_L of the A-module $\mathrm{F}L$ is nonzero.*

Proof. Applying the exact functor $V \otimes_A {}_-$ to the short exact sequence

$$0 \to \mathrm{Rad}(A) \to {}_AA \to L \to 0,$$

we obtain the short exact sequence

$$0 \to V \otimes_A \mathrm{Rad}(A) \to V \to V \otimes_A L \to 0.$$

Note that $V \otimes_A \mathrm{Rad}(A) = R$. Applying to the latter sequence the exact functor $\mathrm{Hom}_{A-}(V, {}_-)$ we obtain the short exact sequence

$$0 \to \mathrm{Hom}_{A-}(V,R) \to \mathrm{Hom}_{A-}(V,V) \to \mathrm{Hom}_{A-}(V, V \otimes_A L) \to 0.$$

By the definition of b, the image of $b \in \mathrm{Hom}_{A-}(V,V)$ does not belong to R and hence b induces a nonzero element $\overline{b} \in \mathrm{Hom}_{A-}(V, V \otimes_A L)$. By adjunction we have the following isomorphism:

$$\mathrm{Hom}_A(L, \mathrm{Hom}_{A-}(V, V \otimes_A L)) \cong \mathrm{Hom}_A(V \otimes_A L, V \otimes_A L),$$

which produces the nonzero endomorphism b_L of $V \otimes_A L$ from our nonzero element $\overline{b}$. The claim follows. □

Set $\overline{A} = A/\mathrm{Rad}(A)$, then $\overline{A}$ is a semisimple algebra. The bimodule $\overline{V} = V/R$ is an $\overline{A}$-bimodule satisfying (5) and we have $\mathrm{Rad}(A)\overline{V} = \overline{V}\mathrm{Rad}(A) = 0$. Hence we have the quotient homomorphism of algebras as follows:

$$\Phi : \mathrm{Hom}_{A-A}(V,V) \to \mathrm{Hom}_{\overline{A}-\overline{A}}(\overline{V}, \overline{V}).$$

Note that the algebra $\overline{A} = Z(\overline{A}) \cong n\Bbbk$ contains as a subalgebra the algebra $\overline{Z(A)} := Z(A)/\mathrm{Rad}(Z(A))$. The algebra $\overline{Z(A)}$ is isomorphic to $k\Bbbk$, where k is the number of connected components of the algebra A. In particular, $\overline{Z(A)} \cong \overline{A}$ if A is a direct sum of local algebras.

Lemma 13. *The kernel of Φ is the radical of $\mathrm{Hom}_{A-A}(V,V)$ and the image of Φ is isomorphic to the algebra $\mathrm{Mat}_{\mathtt{m}\times\mathtt{m}}(\overline{Z(A)})$.*

Proof. The space $\mathrm{Hom}_{A-A}(V,V)$ is a free left $Z(A)$-module of rank $\mathtt{k}$ with the basis $\{b_i\}$ from the above. Since $\overline{A}$ is semisimple, so is $\overline{V}$ and thus both $\mathrm{Rad}(Z(A))$ and the radical of $\mathrm{Hom}_{A-A}(V,V)$ annihilate $\overline{V}$ both from the left and from the right. Therefore the first claim of the lemma follows from the second claim just by counting dimensions.

Similarly to the proof of Lemma 12 one shows that the image X of Φ has dimension $\mathtt{k} \cdot \dim \overline{Z(A)}$ and is a subalgebra of $\mathrm{End}_{\overline{A}}(\overline{V} \otimes_{\overline{A}} \overline{A})$, which corresponds to the embedding $\overline{Z(A)} \subset \overline{A}$ (the algebra $\mathrm{End}_{\overline{A}}(\overline{V} \otimes_{\overline{A}} \overline{A})$ is free both as a left and as a right $\overline{A}$-module). By Lemma 11 we have an isomorphism $\overline{V} \otimes_{\overline{A}} \overline{A} \cong \overline{V} \cong \mathtt{m}\overline{A}$ of $\overline{A}$-modules. Hence the algebra X is a subalgebra of the algebra

$$\mathrm{End}_{\overline{A}}(\mathtt{m}\overline{A}) \cong \mathrm{Mat}_{\mathtt{m}\times\mathtt{m}}(\overline{A}),$$

which corresponds to the embedding $\overline{Z(A)} \subset \overline{A}$. This means that $X \cong \mathrm{Mat}_{\mathtt{m}\times\mathtt{m}}(\overline{Z(A)})$. □

We have $\mathrm{Mat}_{\mathtt{m}\times\mathtt{m}}(\overline{Z(A)}) \cong \mathrm{Mat}_{\mathtt{m}\times\mathtt{m}}(\Bbbk) \otimes \overline{Z(A)}$. Let e_i, $i = 1, \dots, n$, denote the usual primitive diagonal idempotents of $\mathrm{Mat}_{\mathtt{m}\times\mathtt{m}}(\Bbbk)$ such that $\sum_i e_i$ is the identity matrix. By Lemma 13 we can lift the idempotents $e_i \otimes 1$ from $\mathrm{Mat}_{\mathtt{m}\times\mathtt{m}}(\Bbbk) \otimes \overline{Z(A)}$ to $\mathrm{Hom}_{A-A}(V, V)$ modulo the radical (see e.g. [La, 3.6]). Thus we obtain $\mathtt{m}$ orthogonal idempotents in $\mathrm{Hom}_{A-A}(V, V)$, which implies the existence of a decomposition

$$\mathrm{F} = \mathrm{F}_1 \oplus \mathrm{F}_2 \oplus \cdots \oplus \mathrm{F}_{\mathtt{m}}$$

for the functor F. As $\overline{Z(A)}$ is a unital subalgebra of $\overline{A}$, we have an isomorphism $(e_i \otimes 1)\overline{A} \cong \overline{A}$ of left $\overline{A}$-modules. Hence, it follows that

$$\mathrm{F}_i L \cong L \quad \text{for all} \quad i \in \{1, 2, \dots, n\}. \tag{9}$$

From (5) we have $\sum_{i,j} \mathrm{F}_i \circ \mathrm{F}_j \cong \mathtt{k}\mathrm{ID}$. From (9) and the Krull-Schmidt theorem it follows that $\mathrm{F}_i \circ \mathrm{F}_j \cong \mathrm{ID}$ for every i and j. In particular, $\mathrm{F}_i \circ \mathrm{F}_i \cong \mathrm{ID}$, which yields that every F_i is selfadjoint by Proposition 2(v). Now the claim of Theorem 7(ii) follows from Proposition 2. This completes the proof. □

4. External direct sums and tensor products

To construct new solutions to functorial equations one may use the classical constructions of external direct sums and tensor products.

We start with the construction of an external direct sum. Let $g(x), h(x) \in \mathbb{Z}_+(x)$. Assume that for $i = 1, 2$ we have a finite dimensional associative $\Bbbk$-algebra A_i and a (selfadjoint) exact functor F_i on A_i-mod such that $g(\mathrm{F}_i) \cong h(\mathrm{F}_i)$. Set $A = A_1 \oplus A_2$ and let $\mathrm{F} := \mathrm{F}_1 \boxplus \mathrm{F}_2$ denote the external direct sum of F_1 and F_2 (it acts on A-modules componentwise).

Proposition 14. *The functor* F *is a (selfadjoint) exact endofunctor on* A-mod *satisfying* $g(\mathrm{F}) \cong h(\mathrm{F})$.

Proof. The action of F is computed componentwise and hence properties of F follow from the corresponding properties of the F_i's. □

The external tensor product works as follows: Let $g(x), h(x) \in \mathbb{Z}_+(x)$. Assume that we have a finite dimensional associative $\Bbbk$-algebra A and a (selfadjoint) exact functor F on A-mod such that $g(\mathrm{F}) \cong h(\mathrm{F})$. Let B be a finite dimensional associative $\Bbbk$-algebra and ID_B denote the identity functor on B-mod. Consider the algebra $C = A \otimes B$. Then the external tensor product $\mathrm{G} := \mathrm{F} \boxtimes \mathrm{ID}_B$ is an exact endofunctor on C-mod defined as follows: Any $X \in C$-mod can be considered as an A-module with a fixed action of B by endomorphisms ψ_b, $b \in B$. Then the C-module $\mathrm{G}X$ is defined as the A-module $\mathrm{F}X$ with the action of B given by $\mathrm{F}\psi_b$. The action of G on morphisms is defined in the natural way.

Proposition 15. *(i) The functor* G *is selfadjoint if and only if* F *is selfadjoint.*
(ii) There is an isomorphism of functors as follows: $g(\mathrm{G}) \cong h(\mathrm{G})$.

Proof. From the definition of G it follows that the adjunction morphisms $\mathrm{adj} : \mathrm{ID}_A \to \mathrm{FF}$ and $\mathrm{adj}' : \mathrm{FF} \to \mathrm{ID}_A$ induce in the natural way adjunction morphisms $\overline{\mathrm{adj}} : \mathrm{ID}_C \to \mathrm{GG}$ and $\overline{\mathrm{adj}}' : \mathrm{GG} \to \mathrm{ID}_C$, and vice versa. This proves claim (i).

Any isomorphism $g(\mathrm{F}) \cong h(\mathrm{F})$ of functors induces, by the functoriality of F and the definition of G, an isomorphism $g(\mathrm{G}) \cong h(\mathrm{G})$. Claim (ii) follows and the proof is complete. □

5. Selfadjoint idempotents (orthogonal projections)

In this section we consider the equation

$$\mathrm{F} \circ \mathrm{F} \cong \mathrm{F}, \tag{10}$$

which simply means that F is a selfadjoint idempotent (an orthogonal projection). For the theory of $*$-representations of algebras, generated by orthogonal projections, we refer the reader to [Co, KS, KRS] and references therein.

Every decomposition $A \cong B \oplus C$ into a direct sum of algebras (unital or zero) yields a decomposition $A\text{-mod} = B\text{-mod} \oplus C\text{-mod}$. Denote by $\mathfrak{p}_B : A\text{-mod} \to B\text{-mod}$ the natural projection with respect to this decomposition, that is the functor $\mathrm{ID} \boxplus \mathbf{0}$. Our main result in this section is the following:

Theorem 16. *Assume that* F *is a selfadjoint endofunctor on* A-mod *satisfying* (10). *Then there exists a decomposition* $A \cong B \oplus C$ *into a direct sum of algebras (unital or zero) such that* $\mathrm{F} \cong \mathfrak{p}_B$.

Proof. For a simple A-module L set $\mathrm{F}L = X_L$.

Lemma 17. *We have* $X_L = 0$ *or* $X_L \cong L \oplus Y_L$ *such that* $\mathrm{F}Y_L = 0$.

Proof. Assume $X_L \neq 0$. Then we have

$$\begin{array}{rcl} 0 & \neq & \mathrm{Hom}_A(X_L, X_L) \\ & = & \mathrm{Hom}_A(\mathrm{F}L, \mathrm{F}L) \\ \text{(by adjunction)} & = & \mathrm{Hom}_A(L, \mathrm{FF}L) \\ \text{(by (10))} & = & \mathrm{Hom}_A(L, \mathrm{F}L) \\ & = & \mathrm{Hom}_A(L, X_L). \end{array}$$

Similarly, $\mathrm{Hom}_A(X_L, L) \neq 0$, which means that L is both, a submodule and a quotient of X_L. In particular, we have $[X_L] = [L] + z$ for some $z \in [A]$.

Further, we have

$$[X_L] = [\mathrm{F}L] \stackrel{(10)}{=} [\mathrm{F}^2 L] = [\mathrm{F}X_L] = [\mathrm{F}][X_L] = \\ = [\mathrm{F}]([L] + z) = [\mathrm{F}][L] + [\mathrm{F}]z = [\mathrm{F}L] + [\mathrm{F}]z = [X_L] + [\mathrm{F}]z.$$

This yields $[\mathrm{F}]z = 0$. In particular, L occurs with multiplicity one in X_L and hence, by the previous paragraph, $X_L \cong L \oplus Y_L$ for some Y_L. We further have $[Y_l] = z$ and thus $\mathrm{F}Y_L = 0$ follows from $[\mathrm{F}]z = 0$. This completes the proof. □

Lemma 18. *For every L such that $\mathrm{F}L \neq 0$ we have $Y_L = 0$.*

Proof. Assume that $Y_L \neq 0$ and let L' be a simple submodule of Y_L. Then $\mathrm{F}L' = 0$ by Lemma 17, in particular, $L' \neq L$. Hence we have

$$\begin{array}{rcl} 0 & \neq & \mathrm{Hom}_A(L', Y_L) \\ & = & \mathrm{Hom}_A(L', Y_L \oplus L) \\ & = & \mathrm{Hom}_A(L', \mathrm{F}L) \\ \text{(by adjunction)} & = & \mathrm{Hom}_A(\mathrm{F}L', L) \\ \text{(by Lemma 17)} & = & 0. \end{array}$$

The obtained contradiction completes the proof. □

From Lemmata 17 and 18 it follows that the matrix M_{F} is diagonal with zeros and ones on the diagonal, and the ones correspond to exactly those simple A-modules, which are not annihilated by F. Without loss of generality we may assume that the simple modules not annihilated by F are $L_1, L_2, \dots, L_k$ for some $k \in \{0, 1, \dots, n\}$. From Lemma 8 we also obtain $M_{\mathrm{F}} = N_{\mathrm{F}}$, that is

$$\mathrm{F}P_i = \begin{cases} P_i, & i \leq k; \\ 0, & i > k. \end{cases}$$

As any simple module is sent to a simple module or zero, it follows that for $i \leq k$ all simple subquotients of P_i have the form L_j, $j \leq k$; and for $i > k$ all simple subquotients of P_i have the form L_j, $j > k$. Therefore there is a decomposition $A \cong B \oplus C$, where

$$B \cong \mathrm{End}_A(P_1 \oplus P_2 \oplus \dots \oplus P_k)^{\mathrm{op}}, \quad C \cong \mathrm{End}_A(P_{k+1} \oplus P_{k+2} \oplus \dots \oplus P_n)^{\mathrm{op}}.$$

The adjunction morphism $\mathrm{ID} \to \mathrm{F}^2 \cong \mathrm{F}$ is nonzero on all L_i, $i \leq k$, and hence is an isomorphism as $\mathrm{F}L_i \cong L_i$ by Lemmata 17 and 18. By induction on the length of a module and the Five Lemma it follows that the adjunction morphism is an isomorphism on all B-modules (see e.g. [Ma, 3.7] for details). Therefore F is isomorphic to the identity functor, when restricted to B-mod. By the definition of C, the functor F is the zero functor on C-mod. The claim of the theorem follows. □

Corollary 19. *If A is connected then the only selfadjoint solutions to* (10) *are the identity and the zero functors.*

Proof. If A is connected and $A \cong B \oplus C$, then either B or C is zero. Thus the statement follows directly from Theorem 16. □

Corollary 20. *If* F *and* G *are two selfadjoint solutions to* (10)*, then* $\mathrm{F} \circ \mathrm{G} \cong \mathrm{G} \circ \mathrm{F}$*.*

Proof. Define:

$$\begin{array}{rcl} X_{00} & = & \{i \in \{1, 2, \dots, n\} : \mathrm{F}L_i \neq 0, \mathrm{G}L_i \neq 0\}, \\ X_{10} & = & \{i \in \{1, 2, \dots, n\} : \mathrm{F}L_i = 0, \mathrm{G}L_i \neq 0\}, \\ X_{01} & = & \{i \in \{1, 2, \dots, n\} : \mathrm{F}L_i \neq 0, \mathrm{G}L_i = 0\}, \\ X_{11} & = & \{i \in \{1, 2, \dots, n\} : \mathrm{F}L_i = 0, \mathrm{G}L_i = 0\}. \end{array}$$

Then $\{1, 2, \dots, n\}$ is a disjoint union of X_{ij}, $i, j \in \{0, 1\}$. For $i, j \in \{0, 1\}$ set

$$A_{ij} := \mathrm{End}_A(\oplus_{s \in X_{ij}} P_s)^{\mathrm{op}}.$$

Similarly to the proof of Theorem 16 one obtains that $A \cong \oplus_{i,j=0}^{1} A_{i,j}$ and, moreover, that both $\mathrm{F} \circ \mathrm{G}$ and $\mathrm{G} \circ \mathrm{F}$ are isomorphic to $\mathfrak{p}_{A_{00}}$ with respect to this decomposition. The claim follows. □

Selfadjointness of F is important for the claim of Theorem 16. Here is an example of an exact, but not selfadjoint, functor satisfying (10), which is not of the type $\mathfrak{p}_B$: Let $A = \Bbbk \oplus \Bbbk$, then an A-module is just a collection (X, Y) of two vector spaces. Define the functor F as follows: $\mathrm{F}(X, Y) := (X \oplus Y, 0)$ with the obvious action on morphisms. Then F satisfies (10) but is not selfadjoint. In fact, for any idempotent matrix $M \in \mathrm{Mat}_{n \times n}(\mathbb{Z}_+)$ one can similarly define an exact endofunctor F on A-mod, where

$$A = \underbrace{\Bbbk \oplus \Bbbk \oplus \dots \oplus \Bbbk}_{n \text{ summands}},$$

such that $M_{\mathrm{F}} = M$ (see Section 7 for more details).

There are also many natural idempotent functors, which are exact on only one side. For example, for any $X \subset \{1, 2, \dots, n\}$ one could define an idempotent right exact (but, in general, not left exact) endofunctor Z_X on A-mod as follows: $\mathrm{Z}_X N$ is the maximal quotient of N, whose simple subquotients are all isomorphic to L_i, $i \in X$. The latter functors appear in Lie Theory, see e.g. [MS2].

6. Functors generating a cyclic semigroup

Proposition 21. *Let* F *be a selfadjoint endofunctor on* A-mod*. If* $\mathrm{F}^k = 0$ *for some* $k \in \mathbb{N}$*, then* $\mathrm{F} = 0$*.*

Proof. The claim is obvious for $k = 1$. Assume that $k = 2$. Then $\mathrm{F}^2 = 0$. The condition $\mathrm{F} \neq 0$ is equivalent to the condition $\mathrm{F}L_i \neq 0$ for some $i \in \{1, 2, \dots, n\}$. If $\mathrm{F}L_i \neq 0$, then, using the adjunction, we get

$$0 \neq \mathrm{Hom}_A(\mathrm{F}L_i, \mathrm{F}L_i) \cong \mathrm{Hom}_A(L_i, \mathrm{F}\mathrm{F}L_i).$$

However, $\mathrm{F}\mathrm{F}L_i = 0$ as $\mathrm{F}^2 = 0$, a contradiction. Therefore $\mathrm{F} = 0$.

Now we proceed by induction on k. Assume $k > 2$. Then $\mathrm{F}^k = 0$ implies $\mathrm{F}^{2(k-1)} = (\mathrm{F}^{k-1})^2 = 0$. As F^{k-1} is selfadjoint, by the above we have $\mathrm{F}^{k-1} = 0$. Now $\mathrm{F} = 0$ follows from the inductive assumption. □

Proposition 22. *Let* F *be a selfadjoint endofunctor on* A-mod *such that*

$$\mathrm{F}^k \cong \mathrm{F}^m \tag{11}$$

for some $k, m \in \mathbb{N}$, $k > m \geq 1$.

(i) If $k - m$ *is odd, then* $\mathrm{F}^2 \cong \mathrm{F}$ *(and, conversely, any* F *satisfying* $\mathrm{F}^2 \cong \mathrm{F}$ *obviously satisfies* $\mathrm{F}^k \cong \mathrm{F}^m$*).*

(ii) If $k - m$ *is even, then there is a decomposition* $A \cong B \oplus C$ *into a direct sum of algebras (unital or zero) and an algebra automorphism* $\varphi : B \to B$ *such that* φ^2 *is inner and* $\mathrm{F} \cong \mathrm{F}_\varphi \boxplus \mathbf{0}$.

Proof. From (11) it follows that $\mathrm{F}^{m+i(k-m)} \cong \mathrm{F}^m$ for all $i \in \mathbb{N}$. We can choose i such that $s := i(k - m) > m$. Applying F^{s-m} to $\mathrm{F}^{m+s} \cong \mathrm{F}^m$ we get $\mathrm{F}^{2s} \cong \mathrm{F}^s$. As F^s is selfadjoint, from Theorem 16 we obtain a decomposition $A \cong B \oplus C$ into a direct sum of algebras (unital or zero) such that $\mathrm{F}^s \cong \mathfrak{p}_B$. We have

$$A\text{-mod} \cong B\text{-mod} \oplus C\text{-mod}.$$

Lemma 23. *We have* $\mathrm{F}N = 0$ *for any* $N \in C$-mod.

Proof. We prove that $\mathrm{F}^i N = 0$ by decreasing induction on i. As $\mathrm{F}^s \cong \mathfrak{p}_B$, we have $\mathrm{F}^s N = 0$, which is the basis of our induction. For $i \in \{1, 2, \dots, s-1\}$ we have, by adjunction,

$$\mathrm{Hom}_A(\mathrm{F}^i N, \mathrm{F}^i N) \cong \mathrm{Hom}_A(N, \mathrm{F}^{2i} N).$$

From the inductive assumption we have $\mathrm{F}^{2i} N = 0$ which implies $\mathrm{Hom}_A(N, \mathrm{F}^{2i} N) = 0$ and hence $\mathrm{F}^i N = 0$. □

Lemma 24. *The functor* F *preserves* B-mod.

Proof. For $N \in B$-mod we have, by adjunction,

$$\mathrm{Hom}_A({}_C C, \mathrm{F}N) \cong \mathrm{Hom}_A(\mathrm{F}_C C, N) \overset{\text{Lemma 23}}{\cong} 0.$$

The claim follows. □

From Lemmata 23 and 24 we may write $\mathrm{F} = \mathrm{G}_B \boxplus \mathbf{0}$, where G_B is a selfadjoint endofunctor on B-mod. From $\mathrm{F}^s \cong \mathfrak{p}_B$ we obtain $\mathrm{G}_B^s \cong \mathrm{ID}$. By Proposition 2, the latter yields $\mathrm{G}_B \cong \mathrm{F}_\varphi$ for some algebra automorphism $\varphi : B \to B$ such that φ^2 is inner.

Note that $\mathrm{F}_\varphi^2 \cong \mathrm{ID}$. Therefore in the case when $k - m$ is odd, we must have that already $F_\varphi \cong \mathrm{ID}$, which implies that $\mathrm{F} \cong \mathfrak{p}_B$. It is easy to see that $\mathfrak{p}_B$ satisfies (11).

In the case when $k - m$ is even, it is easy to check that every $\mathrm{F}_\varphi \boxplus \mathbf{0}$, for φ as above, satisfies (11). The claim follows. □

7. Semisimple algebras

For a semisimple algebra $A \cong \oplus_{i=1}^{n}\Bbbk$ there is a natural bijection between isomorphism classes of endofunctors on A-mod and matrices in $\mathrm{Mat}_{n\times n}(\mathbb{Z}_+)$. The correspondence is given as follows: The endofunctor F on A-mod is sent to the matrix M_{F}. The inverse of this map is defined as follows: Denote by $\Bbbk_{(i)}$, $i = 1,\dots,n$, the i-th simple component of the algebra A (i.e. $A = \Bbbk_{(1)} \oplus \cdots \oplus \Bbbk_{(n)}$). Then the matrix $X = (\mathtt{x}_{i,j})_{i,j=1,\dots,n}$ is sent to the direct sum (over all i and j) of the functors $\mathtt{x}_{i,j}\mathrm{ID} : \Bbbk_{(j)}\text{-mod} \to \Bbbk_{(i)}\text{-mod}$. We have

Proposition 25. *Let $A \cong \oplus_{i=1}^{n}\Bbbk$.*

(i) An endofunctor F on A-mod is selfadjoint if and only if M_{F} is symmetric.
(ii) Let $g(x), h(x) \in \mathbb{Z}_+(x)$. Then there is a one-to one correspondence between the isomorphism classes of (selfadjoint) endofunctors F on A-mod satisfying $g(\mathrm{F}) \cong h(\mathrm{F})$ and (symmetric) solutions (in $\mathrm{Mat}_{n\times n}(\mathbb{Z}_+)$) of the matrix equation $g(x) = h(x)$.

Proof. Since over A simple modules are projective, claim (i) follows from Lemma 8. Claim (ii) follows from (i), the complete reducibility of functors on semisimple algebras and the previous paragraph. □

In light of Proposition 25 the problem we consider in this paper may be viewed as a kind of a categorical generalization of the problem of solving matrix equations. From Proposition 25 we have the following general criterion for solubility of functorial equations:

Corollary 26. *Let $g(x), h(x) \in \mathbb{Z}_+(x)$. Then the following conditions are equivalent:*

(a) There is a finite dimensional basic $\Bbbk$-algebra A with n isomorphism classes of simple modules and an exact endofunctor F of A-mod such that $g(\mathrm{F}) \cong h(\mathrm{F})$.
(b) There is a matrix $X \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_+)$ such that $g(X) = h(X)$.

Proof. If A and F are as in (a), then M_{F} is a solution to the matrix equation $g(x) = h(x)$. Hence (a) implies (b).

On the other hand, that (b) implies (a) in the case of a semisimple algebra A follows from Proposition 25. This completes the proof. □

Note that a tensor product of a semisimple algebra and a local algebra is a direct sum of local algebras. Therefore we would like to finish this section with the following observation, which might be used for reduction of certain classification problems to corresponding problems over semisimple algebras.

Proposition 27. *Let A be a finite-dimensional algebra and $\mathrm{F}_1,\dots,\mathrm{F}_k$ be a collection of selfadjoint endofunctors on A-mod such that the following conditions are satisfied:*

(a) For every $i = 1, \dots, k$ *we have* $M_{\mathrm{F}_i} = M^t_{\mathrm{F}_i}$.
(b) For some field $\mathbb{K}$ *of characteristic zero the space* $\mathbb{K} \otimes_{\mathbb{Z}} [A]$ *does not contain any proper subspace invariant under all* $[\mathrm{F}_i]$.

Then A *is a direct sum of local algebras of the same dimension.*

Proof. Let C denote the Cartan matrix of A (i.e. the matrix of multiplicities of simple modules in projective modules). Then $[\mathrm{F}_i]C = C[\mathrm{F}_i]$ for all $i = 1, \dots, k$ by Lemma 9 and condition (a). Since the representation $\mathbb{K} \otimes_{\mathbb{Z}} [A]$ of the associative algebra, generated by the $[\mathrm{F}_i]$, $i = 1, \dots, k$, is irreducible by (b), from the Schur Lemma it follows that C is a multiple of the identity matrix. The claim follows. □

8. Restriction to centralizer subalgebras

Let X be a projective A-module and $B = \mathrm{End}_A(X)^{\mathrm{op}}$ (the corresponding centralizer subalgebra of A). Then X has the natural structure of an A-B-bimodule. Denote by $\mathrm{add}(X)$ the additive closure of X, that is the full subcategory of A-mod, which consists of all modules Y, isomorphic to direct sums of (some) direct summands of X. Consider the full subcategory $\mathcal{X} = \mathcal{X}_X$ of A-mod, which consists of all modules Y admitting a two step resolution

$$X_1 \to X_0 \to Y \to 0, \qquad X_0, X_1 \in \mathrm{add}(X). \tag{12}$$

The functor $\Phi := \mathrm{Hom}_A(X, {}_-) : \mathcal{X} \to B\text{-mod}$ is an equivalence, see [Au, § 5].

Proposition 28. *Assume that* F *is a selfadjoint endofunctor on* A-mod *such that* $\mathrm{F}X \in \mathrm{add}(X)$. *Then the following holds:*

(i) The functor F *preserves* $\mathcal{X}$ *and induces (via* Φ*) a selfadjoint endofunctor* $\overline{\mathrm{F}}$ *on* B-mod.
(ii) If $g(x), h(x) \in \mathbb{Z}_+[x]$ *and* $g(\mathrm{F}) \cong h(\mathrm{F})$, *then* $g(\overline{\mathrm{F}}) \cong h(\overline{\mathrm{F}})$.

Proof. Applying F to the exact sequence (12) we obtain an exact sequence

$$\mathrm{F}X_1 \to \mathrm{F}X_0 \to \mathrm{F}Y \to 0.$$

Here both $\mathrm{F}X_0$ and $\mathrm{F}X_1$ are in $\mathrm{add}(X)$ by assumption and hence $\mathrm{F}Y \in \mathcal{X}$. Therefore F preserves $\mathcal{X}$ and hence $\overline{\mathrm{F}} := \Phi\mathrm{F}\Phi^{-1}$ is a selfadjoint endofunctor on B-mod. This proves claim (i). Claim (ii) follows from the definition of $\overline{\mathrm{F}}$ by restricting any isomorphism $g(\mathrm{F}) \cong h(\mathrm{F})$ to the subcategory $\mathcal{X}$, which is preserved by both $g(\mathrm{F})$ and $h(\mathrm{F})$ by claim (i). This completes the proof. □

Corollary 29. *Assume that* X *is a multiplicity free direct sum of all indecomposable projective-injective* A*-modules and* $X \neq 0$. *Then we have the following:*

(i) Any selfadjoint endofunctor F *on* A-mod *induces a selfadjoint endofunctor* $\overline{\mathrm{F}}$ *on* B-mod.

(ii) The map $\mathrm{F}\mapsto\overline{\mathrm{F}}$ *is functorial in* F.

Proof. From the definition of X we have that the category $\mathrm{add}(X)$ is just the full subcategory of A-mod consistsing of all projective-injective modules. If F is a selfadjoint endofunctor on A-mod, then F preserves both projective and injective modules and hence preserves $\mathrm{add}(X)$. Therefore claim (i) follows from Proposition 28(i). Up to conjugation with the equivalence Φ, the map $\mathrm{F}\mapsto\overline{\mathrm{F}}$ is just the restriction map to an invariant subcategory, which is functorial. □

Until the end of this section we assume that X is projective-injective. Recall (see [Ta, KSX, MS5]) that A is said to have the *double centralizer property* for X provided that there is an exact sequence

$$ {}_AA\hookrightarrow X_0\overset{\alpha}{\to}X_1,\qquad X_0,X_1\in\mathrm{add}(X). \tag{13}$$

The name comes from the observation, see [Ta], that in this case the actions of A and B on X are exactly the centralizers of each other. Examples of such situations include blocks of various generalizations of the BGG category $\mathcal{O}$, see [MS5] for details. The following result can be seen as a generalization of [St2, Theorem 1.8], where a similar result was obtained for projective functors on the category $\mathcal{O}$ (and its parabolic version).

Theorem 30. *Assume that* X *is projective-injective and that* A *has the double centralizer property for* X*. Then the functor* $\mathrm{F}\mapsto\overline{\mathrm{F}}$ *from Corollary 29 is full and faithful.*

Proof of faithfulness. Let F and G be two selfadjoint endofunctors on A-mod and $\xi:\mathrm{F}\to\mathrm{G}$ be a natural transformation. Assume that $\overline{\xi}:\overline{\mathrm{F}}\to\overline{\mathrm{G}}$ is zero. Since both F and G are exact, from (13) we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{F}\,{}_AA & \hookrightarrow & \mathrm{F}X_0 & \longrightarrow & \mathrm{F}X_1\\
 & & \downarrow{\scriptstyle \xi_{{}_AA}} & & \downarrow{\scriptstyle \xi_{X_0}} & & \downarrow{\scriptstyle \xi_{X_1}}\\
0 & \longrightarrow & \mathrm{G}\,{}_AA & \hookrightarrow & \mathrm{G}X_0 & \longrightarrow & \mathrm{G}X_1
\end{array}$$

By assumption, $\overline{\xi}$ is zero, which means that both ξ_{X_0} and ξ_{X_1} are zero. Therefore $\xi_{{}_AA}$ is zero as well.

Now for any $M\in A$-mod consider the first two steps of the projective resolution of M:

$$P_1\to P_0\to M\to 0. \tag{14}$$

Since both F and G are exact, from (14) we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{F}P_1 & \longrightarrow & \mathrm{F}P_0 & \longrightarrow & \mathrm{F}M & \longrightarrow & 0\\
\downarrow{\scriptstyle \xi_{P_1}} & & \downarrow{\scriptstyle \xi_{P_0}} & & \downarrow{\scriptstyle \xi_{M}} & & \\
\mathrm{G}P_1 & \longrightarrow & \mathrm{G}P_0 & \longrightarrow & \mathrm{G}M & \longrightarrow & 0
\end{array}$$

As $\xi_{{}_AA}$ is zero by the previous paragraph and $P_0, P_1 \in \mathrm{add}({}_AA)$, we have that both ξ_{P_0} and ξ_{P_1} are zero. Therefore ξ_M is zero as well. This shows that the natural transformation ξ is zero, which establishes faithfullness of the functor $\mathrm{F} \mapsto \overline{\mathrm{F}}$. $\square$

Proof of fullness. Let F and G be two selfadjoint endofunctors on A-mod and $\xi : \overline{\mathrm{F}} \to \overline{\mathrm{G}}$ be a natural transformation. Then we have the following commutative diagram:

$$\begin{array}{ccc}
\overline{\mathrm{F}}\Phi X_0 & \xrightarrow{\overline{\mathrm{F}}\Phi\alpha} & \overline{\mathrm{F}}\Phi X_1 \\
\downarrow{\scriptstyle \xi_{\Phi X_0}} & & \downarrow{\scriptstyle \xi_{\Phi X_1}} \\
\overline{\mathrm{G}}\Phi X_0 & \xrightarrow{\overline{\mathrm{G}}\Phi\alpha} & \overline{\mathrm{G}}\Phi X_1
\end{array}$$

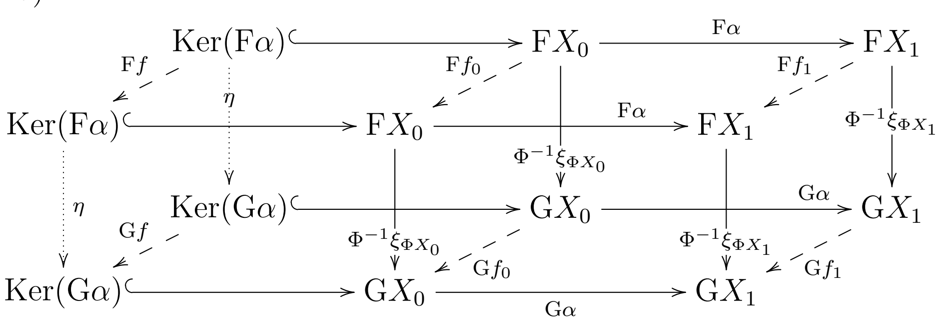
Applying Φ^{-1} we obtain the following diagram, the solid part of which commutes:

(15)
$$\begin{array}{ccccc}
\mathrm{F}{}_AA & \hookrightarrow & \mathrm{F}X_0 & \xrightarrow{\mathrm{F}\alpha} & \mathrm{F}X_1 \\
\vdots{\scriptstyle \eta} & & \downarrow{\scriptstyle \Phi^{-1}\xi_{\Phi X_0}} & & \downarrow{\scriptstyle \Phi^{-1}\xi_{\Phi X_1}} \\
\mathrm{G}{}_AA & \hookrightarrow & \mathrm{G}X_0 & \xrightarrow{\mathrm{G}\alpha} & \mathrm{G}X_1
\end{array}$$

Because of the commutativity of the solid part, the diagram extends uniquely to a commutative diagram by the dotted arrow η. We claim that η is, in fact, a bimodule homomorphism. Indeed, any homomorphism $f : {}_AA \to {}_AA$ can be extended, by the injectivity of X, to a commutative diagram as follows:

(16)
$$\begin{array}{ccccc}
{}_AA & \hookrightarrow & X_0 & \xrightarrow{\alpha} & X_1 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} \\
{}_AA & \hookrightarrow & X_0 & \xrightarrow{\alpha} & X_1
\end{array}$$

Consider the following diagram:
(17)



The upper face of the diagram (17) commutes as it coincides with the image of the commutative diagram (16) under F. Similarly, the lower face of the diagram (17) commutes as it coincides with the image of the commutative diagram (16) under G. The front and the back faces coincide with (15) and hence commute. The right and the middle square sections commute as ξ is a natural transformation. This implies that the whole diagram commutes, showing that η is indeed a bimodule map from F to G.

This means that η defines a natural transformation from F to G. By construction, we have $\xi = \overline{\eta}$, which proves that the functor $\mathrm{F} \mapsto \overline{\mathrm{F}}$ is full. □

Unfortunately, the functor $\mathrm{F} \mapsto \overline{\mathrm{F}}$ from Corollary 29 is not dense (in particular, not an equivalence between the monoidal categories of selfadjoint endofunctors on A-mod and B-mod) in the general case. Let G be a selfadjoint endofunctors on B-mod and assume that $\mathrm{G} = \overline{\mathrm{F}}$ for some selfadjoint endofunctors on A-mod. Then from (13) we have

$$\mathrm{F}_A A = \mathrm{Ker}(\Phi^{-1}\mathrm{G}\Phi\alpha) \tag{18}$$

(as a bimodule, with the induced action on morphisms), which uniquely defines the functor F (see [Ba, Chapter II]). However, here is an example of A, X and G for which the bimodule $\mathrm{Ker}(\Phi^{-1}\mathrm{G}\Phi\alpha)$ defines only a right exact (and hence not selfadjoint) functor:

Example 31. Let A be the algebra of the following quiver with relations:

$$1 \circlearrowleft x \qquad\qquad 2 \underset{b}{\overset{a}{\rightleftarrows}} 3 \qquad\qquad ab = x^2 = 0$$

The indecomposable projective A-modules look as follows:

$$P_1:\ \begin{array}{c}1\\ \downarrow{\scriptstyle x}\\ 1\end{array} \qquad\qquad P_2:\ \begin{array}{c}2\\ \downarrow{\scriptstyle a}\\ 3\\ \downarrow{\scriptstyle b}\\ 2\end{array} \qquad\qquad P_3:\ \begin{array}{c}3\\ \downarrow{\scriptstyle b}\\ 2\end{array}$$

The modules P_1 and P_2 are injective, so we take $X = P_1 \oplus P_2$ and have that B is isomorphic to the algebra of the following quiver with relations:

$$1 \circlearrowleft x \qquad\qquad 2 \circlearrowleft y \qquad\qquad x^2 = y^2 = 0 \tag{19}$$

(here $y = ba$). The double centralizer property is guaranteed by the fact that the first two steps of the injective coresolution of P_3 are as follows:

$$0 \to P_3 \to P_2 \xrightarrow{\beta} P_2,$$

where β is the right multiplication with the element ba. Let $\varphi : B \to B$ be the involutive automorphism of B given by the automorphism of the quiver (19) swapping the vertices. Then $\mathrm{G} := {}_{\varphi}B \otimes_B {}_-$ is a selfadjoint autoequivalence of B-mod (see Proposition 2). Assume that F is a right exact endofunctor on A-mod given by (18). Then the restriction of F to add(X) is isomorphic to G, which implies $\mathrm{F}P_1 \cong P_2$. For $i = 1, 2, 3$ we denote by L_i the simple head of P_i. Applying F to the exact sequence $P_1 \to P_1 \twoheadrightarrow L_1$, we get the exact sequence $P_2 \to P_2 \twoheadrightarrow \mathrm{F}L_1$, which implies that the module $\mathrm{F}L_1$ is isomorphic to the following module:

$$N: \quad \begin{array}{c} 2 \\ \downarrow a \\ 3 \end{array}$$

Now, applying F to the short exact sequence $L_1 \hookrightarrow P_1 \twoheadrightarrow L_1$ we obtain the sequence

$$N \to P_2 \twoheadrightarrow N,$$

which is not exact. This means that F is not exact and thus cannot be selfadjoint.

It would be interesting to know when the functor $\mathrm{F} \mapsto \overline{\mathrm{F}}$ from Corollary 29 is dense.

9. Invariant Serre subcategories and quotients

For $S \subset \{1, 2, \dots, n\}$ set $S' = \{1, 2, \dots, n\} \setminus S$ and let $\mathcal{N}_S$ denote the full subcategory of A-mod, which consists of all modules N for which $[N : L_i] \neq 0$ implies $i \in S$. Then $\mathcal{N}_S$ is a Serre subcategory of A-mod and, moreover, any Serre subcategory of A-mod equals $\mathcal{N}_S$ for some S as above. Both $\mathcal{N}_S$ and the quotient $\mathcal{Q}_S := A\text{-mod}/\mathcal{N}_S$ are abelian categories. Recall (see e.g. [Ga, Chapter III] or [Fa, Chapter 15]) that the quotient $\mathcal{Q}_S$ has the same objects as A-mod and for objects M, N we have

$$\mathrm{Hom}_{\mathcal{Q}_S}(M, N) = \varinjlim \mathrm{Hom}_A(M', N/N'),$$

where $M' \subset M$ and $N' \subset N$ are such that $M/M', N' \in \mathcal{N}_S$. As we are working with finite dimensional modules, the space $\mathrm{Hom}_{\mathcal{Q}_S}(M, N)$ can be alternatively described as follows: For a module M let M^- denote the smallest submodule of M such that $M/M^- \in \mathcal{N}_S$ and M^+ denote the largest submodule of M such that $M^+ \in \mathcal{N}_S$. Then we have

$$\mathrm{Hom}_{\mathcal{Q}_S}(M, N) = \mathrm{Hom}_A((M^- + M^+)/M^+, (N^- + N^+)/N^+).$$

For $S \subset \{1, 2, \dots, n\}$ define $P_S := \oplus_{i \in S} P_i$ and $B_S := \mathrm{End}_A(P_S)^{\mathrm{op}}$. If S is nonempty, let I_S denote the trace of the module $P_{S'}$ in ${}_AA$. Then I_S is obviously an ideal in A, so we can define the quotient algebra $D_S := A_S/I_S$.

Proposition 32. *For any $N \in \mathcal{N}_S$ we have $I_S N = 0$, so such N becomes a D_S-module. This defines an equivalence $\mathcal{N}_S \cong D_S$-mod.*

Proof. The quotient map $A \twoheadrightarrow D_S$ defines a full and faithful embedding of D_S-mod into A-mod and the image of this embedding consists exactly of $N \in A$-mod such that $I_S N = 0$.

If $N \in \mathcal{N}_S$, then $\mathrm{Hom}_A(P_{S'}, N) = 0$ by the definition of $\mathcal{N}_S$, which implies $I_S N = 0$. Conversely, if $N \in A$-mod is such that $N \neq 0$, $I_S N = 0$, then $\mathrm{Hom}_A(P_{S'}, N) = 0$ and hence every composition subquotient of N is isomorphic to some L_i, $i \in S$. This means that $\mathcal{N}_S$ coincides with the image of D_S-mod in A-mod and the claim follows. □

Proposition 33. *Let $S \subsetneq \{1, 2, \dots, n\}$. Then we have equivalences $\mathcal{Q}_S \cong \mathcal{X}_{P_{S'}} \cong B_{S'}$-mod.*

Proof. That $\mathcal{X}_{P_{S'}}$ is equivalent to $B_{S'}$-mod follows from [Ba, Chapter II] (see also [Au, § 5]). Let us show that the embedding of $\mathcal{X}_{P_{S'}}$ to A-mod induces an equivalence $\mathcal{Q}_S \cong \mathcal{X}_{P_{S'}}$ via the canonical quotient map $A\text{-mod} \twoheadrightarrow \mathcal{Q}_S$. Let $\Psi : \mathcal{X}_{P_{S'}} \hookrightarrow A\text{-mod} \twoheadrightarrow \mathcal{Q}_S$ denote the corresponding functor.

If $M \in \mathcal{X}_{P_{S'}}$ and $M' \subset M$, then M/M' is a quotient of some module from $\mathrm{add}(P_{S'})$. Hence $M/M' \notin \mathcal{N}_S$ unless $M/M' = 0$.

Lemma 34. *For $M \in \mathcal{X}_{P_{S'}}$ we have $\mathrm{Ext}^1_A(M, Z) = 0$ for any $Z \in \mathcal{N}_S$.*

Proof. Let $X_1 \to X_0 \twoheadrightarrow M$ be the first two steps of the projective resolution of M, given by (12). Then $\mathrm{Hom}_A(X_1, Z) = 0$ (as the head of X_1 contains only L_j, $j \in S'$, while all composition subquotients of Z are of the form L_i, $i \in S$) and the claim follows. □

If $M \in \mathcal{X}_{P_{S'}}$ and $N' \subset N$ is such that $N' \in \mathcal{N}_S$, then $\mathrm{Hom}_A(M, N') = 0$ and $\mathrm{Ext}^1_A(M, N') = 0$ (the latter by Lemma 34). Therefore $\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_A(M, N/N')$. Combining this with the paragraph before Lemma 34 we have $\mathrm{Hom}_A(M, N) = \mathrm{Hom}_A(M', N/N')$ in the case $M, N \in \mathcal{X}_{P_{S'}}$. This yields

$$\mathrm{Hom}_{\mathcal{Q}_S}(M, N) = \mathrm{Hom}_A(M, N) \quad \text{for all} \quad M, N \in \mathcal{X}_{P_{S'}}.$$

This means that the functor Ψ is full and faithfull. It is left to prove that Ψ is dense.

Let N be an A-module and N' be the trace of $P_{S'}$ in N. Take a projective cover $X_0 \twoheadrightarrow N'$, where $X_0 \in \mathrm{add}(P_{S'})$, let Q be the kernel of this epimorphism and Q' be the trace of $P_{S'}$ in Q. Define $M = X_0/Q'$ and $M' = Q/Q' \subset M$. Then $M', N/N' \in \mathcal{N}_S$ and $M/M' \cong X_0/Q \cong N'$ by definition. Let $\varphi : M \to N$ be the composition of the natural maps $M \to N' \hookrightarrow N$. Let $\psi : N' \to M/M'$ be the inverse of the natural isomorphism $M/M' \xrightarrow{\sim} N'$. Then both $\varphi \in \mathrm{Hom}_{\mathcal{Q}_S}(M, N)$ and $\psi \in \mathrm{Hom}_{\mathcal{Q}_S}(N, M)$ and it is straightforward to check that φ and ψ are mutually inverse isomorphisms. This means that N is isomorphic in $\mathcal{Q}_S$ to $M \in \mathcal{X}_{P_{S'}}$ and hence the functor Ψ is dense. This completes the proof. □

Proposition 33 can be deduced from the results described in [Fa, Chapter 15]. However, it is shorter to prove it in the above form than to introduce all the notions and notation necessary for application of

[Fa, Chapter 15]. The correspondence $N \mapsto M$ from the last paragraph of the proof of Proposition 33 is functorial. The module M is called the *partial coapproximation* of N with respect to $\mathcal{X}_{P_{S'}}$, see [KM, 2.5] for details. From Proposition 33 it follows that $\mathcal{S}$-subcategories of the BGG category $\mathcal{O}$ associated with parabolic $\mathfrak{sl}_2$-induction (see [FKM1, FKM2]) can been regarded as quotients of blocks of the usual category $\mathcal{O}$ modulo the corresponding parabolic subcategory (in the sense of [RC]). In the general case $\mathcal{S}$-subcategories of $\mathcal{O}$ are quotient categories as well (however, modulo a subcategory, which properly contains the corresponding parabolic subcategory). In fact, the latter can be deduced combining several known results from the literature ([BG], [Ja, Kapitel 6] and [KoM]).

Corollary 35. *Let* F *be a selfadjoint endofunctor on* A-mod *and* $S \subsetneq \{1, 2, \dots, n\}$ *be such that the linear span of* $[L_i]$, $i \in S$, *in invariant under* $[\mathrm{F}]$.

(i) The functor F *preserves the category* $\mathcal{N}_S$ *and hence induces, via restriction and the equivalence from Proposition 32, a selfadjoint endofunctor* $\hat{\mathrm{F}}$ *on* D_S-mod.

(ii) The functor F *preserves the category* $\mathcal{X}_{P_{S'}}$ *and hence induces, via restriction and the equivalence from Proposition 33, a selfadjoint endofunctor* $\overline{\mathrm{F}}$ *on* $B_{S'}$-mod.

(iii) If $g(x), h(x) \in \mathbb{Z}_+[x]$ *and* $g(\mathrm{F}) \cong h(\mathrm{F})$, *then* $g(\hat{\mathrm{F}}) \cong h(\hat{\mathrm{F}})$ *and* $g(\overline{\mathrm{F}}) \cong h(\overline{\mathrm{F}})$.

Proof. The functor F preserves the category $\mathcal{N}_S$ by our assumptions and claim (i) follows.

For $i \in S$ we have

$$\mathrm{Hom}_A(\mathrm{F}P_{S'}, L_i) \cong \mathrm{Hom}_A(P_{S'}, \mathrm{F}L_i) = 0$$

as $\mathrm{F}L_i \in \mathcal{N}_S$ by claim (i). This means that $\mathrm{F}P_{S'} \in \mathrm{add}(P_{S'})$ and claim (ii) follows from Proposition 28.

Any isomorphism $g(\mathrm{F}) \cong h(\mathrm{F})$ induces, by restriction to $\mathcal{N}_S$ and $\mathcal{X}_{P_{S'}}$, isomorphisms $g(\hat{\mathrm{F}}) \cong h(\hat{\mathrm{F}})$ and $g(\overline{\mathrm{F}}) \cong h(\overline{\mathrm{F}})$, respectively. This proves claim (iii) and completes the proof. □

From the proof of Corollary 35 it follows that in the case when a selfadjoint endofunctor F on A-mod preserves the category $\mathrm{add}(P_S)$ for some nonempty $S \subset \{1, 2, \dots, n\}$, then F preserves the category $\mathcal{N}_{S'}$ as well.

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