Generalized Verma Modules

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1 Introduction

The foundations of the theory of highest weight modules over simple complex finite-dimensional Lie algebras due to the original paper by Verma ([V]), where the family of universal highest weight modules (Verma modules) was introduced and studied. Although in [V] only basic properties of Verma modules were studied, it was clear, that Verma modules should have some deep connection with the general theory of Lie algebras, Weyl groups, flag manifolds and so on.

The major famous results about Verma modules were obtained in a series of celebrated papers ([BGG1, BGG2, BGG3]) by Bernstein - Gelfand - Gelfand (BGG). In a few words, these three results describe the following: The first one, usually called the BGG Theorem, gives, in terms of the Weyl group action on the weight space, a criterion for the existence of a Verma submodule in a Verma module. The second one presents a combinatorially constructed resolution of a simple finite-dimensional module by Verma modules. This resolution is usually called the BGG resolution. The last result shows, that in a natural category, which is now known as category $\mathcal{O}$, Verma modules play the "intermediate" role between simple and projective objects.

It is quite difficult to overestimate the influence of these results on modern representation theory of Lie algebras. The bibliography of papers on this and related subjects is enormous (the reader can consult, for example, resent monographs [MP] or [Jo1], although it is impossible to find the complete bibliography anywhere outside Math. Reviews). The results by BGG not only led to the development of new branches of algebra (like quasi-hereditary algebras), but also found applications in functional analysis, combinatorics, modern physics and so on.

During the 30 years since the paper of Verma, many famous and deep results about Verma modules were obtained. It is worth mentioning, for example, the Kazhdan-Lusztig Theorem ([KL, BB1, BrKa]) or Soergel's description of the category $\mathcal{O}$ ([S1]). Some results have been generalized to certain infinite-dimensional Lie algebras, e.g. (affine) Kac-Moody Lie algebras, Virasoro algebra, quantum algebras, Yangians and so forth. So far the theory of Verma modules is not completed and there are many interesting unsolved questions and problems.

There have been several attempts to generalize the theory of Verma modules, and one of the most natural of them is to generalize Verma modules themselves. This can be done in different ways (compare, for example, [Gy, L1, RC]). Generalized Verma modules (GVM) were studied from different points of view and many properties of classical Verma modules were established for or generalized to GVMs. The aim of this manuscript is to present a systematic study of GVM obtained by so-called parabolic induction for a parabolic subalgebra of a simple Lie algebra and to give a historical overview of the development of the subject.

In particular, we present an analogue of the BGG Theorem (in partial cases, covering explicitly the case of induction from a weight $sl(2,\mathbb{C})$-module), an analogue of the BGG resolution for simply laced algebras, an analogue of the Kazhdan-Lusztig Theorem and an analogue of category $\mathcal{O}$, including Soergel's type description for its blocks. In order to avoid technical calculations (which are quite difficult in some cases), we present mostly the ideological foundations of the proofs. For complete details the reader can consult the corresponding original paper.

The results, described here, were obtained by several authors during the last 14 years (see [CF, F1, F2, FM1, FM2, FM3, FM4, FKM1, FKM2, FKM3, KlMa, KoMa, KM1, KM2, KM3, KM4, KM5, M3, M4, MO, MT]).

This manuscript was written during the authors stay as an Alexander von Humboldt fellow at Universität Bielefeld. The finansial support of Humboldt Foundation and the hospitality of Bielefeld University are gratefully acknowledged. Finally, I want to thank all my colleagues for their help in my research work, leading to the appearance of this manuscript. I am especially indebt to Prof. M.Klucznik for his generous help in preparation of this manuscript.

2 Generalities on Lie algebras

In this Chapter we give a brief synopsis of facts and notations used in general theory of Lie gorups, Lie algebras and their representations. As some preliminary and introductory text-books one can use [BK, BKK, D, J, Ka, Se].

2.1 Lie algebras and modules over Lie algebras

We fix throughout the complex field $\mathbb{C}$ and note that instead one can work over arbitrary algebraically closed field of characteristic zero. Let $\mathbb{Z}$ denote the ring of integers, $\mathbb{Z}_+$ the set of all non-negative integers and $\mathbb{N}$ the set of all positive integers.

A *Lie algebra*, $\mathfrak{G}$, is a $\mathbb{C}$-vectorspace equipped with a binary bilinear operation $[\cdot,\cdot] : \mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$, which satisfies the following two conditions:

L1. $[x,y]=-[y,x]$ for all $x,y\in\mathfrak{G}$.

L2. $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ for all $x,y,z\in\mathfrak{G}$ (Jacobi identity).

As we work over the field of characteristic zero, L1 is equivalent to the following: $[x,x]=0$ for all $x\in\mathfrak{G}$ (in fact, here we really need that the characteristic is different from 2). The standard examples of Lie algebras are *commutative* or *abelian Lie algebras*, in which $[x,x]=0$ for any $x\in\mathfrak{G}$, and Lie algebras associated with associative algebras. The last ones are constructed as follows: fix an associative algebra, A, and define on the underlined vectorspace $\mathfrak{G}_A=A$ a new operation via $[x,y]=xy-yx$. Then $\mathfrak{G}_A$ is a Lie algebra called *the Lie algebra, associated with* A. If we will have to distinguish the Lie brackets in different Lie algebras, we will add the algebra as a subscript denoting $[\cdot,\cdot]_{\mathfrak{G}}$ the Lie brackets of a Lie algebra $\mathfrak{G}$. We also introduce two standard notation: for a $\mathbb{C}$-vectorspace V the Lie algebras associated with assosiative algebras $gl(V)$ and $sl(V)$ will be denoted by $\mathfrak{gl}(V)$ and $\mathfrak{sl}(V)$ respectively.

For a given Lie algebras $\mathfrak{G}$ and $\mathfrak{A}$, a *homomorphism* (of Lie algebras) from $\mathfrak{G}$ to $\mathfrak{A}$ is a linear map, f, which preserves the Lie brackets, that is $f([x,y]_{\mathfrak{G}})=[f(x),f(y)]_{\mathfrak{A}}$ for all $x,y\in\mathfrak{G}$. As usual, a bijective (resp. surjective, resp. injective) homomorphism will be called an *isomorphism* (resp. *epimorphism*, resp. *monomorphism*).

Let $\mathfrak{G}$ be a Lie algebra. A $\mathfrak{G}$*-module*, V, is a $\mathbb{C}$-vectorspace on which the elements of $\mathfrak{G}$ act by linear transformations and the following condition is satisfied: $[x,y](v)=x(y(v))-y(x(v))$ for all $x,y\in\mathfrak{G}$ and $v\in V$. The last is equivalent to requiring that the map $f:\mathfrak{G}\to\mathfrak{gl}(V)$, $f:x\to x(\cdot)\in\mathfrak{gl}(V)$ is a homomorphism of Lie algebras. Such map is called a *representation* of $\mathfrak{G}$, in other words, modules and representations are the same things from a little bit different points of view. The most natural example of a $\mathfrak{G}$ module is $\mathfrak{G}$ acting on itself via the Lie bracket $x\mapsto \mathrm{ad}\ x=[x,\cdot]$. This module is called the *adjoint module* (resp. *adjoint representation*). In what follows we will often omit the brackets in the expression $x(v)$ in order to simplify notation. As usual, a *submodule* of a module is a subspace closed under the action of all elements of the Lie algebra. A module, V, is called *simple* (or *irreducible*) if it does not have any submodules except V and 0 (*trivial* or *non-proper submodules*), and *reducible* otherwise. A module is called *indecomposable* if it can not be decomposed into a direct sum of non-trivial submodules, and *decomposable* otherwise.

If V and M are modules over a Lie algebra $\mathfrak{G}$, then, via $g \mapsto g \oplus g$, $V \oplus M$ is maid into a $\mathfrak{G}$-module, called a *direct sum* of V and M. Analogously, via $g \mapsto g \otimes 1 + 1 \otimes g$, $V \otimes M$ is maid into a $\mathfrak{G}$-module, called a *tensor product* of V and M.

2.2 Nilpotent, solvable, semi-simple, simple and reductive Lie algebras

Let $\mathfrak{G}$ be a Lie algebra. An *ideal*, I, of $\mathfrak{G}$ is a subspace of $\mathfrak{G}$ such that $[x, y] \in I$ for any $x \in \mathfrak{G}$ and $y \in I$. Clearly, any ideal of $\mathfrak{G}$ is a Lie subalgebra of $\mathfrak{G}$. Given $\mathfrak{G}$ one can associate with it the *derived algebra* $\mathfrak{G}^1 = \mathfrak{G}^{(1)} = [\mathfrak{G}, \mathfrak{G}]$. Clearly, $\mathfrak{G}^1$ is an ideal of $\mathfrak{G}$. Now we can define inductively $\mathfrak{G}^i = [\mathfrak{G}, \mathfrak{G}^{i-1}]$ and $\mathfrak{G}^{(i)} = [\mathfrak{G}^{(i-1)}, \mathfrak{G}^{(i-1)}]$ both being ideals in $\mathfrak{G}$. The algebra $\mathfrak{G}$ is called *nilpotent* (resp. *solvable*) if there exists $i \in \mathbb{N}$ such that $\mathfrak{G}^i = 0$ (resp. $\mathfrak{G}^{(i)} = 0$). As $\mathfrak{G}^{(i)} \subset \mathfrak{G}^i$, any nilpotent Lie algebra is solvable. The classical examples of nilpotent (resp. solvable) Lie algebras are Lie algebras associated with associative algebras of strictly upper triangular (resp. upper triangular) $n \times n$ matrices. This also shows that, in general, a solvable Lie algebra is not nilpotent. We also note that any abelian Lie algebra is both nilpotent and solvable.

The main structure result about nilpotent Lie algebras is the *Engel's Theorem* ([D, Theorem 1.3.15]), which states that a Lie algebra, $\mathfrak{G}$, is nilpotent if and only if each element of $\mathfrak{G}$ is ad-nilpotent.

For solvable Lie algebras, the main result is not structural but relates to the representation theory. This is the *Lie's Theorem* ([D, Theorem 1.3.12]), which claims that for any solvable Lie algebra $\mathfrak{G}$ and any finite-dimensional $\mathfrak{G}$-module V there is a non-zero element in V, which is an eigenvector for all elements of $\mathfrak{G}$. In particular, any simple finite-dimensional $\mathfrak{G}$-module is one-dimensional.

A Lie algebra, $\mathfrak{G}$, is called *simple* if it is not abelian and does not contain any proper ideals. A Lie algebra, $\mathfrak{G}$, is called *semisimple* if it is non-zero and has no abelian non-zero ideals. A classical example of a simple Lie algebra is $\mathfrak{sl}(n, \mathbb{C})$. Each simple Lie algebra is semi-simple by definition. A Lie algebra, $\mathfrak{G}$, is called *reductive* if it is a direct sum of a semi-simple Lie algebra and a commutative Lie algebra.

Any finite-dimensional Lie algebra, $\mathfrak{G}$, has a unique maximal solvable ideal, $R_{\mathfrak{G}}$, called the *radical* of $\mathfrak{G}$. The principal structure theorem of semi-simple finite-dimensional Lie algebras claims that a finite-dimensional Lie algebra is semi-simple if and only if its radical is zero if and only if it is a direct sum of simple Lie algebras (the last decomposition is unique up to a permutation of the components), see [D, Theorem 1.5.2].

2.3 Classification of simple finite-dimensional complex Lie algebras

The iso-classes of simple finite-dimensional complex Lie algebras are in bijective correspondence with finite reduced indecomposable root systems (see [BK, D, Se]). In this section we will briefly describe this correspondence. We start with a definition of a root system (see [D, Appendix]).

A subset, Δ, of a finite-dimensional vectorspace, V, is called a *reduced root system* if the following conditions are satisfied:

RS1. Δ is finite, does not contain 0, and generates V.

RS2. For all $\alpha \in \Delta$, there exists an $\alpha^\vee \in V^*$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and Δ is stable under the reflection $s_\alpha : \nu \to \nu - \langle \alpha, \alpha^\vee \rangle \alpha$ (the element $\alpha^\vee$ is then unique).

RS3. For all $\alpha \in \Delta$, we have $\alpha^\vee(\Delta) \in \mathbb{Z}$.

RS4. If $\alpha \in \Delta$, the only elements of Δ which are proportional to α are $\pm\alpha$.

Elements of Δ are colled *roots*. The group W of automorphisms of V generated by s_α is called the *Weyl group* of Δ. Any reduced root system contains a basis, that is a finite subset such that any root can be written as a linear combination of basic roots with either only integer non-negative coefficients or only integer non-positive coefficients. Each two basis are conjugated by the Weyl group (as sets).

With a fixed basis π of a reduced root system Δ one associates the Dynkin diagram of Δ (it does not depend on the choice of π since all basis are W-conjugated) defined as follows. This diagram is a "graph" with π as the set of vertices. Two vertices α_i and α_j are connected with $n_{i,j}$ lines (bons), where $n_{i,j} = \langle \alpha_i, \alpha_j^\vee \rangle \cdot \langle \alpha_j, \alpha_i^\vee \rangle$. There is an arrow from α_j to α_i if $|\langle \alpha_j, \alpha_i^\vee \rangle| > 1$. A reduced root system is called *indecomposable* if the corresponding Dynkin diagram is connected. Two root systems are isomorphic if and only if their Dynkin diagrams are isomorphic. The Dynkin diagrams of indecomposable reduced root systems are very well known and form four series A_n, B_n, C_n, D_n and 5 exceptional diagrams G_2, F_4, E_6, E_7, E_8 (e.g. see [BK, Section 9]).

Given an indecomposable reduced root system, Δ, one constructs a simple Lie algebra, $\mathfrak{G}_\Delta$, via the following *Serre's construction*. Let π be a basis of Δ. The Lie algebra $\mathfrak{G}_\Delta$ will be generated by X_α, Y_α and H_α, $\alpha \in \pi$ with relations

S1. $[H_\alpha, H_\beta] = 0$ for all $\alpha, \beta \in \pi$.

S2. $[X_\alpha, Y_\beta] = \delta_{\alpha,\beta} H_\alpha$ for all $\alpha, \beta \in \pi$ (here $\delta_{\alpha,\beta}$ is the Kronecker symbol).

S3. $[H_\alpha, X_\beta] = \langle \beta, \alpha^\vee \rangle X_\beta$ and $[H_\alpha, Y_\beta] = -\langle \beta, \alpha^\vee \rangle Y_\beta$ for all $\alpha, \beta \in \pi$.

S4. $(\text{ad } X_\alpha)^{-\langle \beta, \alpha^\vee \rangle + 1}(X_\beta) = 0$ and $(\text{ad } Y_\alpha)^{-\langle \beta, \alpha^\vee \rangle + 1}(Y_\beta) = 0$ for all $\alpha, \beta \in \pi$.

Serre's Theorem says that $\mathfrak{G}_\Delta$ is a simple finite-dimensional Lie algebra.

Conversely, with each simple finite-dimensional complex Lie algebra we should associate a root system. Let $\mathfrak{G}$ be a Lie algebra. A nilpotent subalgebra, $\mathfrak{H}$, of $\mathfrak{G}$, which is equal to its normalizer in $\mathfrak{G}$, is called a *Cartan subalgebra* of $\mathfrak{G}$. A Cartan subalgebra of $\mathfrak{G}$ is a maximal nilpotent Lie subalgebra of $\mathfrak{G}$ ([D, Theorem 1.9.4]). If $\mathfrak{G}$ is finite-dimensional, Cartan subalgebras exist ([D, Theorem 1.9.9]) and any two Cartan subalgebras are conjugated by an elementary automorphism of $\mathfrak{G}$ ([D, Theorem 1.9.11]). For a semi-simple finite-dimensional $\mathfrak{G}$, any Cartan subalgebra, $\mathfrak{H}$, of $\mathfrak{G}$ is, in fact, maximal commutative and all its elements are ad-semisimple ([D, Theorem 1.10.6]). Fix a simple complex finite-dimensional Lie algebra $\mathfrak{G}$, a Cartan subalgebra, $\mathfrak{H}$, in $\mathfrak{G}$ and denote by $\Delta = \Delta(\mathfrak{G}, \mathfrak{H})$ the subset of $\mathfrak{H}^*$ consisting of all those non-zero α for which there exists an element $x \in \mathfrak{G}$ such that $[h, x] = \alpha(h)x$ for all $h \in \mathfrak{H}$. Then Δ is an indecomposable reduced root system in $\mathfrak{H}^*$ ([D, Proposition 1.10.7]). Moreover, this map (from $\mathfrak{G}$ to Δ) is inverse to the above map from Δ to $\mathfrak{G}_\Delta$.

Let $\mathfrak{G}$ be a (semi)-simple finite-dimensional complex Lie algebra with a fixed Cartan subalgebra, $\mathfrak{H}$, and the corresponding root system $\Delta \subset \mathfrak{H}^*$ with a fixed basis, π. Then in $\mathfrak{G}$ one can choose a *Weyl-Chevalley basis* (here basis means basis as a vectorspace), X_α, $\alpha \in \Delta$, H_α, $\alpha \in \pi$. In this basis one has $[H_\alpha, H_\beta] = 0$, $[H_\alpha, X_\beta] = \langle \beta, \alpha^\vee \rangle X_\beta$ and $[X_\alpha, X_\beta] = c_{\alpha,\beta} X_{\alpha+\beta}$ with integer non-zero $c_{\alpha,\beta}$ if $\alpha + \beta \in \Delta$ or $[X_\alpha, X_\beta] = 0$ if $\alpha + \beta \notin \Delta$. There is a canonical antiinvolution on $\mathfrak{G}$ associated with this basis, the *Chevalley antiinvolution* σ. It maps each X_α to $X_{-\alpha}$ and fixes all H_α. For example, the non-diagonal matrix units and elements $e_{i,i} - e_{i+1,i+1}$ form a Weyl-Chevalley basis of $\mathfrak{sl}(n, \mathbb{C})$ and the corresponding Chevalley involution is just the transposition of matrices.

Fix $\mathfrak{G}$, $\mathfrak{H}$, Δ and π as above. For $\alpha \in \Delta$ denote by $\mathfrak{G}_\alpha$ the set of all elements $x \in \mathfrak{G}$ on which $\mathfrak{H}$ acts via α ($\mathfrak{G}_\alpha$ is called a *root subspace* of $\mathfrak{G}$). We set $\mathfrak{G}_0 = \mathfrak{H}$ and have that for $\alpha \in \Delta$ the space $\mathfrak{G}_\alpha$ is one-dimensional. The Weyl-Chevalley basis is compatible with the direct sum decomposition $\mathfrak{G} = \mathfrak{H} \oplus (\oplus_{\alpha\in\Delta} \mathfrak{G}_\alpha)$. Let $\Delta = \Delta_+ \cup \Delta_-$ be the decomposition of Δ into a disjoint union of *positive* (Δ_+) and *negative* (Δ_-) roots with respect to π. Define $\mathfrak{N}_\pm = \oplus_{\alpha\in\Delta_\pm} \mathfrak{G}_\alpha$. Then $\mathfrak{G} = \mathfrak{N}_- \oplus \mathfrak{H} \oplus \mathfrak{N}_+$ is a classical *triangular decomposition* of $\mathfrak{G}$. The algebras $\mathfrak{N}_\pm$ are nilpotent. A *Borel subalgebra*, $\mathfrak{B} = \mathfrak{H} \oplus \mathfrak{N}_+$, is a solvable Lie algebra.

2.4 Contragradient algebras, Kac-Moody algebras and Lie algebras with triangular decomposition

There are several ways to define a reasonable generalization of simple finite-dimensional Lie algebras. The most famous one leads to Kac-Moody Lie algebras which we are going to define in this section. As somewhat related we will also recall the definitions of contragradient Lie algebras and Lie algebras with triangular decomposition.

So, we move towards the definition of Kac-Moody Lie algebras and start with contragradient (or Chevalley) algebras. Let $A = (a_{i,j})$ be an $n \times n$ matrix with complex entries. With A one associates a *contragradient Lie algebra*, $\mathfrak{G}(A)$, uniquely defined by the following properties ([Ka, Proposition 4]):

CG1. $\mathfrak{G}(A)$ contains an abelian diagonalizable subalgebra, $\mathfrak{H}$, such that $\mathfrak{G}(A) = \oplus_{\alpha\in\mathfrak{H}^*}\mathfrak{G}_\alpha$, where $\mathfrak{G}_0 = \mathfrak{H}$ and $\mathfrak{G}_\alpha = \{x \in \mathfrak{G}(A) \,|\, [h, x] = \alpha(h)x, h \in \mathfrak{H}\}$.

CG2. Any ideal of $\mathfrak{G}(A)$ which intersects $\mathfrak{H}$ trivially is zero.

CG3. There exists a linearly independent system of linear functions $\alpha_1, \dots, \alpha_n \in \mathfrak{H}^*$ and system of elements $e_1, \dots, e_n$, $f_1, \dots, f_n$ of $\mathfrak{G}(A)$ such that

- (i) $\mathfrak{G}_{\alpha_i} = \mathbb{C}e_i$, $\mathfrak{G}_{-\alpha_i} = \mathbb{C}f_i$, $i = 1, \dots, n$.
- (ii) $[e_i, f_j] = 0$ for $i \neq j$.
- (iii) All e_i, all f_j and $\mathfrak{H}$ generate $\mathfrak{G}(A)$.
- (iv) The elements $h_i = [e_i, f_i]$ are linearly independent.
- (v) $\alpha_j(h_i) = a_{i,j}$ for all i, j.
- (vi) If $h \in \mathfrak{H}$ is such that $\alpha_i(h) = 0$ for all i, then $h \in \sum_{i=1}^n \mathbb{C}h_i$.

To obtain a *Kac-Moody Lie algebra* we should choose A to be a *generalized Cartan matrix*, that is $a_{i,i} = 2$, $a_{i,j} \in -\mathbb{Z}_+$ for $i \neq j$ and $a_{i,j} = 0$ implies $a_{j,i} = 0$ for all i, j. It is known that in the case, when A is a *symmetrizable* (i.e. there exists a diagonal non-degenrate matrix D such that DA is symmetric) generalized Cartan matrix, $\mathfrak{G}(A)$ admits a Serre's type presentation ([MP, Chapter 4]). Any contragradient Lie algebra $\mathfrak{G}(A)$ has the Chevalley antiinvolution σ, which interchanges e_i and f_i and stabilizes $\mathfrak{H}$. A classical textbook about structure and representation theory of Kac-Moody algebras is [Ka].

Let $\mathfrak{G}$ be a Lie algebra. A *triangular decomposition* of $\mathfrak{G}$ is a 4-tuple $(\mathfrak{H}, \mathfrak{G}_+, Q_+, \sigma)$, where $\mathfrak{H}$ is an abelian finite-dimensional subalgebra of $\mathfrak{G}$, $\mathfrak{G}_+$ is a subalgebra of $\mathfrak{G}$, Q_+ is a free additive subsemigroup $\oplus_{j\in J}\mathbb{Z}_+\alpha_j \setminus \{0\}$ of $\mathfrak{H}^*$ with linearly independent α_j, and σ is an antiinvolution on $\mathfrak{G}$ such that

TD1. $\mathfrak{G}_+$ admits a decomposition $\mathfrak{G}_+ = \oplus_{\alpha\in Q_+}\mathfrak{G}_+^\alpha$, with $\mathfrak{G}_+^\alpha = \{x \in \mathfrak{G}_+ \,|\, [h, x] = \alpha(h)x, h \in \mathfrak{H}\}$.

TD2. $\mathfrak{G} = \sigma(\mathfrak{G}_+) \oplus \mathfrak{H} \oplus \mathfrak{G}_+$.

If $\mathfrak{G}$ is a semi-simple finite-dimensional Lie algebra with a fixed Cartan subalgebra $\mathfrak{H}$ and a basis π of Δ, $(\mathfrak{H}, \mathfrak{N}_+, \mathbb{Z}_+\Delta_+ \setminus \{0\}, \sigma)$ is a triangular decomposition of $\mathfrak{G}$ (in other words, the classical triangular decomposition of $\mathfrak{G}$ gives rise to a triangular decomposition in the sense above). If $\mathfrak{G}(A)$ is a contragradient algebra as above, $\mathfrak{G}_+$ its subalgebra generated by e_i and Q_+ the semigroup generated by α_i, $(\mathfrak{H}, \mathfrak{G}_+, Q_+, \sigma)$ is a triangular decomposition of $\mathfrak{G}(A)$. The class of algebras with triangular decompposition is bigger than that of contragradient algebras, since, for example, the Virasoro algebra is not contragradient but has a triangular decomposition. More about Lie algebras with triangular decomposition can be found in a very good textbook [MP].

In what follows we will mostly work over a simple finite-dimensional Lie algebra and in this case we will not need abstract machinery of triangular decomposition in the above sence. It will be necessary only in Chapter 9.

2.5 Weight modules

There is a nice common property of semi-simple finite dimensional algebras, Kac-Moody algebras and algebras with triangular decomposition – they have a reach theory of weight modules. As the notion of a Lie algebra with triangular decomposition is the most general one, we will give all definitions in this case.

We start from a general definition of a weight module. Let $\mathfrak{G}$ be a Lie algebra and $\mathfrak{H}$ be an abelian subalgebra of $\mathfrak{G}$. A $\mathfrak{G}$-module M will be called $\mathfrak{H}$-*weight* if it decomposes into a direct sum of its $\mathfrak{H}$-*weight (sub)spaces* $M_\lambda = \{v \in M \mid hv = \lambda(h)v, h \in \mathfrak{H}\}$, $\lambda \in \mathfrak{H}^*$. The set of all non-zero $\mathfrak{H}$-weights of M is called the $\mathfrak{H}$-*support* of M and is denoted by $\mathrm{supp}_{\mathfrak{H}} M$. For example, if $\mathfrak{G}$ is a Lie algebra with a triangular decompositiopn $(\mathfrak{H}, \mathfrak{G}_+, Q_+, \sigma)$, then, by definition, $\mathfrak{G}$ itself is an $\mathfrak{H}$-weight module under the adjoint action. Usually, if $\mathfrak{H}$ is a fixed Cartan subalgebra of $\mathfrak{G}$ we will omit it in notation and refer to the corresponding $\mathfrak{H}$-weight modules simply as to *weight* modules.

The main property of the weight modules over Lie algebras with triangular decomposition is that they are graded with respect to the Q-grading of $\mathfrak{G}$ defined by $Q_+ \cup -Q_+$. Indeed, let Q denote the abelian additive subgroup in $\mathfrak{H}$ generated by Q_+. Then a natural decomposition $\mathfrak{G} = \oplus_{\alpha \in Q} \mathfrak{G}^\alpha$ defines on $\mathfrak{G}$ the structure of a Q-graded Lie algebra, that is $[\mathfrak{G}^\alpha, \mathfrak{G}^\beta] \subset \mathfrak{G}^{\alpha+\beta}$. If M is a weight $\mathfrak{G}$-module then $\mathfrak{G}^\alpha M_\lambda \subset M_{\alpha+\lambda}$ and hence M is a graded $\mathfrak{G}$-module with respect to the Q-grading. Any homomorphism of weight $\mathfrak{G}$-modules is automatically graded of degree 0.

There exist a lot of examples of weight modules over Lie algebras with triangular decomposition. Here we disscuss only the most classical one. Let $\mathfrak{G}$ be a semi-simple finite-dimensional Lie algebra with a fixed triangular decomposition $(\mathfrak{H}, \mathfrak{N}_+, Q_+, \sigma)$. Then any simple finite-dimensional $\mathfrak{G}$-module is, clearly, a weight module. By the *Weyl Theorem* ([Se, Section 7]), any finite-dimensional module over $\mathfrak{G}$ is *completely reducible*, that is a direct sum of simple modules. Hence, any finite-dimensional $\mathfrak{G}$-module is weight.

If V is a weight $\mathfrak{G}$-module, the *(formal) character* $\mathrm{ch}(V)$ is a formal expression

$$\sum_{\lambda \in \mathfrak{H}^*} \dim(V_\lambda) e^\lambda.$$

The characters behave well in direct sums, weight extensions and tensor products. In particular, $\mathrm{ch}(V_1 \oplus V_2) = \mathrm{ch}(V_1) + \mathrm{ch}(V_2)$ and $\mathrm{ch}(V_1 \otimes V_2) = \mathrm{ch}(V_1) \times \mathrm{ch}(V_2)$, see [D, Section 7.5].

2.6 Universal enveloping algebras

Let $\mathfrak{G}$ be a Lie algebra and V be a $\mathfrak{G}$-module. Then any $x, y \in \mathfrak{G}$ are linear operators on V. Unfortunately, in general, the usual composition $x \circ y$ of these two operators does not represent any element from $\mathfrak{G}$. In other words, if we consider V as a representation $f : \mathfrak{G} \to \mathfrak{gl}(V)$, the image of f is not closed under taking compositions of linear operators (associative structure on $gl(V)$). By definition, this image is only closed under taking commutators $[x, y]$ (Lie structure on $\mathfrak{gl}(V)$). This is not quite good, since usual methods of linear algebra strongly rely on composition of linear operators. In order to improve the situation, we want to embed $\mathfrak{G}$ into a bigger associative algebra, $U(\mathfrak{G})$, such that any $\mathfrak{G}$-module can be canonically extended to a $U(\mathfrak{G})$-module by taking all possible compositions of the operators $x \in \mathfrak{G}$. This leads us to the notion of the universal enveloping algebra of $\mathfrak{G}$ (see [D, Chapter 2]).

Define the *universal enveloping algebra* $U(\mathfrak{G})$ of $\mathfrak{G}$ as the quotient of the tensor algebra $T(\mathfrak{G}) = \oplus_{i\in\mathbb{Z}_+}\mathfrak{G}^{\otimes i}$ over the ideal generated by all elements $xy - yx - [x, y]$, $x, y \in \mathfrak{G}$. $U(\mathfrak{G})$ has a natural associative structure inherited from $T(\mathfrak{G})$. The main result about universal enveloping algebras is the famous *Poincaré-Birkhoff-Witt Theorem* (PBW-Theorem) claiming that choosing a $\mathbb{C}$-basis, $\{g_i \mid i \in I\}$, indexed by a totally ordered set I, the set of all monomials $\{g_{i_1} \dots g_{i_k} \mid k \geqslant 0, i_1 \leqslant i_2 \leqslant \dots \leqslant i_k\}$ forms a $\mathbb{C}$-basis in $U(\mathfrak{G})$. In particular, a natural map $\mathfrak{G} \to U(\mathfrak{G})$ is an injective Lie algebra homomorphism.

Any $U(\mathfrak{G})$-module is a $\mathfrak{G}$-module just by restriction. As $T(\mathfrak{G})$ is generated by $\mathfrak{G}$, composing elements $x \in \mathfrak{G}$, any $\mathfrak{G}$-module uniquely extends to a $U(\mathfrak{G})$-module. Moreover, this is a canonical exact equivalence of module categories.

$U(\mathfrak{G})$ admits a natural filtration by degree of monomials. From the PBW-Theorem it also follows that the associated graded algebra is commutative and isomorphic to the polynomial algebra in $|I|$ variables. In particular, this means that $U(\mathfrak{G})$ is *almost commutative*. We will denote by $Z(\mathfrak{G})$ the center of $U(\mathfrak{G})$. It is known that if $\mathfrak{G}$ is semi-simple finite-dimensional with a fixed Cartan subalgebra, $\mathfrak{H}$, $Z(\mathfrak{G})$ is isomorphic to a polynomial algebra in $\dim\mathfrak{H}$ variables ([D, Section 7.4]).

2.7 Classical $sl(2, \mathbb{C})$-theory

We finish this Chapter recalling some results from the classical representation theory of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ (see [Se, Section 4]). Let $\mathfrak{G} = \mathfrak{sl}(2, \mathbb{C})$ with the standard basis

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This is a Weyl-Chevalley basis with relations: $[X, Y] = H$, $[H, X] = 2X$ and $[H, Y] = -2Y$. The one-dimensional subalgebra generated by H is a Cartan subalgebra, and $\mathfrak{H}^*$ can be identified with $\mathbb{C}$. The corresponding root system Δ then contains two roots ± 2. Choosing $\pi = \{\alpha = 2\}$ we have that $\mathfrak{G}_\alpha = \mathfrak{N}_+$ is spanned by X and $\mathfrak{G}_{-\alpha} = \mathfrak{N}_-$ is spanned by Y. The Chevalley aniinvolution sends X to Y and fixes H. The center $Z(\mathfrak{G})$ of $U(\mathfrak{G})$ is a polynomial algebra in one variable $\mathfrak{c} = (H + 1)^2 + 4YX$. The last element is called a *Caismir element*.

Simple finite-dimensional $\mathfrak{G}$-modules are parametrized by their dimensions, that is for each $n \in \mathbb{N}$ there exists a unique simple n-dimensional $\mathfrak{G}$-module V_n. V_n has a basis, v_i, $i \in \{-n + 1, -n + 3, \dots, n - 3, n - 1\}$ and the action of generating elements of $\mathfrak{G}$ in this basis is given by

$$Yv_i = v_{i-2}, \quad Hv_i = iv_i, \quad Xv_i = \frac{1}{4}(n^2 - (i + 1)^2)v_{i+2}.$$

The element v_{n-1} is a *highest weight element* of V_n, that is $Xv_{n-1} = 0$. The element v_{-n+1} is a *lowest weight element* of V_n, that is $Yv_{n-1} = 0$. The unique eigenvalue of $\mathfrak{c}$ on V_n equals i^2.

The module V_n is the unique simple quotient of the Verma module $M(n)$, which has a basis, v_i, $i \in n-1-2\mathbb{Z}_+$ and the action of generating elements of $\mathfrak{G}$ in this basis is given by

$$Yv_i = v_{i-2}, \quad Hv_i = iv_i, \quad Xv_i = \frac{1}{4}(n^2 - (i+1)^2)v_{i+2}.$$

$M(n)$ is a weight module generated by a highest weight element v_{n-1}. Each $M(n)$, $n \in \mathbb{N}$ has a unique non-trivial submodule, which is isomorphic to $M(-n)$. If we define $M(n)$ for $n \in \mathbb{C} \setminus \mathbb{Z}$ in the same way as above, the result will be a simple $\mathfrak{G}$-module.

3 Verma modules

Before introducing generalized Verma modules, it is natural to discuss classical Verma modules, which we are going to define in this Chapter. We restrict us to the case of simple complex finite-dimensional Lie algebras and list the main properties of Verma modules. We follow closely the classical textbooks [D, Jo1, MP] for basic definitions and results. More advanced facts are available only as research papers and we will give precise references in each case.

3.1 Definition

Let $\mathfrak{G}$ be a simple complex finite-dimensional Lie algebra with a fixed Cartan subalgebra $\mathfrak{H}$; $\Delta \subset \mathfrak{H}^*$ the corresponding root system; π a basis od Δ; $\Delta = \Delta_+ \cup \Delta_-$ the decomposition with respect to $\mathfrak{P}$; $\mathfrak{G} = \mathfrak{N}_- \oplus \mathfrak{H} \oplus \mathfrak{N}_+$ the corresponding classical triangular decomposition of $\mathfrak{G}$; σ the Chevalley antiinvolution; X_α, $\alpha \in \Delta$, H_α, $\alpha \in \pi$ a fixed Weyl-Chevalley basis; W the Weyl group and ρ half the sum of positive roots. For $\alpha \in \Delta$ let $\mathfrak{G}_\alpha$ denote the corresponding root subspace of $\mathfrak{G}$. W acts on $\mathfrak{H}^*$ in a natural way as a Weyl group. Define the dot-action of W on $\mathfrak{H}^*$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$. For $\alpha \in \Delta$, s_α denotes the corresponding reflection. Let $(\cdot\,,\cdot)$ be the standard W-invariant form on $\mathfrak{H}^*$. The corresponding dual form on $\mathfrak{H}$ will be also denoted by $(\cdot\,,\cdot)$.

Fix $\lambda \in \mathfrak{H}^*$ and consider a one-dimensional $\mathfrak{B} = \mathfrak{H} \oplus \mathfrak{N}_+$-module, $\mathbb{C} = \mathbb{C}_\lambda$, such that $(a+h)(z) = (\lambda - \rho)(h)z$ for all $a \in \mathfrak{N}_+$, $h \in \mathfrak{H}$ and $z \in \mathbb{C}$. The module

$$M(\lambda) = U(\mathfrak{G}) \bigotimes_{U(\mathfrak{B})} \mathbb{C}_\lambda$$

is called the *Verma module*, associated with $\mathfrak{G}$, $\mathfrak{H}$, π and λ (see [D, Chapter 7]).

We note, that there are two standard ways to parameterize the module $M(\lambda)$. They differ by the replacement of $\lambda - \rho$ with λ in the above definition of $\mathbb{C}_\lambda$. Both possibilities occur in the literature; we have chosen to follow the convention of Dixmier in his standard reference [D].

3.2 Basic properties

Originally, Verma modules appeared as the universal highest weight modules. The first branch of basic properties of $M(\lambda)$ is connected precisely with this notion.

Let M be a weight $\mathfrak{G}$-module. A non-zero element, $v \in V_\mu$, is called a *highest weight element* provided $\mathfrak{N}_+ v = 0$. A weight $\mathfrak{G}$-module, M, is called a *highest weight module* if M is generated by a highest weight vector, whose weight is called the *highest weight* of M. We also recall the standard partial order $\leqslant$ (which depends on π) on $\mathfrak{H}^*$ defined as follows: $\mu_1 \leqslant \mu_2$ if and only if $\mu_2 - \mu_1$ can be written as a linear combination of positive roots with non-negative integer coefficients. Define the *Kostant partition function* $P : \mathfrak{H}^* \to \mathbb{Z}_+$ as the number of different decompositions of $\mu \in \mathfrak{H}^*$ in a linear combination of positive roots with non-negative integer coefficients. Clearly $P(\mu) > 0$ if and only if $0 \leqslant \mu$.

Theorem 3.2.1. *1. $M(\lambda)$ is a highest weight module with the highest weight $\lambda - \rho$.*

2. $M(\lambda)$ is a $U(\mathfrak{N}_-)$-free module of rank 1. In particular, $M(\lambda)$ is isomorphic to $U(\mathfrak{N}_-)$ as a vector space (graded by weights).

3. $\mathrm{supp}(M(\lambda))$ *coinsides with the set of all $\mu \in \mathfrak{H}^*$ such that $\mu \leqslant \lambda - \rho$. Moreover,* $\dim(M(\lambda)_\mu) = P(\lambda - \rho - \mu)$ *for any $\mu \in \mathfrak{H}^*$.*

4. Any highest weight $\mathfrak{G}$-module with the highest weight $\lambda - \rho$ is a quotient of $M(\lambda)$.

5. $M(\lambda)$ has a unique simple quotient, $L(\lambda)$.

6. $M(\lambda)$ has a central character, χ_λ.

7. Any endomorhpism of $M(\lambda)$ is scalar.

The complete proof can be found, for example, in [D, Chapter 7]. We note that, under our definition, the parameter λ of $M(\lambda)$ does not coincide with the highest weight of $M(\lambda)$, which is $\lambda - \rho$. Of course one can parameterize Verma modules by their highest weights. This leads to the second way of describing them (as was mentioned above). Further we will see that our way is a bit more convenient for some properties of Verma modules. We also note the the fifth statement of Theorem 3.2.1 is usually called the universal property of Verma modules. From this statement it follows, for example, that any simple finite-dimensional $\mathfrak{G}$-module is a quotient of an appropriate Verma module (and hence coincides with the corresponding $L(\lambda)$). The set of λ such that $L(\lambda)$ is finite-dimensional is usually denoted P^{++} and called the set of *regular dominant integral* parameters (see [D, Section 7.2]).

The next collection of properties of Verma modules is closely related to the structure of $M(\lambda)$.

Theorem 3.2.2. *1. $\chi_\lambda = \chi_\mu$ if and only if $\lambda \in W(\mu)$.*

2. For any $\chi \in Z(\mathfrak{G})^*$ there exists $\lambda \in \mathfrak{H}^*$ such that $\chi = \chi_\lambda$.

3. $M(\lambda)$ has a composition series.

4. $\dim \mathrm{Hom}_{\mathfrak{G}}(M(\lambda), M(\mu)) \leqslant 1$ *and any non-zero homomorphism from this space is injective.*

Again, the proof can be found in [D, Chapter 7]. The first two statements of this Theorem are easy corollaries of the famous Harish-Chandra isomorphism Theorem, which states that, under the Harish-Chandra homomorphism φ, $Z(\mathfrak{G})$ maps isomorphically onto the algebra $S(\mathfrak{H})^{(W,\cdot)}$ of polynomials in $\mathfrak{H}$ invariant under the dot-action of the Weyl group. Composing φ with the shift of indetermines in $\mathfrak{H}$, associated with ρ, we obtain a new map, whose image coincide with the algebra $S(\mathfrak{H})^W$ of usual invariants under the standard action of W (see [D, Section 7.4]).

3.3 BGG Theorem and Shapovalov form

The following Theorem is the celebrated BGG criterion for the existence of a non-trivial homomorphism between two Verma modules ([BGG2]). The sufficiency of it was proved by Verma ([V]), who also conjectured the necessity part.

Theorem 3.3.1. *For $\lambda, \mu \in \mathfrak{H}^*$ the following are equivalent*

1. *$M(\mu) \subset M(\lambda)$.*

2. *$L(\mu)$ is a subquotient of $M(\lambda)$.*

3. *There exist a sequence α_1, α_2, $\ldots$, α_k of positive roots such that*

$$\mu \leqslant s_{\alpha_1}(\mu) \leqslant s_{\alpha_2}(s_{\alpha_1}(\mu)) \leqslant s_{\alpha_k}(\ldots(s_{\alpha_2}(s_{\alpha_1}(\mu)))\ldots) = \lambda.$$

The proof can be found in [D, MP, BGG2] ([D, Section 7.6] is the most detailed). From this Theorem one can easily deduce the following criterion for $M(\lambda)$ to be simple.

Corollary 3.3.1. *$M(\lambda)$ is simple if and only if $s_\alpha(\lambda) \nleqslant \lambda$ for any $\alpha \in \Delta_+$.*

Originally, the proof of Theorem 3.3.1 crucially used the first statement from Theorem 3.2.2. Later ([KK]), an elementary proof was found. It uses the notion of the Shapovalov form on $M(\lambda)$, defined in [Sh]. Denote by v_λ a canonical generator of the module $\hat{M}(\lambda) = M(\lambda+\rho)$ (in what follows we will keep this notation). According to Theorem 3.2.1, any element of $\hat{M}(\lambda)$ can be written as uv_λ, $u \in U(\mathfrak{N}_-)$. Define

$$F(u_1 v_\lambda, u_2 v_\lambda) = \varphi(\mathrm{proj}_{U_0}(\sigma(u_1)u_2))(\lambda),$$

where U_0 denotes the centralizer of $\mathfrak{H}$ in $U(\mathfrak{G})$. F is called the *Shapovalov form* on $\hat{M}(\lambda)$.

Theorem 3.3.2.

1. *F is a symmetric contravariant bilinear form on $\hat{M}(\lambda)$.*

2. *The weight subspaces of $\hat{M}(\lambda)$ are orthogonal with respect to F.*

3. *The radical of F coincides with the maximal submodule of $\hat{M}(\lambda)$.*

4. *$\hat{M}(\lambda)$ is simple if and only if F is non-degenerate.*

5. $$\det F|_{\hat{M}(\lambda)_\mu} = \prod_{\alpha \in \Delta_+} \prod_{n=1}^{\infty} (\lambda(H_\alpha) + \rho(H_\alpha) - n(\alpha,\alpha)/2)^{P(\lambda-\mu-n\alpha)}.$$

Theorem 3.3.1 is an easy corollary of Theorem 3.3.2. But the proof of Theorem 3.3.2 in [Sh] was based on Theorem 3.3.1. The proof in [KK] is quite elementary (but not easy!) and is independent of Theorem 3.3.1 (and even of Harish-Chandra Theorem). Hence the last gives us an elementary proof of Theorem 3.3.1.

3.4 Weyl and Demazure character formulae and Schubert filtration

As it was mentioned, simple finite-dimensional $\mathfrak{G}$-modules are $L(\lambda)$, $\lambda \in P^{++}$. There are a lot of things known for such $L(\lambda)$. Here we are going to disscuss the most famous ones – Weyl and Demazure character formulae and the corresponding Schubert filtration.

The first famous result about simple finite-dimensional $\mathfrak{G}$-modules is the celebrated *Weyl character formula*, [D, Theorem 7.5.9].

Theorem 3.4.1. *Let $\lambda \in P^{++}$. Then*

$$\mathrm{ch}(L(\lambda)) = \left(\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda)} \right) \left(\sum_{w \in W} (-1)^{l(w)} e^{w(\rho)} \right)^{-1}.$$

Demazure character formula is an improvement of the Weyl character formula. To write it we need some additional notation. For $\alpha \in \pi$ we set $d_\alpha = (1 - e^{-\alpha})^{-1}(1 - e^{-\alpha} s_\alpha)$, where we consider s_α as an operator on the ring of characters via $s_\alpha(e^\lambda) = e^{s_\alpha(\lambda)}$.

Theorem 3.4.2. *Let $\lambda \in P^{++}$ and $w_0 = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}$ be a reduced decomposition of the longest element $w_0 \in W$. Then*

$$\mathrm{ch}(L(\lambda)) = d_{\alpha_1} d_{\alpha_2} \dots d_{\alpha_k} e^{\lambda - \rho}.$$

Proof can be found in [Jo1, A] or [Z2, Chapter 2]. Associated with this character formula there is a canonical filtration of $L(\lambda)$ as $U(\mathfrak{B})$-module, called *Schubert filtration* (see [Jo1, Z2]).

Theorem 3.4.3. *Let $\lambda \in P^{++}$ and $w_0 = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}$ be a reduced decomposition of the longest element $w_0 \in W$. Then thre exists a filtration $L(\lambda) = L_1 \subset L_2 \subset ... \subset L_k \subset L_{k+1} = 0$ of $U(\mathfrak{B})$-modules such that for $i = 1, 2, \dots, k$ holds*

$$\mathrm{ch}(L_i) = d_{\alpha_i} d_{\alpha_{i+1}} \dots d_{\alpha_k} e^{\lambda - \rho}.$$

We also note that all $U(\mathfrak{B})$-modules L_i mentioned in Theorem 3.4.3 are cyclic and any canonical generator of L_i is a highest weight element of $L(\lambda)$ with respect to some other choice of the basis in Δ.

3.5 Composition multiplicities

Theorem 3.3.1 gives us a criterion for $L(\mu)$ to be a subquotient of $M(\lambda)$, or, in other words, for the corresponding *composition multiplicity*, that is the number of times $L(\mu)$ occurs as a quotient in a composition series of $M(\lambda)$, which is usually denoted by $(M(\lambda) : L(\mu))$, to be positive (or $\neq 0$). A natural question to determine $(M(\lambda) : L(\mu))$ completely seems to be the most difficult part of the theory of Verma modules. This is the content of the famous Kazhdan-Lusztig Congecture ([KL]), which has been proved for semi-simple finite-dimensional $\mathfrak{G}$ in [BB1, BrKa]. To formulate the statement we need some preparation.

Let W be a Coxeter group, S the set of simple reflections, l the corresponding length function (see [H1]) and $<$ the Bruhat order on W. Set $\mathcal{L} = \mathbb{Z}[v, v^{-1}]$. Define the *Hecke algebra* $\mathcal{H}(W, S)$, associated with W and S, as a free $\mathcal{L}$-module with a basis T_w, $w \in W$ and associative multiplication defined by

HA1. $T_x T_y = T_{xy}$, if $l(xy) = l(x)l(y)$.

HA2. $T_s^2 = v^{-2}T_1 + (v^{-2} - 1)T_s$, $s \in S$.

Of course, one has to check the existence of such structure and we refer the reader to [KL] for this. Set $H_w = v^{l(w)}T_w$, $w \in W$. One has $H_x H_y = H_{xy}$ if $l(xy) = l(x) + l(y)$ and $H_s^2 = 1 + (v^{-1} - v)H_s$, $s \in S$. In particular, $H_s^{-1} = H_s + (v - v^{-1})$ and H_s is invertible for $s \in S$. Using $H_x H_y = H_{xy}$ one derives that H_w is invertible for any $w \in W$. There exists a skew-linear (i.e. $v \mapsto v^{-1}$) involution $d : \mathcal{H} \to \mathcal{H}$, which sends $H_w \mapsto (H_{w^{-1}})^{-1}$, $w \in W$ and is a ring homomorphism. Further, for any $w \in W$ there exists a unique $\underline{H}_w$ such that $d(\underline{H}_w) = \underline{H}_w$ and $\underline{H}_w = H_w + \sum_{y<w} h_{y,w}(v)H_y$, where $h_{y,w}(v) \in v\mathbb{Z}[v]$, moreover, $\underline{H}_w$, $w \in W$ form a basis of $\mathcal{H}$ (see [KL]). For $x, y \in W$ set $P_{y,x} = v^{l(y)-l(x)}h_{y,x}$, which is known to be an element of $\mathbb{Z}[v]$ and is usually called a *Kazhdan-Lusztig polynomial.* Then the Kazhdan-Lusztig conjecture states the following.

Theorem 3.5.1. *Let W be the Weyl group of $\mathfrak{G}$ and S the set of simple reflections with respect to π. Let $\lambda \in \mathfrak{H}^*$ be integral dominant and $x, y \in W$. Then $(M(x(\lambda)) : L(y(\lambda))) = P_{x,y}(1)$.*

Proof, even for finite-dimensional $\mathfrak{G}$, is highly non-trivial and can be found in [BB1, BrKa].

3.6 BGG resolution

As was mentioned, $L(\lambda)$ is finite-dimensional if and only if $\lambda \in P^{++}$, moreover, each simple finite-dimensional module is of the form $L(\lambda)$ for an appropriate $\lambda \in P^{++}$. Let $l : W \to \mathbb{Z}_+$ denote the standard length function with respect to π. Fix $\lambda \in P^{++}$. For $i = 0, 1, \dots, |\Delta_+|$ denote by C_i the direct sum of all $M(w(\lambda))$, with $l(w) = i$ (we will also denote by W_i the set of such w). According to Theorem 3.2.2, any homomorphism $d_i : C_i \to C_{i-1}$ can be defined via a complex matrix $(d(i)_{x,y})_{x\in W_{i-1}}^{y\in W_i}$. According to [BGG1] or [RC], one can choose $d(i)_{x,y}$ simultaneously such that $d(i)_{x,y} \neq 0$ if and only if $x < y$ with respect to the Bruhat order on W and $d_{i-1} \circ d_i = 0$ for all i.

Theorem 3.6.1. *The sequence*

$$0 \to C_{|\Delta_+|} \xrightarrow{d_{|\Delta_+|}} C_{|\Delta_+|-1} \xrightarrow{d_{|\Delta_+|-1}} \dots C_1 \xrightarrow{d_1} M(\lambda) \xrightarrow{p} L(\lambda) \to 0,$$

where p is the canonical projection, is exact.

The exact sequence from Theorem 3.6.1 is known as the *BGG resolution* of $L(\lambda)$. The proof of Theorem 3.6.1 can be found in [BGG1]. Some interesting facts about it also can be found in [L1, RC, RCW1, GJ]. We have to note that sometimes this BGG resolution is called the *strong* BGG resolution in order to distinguish it from another (quite similar) resolution, constructed in [BGG1] by using the cohomology of $\mathfrak{G}$. The equivalence of both resolutions was proved in [RC].

3.7 Category $\mathcal{O}$

One of the most important structures arising in the representation theory of $\mathfrak{G}$ is the so-called category $\mathcal{O}$, defined and investigated in [BGG3]. In particular, it led to the appearence of such new objects in modern algebra, as quasi-hereditary algebras, highest weight categories, BGG catregories and so on (see [CPS1, I2]).

Let $\mathcal{O}$ denote a full subcategory of the category of all $\mathfrak{G}$-modules, which consists of those modules M, which are

- finitely generated;
- $\mathfrak{H}$ diagonalizable;
- $\mathfrak{N}_+$ finite (i.e. $U(\mathfrak{N}_+)v$ is finite-dimensional for any $v \in M$).

Theorem 3.7.1. *1. $\mathcal{O}$ is closed under taking submodules, quotients, finite direct sums and under tensoring with finite-dimensional modules.*

2. All Verma modules belong to $\mathcal{O}$.

3. $L(\lambda)$, $\lambda \in \mathfrak{H}^*$ exhaust the set of simple modules in $\mathcal{O}$.

4. Any module in $\mathcal{O}$ has a composition series.

5. $\mathcal{O} = \oplus_{\chi \in Z(\mathfrak{G})^*} \mathcal{O}(\chi)$, where $\mathcal{O}(\chi)$ is a full subcategory consisting of all modules, which are annihilated by a big enough power of $z - \chi(z)$ for all $z \in Z(\mathfrak{G})$.

The proof can be found in [BGG3] or in [MP]. It is quite easy. The first non-trivial property of $\mathcal{O}$ states that it can be decomposed into a direct sum of full subcategories, each of which is equivalent to the category of (finite-dimensional) modules over a finite-dimensional algebra.

Theorem 3.7.2. *1. $\mathcal{O}$ has enough projective modules (i.e. any module in $\mathcal{O}$ is a quotient of a projective module in $\mathcal{O}$).*

2. There is a bijection between simple modules and indecomposable projective modules in $\mathcal{O}$. We will denote by $P(\lambda)$ the projective cover of $L(\lambda)$, $\lambda \in \mathfrak{H}^*$.

3. Each $\mathcal{O}(\chi)$ is equivalent to the category of (finite-dimensional) modules over a finite-dimensional algebra.

Again, the proof can be found in [BGG3] or in [MP]. The next Theorem is the celebrated BGG reciprocity principle for $\mathcal{O}$.

Theorem 3.7.3. *1. Any projective module in $\mathcal{O}$ admits a* Verma flag, *that is a filtration, whose subquotients are Verma modules.*

2. For any $\lambda, \mu \in \mathfrak{H}^*$ holds $[P(\lambda) : M(\mu)] = (M(\mu) : L(\lambda))$, where $[P(\lambda) : M(\mu)]$ denotes the number of occurrences of $M(\mu)$ in a Verma flag of $P(\lambda)$. In particular, the last is a well-defined number (i.e. does not depend on a Verma flag).

Roughly speaking, Teorem 3.7.3 shows that Verma modules play in $\mathcal{O}$ a role of intermediate modules between simples and projectives. The proof can be found in [BGG3] or in [MP].

Theorems 3.7.2 and 3.7.3 were a motivation for introducing in [CPS1] the class of quasi-hereditary algebras. Let A be a finite-dimensional associative algebra and I be a set parametrizing simple A-modules L_i, $i \in I$. Denote by P_i the projective cover of L_i and let $\leqslant$ be a partial order on I. The algebra A is called *quasi-hereditary* (with respect to $\leqslant$) if there exists a choise of A-modules, M_i, $i \in I$ such that

QH1. M_i surjects onto L_i and the kernel of this map is filtered by L_j, $j < i$.

QH2. P_i surjects onto M_i and the kernel of this map is filtered by M_j, $i < j$.

Considering $\mathcal{O}(\chi)$ with the partial order coming from the Bruhat order on W, one easily derives that any finite-dimensional associative algebra associated with $\mathcal{O}(\chi)$ is quasi-hereditary. Theory of quasi-hereditary algebras is relatively well-developed, in particular, there is an abstract analogue of the BGG reciprocity and a lot of other nice properties (see [CPS1, R3, I2, KlKo] and references therein).

The next natural question for the category $\mathcal{O}$ is to give a precise (combinatorial) description of the algebras which appear in Theorem 3.7.2. This was solved (much later) by Soergel in [S1]. In order to state Soergel's results, we need a bit more notation.

Denote by C the quotient of the polynomial algebra $S(\mathfrak{H})$ over the ideal generated by all non-constant homogeneous polynomials in $S(\mathfrak{H})$, invariant under the action of W (we emphasize that here we mean the ordinary action). The algebra C is usually called the *coinvariant algebra*. Let $\lambda_0 \in P^{++}$ be such that the module $L(\lambda_0)$ is one-dimensional (trivial module). Then the corresponding block $\mathcal{O}(\chi_{\lambda_0})$ is called *principal*. Let w_0 denote the unique longest element in W ($l(w_0) = |\Delta_+|$). The projective module $P(w_0(\lambda_0))$ is usually called the *big projective module*. It is easy to see that $[P(w_0(\lambda_0)) : M(w(\lambda_0))] = 1$ for any $w \in W$ and $[P(w(\lambda_0)) : M(w_0(\lambda_0))] = 0$ for any $w \neq w_0$.

Theorem 3.7.4. *1. The endomorphism algebra of $P(w_0(\lambda_0))$ is isomorphic to C.*

2. The finite-dimensional algebra, associated with $\mathcal{O}(\chi_{\lambda_0})$, coinsides with the endomorphism algebra of $P(w_0(\lambda_0))$, viewed as a module over its endomorphism algebra (i.e. viewed as a C-module).

3. If $\mu_1, \mu_2 \in P^{++}$, then $\mathcal{O}(\chi_{\mu_1})$ and $\mathcal{O}(\chi_{\mu_1})$ are equivalent.

4. Any $\mathcal{O}(\chi)$ is equivalent to some $\mathcal{O}(\chi_\lambda)$ with an integral λ with respect to the semisimple part of the Levi factor of a parabolic subalgebra of $\mathfrak{G}$.

The proof can be found in [S1, Be]. The second statement of this Theorem is know as the *double centralizer property.* The third statement is standard and was stated (without proof) in [BGG3]. From the last statement it follows, in particular, that for a fixed algebra $\mathfrak{G}$ the set of non-isomorphic finite-dimensional algebras arizing from the corresponding $\mathcal{O}$ is finite and that Kazhdan-Lusztig Theorem describes the composition multiplicities fot all Verma modules, not necessarily having an integral parameter.

The Chevalley antiinvolution on $\mathfrak{G}$ leads to a natural duality, $*$, on $\mathcal{O}$ (by a *duality* we mean a contravariant exact functor, which preserves simple objects). Let $\mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\nabla)$) denote the full subcategory of $\mathcal{O}$ which consists of all modules admitting a Verma flag (resp. *dual Verma flag*, that is a flag, whose subquotients are isomorphic to $M(\lambda)^*$, $\lambda \in \mathfrak{H}^*$). Then, according to the general result of Ringel for quasi-hereditary algebras ([R1]), the indecomposable modules in $\mathcal{F} = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ are naturally parametrized by $\lambda \in \mathfrak{H}^*$ (i.e. there is a natural bijection between simple modules in $\mathcal{O}$ and indecomposable modules in $\mathcal{F}$), moreover $\mathcal{F}$ coincides with the full subcategory consisting of all direct sums of these indecomposable modules. We will denote by $T(\lambda)$ the indecomposable module in $\mathcal{F}$ which corresponds to $\lambda \in \mathfrak{H}^*$ (this means that any Verma flag of $T(\lambda)$ starts with $M(\lambda)$). $T(\lambda)$ is usually called a *tilting module.* If we restrict our considiration to $\mathcal{O}(\chi_\lambda)$, $\lambda \in \mathfrak{H}^*$ and denote by W_λ the subgroup of W stabilizing λ, then the direct sum $T = \oplus_{w \in W/W_\lambda} T(w(\lambda))$ is called the *characteristic tilting module* of $\mathcal{O}(\chi_\lambda)$. In this case $\mathcal{F}(\chi_\lambda)$ coinsides with $\mathrm{add}\, T$ (see [KlKo]). Having this characteristic tilting module T one can consider its endomorphism algebra, which is known (for our example, or in general, for quasi-hereditary algebras) to be a quasi-hereditary algebra ([R1, R2]). This algebra is called the *Ringel dual* for the algebra, corresponding to $\mathcal{O}(\chi_\lambda)$.

Theorem 3.7.5. *1. The principal block $\mathcal{O}(\chi_{\lambda_0})$ of $\mathcal{O}$ is its own Ringel dual (i.e. the corresponding algebra is isomorphic to its Ringel dual).*

2. For $x, y \in W$ holds $[T(x(\lambda_0)) : M(y(\lambda_0))] = (M(yw_0(\lambda_0)) : L(xw_0(\lambda_0)))$.

This Theorem is a recent result of Soergel ([S4]), where the reader can find a proof of both statements. Some more interesting results about category $\mathcal{O}$ and related structures can be found in [BG, BC1, BC2, BGS, H2, CI, CS, ES, I1, I3, I4, IS, MP, RCW2, S2, S3].

3.8 Loewy series

The last basic fact about Verma modules, which we are going to discuss here is a description of the Loewy series, in particular, rigidity of Verma modules, established in [I1].

If M is a module of finite length, a *Loewy filtration* of M is a filtration of shortest possible length with semi-simple subquotients. Among such filtrations there is one which contains any other term-by-term, the *socle filtration*

$$0 \subset \operatorname{soc}(M) \subset \operatorname{soc}(M/\operatorname{soc}(M)) + \operatorname{soc}(M) \subset \dots,$$

and one which is contained in any other, the *radical filtration*

$$\dots \subset \operatorname{rad}(\operatorname{rad}(M)) \subset \operatorname{rad}(M) \subset M.$$

The length of a Loewy filtration of M is called the *Loewy length* of M and, by definition, does not depend on the choice of a Loewy filtration. M is called *rigid* if the socle and the radical filtrations of M coincide, equivalentely, if there exists a unique Loewy filtration of M. For $i \in \mathbb{N}$ we denote by $\operatorname{soc}^i(M)$ the entries of the socle filtration of M and set $\operatorname{soc}_i(M) = \operatorname{soc}^i(M)/\operatorname{soc}^{i-1}(M)$.

Theorem 3.8.1. *1. Any Verma module is rigid.*

2. *For an integral antidominant regular λ and $w \in W$ the Loewy length of $M(w(\lambda))$ equals $l(w)+1$.*

3. *Let λ be antidominant integral and regular. Then $P_{x,y}(v) = \sum_i (\operatorname{soc}_{l(y)+1+2i}(M(x(\lambda))) : L(y(\lambda)))v^i$.*

The proof can be found in [I1]. We only note that, in particular, the third statement explains the "structure" of the Kazhdan-Lusztig polynomials, whose coefficients count the number of occurrences of a simple module in the layers of the unique Loewy filtration of a Verma module.

4 Introducing Generalized Verma Modules

In this Chapter we introduce the main object of our interest: Generalized Verma modules. We establish their basic properties and introduce one important tool: the generalized Harish-Chandra homomorphism.

4.1 Definition

Let $\mathcal{P} \supset \mathfrak{B}$ be a parabolic subalgebra of $\mathfrak{G}$ which contains $\mathfrak{B}$. Denote by $\mathfrak{N}$ the nilpotent radical of $\mathcal{P}$ and by $\mathfrak{A}'$ the Levi factor of $\mathcal{P}$. Then $\mathfrak{N} \subset \mathfrak{N}_+$, $\mathfrak{A}'$ is a reductive Lie algebra, which can be decomposed $\mathfrak{A}' = \mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}$, where $\mathfrak{A}$ is semi-simple and $\mathfrak{H}_{\mathfrak{A}} \subset \mathfrak{H}$ is abelian and central in $\mathfrak{A}'$. Let V be an $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable $\mathfrak{A}'$-module. Set $\mathfrak{N}V = 0$. In this way we turn V into a $\mathcal{P}$-module. Now we can use the induction from $\mathcal{P}$ to $\mathfrak{G}$ and obtain the module

$$M_{\mathcal{P}}(V) = U(\mathfrak{G}) \bigotimes_{U(\mathcal{P})} V,$$

associated with $\mathfrak{G}$, $\mathcal{P}$ and V. This induction, which is associated with a parabolic subalgebra $\mathcal{P}$ of $\mathfrak{G}$ is usually called a *parabolic induction.*

If the module V is simple we will call $M_{\mathcal{P}}(V)$ a *Generalized Verma module* (GVM). If the parabolic algebra $\mathcal{P}$ is fixed or clear from the context we will usually omit it in the notation and will simply write $M(V)$.

We will also need some more notation. Let $\Delta(\mathfrak{A}) \subset \Delta$ be the root system of $\mathfrak{A}$. Then $\pi(\mathfrak{A}) = \pi \cap \Delta(\mathfrak{A})$ is a basis in $\Delta(\mathfrak{A})$ and $\Delta(\mathfrak{A})_{\pm} = \Delta_{\pm} \cap \Delta(\mathfrak{A})$ are the corresponding sets of positive (negative) roots with respect to $\pi(\mathfrak{A})$. Denote by $W(\mathfrak{A})$ the Weyl group of $\Delta(\mathfrak{A})$, which is a subgroup of W. Let $\mathfrak{N}(\mathfrak{A})$ denote the subalgebra of $\mathfrak{N}_-$ generated by X_α, $\alpha \in (\Delta_- \setminus \Delta(\mathfrak{A}))$. Clearly, $\mathfrak{N}(\mathfrak{A})$ is the image of $\mathfrak{N}$ under σ.

Denote by $\mathfrak{H}(\mathfrak{A})$ the intersection $\mathfrak{H} \cap \mathfrak{A}$. Since $\mathfrak{H}_{\mathfrak{A}}$ is central in $\mathfrak{A}'$, we have that $\mathfrak{H}_{\mathfrak{A}}$ coincides with the orthogonal complement to $\mathfrak{H}(\mathfrak{A})$ in $\mathfrak{H}$ with respect to $(\cdot\,,\cdot)$. Hence we can identify $\mathfrak{H}(\mathfrak{A})^*$ with the subspace of $\mathfrak{H}^*$, generated by all $\alpha \in \Delta(\mathfrak{A})$ and $\mathfrak{H}_{\mathfrak{A}}^*$ with the orthogonal complement to $\mathfrak{H}(\mathfrak{A})^*$ in $\mathfrak{H}^*$. Let $p_{\mathfrak{A}} : \mathfrak{H}^* \to \mathfrak{H}^*$ denote the projection on $\mathfrak{H}_{\mathfrak{A}}^*$ with respect to $\mathfrak{H}(\mathfrak{A})^*$. $\mathfrak{H}_{\mathfrak{A}}^*$ inherits a natural partial order from the order $\leqslant$ on $\mathfrak{H}^*$. We will denote this order by the same symbol and set $\mu_1 \leqslant \mu_2$ for $\mu_1, \mu_2 \in \mathfrak{H}_{\mathfrak{A}}^*$ if there exists $\mu_3 \in \mathfrak{H}^*$ such that $p_{\mathfrak{A}}(\mu_3) = \mu_2$ and $\mu_1 \leqslant \mu_3$ (as elements in $\mathfrak{H}^*$).

4.2 Basic properties

For this Section we fix a GVM $M(V) = M_{\mathcal{P}}(V)$ and keep the notation of Section 4.1. Since V is a simple $\mathfrak{A}'$-module and $\mathfrak{H}_{\mathfrak{A}}$ is abelian and central, $\mathfrak{H}_{\mathfrak{A}}$ acts on V by scalars, i.e. via some $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$. So for the rest of this section we fix V, $M(V)$ and λ above.

Let M be an $\mathfrak{H}_{\mathfrak{A}}$-weight $\mathfrak{G}$-module. A non-zero element, $v \in M_\lambda$, $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$ will be called an $\mathfrak{A}$-*highest weight element* (or an $\mathfrak{A}$-*primitive element*, or just a *semiprimitive element*, if $\mathfrak{A}$ is clear from the context) provided $\mathfrak{N}v = 0$. An $\mathfrak{H}_{\mathfrak{A}}$-weight $\mathfrak{G}$-module, M, will be called an $\mathfrak{A}$-*highest weight module* if it is generated by an $\mathfrak{A}$-highest weight vector, v, such that $U(\mathfrak{A})v$ is a simple $\mathfrak{A}$-module. The $\mathfrak{A}$-weight of v will be called the $\mathfrak{A}$-*highest weight* of M.

Proposition 4.2.1. *1. $M(V)$ is an $\mathfrak{A}$-highest weight module with the $\mathfrak{A}$-highest weight λ. Moreover $M(V)_\lambda \simeq V$ as an $\mathfrak{A}$-module.*

2. $M(V)$ is a $U(\mathfrak{N}(\mathfrak{A}))$-free module and any basis of V is a $U(\mathfrak{N}(\mathfrak{A}))$-basis of $M(V)$.

3. $\mathrm{supp}_{\mathfrak{H}_\mathfrak{A}}(M(V))$ coincides with the set of all $\mu \in \mathfrak{H}_\mathfrak{A}^*$ such that $\mu \leqslant \lambda$.

4. Any $\mathfrak{A}$-highest weight module M such that λ is the $\mathfrak{A}$-highest weight of M and $M_\lambda \simeq V$ (as an $\mathfrak{A}$-module) is a quotient of $M(V)$.

5. $M(V)$ has a unique simple quotient, $L(V) = L_\mathcal{P}(V)$.

6. $\mathrm{End}_\mathfrak{G}(M(V)) \simeq \mathrm{End}_\mathfrak{A}(V)$. In particular, any $\mathfrak{G}$-endomorphism of $M(V)$ is scalar.

Proof. The second statement follows from PBW Theorem. The fourth statement follows from the universal property of the tensor product. The last statement follows from [D, Proposition 2.6.5]. The rest is trivial. $\square$

The forth statement of Proposition 4.2.1 is nothing more than the universal property of a GVM. One sees that Proposition 4.2.1 is almost analogous to Theorem 3.2.1, although one statement is missing. We will discuss it in the next Section.

The last basic property of a GVM, which will be explained in this Section is connected with its $\mathfrak{H}_\mathfrak{A}$-weight structure. Consider $U(\mathfrak{G})$ as an $\mathfrak{A}$-module under the adjoint action. Then this module is $\mathfrak{H}_\mathfrak{A}$-weight. Given an $\mathfrak{H}_\mathfrak{A}$ weight λ of $U(\mathfrak{G})$, we will denote by $U(\mathfrak{G})^\lambda$ the vector space, generated by all monomials in $U(\mathfrak{G})_\lambda$, which do not contain any factor from $\mathfrak{A}$. Clearly, $U(\mathfrak{G})^\lambda$ is finite-dimensional and is an $\mathfrak{A}$-submodule of $U(\mathfrak{G})$.

Lemma 4.2.1. *Let λ be the $\mathfrak{A}$-highest weight of $M_\mathcal{P}(V)$ and $\mu \in \mathfrak{H}_\mathfrak{A}^*$. Then $M_\mathcal{P}(V)_\mu$ is isomorphic to $U(\mathfrak{G})^{\mu-\lambda} \otimes V$ as a $\mathfrak{A}$-module.*

Proof. Since $M_\mathcal{P}(V) = U(\mathfrak{G}) \otimes_{U(\mathcal{P})} V$, we have $M_\mathcal{P}(V)_\mu = U(\mathfrak{G})^{\mu-\lambda} \otimes V$ as a vector space. If we consider $U(\mathfrak{G})^{\mu-\lambda}$ as a $\mathfrak{A}$-module under adjoint action, the equality above becomes an isomorphism of $\mathfrak{A}$-modules. $\square$

4.3 Generalized Harish-Chandra homomorphism

One of the most important tools in studing of GVMs is the so-called generalized Harish-Chandra homomorphism, defined and studied in [DOF3]. We will follow [DOF3] in this Section so the reader can consult [DOF3] for any technical details which will be missed.

Again, we fix a GVM $M(V)$ as in the previous section. The reason to introduce a generalization of the Harish-Chandra homomorphism is to study the action of $Z(\mathfrak{G})$ on $M(V)$. This will be explained in this Section.

Define $L(\mathfrak{A}) = U_0 \cap U(\mathfrak{G})\mathfrak{N}$. Then it is clear that $L(\mathfrak{A})$ is a two-sided ideal in U_0, $L(\mathfrak{A}) = U_0 \cap \mathfrak{N}(\mathfrak{A})U(\mathfrak{G})$ and $U_0 = L(\mathfrak{A}) \oplus (U(\mathfrak{A})_0 \otimes U(\mathfrak{H}_{\mathfrak{A}}))$. We define the *generalized Harish-Chandra homomorphism* (or *$\mathfrak{A}$-Harish-Chandra homomorphism*) $\varphi_{\mathfrak{A}}$ as the projection of U_0 on $U(\mathfrak{A})_0 \otimes U(\mathfrak{H}_{\mathfrak{A}})$ with respect to $L(\mathfrak{A})$. We note that $\varphi_{\mathfrak{A}}$ is completely determined by a subset $\pi(\mathfrak{A})$ of π and if $\pi(\mathfrak{A}) = \varnothing$, i.e. $\mathfrak{A} = 0$, then $\varphi_{\mathfrak{A}}$ coincides with the classical Harish-Chandra homomorphism φ (see [D, Chapter 7]).

The next lemma explains an importance of $\mathfrak{A}$-Harish-Chandra homomorphism in the study of $\mathfrak{A}$-highest weight modules, in particular, in the study of GVMs.

Lemma 4.3.1. *Let M be a $\mathfrak{A}$-highest weight module with the $\mathfrak{A}$-highest weight $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$. Then for any $u \in U_0$ and for any $v \in M_\lambda$ holds $uv = \varphi_{\mathfrak{A}}(u)v$.*

Proof. Follows directly form the definitions of $\mathfrak{A}$-highest weight and $\mathfrak{A}$-Harish-Chandra homomorphism. $\square$

Since $Z(\mathfrak{G}) \subset U_0$, we can compute the action of any $z \in Z(\mathfrak{G})$ on a GVM $M(V)$ directly by an application of Lemma 4.3.1. Hence, it is necessary to study the image of $Z(\mathfrak{G})$ under $\varphi_{\mathfrak{A}}$.

Let $\rho_{\mathfrak{A}}$ denote half the sum of all positive roots in $\Delta(\mathfrak{A})$ and let $\rho^{\mathfrak{A}} = \rho - \rho_{\mathfrak{A}}$. Denote by γ the automorphism of $S(\mathfrak{H})$ defined by $\gamma(f)(\lambda) = f(\lambda - \rho)$ (we say that γ is the *shift by* ρ). Analogously, let $\gamma_{\mathfrak{A}}$ be the automorphism of $S(\mathfrak{H}(\mathfrak{A}))$ defined by $\gamma_{\mathfrak{A}}(f)(\lambda) = f(\lambda - \rho_{\mathfrak{A}})$. Let $\gamma^{\mathfrak{A}} = \gamma|_{S(\mathfrak{H}_{\mathfrak{A}})}$.

Denote by i the restriction of the composition $(1 \otimes \gamma^{\mathfrak{A}}) \circ \varphi_{\mathfrak{A}} : U_0 \to U(\mathfrak{A})_0 \otimes U(\mathfrak{H}_{\mathfrak{A}})$ to $Z(\mathfrak{G})$ and by $i_{\mathfrak{A}}$ the restriction of $(\gamma_{\mathfrak{A}} \circ \varphi) \otimes 1 : U(\mathfrak{A})_0 \otimes S(\mathfrak{H}_{\mathfrak{A}}) \to S(\mathfrak{H}(\mathfrak{A})) \otimes S(\mathfrak{H}_{\mathfrak{A}})$ to $Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}})$. Let $m : S(\mathfrak{H}(\mathfrak{A})) \otimes S(\mathfrak{H}_{\mathfrak{A}}) \to S(\mathfrak{H})$ be a canonical isomorphism. For a semisimple Lie algebra A with a fixed Cartan subalgebra we will denote by a_A the canonical inclusion of $Z(A)$ into $U(A)_0$.

Theorem 4.3.1. *The following diagrams commute:*

1.

$$\begin{tikzcd}
U_0 \arrow[r, "(1\otimes\gamma^{\mathfrak{A}})\circ\varphi_{\mathfrak{A}}"] \arrow[d, "\gamma\circ\varphi"'] & U(\mathfrak{A})_0 \otimes S(\mathfrak{H}_{\mathfrak{A}}) \arrow[d, "(\gamma_{\mathfrak{A}}\circ\varphi)\otimes 1"] \\
S(\mathfrak{H}) & S(\mathfrak{H}(\mathfrak{A})) \otimes S(\mathfrak{H}_{\mathfrak{A}}) \arrow[l, "m"']
\end{tikzcd}$$

2.

$$\begin{tikzcd}
Z(\mathfrak{G}) \arrow[r, "i"] \arrow[d, "a_{\mathfrak{G}}"'] & Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}}) \arrow[r, "m\circ i_{\mathfrak{A}}"] \arrow[d, "a_{\mathfrak{A}}\otimes 1"] & S(\mathfrak{H}) \arrow[d, "\mathrm{id}"] \\
U_0 \arrow[r, "(1\otimes\gamma^{\mathfrak{A}})\circ\varphi_{\mathfrak{A}}"] & U(\mathfrak{A})_0 \otimes S(\mathfrak{H}_{\mathfrak{A}}) \arrow[r, "m\circ((\gamma_{\mathfrak{A}}\circ\varphi)\otimes 1)"] & S(\mathfrak{H})
\end{tikzcd}$$

3.

$$\begin{array}{ccc} Z(\mathfrak{G}) & \xrightarrow{i} & Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}}) \\ {\scriptstyle \gamma\circ\varphi}\downarrow & & \downarrow{\scriptstyle (\gamma_{\mathfrak{A}}\circ\varphi)\otimes 1} \\ S(\mathfrak{H})^W & \longrightarrow & S(\mathfrak{H})^{W(\mathfrak{A})} \end{array}$$

Proof. Direct verification, which uses Harish-Chanda isomorphism Theorem ([D, Theorem 7.4.5]). □

Corollary 4.3.1. $\varphi_{\mathfrak{A}}(Z(\mathfrak{G})) \subset Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}})$.

Hence, the action of $Z(\mathfrak{G})$ on $M(V)$ can be computed from the action of $Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}})$ on V. Since V is a simple $\mathfrak{A}'$-module, $\mathfrak{H}_{\mathfrak{A}}$ and hence $S(\mathfrak{H}_{\mathfrak{A}})$ acts on V by scalars, which correspond to $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$. Since V is a simple $\mathfrak{A}$-module, we can apply Quillen's Lemma ([D, Proposition 2.6.8]) and obtain that $Z(\mathfrak{A})$ acts on V by scalars. Hence $Z(\mathfrak{G})$ acts on $M(V)$ by scalars. This means that we have already proved the following result.

Corollary 4.3.2. *M(V) admits a central character.*

Having the map $i : Z(\mathfrak{G}) \to Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}})$, we can define a natural dual map $i^* : (Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}}))^* \to Z(\mathfrak{G})^*$. One more important consequence of Theorem 4.3.1 and the classical Galois theory is the following statement.

Corollary 4.3.3. *For each $\chi \in Z(\mathfrak{G})^*$ the set $(i^*)^{-1}(\chi)$ is finite. Moreover $|(i^*)^{-1}(\chi)|$ is less or equal to the index of $W(\mathfrak{A})$ in W.*

5 Motivation and Tools

In this Chapter we present some motivation for why one should be interested in studying GVMs. In particular, GVMs appear in different generalizations of the $\mathcal{O}$ category, e.g. in the category $\mathcal{O}_S$ introduced by Rocha-Caridi ([RC]) or in the category $\mathcal{O}^\alpha$ introduced by Coleman and Futorny ([CF]). GVMs also play an important role in the classification of all simple weight $\mathfrak{G}$-modules.

In the last two sections of this chapter we recall a Theorem of Kostant ([Ko2, Theorem 5.1]) and a localization of $U(\mathfrak{G})$ introduced by Mathieu ([Ma]). They will be essential tools in our study.

5.1 Classification of simple weight modules

The main problem in the representation theory is to classify the representations of some algebraic object. Usually, this problem is very difficult. For example, in some cases it is equivalent to the problem of classification of all representations for all objects of the same kind. Such a situation is called *wild* (of course this is not a formal definition, see, for example, [Dr]). Thus usually one restricts the general problem to classification of some reasonable classes of representations. One of the most natural classes of representations of a simple complex finite-dimensional Lie algebra is the class of all weight modules. For example, all finite-dimensional modules ($L(\lambda)$, $\lambda \in P^{++}$) are weight modules. Hence, the problem to clasify all weight (in particular, all simple weight) $\mathfrak{G}$-modules is natural and very important. In this section we describe, what is known here and how this problem relates to the study of GVMs.

First of all about the history of the question. A classification of all simple $\mathfrak{G}$-modules is known only for $\mathfrak{G} = \mathfrak{sl}(2, \mathbb{C})$ ([Bl] or [B] for a more general result). A classification of simple weight $\mathfrak{G}$-modules with finite-dimensional weight spaces was recently completed by Mathieu ([Ma]), some partial results towards it were obtained in [F3, BLF] (see also references therein). For weight modules with infinite-dimensional weight spaces the problem is still open, however there are some partial results, see [F3, DOF1].

Now, about the problem itself. One of the most important results for the solution of this problem is the Theorem, proved independently in some special cases by Fernando ([Fe]) and Futorny ([F4]) and then in a complete generality in [CFO] (using a computer). Recently an easy proof was found in [DMP]. To state the theorem, we have to introduce the notion of a dense module.

Recall that $\mathfrak{G}$ and $\mathfrak{H}$ are fixed. A weight $\mathfrak{G}$-module, M, is called *dense* if $\mathrm{supp}(M)$ coincides with the set $\lambda + \mathbb{Z}\Delta$ for some $\lambda \in \mathfrak{H}^*$. It is clear, that if M is indecomposable, then $\mathrm{supp}(M)$ is a subset of some $\lambda + \mathbb{Z}\Delta$. Hence, for an indecomposable M, dense means, that M has the maximal possible support. Now the mentioned result (which we will call the Fernando-Futorny Theorem) is the following.

Theorem 5.1.1. *Let M be a simple weight $\mathfrak{G}$-module. Then M is either dense or there exists a parabolic subalgebra $\mathcal{P} \supset \mathfrak{B}$ in Γ with the Levi decomposition $\mathcal{P} = \mathfrak{A}' \oplus \mathfrak{N}$ and a simple dense module V over $\mathfrak{A}'$, such that $M \simeq L_{\mathcal{P}}(V)$.*

We note, that the module V, given in Theorem 5.1.1 is automatically $\mathfrak{H}(\mathfrak{A})$-weight. This result is quite non-trivial, especially if one looks at [CFO] and the computer list presented there. The proof in [DMP] is tricky and uses both ideas from [Fe] and [F4].

In principal, Theorem 5.1.1 decomposes our classification problem into two parts. The first one: to classify all dense modules. Exactly this was done in [Ma] for weight modules with finite-dimensional weight spaces. The second problem is to study modules $L_{\mathcal{P}}(V)$, or more general, modules $M_{\mathcal{P}}(V)$, which are GVMs, associated with $\mathfrak{H}(\mathfrak{A})$-weight modules. We also note, that some information about $L_{\mathcal{P}}(V)$, in the case when V has finite-dimensional $\mathfrak{A}$-weight spaces was also obtained in [Ma]. In fact, the character of $L_{\mathcal{P}}(V)$ was computed, but not directly, only by reducing the problem to the character of some $L(\lambda)$.

In principal, the claim that Theorem 5.1.1 reduces the classification of simple weight modules do the classification of simple dense modules is a bit unfair, since given $\mathcal{P}$ and a simple $\mathfrak{A}$-weight module V the construction of $L_{\mathcal{P}}(V)$ may be a very difficult problem. This is not easy even for Verma modules, since it involves at least a description of a maximal submodule in $M(\lambda)$, which is far from being trivial in general. Anyway, Theorem 5.1.1 presents a good motive to study GVMs and the corresponding simple quotients.

5.2 Category $\mathcal{O}_S$

The Verma modules appear as important intermediate objects in the category $\mathcal{O}$. Analogously, GVMs appear in different generalizations of $\mathcal{O}$. One of them, which we will describe in this Section was introduced and studied by Rocha-Caridi in [RC]. We fix notation from Section 4.1. Original notation $\mathcal{O}_S$ relates to a subset S of π, which defines the semisimple part of the Levi factor of $\mathcal{P}$. In our situation $S=\pi(\mathfrak{A})$, but we retain the classical notation for the whole category.

Define $\mathcal{O}_S$ to be the full subcategory of the category of $\mathfrak{G}$-modules consisting of the modules M such that

- M is finitely generated;
- M is a direct sum of simple finite-dimensional $\mathfrak{A}'$-modules, when viewed as an $\mathfrak{A}'$-module;
- M is $\mathfrak{N}$-finite, i.e. $U(\mathfrak{N})v$ is finite-dimensional for any $v\in M$.

We note, that the second condition is equivalent to saying, that M is $\mathfrak{H}$-diagonalizable and is a direct sum of finite-dimensional $\mathfrak{A}$-modules, when viewed as an $\mathfrak{A}$-module.

It happened that $\mathcal{O}_S$ has properties, similar to those of $\mathcal{O}$.

Theorem 5.2.1. *1. $\mathcal{O}_S$ is a full subcategory of $\mathcal{O}$.*

2. $\mathcal{O}_S$ is closed under taking submodules, quotients, finite direct sums and under tensoring with finite-dimensional modules.

3. *All GVMs $M_{\mathcal{P}}(V)$, where V is finite-dimensional, belong to $\mathcal{O}_S$.*

4. *$L_{\mathcal{P}}(V)$, V is simple finite-dimensional, exhaust the set of simple modules in $\mathcal{O}_S$.*

5. *Any module in $\mathcal{O}_S$ has a composition series.*

6. *$\mathcal{O}_S = \oplus_{\chi \in Z(\mathfrak{G})^*} \mathcal{O}_S(\chi)$, where $\mathcal{O}_S(\chi)$ is a full subcategory consisting of all modules, which are annihilated by a big enough power of $z - \chi(z)$ for all $z \in Z(\mathfrak{G})$.*

As in $\mathcal{O}$, category $\mathcal{O}_S$ has a block decomposition, with each block being a module category over a finite-dimensional algebra.

Theorem 5.2.2. *1. $\mathcal{O}_S$ has enough projective modules (i.e. any module in $\mathcal{O}_S$ is a quotient of a projective module in $\mathcal{O}_S$).*

2. *There is a bijection between simple modules and indecomposable projective modules in $\mathcal{O}_S$. We will denote by $P(L)$ the projective cover of a simple module $L \in \mathcal{O}_S$.*

3. *Each $\mathcal{O}_S(\chi)$ is equivalent to the category of (finite-dimensional) modules over a finite-dimensional algebra.*

And finally, there is an analogue of BGG reciprocity.

Theorem 5.2.3. *1. Any projective module in $\mathcal{O}_S$ admits a generalized Verma flag (i.e. a filtration, whose subquotients are $M_{\mathcal{P}}(V)$, V finite-dimensional).*

2. *For any finite-dimensional $\mathfrak{A}'$-modules V_1 and V_2 holds $[P(L_{\mathcal{P}}(V_1)) : M_{\mathcal{P}}(V_2)] = (M_{\mathcal{P}}(V_2) : L_{\mathcal{P}}(V_1))$, where $[P(L_{\mathcal{P}}(V_1)) : M_{\mathcal{P}}(V_2)]$ denotes the number of occurrences of $M_{\mathcal{P}}(V_2)$ in a generalized Verma flag of $P(L_{\mathcal{P}}(V_1))$. In particular, the last is a well-defined number (i.e. does not depend on a generalized Verma flag).*

The proofs of all theorems can be found in [RC], for a more general context see also [RCW2, MP].

The finite-dimensional algebras, arising in $\mathcal{O}_S$ are related the the algebras, arizing from $\mathcal{O}$. In fact, they are Koszul dual to algebras corresponding to the singular blocks of $\mathcal{O}$ ([BGS]). From Theorem 5.2.2 and Theorem 5.2.3 it follows easily, that the blocks of $\mathcal{O}_S$ correspond to quasi-hereditary algebras, so one can construct, for example, tilting modules. But so far this theory is not completed.

We see, that GVMs naturally occur and play an important role in the category $\mathcal{O}_S$.

5.3 Category $\mathcal{O}^\alpha$

GVMs appear also in another generalization of $\mathcal{O}$, proposed by Coleman and Futorny in [CF]. Assume that $\mathfrak{A}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Then $\pi(\mathfrak{A})$ contains the unique element, say α. This is why we will denote the category, associated with such $\mathcal{P}$ by $\mathcal{O}^\alpha$. To proceed we need one more notation, introduced in [CF]. A weight $\mathfrak{G}$-module M with finite-dimensional weight spaces will be called *α-stratified* provided X_α and $X_{-\alpha}$ act injectively (and hence bijectively) on M. We also fix the quadratic Casimir operator C of $U(\mathfrak{G})$ ([Ka, p.22]).

We define $\mathcal{O}^\alpha$ the be the full subcategory of the category of $\mathfrak{G}$-modules, consisting of all modules M such that

- M is finitely generated;
- M is $\mathfrak{H}$-diagonalizable;
- M is α-stratified;
- M is $\mathfrak{N}$-finite, i.e. $U(\mathfrak{N})v$ is finite-dimensional for all $v \in M$;
- M is C-diagonalizable.

The fifth condition seems to be quite disputable. The reason to add it in the definition above was to avoid the self-extensions of GVMs inside $\mathcal{O}^\alpha$. But it happens that exacly this condition causes difficulty in naturally generalizing some of the properties of category $\mathcal{O}$ to $\mathcal{O}^\alpha$. We improve the situation later in Chapter 12. Nevertheless, now we list the basic properties of $\mathcal{O}^\alpha$.

Theorem 5.3.1. *1. $\mathcal{O}^\alpha$ is closed under taking submodules, quotients and finite direct sums.*

2. All GVMs $M_{\mathcal{P}}(V)$, where V is a simple α-stratified $\mathfrak{A}$-module, belong to $\mathcal{O}^\alpha$.

3. $L_{\mathcal{P}}(V)$, where V is a simple α-stratified $\mathfrak{A}$-module, exhaust the set of all simple modules in $\mathcal{O}^\alpha$.

4. Any module in $\mathcal{O}^\alpha$ has a composition series.

5. $\mathcal{O}^\alpha = \oplus_{\chi \in Z(\mathfrak{G})^*} \mathcal{O}^\alpha(\chi)$, where $\mathcal{O}^\alpha(\chi)$ is a full subcategory consisting of all modules, which are annihilated by a big enough power of $z - \chi(z)$ for all $z \in Z(\mathfrak{G})$.

The proof can be found in [CF, Section 4].

First, we note that, unlike $\mathcal{O}$ and $\mathcal{O}_S$, the category $\mathcal{O}^\alpha$ is not closed under tensoring with finite-dimensional modules. The reason is that such a tensor product does not preserve the fifth condition (but it preserves all other conditions).

Secondly, all simple α-stratified $\mathfrak{A}$-modules can be easily classified. This is a part of classical $\mathfrak{sl}(2, \mathbb{C})$ theory. We will need this later, so now we present a construction of all such modules.

Fix $a, b \in \mathbb{C}$ and recall the Casimir element $\mathfrak{c} = (H_\alpha + 1)^2 + 4X_{-\alpha}X_\alpha$ of $U(\mathfrak{A})$ (see Section 2.7). Denote by $V(a, b)$ the $\mathfrak{A}$-module with the basis v_i, $i \in \mathbb{Z}$ and the action of generators of $\mathfrak{A}$ defined as follows:

$$H_\alpha v_i = (a + 2i)v_i, \qquad X_{-\alpha} v_i = v_{i-1}, \qquad X_\alpha v_i = \frac{1}{4}(b - (a + 2i + 1)^2)v_{i+1}.$$

We note that b is a unique eigenvalue of $\mathfrak{c}$ on $V(a, b)$.

Proposition 5.3.1. *1. $V(a,b)$ is an indecomposable dense weight module of length $\leqslant 3$ with one dimensional weight spaces.*

2. $V(a,b)$ is simple if and only if $V(a,b)$ is α-stratified if and only if $b \neq (a+2j+1)^2$ for all $j \in \mathbb{Z}$.

3. The set of simple $V(a,b)$ exhaust the set of simple α-stratified $\mathfrak{A}$-modules.

4. $V(a,b) \simeq V(a',b')$ if and only if $b = b'$ and $a = a' + 2k$ for some $k \in \mathbb{Z}$.

Proof. The first and the second statements follow directly form the definition of $V(a,b)$. To prove the rest we recall one known fact ([DOF3, Lemma 3]), that for any simple $\mathfrak{H}(\mathfrak{A})$-weight $\mathfrak{A}$-module M any weight space M_λ, $\lambda \in \mathfrak{H}(\mathfrak{A})^*$ is a simple $U(\mathfrak{A})_0$-module and any simple $\mathfrak{H}(\mathfrak{A})$-weight $U(\mathfrak{A})_0$-module N has a unique extension to a simple $\mathfrak{H}(\mathfrak{A})$-weight $\mathfrak{A}$-module. The algebra $U(\mathfrak{A})_0$ is generated by H_α and $\mathfrak{c}$ and thus is commutative. Hence, any simple $\mathfrak{H}(\mathfrak{A})$-weight $U(\mathfrak{A})_0$-module is one-dimensional and thus, coincides with some $V(a,b)_\lambda$ for some $a, b \in \mathbb{C}$ and $\lambda \in \mathfrak{H}(\mathfrak{A})^*$. Now the third statement follows from the first and the second ones. The rest is obvious. $\square$

Using Proposition 5.3.1, one can characterize $V(a,b)$ as a unique $\mathfrak{A}$-module, having the following properties:

- a is an eigenvalue of H_α on $V(a,b)$;
- b is the eigenvalue of $\mathfrak{c}$ on $V(a,b)$;
- $V(a,b)$ is dense;
- $X_{-\alpha}$ acts bijectively (or injectively) on $V(a,b)$;
- all weight spaces of $V(a,b)$ are one-dimensional.

We will need this characterization later.

We return to the category $\mathcal{O}^\alpha$. We have already established its basic properties. The second step is the result about finite-dimensional algebras arising from $\mathcal{O}^\alpha$.

Theorem 5.3.2. *1. $\mathcal{O}^\alpha$ has enough projective modules (i.e. any module in $\mathcal{O}^\alpha$ is a quotient of a projective module in $\mathcal{O}^\alpha$).*

2. There is a bijection between simple modules and indecomposable projective modules in $\mathcal{O}^\alpha$.

3. Each $\mathcal{O}^\alpha(\chi)$ can be decomposed into a direct sum of full subcategories, each of which is equivalent to the category of (finite-dimensional) modules over a finite-dimensional algebra.

The proof of this fact is essentialy [CF, Theorem 4.7]. Again, one can see some differences between $\mathcal{O}$ and $\mathcal{O}_S$. The direct summand $\mathcal{O}^\alpha(\chi)$ of $\mathcal{O}^\alpha$ has infinitely many simple objects, hence one should decompose it further in order to reduce the situation to finite-dimensional algebras. The fifth condition from the definition of $\mathcal{O}^\alpha$ causes the fact that the finite-dimensional algebras arising from $\mathcal{O}^\alpha$ are not quasi-hereditary in general (at least in a natural presentation). The problem is that the projective modules in $\mathcal{O}^\alpha$ do not always admit a generalized Verma flag (i.e. a filtration, with quotients of the form $M_{\mathcal{P}}(\mathrm{V(a,b)})$). We will improve this situation in Chapter 12.

Category $\mathcal{O}^\alpha$ is another generalization of $\mathcal{O}$, in which some GVMs appear as natural objects, and whose simple objects are exhaust by the unique simple quotients of GVMs.

We hope that we have presented enough motivation to claim that the study of GVMs is an interesting and important problem in representation theory of Lie algebras.

5.4 Kostant's Theorem

We have already mentioned that a lot of information about $\mathfrak{G}$-modules, in particular, about Verma modules or GVMs, can be obtained, studying the action of $Z(\mathfrak{G})$ on these modules. For example, this was a motivation for introducing the generalized Harish-Chandra homomorphism in Section 4.3 (and in [DOF3]). Further, one of the most powerful technical tools for studying Verma modules is the Jantzen *translation functor*, which is the composition of the tensor product with a finite-dimensional module followed by the projection on $\mathcal{O}(\chi)$. This motivates the exploration of how a central character behaves under tensoring with a finite-dimensional module. This is the content of a famous result of Kostant ([Ko2, Theorem 5.1]). This Theorem will be an important statement for us and we present a complete formulation of it, and will call it Kostant's Theorem in what follows.

We recall, that, according to Quillen's Lemma, any simple $\mathfrak{G}$-module M has a central character. This means that for some $\chi = \chi_M \in Z(\mathfrak{G})^*$ holds: $zv = \chi(z)v$ for all $v \in M$, $z \in Z(\mathfrak{G})$. We also recall, that according to Theorem 3.2.2, for any $\chi \in Z(\mathfrak{G})^*$ there exists $\lambda \in \mathfrak{H}^*$ such that $\chi = \chi_\lambda$, i.e. χ is the central character of $M(\lambda)$.

Theorem 5.4.1. *Let M be a simple $\mathfrak{G}$-module having a central character $\chi_M = \chi_\lambda$ for some $\lambda \in \mathfrak{H}^*$ and F be a finite-dimensional $\mathfrak{G}$-module. Then for any $z \in Z(\mathfrak{G})$ the element*

$$\prod_{\mu \in \mathrm{supp}(F)} (z - \chi_{\lambda+\mu}(z))$$

annihilates $M \otimes F$.

We remark, that under our notations $\mathrm{supp}(F)$ is an ordinary set and not a multi-set. Hence any weight of F is taken only once in the product above. A proof can be found in [Ko2]. We emphasize one useful corollary of this Theorem.

Corollary 5.4.1. *Keep the notation of Theorem 5.4.1 and assume that for any $\mu_1, \mu_2 \in \mathrm{supp}(F)$, $\mu_1 \neq \mu_2$ holds $\chi_{\lambda+\mu_1} \neq \chi_{\lambda+\mu_2}$. Then $M \otimes F$ is $Z(\mathfrak{G})$-diagonalizable.*

In particular, the last means that for almost all (with respect to, say, Lebesgue measure) $\lambda \in \mathfrak{H}^*$ the module $M(\lambda) \otimes F$ is $Z(\mathfrak{G})$-diagonalizable for all finite-dimensional F.

Theorem 5.4.1 can be easily applied for example, to the modules $V(a, b)$ introduced in Section 5.3. Fix $a, b \in \mathbb{C}$ and a finite-dimensional module F of dimension n. Since F is n-dimensional, its highest weight is $n-1$ and the eigenvalue of $\mathfrak{c}$ on F equals n^2. The eigenvalue of $\mathfrak{c}$ on $V(a, b)$ equals b, which coincides with the eigenvalue of $\mathfrak{c}$ on the Verma module $M(\sqrt{b}-1)$ (since $\sqrt{b}$ is defined up to a sign, there can exist, in general, two Verma modules having the same eigenvalue of $\mathfrak{c}$). According to Theorem 5.4.1, the element

$$\prod_{i=0}^{n-1}(\mathfrak{c}-(\sqrt{b}-n+2i+1)^2)$$

annihilates $F \otimes V(a, b)$. Hence, the only subquotient that can appear in $F \otimes V(a, b)$ are subquotients of $V(a, (\sqrt{b}-n+2i+1)^2)$, $i = 0, 1, \dots, n-1$. Moreover, if $V(a, b)$ is simple and b is not the square of an integer, the module $F \otimes V(a, b)$ is completely reducible. One more thing, if $V(a, b)$ is simple, then any indecomposable submodule of $F \otimes V(a, b)$ is of the length 1 or 2, and any such submodule of the length 2 is a self-extension of some $V(a, k^2)$, $k \in \mathbb{Z}$.

Comparing the dimensions of the weight spaces one can show (we will do it later on in Section 6.7), that $F \otimes V(a, b)$ has precisely $\dim(F)$ subquotients (counted with multiplicities), which has the form $V(a, b')$. This is a nice property, which has no analogue for a finite-dimensional substitution of $V(a, b)$. Really, the length of $F \otimes E$ for F, E finite-dimensional can be smaller than $\dim(F)$.

We also note that for a general simple M it is not known if the module $F \otimes M$ (F, finite-dimensional) has finite length. This was proved by Kostant ([Ko2]) for simple Harish-Chandra modules (i.e. modules coming from group representations), but in general the question is open. We do not know any counterexample, but we also can not prove this, so we will have to be careful later, especially in Chapter 12.

5.5 Mathieu's localization

Another important tool for us will be a special localization of the universal enveloping algebra together with a family of automorphisms, introduced by Mathieu in [Ma]. As mentioned above, Mathieu studied weight modules with finite-dimensional weight spaces, so his construction works perfectly only in that case. This means that we will not be able to apply it everywhere. Nevertheless, we will use it for the study of α-stratified modules (it can also be applied in a more general situation, which we will not discuss here). We also note, that we will not present Mathieu's construction in complete generality. Since we are going to use it only for α-stratified modules, we will define everything to cover only this case. For the general case we refer the reader to [Ma].

So, assume that $\mathfrak{A}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ and $\pi(\mathfrak{A}) = \{\alpha\}$. The main object of our interests here will be the element $X_{-\alpha}$. Since $\mathfrak{G}$ is a simple finite-dimensional complex Lie algebra, $X_{-\alpha}$ is locally ad-nilpotent on $\mathfrak{G}$ and thus on $U(\mathfrak{G})$. Hence, the multiplicative set

S of powers of $X_{-\alpha}$ in $U(\mathfrak{G})$ satisfies the Ore conditions for localizability (see also [Ma, Lemma 4.2]). Denote by U_S the corresponding localization of U. The following proposition presents a nice family of automorphisms of U_S. This is a partial case of [Ma, Lemma 4.3] and we have just copied the proof from there.

Proposition 5.5.1. *There exists a unique family of automorphisms, $\theta_x : U_S \to U_S$, $x \in \mathbb{C}$, such that*

- *$\theta_x(u) = X_{-\alpha}^x u X_{-\alpha}^{-x}$ for all integer x;*
- *the map $x \mapsto \theta_x(u)$ is polynomial in x for any $u \in U_S$.*

Proof. Clearly, $X_{-\alpha}$ is locally ad-nilpotent on U_S. Thus for a fixed $u \in U_S$ there is a positive integer, N, such that $\mathrm{ad}\,(X_{-\alpha})^{N+1}(u) = 0$. For $x \geqslant 0$ set

$$\theta_x(u) = \sum_{i=0}^{N} (-1)^i \binom{x}{i} \mathrm{ad}\,(X_{-\alpha})^i(u) X_{-\alpha}^{-i}.$$

As the binomial coefficient can be extended to a polynomial on $\mathbb{C}$, we can polynomially extend θ_x for all $x \in \mathbb{C}$. Uniqueness follows from the uniqueness of the polynomial extension. $\square$

Proposition 5.5.1 allows one to twist special $U(\mathfrak{G})$-modules by θ_x. Indeed, let M be a $\mathfrak{G}$-module on which $X_{-\alpha}$ acts bijectively (for example, M can be α-stratified). Then M extends to a U_S-module $\hat{M}$ in a natural way and we can apply θ_x, $x \in \mathbb{C}$ to obtain the twisted U_S-module $\theta_X(\hat{M})$. Then we can restrict the result on $U(\mathfrak{G})$ and obtain a new $U(\mathfrak{G})$ module, which we will denote by $\theta_x(M)$. We note, that if we start from an α-stratified module M the result $\theta_x(M)$ is not necessary α-stratified. This means that it is possible that the action of X_α on $\theta_x(M)$ is not injective (the action of $X_{-\alpha}$ on $\theta_x(M)$ is bijective by construction). In a more general context one can start from an arbitrary module M and use an induction from $U(\mathfrak{G})$ to U_S, but if $X_{-\alpha}$ acts not injectively on M, some element of M will be annihilated during the induction process. We emphasize the following important property of θ_x.

Lemma 5.5.1. *Let M be a weight $\mathfrak{G}$-module on which $X_{-\alpha}$ acts bijectively and $\lambda \in \mathfrak{H}^*$. Then for any $h \in \mathfrak{H}$, $v \in M_\lambda$ and $x \in \mathbb{C}$ holds $\theta_x(h)v = (\lambda + x\alpha)(h)v$. In particular, $\mathrm{supp}(\theta_x(M)) = \mathrm{supp}(M) + x\alpha$.*

Proof. Suppose that $x \in \mathbb{Z}$. Then $\theta_x(h)v = X_{-\alpha}^x h X_{-\alpha}^{-x} v = (h + x\alpha(h))v = (\lambda + x\alpha)(h)v$. Since θ_x is polynomial in x, $\theta_x(h)v = (\lambda + x\alpha)(h)v$ for any $x \in \mathbb{C}$. The statement about the support is now trivial. $\square$

Lemma 5.5.2. *$\theta_x(z) = z$ for any $x \in \mathbb{C}$ and any $z \in Z(\mathfrak{G})$.*

Proof. Again the statement is trivial for $x \in \mathbb{Z}$. For arbitrary x everything follows now from polynomiality of θ_x. $\square$

As an example, one can take $\mathfrak{G} = \mathfrak{A}$ and $M = V(a, b)$, the module constructed in the Section 5.3. Then $X_{-\alpha}$ acts bijectively on $V(a, b)$ (but $V(a, b)$ is not necessary α-stratified) and one can consider $\theta_x(V(a, b))$. Clearly from Proposition 5.3.1 it follows that $\theta_x(V(a, b)) \simeq V(a', b')$ for some $a', b' \in \mathbb{C}$. By Lemma 5.5.2, $b' = b$. By Lemma 5.5.1, $a' = a + 2x$. Hence $\theta_x(V(a, b)) \simeq V(a + 2x, b)$. Further we will use θ_x to α-stratified module, and will see that a lot of information can be derived from the last calculation with θ_x.

6 An analogue of the BGG Theorem, I

We begin the study of GVMs with an attempt to generalize the BGG Theorem about the existence of non-trivial homomorphisms between Verma modules (Theorem 3.3.1). In this Chapter we cover the case $\mathfrak{A} \simeq \mathfrak{sl}(2,\mathbb{C})$, which was worked out in [M1, M3, FM1, KM1]. We note, that the results of this chapter will be generalized and reproved in Chapter 9 in a much easier way. Nevertheless we will present the scheme of the original proof here, because the same scheme can be applied for another situation (Chapter 8), which will not be covered in Chapter 9.

As a first step, we present one general sufficient condition for simplicity of $M_{\mathcal{P}}(V)$ ([KM4]).

6.1 Naive sufficient condition for simplicity

In this section with each GVM we associate a Verma module. We will prove (Theorem 6.1.1), that under some conditions, the simplicity of this Verma module implies the simplicity of the original GVM. Finally, we conjecture that this is the case for any GVM. This is the easiest result about the structure of a GVM and it is based on a study of properties of the generalized Harish-Chandra homomorphism, combined with Kostant's Theorem. We retain the notation from Chapter 4.

Consider a GVM, $M_{\mathcal{P}}(V)$. Let χ_V be the central character of V, where V is considered as a $\mathfrak{A}$-module. Assume that $\mathfrak{H}_{\mathfrak{A}}$ acts on V via $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$ (or, equivalently, λ is the $\mathfrak{A}$-highest weight of $M_{\mathcal{P}}(V)$). Suppose that $\chi_V = \chi_\mu$, $\mu \in \mathfrak{H}(\mathfrak{A})^*$ and μ belongs to the closure of the antidominant Weyl chamber. Now we can consider the module $M_{\mathcal{P}}(M(\mu))$, which is, in fact, the Verma module $M(\nu)$, $\nu \in \mathfrak{H}^*$, such that $p_{\mathfrak{A}}(\nu) = \lambda + \rho^{\mathfrak{A}}$ and $\nu - p_{\mathfrak{A}}(\nu) = \mu$. Set $M(\nu) = \mathfrak{f}(M_{\mathcal{P}}(V))$. We remark, that $M(\mu)$ is a simple Verma module over $\mathfrak{A}$, since μ is chosen to be in the closure of the antidomimant Weyl chamber.

Theorem 6.1.1. *Suppose that* $\mathrm{supp}(M(\nu)) \cap W(\nu) - \rho = \{\nu - \rho\}$. *Then* $M_{\mathcal{P}}(V)$ *is simple.*

We remark, that $\mathrm{supp}(M(\nu)) \cap W(\nu) - \rho = \{\nu - \rho\}$ automatically implies that $M(\nu)$ is simple, according to the BGG Theorem (Theorem 3.3.1).

Proof. In this proof all weight subspaces will be taken with respect to the abelian algebra $\mathfrak{H}_{\mathfrak{A}}$.

Consider $M(\nu)$ as a GVM $M_{\mathcal{P}}(M(\mu))$. Suppose that $M_{\mathcal{P}}(V)$ is not simple and has a non-trivial submodule N. Clearly, N contains a non-zero $\mathfrak{A}$-primitive element $v \in M_{\mathcal{P}}(V)_\xi$, for some $\xi \in \mathfrak{H}_{\mathfrak{A}}^*$ such that $\lambda \neq \xi$. By Lemma 4.2.1, there exists a finite-dimensional $\mathfrak{A}$-module F, such that $M_{\mathcal{P}}(V)_\xi$ is isomorphic to $F \otimes V$, as an $\mathfrak{A}$-module. Applying Kostant's Theorem, can assume that $Z(\mathfrak{A})$ acts on v via some character, say χ.

Consider a $\mathfrak{A}$ module $M_{\mathcal{P}}(M(\mu))_\xi$. Clearly, it is isomorphic to $F \otimes M(\mu)$ as a $\mathfrak{A}$-module. Moreover, since $M(\mu)$ is a Verma module, then $M(\mu) \otimes F$ has a Verma flag ([D, Lemma 7.6.14]). Comparing the filtration in [D, Lemma 7.6.14] with Kostant's Theorem, we conclude that necessarily one of the Verma sub-quotients of $M_{\mathcal{P}}(M(\mu))_{\xi}$, say $M(\eta)$, has the central character χ.

Consider the GVM $M_{\mathcal{P}}(M(\eta))$ with the $\mathfrak{A}$-highest weight ξ (clearly, $M_{\mathcal{P}}(M(\eta))$ is a Verma module over $\mathfrak{G}$), and let w be its canonical generator. Since w is a $\mathfrak{A}$-primitive element, we can calculate the action of $Z(\mathfrak{G})$ on w in terms of χ and ξ using the generalized Harish-Chandra homomorphism (see Lemma 4.3.1). Moreover, the same can be done for v. This implies, that the central characters of $M_{\mathcal{P}}(M(\eta))$ and $M_{\mathcal{P}}(V)$ and thus the central characters of $M_{\mathcal{P}}(M(\eta))$ and $M(\nu)$ coincide. Let $M_{\mathcal{P}}(M(\eta)) \simeq M(\nu')$ for some $\nu' \in \mathfrak{H}^*$. From Theorem 3.2.2 we immediately obtain, that $\nu' \in W(\nu)$, which contradicts our conditions. This completes the proof. □

Conjecture 6.1.1. *If $\mathfrak{f}(M_{\mathcal{P}}(V))$ is simple then $M_{\mathcal{P}}(V)$ is simple.*

We also believe that this conjecture is true if one replaces $\mathfrak{G}$ with a symmetrizable Kac-Moody Lie algebra and $\mathcal{P}$ with a standard parabolic subalgebra of $\mathfrak{G}$, such that $\mathfrak{A}$ is finite-dimensional.

6.2 The category $\mathcal{K}^{\alpha}$

In this Section we establish some basic properties of weight GVMs in the case $\mathfrak{A} \simeq \mathfrak{sl}(2,\mathbb{C})$. As in Section 5.3, we assume $\pi(\mathfrak{A}) = \{\alpha\}$. If V is a simple weight $\mathfrak{A}$-module without highest or lowest weights (otherwise we will have $M_{\mathcal{P}}(V)$ is a Verma module), then $V \simeq V(a,b)$ for some $a, b \in \mathbb{C}$, $b \neq (a+2i+1)^2$ for all $i \in \mathbb{Z}$, according to Proposition 5.3.1. In this case both X_{α} and $X_{-\alpha}$ act bijectively on $V(a,b)$. By Lemma 4.2.1, any $M_{\mathcal{P}}(V(a,b))_{\mu}$, $\mu \in \mathfrak{H}^*_{\mathfrak{A}}$ is isomorphic to $F \otimes V(a,b)$ for some finite-dimensional $\mathfrak{A}$-module F. Let $V(a',b')$ be a simple subquotient of $F \otimes V(a,b)$. If we recall our example from Section 5.4, we will see, that $b' \neq (a'+2i+1)^2$ for all $i \in \mathbb{Z}$. Hence $M_{\mathcal{P}}(V(a,b))$ is α-stratified.

It will be more convenient to study modules $M_{\mathcal{P}}(V(a,b))$ for all $a, b \in \mathbb{C}$. From the previous paragraph it follows, that $M_{\mathcal{P}}(V(a,b))$ is α-stratified if and only if $V(a,b)$ is α-stratified. To be able to work with any $a, b \in \mathbb{C}$ we introduce a new category of $\mathfrak{G}$-modules. Let $\mathcal{K}^{\alpha}$ denote the full subcategory of the category of all $\mathfrak{G}$-modules, which consists of all weight modules with finite-dimensional weight spaces, on which $X_{-\alpha}$ acts bijectively. Clearly, all $M_{\mathcal{P}}(V(a,b))$ are objects of $\mathcal{K}^{\alpha}$. In this section, we will study the properties of $M_{\mathcal{P}}(V(a,b))$ inside the category $\mathcal{K}^{\alpha}$.

Lemma 6.2.1. *K^{α} is closed under taking kernels and cokernels of morphisms and under taking finite direct sums.*

Proof. Let $M_1, M_2 \in K^{\alpha}$ and $f : M_1 \to M_2$ be a homormorphism. First we prove that $X_{-\alpha}$ acts bijectively on the kernel of f. Indeed, $X_{-\alpha}$ acts injectively on it, since it is a submodule of M_1. Let $x \in M_1$, $f(x) = 0$ and $y \in M_1$ be such that $X_{-\alpha}y = x$. Then $X_{-\alpha}f(y) = f(x) = 0$ and hence $f(y) = 0$, since $X_{-\alpha}$ acts injectively on M_2. This means that $X_{-\alpha}$ acts bijectively on the kernel of f.

Since M_1 has finite-dimensional weight spaces, we obtain, that $X_{-\alpha}$ also acts bijectively on $M_1/\ker(f)$. Thus, $X_{-\alpha}$ acts bijectively on the image of f. Finally, since M_2 has finite-dimensional weight spaces, we obtain that $X_{-\alpha}$ acts bijectively on the cokernel of f. $\square$

It is convenient to reparameterize our GVMs. The module $M_{\mathcal{P}}(V(a,b))$ is uniquely determined by the $\mathfrak{H}$-weight $\lambda-\rho$ of one of its $\mathfrak{A}$-primitive generators v and by the mentioned $b=p^2$ for some $p\in\mathbb{C}$, which is the eigenvalue of $\mathfrak{c}$ on v. We set $M(\lambda,p)=M_{\mathcal{P}}(V(a,b))$. Under this notation we always have $M(\lambda,p)\simeq M(\lambda,-p)\simeq M(\lambda+k\alpha,p)$ for all $k\in\mathbb{Z}$. Clearly, $M(\lambda,p)$ is α-stratified if and only if $\pm p\neq\lambda(H_\alpha)+2i$ for all $i\in\mathbb{Z}$. We can always choose λ such that $(\lambda-\rho)(H_\alpha)=a$. Sometimes it will be convenient to fix such λ.

Lemma 6.2.2. *Each $M(\lambda,p)$ has a unique simple quotient object $L(\lambda,p)$ in $\mathcal{K}^\alpha$. Moreover, $L(\lambda,p)$ is a simple $\mathfrak{G}$-module if and only if $M(\lambda,p)$ is α-stratified.*

Proof. The proof of the first statement is analogous to the proof of the corresponding statement for Verma modules. The second statement is obvious. $\square$

Let v_i be the basis of $V(a,b)$, defined in Section 5.3. Then, by Proposition 4.2.1 (the second part of it can be easily extended on the case of a non-simple V), $1\otimes v_i$, $i\in\mathbb{Z}$ is a $U(\mathfrak{N}(\mathfrak{A}))$-basis of $M(\lambda,p)$.

Lemma 6.2.3. *For any $w\in M(\lambda,p)$ there exists $i\in\mathbb{Z}$ and $u\in U(\mathfrak{N}_-)$ such that $w=uv_i$.*

Proof. For some $k\in\mathbb{N}$, $a_j\in\mathbb{C}$ and $u_j\in U(\mathfrak{N}(\mathfrak{A}))$, $-k\leqslant j\leqslant k$, we have

$$w=\sum_{j=-k}^{k}(a_ju_jv_j)=\sum_{j=-k}^{k}(a_ju_jX_{-\alpha}^{k-j}v_k)=\left(\sum_{j=-k}^{k}a_ju_jX_{-\alpha}^{k-j}\right)v_k.$$

This completes the proof. $\square$

Proposition 6.2.1. *1. Any two non-trivial subobjects (in K^α) of $M(\lambda,p)$ have a non-trivial intersection. In particular, $M(\lambda,p)$ contains a unique minimal subobject.*

2. $\dim\operatorname{Hom}(M(\lambda,p),M(\mu,q))\leqslant 1$ *and any non-zero homomorphism from this space is a monomorphism.*

Proof. Follows from Lemma 6.2.3 and the fact that $U(\mathfrak{N}_-)$ has no zero divisors by standard arguments (see, for example [D, Proposition 7.6.3,Theorem 7.6.6]). $\square$

6.3 A_2 case

In this Section we explain the results from [F2]. This was the first attempt to study the structure of GVMs in the simplest case, when $\mathfrak{G}$ is isomorphic to $\mathfrak{sl}(3,\mathbb{C})$. In this case we can assume that $\mathfrak{A}\simeq\mathfrak{sl}(2,\mathbb{C})$, $\pi(\mathfrak{A})=\{\alpha\}$, $\pi=\{\alpha,\beta\}$. We retain notation from Section 6.1. Our main result here is the following statement.

Theorem 6.3.1. *Denote* $n^{\pm} = n^{\pm}(\lambda, p) = (\lambda(H_\alpha + 2H_\beta) \pm p)/2$.

1. $M(\mu, q) \subset M(\lambda, p)$ *if and only if* $\mu = \lambda - n\beta + k\alpha$, $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, *and one of the following conditions holds:*

   (i) $n = 0$ *and* $q = \pm p$;

   (ii) $n \in \{n^{\pm}(\lambda, p)\}$ *and* $q^2 = (p \mp n^{\pm}(\lambda, p))^2$.

   Moreover, if $M(\lambda, p)$ *contains a unique non-trivial Verma submodule* $V(\mu, q)$, *then* $M(\lambda, p) \supset M(\mu, q) \supset 0$ *is a composition series of* $M(\lambda, p)$ *(in* $\mathcal{K}^\alpha$*).*

2. *If* $n_1 = n^-(\lambda, p) \in \mathbb{N}$ *and* $n_2 = n^+(\lambda, p) \in \mathbb{N}$, *then*

$$M(\lambda, p) \supset M(\lambda - n_1\beta, p + n_1) \supset M(\lambda - n_2\beta, p - n_2) \supset 0,$$

   moreover, this is a composition series of $M(\lambda, p)$ *(in* $\mathcal{K}^\alpha$*).*

3. *The module* $M(\lambda, p)$ *is a simple object in* $\mathcal{K}^\alpha$ *if and only if* $n^{\pm}(\lambda, p) \notin \mathbb{N}$.

We note, that this statement gives us more information than we have expected. In fact, the statement additionally presents a composition series for $M(\lambda, p)$. This is a phenomena for $\mathfrak{sl}(3, \mathbb{C})$ case and we will not recover it in this Chapter for other cases. One can also note, that the proof of this additional information is ideologically the most difficult part of the whole proof.

Proof. The original proof from [F2] is based on direct technical calculation. We will omit these technical detailes, which can be found in [F2].

Clearly, to prove the theorem, we have to determine the set of $\mathfrak{A}$-primitive elements in $M(\lambda, p)$ (this will be a direct generalization of the BGG Theorem). Suppose that $v_0 \in M(\lambda, p)_{\lambda-\rho}$ (i.e. $(\lambda - \rho)(H_\alpha) = a$). Let $u \in M(\lambda, p)_{\lambda-\rho+k\alpha-n\beta}$, $k \in \mathbb{Z}$, $n \in \mathbb{Z}_+$ be a non-zero element. Then

$$u = \sum_{j=0}^{n} a_j X_{-\alpha-\beta}^{j} X_{-\beta}^{n-j} \otimes v_{j+k}. \tag{1}$$

The element u is $\mathfrak{A}$-primitive if and only if $X_\beta u = X_{\alpha+\beta} u = 0$. Using (1), we can write these conditions as a system of linear equations on a_j. Solving this sistem, we obtain that a non-trivial solution exists if and only if $n = 0$ or $p^2 = (-2n + \lambda(H_\alpha + 2H_\beta))^2$. This, together with Proposition 6.2.1, in fact, proves all statements about the embeddings of GVMs.

To complete the proof we have to show that the natural embeddings of GVMs form a composition series. We will prove this for the second statement. The corresponding result for the first statement can be obtained by the same arguments. We have $M(\lambda - n_1\beta, p + n_1) \subset M(\lambda, p)$, $M(\lambda - n_2\beta, p - n_2) \subset M(\lambda, p)$ and $M(\lambda - n_2\beta, p - n_2) \subset M(\lambda - n_1\beta, p + n_1)$ by virtue of the first part of the Theorem. Now one can easily calculate, that

$\dim(M(\lambda,p)_{\lambda-\rho-n\beta}) = n+1$. Let $L_1 = M(\lambda,p)/M(\lambda-n_1\beta, p+n_1)$. From Proposition 6.2.1 it follows that $\dim(L_1)_{\lambda-\rho-n\beta} = n+1$, if $n < n_1$ and $\dim(L_1)_{\lambda-\rho-n\beta} = n_1$, if $n \geqslant n_1$. Denote by $\mathfrak{G}_1$ the $\mathfrak{sl}(2,\mathbb{C})$ subalgebra of $\mathfrak{G}$, generated by $X_{\pm\beta}$. Since α and β are not orthogonal, from the standard $\mathfrak{sl}(2,\mathbb{C})$ calculation it follows that there exists $i \in \mathbb{Z}$ such that the module $U(\mathfrak{G}_1)v_i$ is simple. Hence, $X_{-\beta}$ acts injectively on $L_{\lambda-\rho+i\alpha-j\beta}$ for any quotient L of $M(\lambda,p)$ and any $j \in \mathbb{Z}_+$. This implies, in particular, that L_1 is simple. By the same arguments the module $L_2 = M(\lambda-n_1\beta, p+n_1)/M(\lambda-n_2\beta, p-n_2)$ is also simple as good as $M(\lambda-n_2\beta, p-n_2)$ itself. This completes the proof. $\square$

On the set of isomorphism classes of all GVMs we can defene a natural partial order with respect to inclusion. According to Theorem 6.3.1, the obtained partially ordered set decomposes into a non-ordered union of chains of lengths 1,2 or 3. This shows that the situation with GVMs is simpler than the analogous situation of Verma modules. For example, for the principal block of $\mathcal{O}$ the corresponding poset of Verma modules is, according to the BGG Theorem, the poset of Bruhat order on the corresponding Weyl group. In particular, the order on this poset is not linear.

6.4 B_2 case

In this Section we present the results, analogous to Theorem 6.3.1 in the case, when $\mathfrak{G}$ is a Lie algebra of type B_2. These results were obtained in [FM1, Section 4], which is, in fact, the most difficult technical part of [FM1]. The proof of these results is absolutely analogous to the proof of Theorem 6.3.1, but involves more complicated calculations. This is why we will omit the proof here and refer the reader to [FM1]. We retain the notations from Section 6.1.

Let again $\pi(\mathfrak{A}) = \{\alpha\}$ and $\pi = \{\alpha, \beta\}$. Unlike the case of $\mathfrak{sl}(3,\mathbb{C})$, the Dynkin diagram of type B_2 has no non-trivial automorphisms and hence we have to consider two different cases:

- α is a short root.
- α is a long root.

We will see that these cases, as well as the corresponding criteria, are quite different.

Consider the first case (α is a short root). Then $\Delta_+ = \{\alpha, \beta, \beta+\alpha, \beta+2\alpha\}$. For $\lambda \in \mathfrak{H}^*$ and $p \in \mathbb{C}$ denote $n^{\pm}(\lambda,p) = (\lambda(H_\alpha + 2H_\beta) \pm p)/2$ and $n(\lambda,p) = n^+(\lambda,p) + n^-(\lambda,p) = \lambda(H_\alpha + 2H_\beta)$.

Theorem 6.4.1. *1. $M(\mu,q) \subset M(\lambda,p)$ if and only if $\mu = \lambda + k\alpha - n\beta$, $k \in \mathbb{Z}$, $n \in \mathbb{Z}_+$ and one of the following conditions holds:*

(i) $n = 0$ and $q = \pm p$;

(ii) $n \in \{n^{\pm}(\lambda,p), n(\lambda,p)\}$ and $q^2 = p^2 + 4n\lambda(H_\alpha + 2H_\beta) - 4n^2$.

2. The module $M(\lambda,p)$ is a simple object in $\mathcal{K}^\alpha$ if and only if $\{n^{\pm}(\lambda,p), n(\lambda,p)\} \cap \mathbb{N} = \varnothing$.

From Theorem 6.4.1 follows immediately, that the poset of GVMs with respect to inclusions is a non-ordered union of chains of lengths 1,2 or 4, which is analogous to the case of $\mathfrak{sl}(3,\mathbb{C})$. As we have already mentioned, the proof of Theorem 6.4.1 can be found in [FM1] and is analogous to the first part of the proof of Theorem 6.3.1. We also note, that unlike Theorem 6.3.1, Theorem 6.4.1 does not give us a composition series of $M(\lambda,p)$.

Now consider the second case (α is a long root). Then $\Delta_+ = \{\alpha,\beta,\beta+\alpha,2\beta+\alpha\}$. For $\lambda\in\mathfrak{H}^*$ and $p\in\mathbb{C}$ denote $n^{\pm}(\lambda,p) = (\lambda(H_\alpha+H_\beta)\pm p)$ and $n(\lambda,p) = n^+(\lambda,p)+n^-(\lambda,p) = 2\lambda(H_\alpha+H_\beta)$.

Theorem 6.4.2. *1. $M(\mu,q)\subset M(\lambda,p)$ if and only if $\mu=\lambda+k\alpha-n\beta$, $k\in\mathbb{Z}$, $n\in\mathbb{Z}_+$ and one of the following conditions holds:*

(i) $n=0$ and $q=\pm p$;

(ii) $n\in\{n^{\pm}(\lambda,p)\}$ and $q^2=(p\mp n^{\pm}(\lambda,p))^2$;

(iii) $n=n(\lambda,p)$, $n/2\in\mathbb{N}$ and $q=\pm p$;

(iv) $n=n(\lambda,p)$, $n/2\notin\mathbb{N}$, $n^{\pm}(\lambda,p)\in\mathbb{N}$ and $q=\pm p$.

2. The module $M(\lambda,p)$ is a simple object in $\mathcal{K}^\alpha$ if and only if $n^{\pm}(\lambda,p)\notin\mathbb{N}$ and $n(\lambda,p)/2\notin\mathbb{N}$.

From Theorem 6.4.2 it follows that the inclusions of GVMs for this case can be more complicated that in previous cases and not only linear posets can occure. Thus, if we assume that $n_1=n^-(\lambda,p)\in\mathbb{N}$ and $n_2=n^+(\lambda,p)\in\mathbb{N}$, $n_2>n_1$ then if $p\in\mathbb{N}$ we have

$$M(\lambda-(n_1+n_2)\beta,p)\subset M(\lambda-n_2\beta,p-n_2)\subset M(\lambda-n_1\beta,p+n_1)\subset M(\lambda,p),$$

and if $p\notin\mathbb{N}$ we have

$$\begin{gathered}M(\lambda-(n_1+n_2)\beta,p)\subset M(\lambda-n_2\beta,p-n_2)\subset M(\lambda,p),\\ M(\lambda-(n_1+n_2)\beta,p)\subset M(\lambda-n_1\beta,p+n_1)\subset M(\lambda,p),\\ M(\lambda-n_1\beta,p+n_1)\not\subset M(\lambda-n_2\beta,p-n_2),\\ M(\lambda-n_2\beta,p-n_2)\not\subset M(\lambda-n_1\beta,p+n_1).\end{gathered}$$

6.5 A new action of the Weyl group and a generalization of the Harish-Chandra Theorem

We have already understood, that the space of parameters for modules $M(\lambda,p)$ is the linear space of all pairs (λ,p), $\lambda\in\mathfrak{H}^*$, $p\in\mathbb{C}$. So far we have not seen why we substituted b by p, $p^2=b$. This will be clarified in this Section. Denote by $\Omega=\mathfrak{H}^*\oplus\mathbb{C}$ the set of parameters of our GVMs. The aim of this Section is to introduce an action of the Weyl group on Ω, which will play an important role in our generalization of the BGG Theorem. For Section 6.5 and Section 6.6 we assume that $\mathfrak{G}$ is a simple finite-dimensional complex Lie algebra and $\mathfrak{G}$ is not of type G_2. The reason is that the action of the Weyl group on

Ω will be defined using rank two calculations, obtained in Theorems 6.3.1, 6.4.1 and 6.4.2. We did not cover the G_2 case there, and so will not be able to consider it here. We will handle the G_2 case in Section 6.7 by different methods, because of the technical difficulties.

Consider the following partition of π: $\pi = \pi_1 \cup \pi_2 \cup \pi_3 \cup \pi_4$, where $\pi_1 = \{\gamma \in \pi \mid \alpha + \gamma \in \Delta, |\alpha| = |\gamma|\}$, $\pi_2 = \{\gamma \in \pi \mid \alpha + \gamma \in \Delta, |\alpha| < |\gamma|\}$, $\pi_3 = \{\gamma \in \pi \mid \alpha + \gamma \in \Delta, |\alpha| > |\gamma|\}$, $\pi_4 = \{\gamma \in \pi \mid \alpha + \gamma \notin \Delta\}$. For $(\lambda, p) \in \Omega$ and $\beta \in \pi \setminus \pi_4$ denote

$$n^{\pm}_{\beta}(\lambda, p) = \begin{cases} \frac{1}{2}(\lambda(H_\alpha + 2H_\beta) \pm p), & \beta \in \pi_1 \cup \pi_2 \\ \lambda(H_\alpha + H_\beta) \pm p, & \beta \in \pi_3 \end{cases}$$

and define three pairs $(\lambda_\beta, p^i_\beta) \in \Omega$, $i = 1, 2, 3$, where $\lambda_\beta = \lambda - n^-_\beta(\lambda, p)\beta$, $p^1_\beta = n^+_\beta(\lambda, p)$, $p^2_\beta = p + 2n^-_\beta(\lambda, p)$, $p^3_\beta = p + n^-_\beta(\lambda, p)$.

For each $\beta \in \pi$ consider $l_\beta \in GL(\Omega)$ such that

$$l_\beta(\lambda, p) = \begin{cases} (\lambda, -p), & \beta = \alpha \\ (s_\beta \lambda, p), & \beta \in \pi_4 \setminus \{\alpha\} \\ (\lambda_\beta, p^i_\beta), & \beta \in \pi_i, \qquad i = 1, 2, 3. \end{cases} \tag{2}$$

If one looks at (2) carefully, it is easy to see, that for rank two algeras of type A_2 or B_2, formulae (2) coincide with those obtained in Theorems 6.3.1, 6.4.1 and 6.4.2. Further we will see that, in fact, (2) defines an action of W on Ω.

For $r \in \mathbb{C}$ denote by Ω_r the afiine hyperplane in Ω, which contains all (λ, p) such that $\lambda = r\alpha + \sum_{\beta \in \pi \setminus \{\alpha\}} a_\beta \beta$, $a_\beta \in \mathbb{C}$. Clearly, $\Omega_r = r\alpha + \Omega_0$, Ω_0 is a subspace of Ω and all Ω_r are invariant under l_β, $\beta \in \pi$.

Lemma 6.5.1. *Suppose that $(\lambda, p) \in \Omega_{r_1}$, $(\mu, q) \in \Omega_{r_2}$ and $r_1 - r_2 \notin \mathbb{Z}$. Then* $\mathrm{Hom}(M(\lambda, p), M(\mu, q)) = 0$.

Proof. Under the conditions of the Lemma, the weight lattices of $M(\lambda, p)$ and $M(\mu, q)$ are different, which completes the proof. $\square$

Clearly, Lemma 6.5.1 can be generalized to any Ext between $M(\lambda, p)$ and $M(\mu, q)$ in the category of weight $\mathfrak{G}$-modules.

Let Δ° be a root system dual to Δ, $\eta' : \Delta \to \Delta^\circ$ be a canonical bijection and $\pi^\circ = \eta'(\pi)$. Construct a map $\eta^\circ_r : \Delta^\circ \to \Omega_r$ as follows. For $\beta \in \pi$ let

$$\eta^\circ_r(\beta^\circ) = \begin{cases} (r\alpha, |\alpha^\circ|^2), & \beta = \alpha \\ ((|\beta^\circ|^2/2)\beta + r\alpha, -\frac{1}{2}|\beta^\circ|^2), & \alpha + \beta \in \Delta, |\alpha| \geqslant |\beta| \\ (\frac{1}{2}\beta + r\alpha, -1), & \alpha + \beta \in \Delta, |\alpha| < |\beta| \\ ((|\beta^\circ|^2/2)\beta + r\alpha, 0), & \alpha + \beta \notin \Delta, \alpha \neq \beta. \end{cases} \tag{3}$$

The formulae (3) define a map from π° to Ω_r, which can be extended to the whole of Δ° by linearity. Define $\eta_r = \eta^\circ_r \circ \eta' : \Delta \to \Omega_r$. For a fixed r we set $\eta = \eta_r$, $\pi_r = \eta(\pi)$, $\Delta_{\alpha,r} = \eta(\Delta)$, $\Delta^+_{\alpha,r} = \eta(\Delta^+)$. Clearly, $\pi_{\alpha,r}$ is a basis of Ω_r.

Define a bilinear form $(\cdot,\cdot)_r : \Omega_r \times \Omega \to \mathbb{C}$ as follows. For $\beta \in \pi_{\alpha,r}$ and $(\lambda,p) \in \Omega$ let

$$(\beta,(\lambda,p))_r = \begin{cases} p, & \eta^{-1}(\beta) = \alpha \\ \lambda(H_{\eta^{-1}(\beta)}), & \eta^{-1}(\beta) \in \pi_4 \setminus \{\alpha\} \\ n^-_{\eta^{-1}(\beta)}(\lambda,p), & \eta^{-1}(\beta) \in \pi \setminus \pi_4. \end{cases} \tag{4}$$

One can extend (4) to the whole space by linearity. It is straightforward, that $(\cdot,\cdot)_r$ is non-degenerate on Ω_r and using this form one verifies that $\Delta_{\alpha,r}$ is a root system in Ω_r, $(\cdot,\cdot)_r$ of the same type as Δ° and $\pi_{\alpha,r}$ is a basis of $\Delta_{\alpha,r}$ (see [FM1, Proposition 5.2,Lemma5.3]).

Lemma 6.5.2. *The rule $s_\beta(\lambda,p) = l_\beta(\lambda,p)$ defines an action of W on Ω.*

Proof. Follows from the discussion above and the fact that the Weyl groups of Δ and Δ° are isomorphic. □

Now we are ready to prove the first important result towards the BGG Theorem: a generalization of the Harish-Chandra Theorem. Consider a map $j : Z(\mathfrak{A}) \otimes S(\mathfrak{H}_\mathfrak{A}) \to \mathbb{C}[t] \otimes S(\mathfrak{H}_\mathfrak{A})$, such that $j(c) = t$, and the algebra $\Lambda = \mathbb{C}[\sqrt{t}] \otimes S(\mathfrak{H}_\mathfrak{A})$, which acts on Ω_r by polynomial functions. Denote by $\Lambda^W \subset \Lambda$ the polynomial functions, invariant under W. Then $\Lambda^W \subset \mathbb{C}[t] \otimes S(\mathfrak{H}_\mathfrak{A})$, since $s_\alpha(\lambda,p) = (\lambda,-p)$.

We know already, that $M(\lambda,p)$ has a central character. We will denote this central character by $\chi_{\lambda,p}$.

Theorem 6.5.1. *$j \circ i : Z(\mathfrak{G}) \to \Lambda^W$ is an isomorphism that does not depend on the choice of π containing α.*

There is a small confusion in the formulation, since we do not yet know if $j \circ i(Z(\mathfrak{G}))$ is contained in Λ^W. We will prove this at the first time.

Proof. We will prove this theorem by reducing it to the classical Harish-Chandra Theorem. As the first part, we prove, that $j \circ i(Z(\mathfrak{G})) \subset \Lambda^W$.

First of all, we remark that for (λ,p), and (μ,q) in Ω_r such that $(\lambda,p) \in W(\mu,q)$ holds $\chi_{\lambda,p} = \chi_{\mu,q}$. Really, it is enough to show this in the case $(\lambda,p) = s_\beta(\mu,q)$ for $\beta \in \pi$. Using the generalized Harish-Chandra homomorphism we see, that the eigenvalue of any $z \in Z(\mathfrak{G})$ on $M(\lambda,p)$ is a polynomial function in (λ,p). Using the classical theory of Verma modules ([D, Chapter 7]) and Theorems 6.3.1, 6.4.1 and 6.4.2 one can find sufficiently many (λ,p) such that $\chi_{(\lambda,p)} = \chi_{s_\beta(\lambda,p)}$ and the statement follows from the polynomiality of the action of $Z(\mathfrak{G})$.

Let $0 \neq z \in Z(\mathfrak{G})$, $w \in W$ and $(\lambda,p) \in \Omega_r$. Then $w(j\circ i)(z)(\lambda,p) = (j\circ i)(z)(w^{-1}(\lambda,p))$. From Lemma 4.3.1 we have that the last expression equals $\chi_{w^{-1}(\lambda,p)}(z) = \chi_{(\lambda,p)}(z) = (j \circ i)(z)(\lambda,p)$. Hence $w(j\circ i)(z) = (j \circ i)(z)$ and we obtain that $j \circ i(Z(\mathfrak{G})) \subset \Lambda^W$.

Now consider the commutative diagram

$$\begin{array}{ccccc} Z(\mathfrak{G}) & \xrightarrow{j\circ i} & \Lambda^W & \xrightarrow{(\varphi\otimes 1)\circ j^{-1}} & S(\mathfrak{H})^W \\ & & \downarrow & & \downarrow \\ & & \mathbb{C}[t] \otimes S(\mathfrak{H}_\mathfrak{A}) & \xrightarrow{(\varphi\otimes 1)\circ j^{-1}} & S(\mathfrak{H})^{W(\mathfrak{A})}. \end{array}$$

Since $((\varphi \otimes 1) \circ j^{-1}) \circ j \circ i$ is an isomorphism, we have that $\ker(j \circ i) = 0$ and $((\varphi \otimes 1) \circ j^{-1})(\Lambda^W) = S(\mathfrak{H})^W$. On the other hand, $(\varphi \otimes 1) \circ j^{-1}$ is an isomorphism on the lower row. Thus $\ker((\varphi \otimes 1) \circ j^{-1}) = 0$, which implies that $j \circ i$ is an epimorphism and hence an isomorphism. It does not depend on the choice of π containing α by the Harish-Chandra Theorem. □

Corollary 6.5.1. *Suppose that $(\lambda, p), (\mu, q) \in \Omega_r$. Then $\chi_{\lambda,p} = \chi_{\mu,q}$ if and only if $(\lambda, p) \in W(\mu, q)$.*

Proof. We have already known the "if" part, so we have to prove the "only if" part. But two different orbits of W on Ω_r can be distinguished by the values of a polynomial function. Summing over W we can assume that this function is W-invariant. Applying Theorem 6.5.1, we obtain that there is a central element (which corresponds to the constructed function), which has different eigenvalues on the corresponding GVMs. □

6.6 BGG Theorem and a simplicity criterion

In the previous Section we defined an action of W on the space Ω of parameters of GVMs and found out that this action is compatible with the central characters of GVMs. This is good evidence to expect that it is possible to generalize the BGG Theorem and give a criterion for the existence of a non-trivial homomorphism between two GVMs in terms of the defined action of W. This will be done in this Section.

Now we have to define a special partial order on Ω_r, which is analogous to the standard partial order on $\mathfrak{H}^*$. For (λ, p), (μ, q) in Ω_r set $(\mu, q) \prec (\lambda, p)$ if there exists $\beta \in \Delta^+_{\alpha,r}$ (here $\Delta^+_{\alpha,r}$ is taken with respect to $\pi_{\alpha,r}$) such that $s_\beta(\lambda, p) = (\mu, q)$ and for $\beta \neq \eta(\alpha)$ holds $(\beta, (\lambda, p))_r \in \mathbb{N}$. Consider the transitive closure of $\prec$, which we will also denote by $\prec$. The main result of this Section is the following Theorem, wich generalizes the BGG Theorem to the case of GVMs of the form $M(\lambda, p)$.

Theorem 6.6.1. *Let (λ, p) and (μ, q) be elements from Ω_r. Then the following statements are equivalent:*

1. *$M(\mu, q) \subset M(\lambda, p)$.*
2. *$L(\mu, q)$ is a subquotient of $M(\lambda, p)$.*
3. *$(\mu, q) \prec (\lambda, p)$.*

The detailed proof of this Theorem can be found in [FM1, Section 7]. In fact, it follows the general line of the original proof of BGG from [BGG2] (see also [D, Theorem 7.6.23]). In order, to omit the standard technical details here we will explain only main ideological steps of this proof.

Proof. In the first part of the proof one has to show that the third condition implies the first one. For this it is enough to show, that if $\beta \in \Delta^+_{\alpha,r}$, $\beta \neq \eta(\alpha)$ such that $(\beta, (\lambda, p)) = n \in \mathbb{N}$, then $M(s_\beta(\lambda, p)) \subset M(\lambda, p)$. If one fixes n, then it is easy to see that the set of those (λ, p), such that $M(s_\beta(\lambda,p)) \subset M(\lambda,p)$ is Zariski closed in Ω_r ([FM1, Proposition 7.1]). Now one has only to find sufficiently many (λ,p) such that the inclusion $M(s_\beta(\lambda,p)) \subset M(\lambda,p)$ is obvious. This can be easily done if one considers (λ,p), which correspond to non α-stratified GVMs. To the highest weight submodules of such GVMs one can apply the classical BGG Theorem, which guarantees the desired inclusions ([FM1, Lemma 7.3]).

Note that the first condition easily implies the second one. Thus, in the second part of the proof one has to show that the second condition implies the third one. For this one can follow the original proof of BGG, the main tool of which is tensoring with finite-dimensional modules. This proof can be directly rewritten in our case, if we understand, how GVMs $M(\lambda,p)$ behave when we tensor them with a finite-dimensional module.

The key statement here is the following observation ([M3, Lemma 4.5]) for $\mathfrak{sl}(2,\mathbb{C})$: Let F be an n-dimensional simple $\mathfrak{sl}(2,\mathbb{C})$-module. Then the module $F\otimes V(a,p^2)$ has length n in the category $\mathcal{K}^\alpha$. Moreover, the list of simple subquotients of $F\otimes V(a,p^2)$ is the following: $V(a,(p-n+1)^2)$, $V(a,(p-n+3)^2)$, $\dots$, $V(a,(p+n-1)^2)$. We note that it is possible, that some subquotients of the above list are isomorphic. This means that they occur in a composition series with the corresponding multiplicity. The proof of this fact is based on the idea, that one can substitute $V(a,b)$ with a (big enough) simple finite-dimensional module, namely, with a simple finite-dimensional module F', such that $\dim(F') > 3n$. Then the statement follows directly from the Littlewood-Richardson rule. This implies the statement for $V(a',b')$, which has F' as a subquotient. Now one has just to note, that the action of generators of $\mathfrak{sl}(2,\mathbb{C})$ is defined on the basis of $V(a,b)$ via some polynomials and the set of possible F' is infinite.

As soon as we know the result above, we can construct a filtration of $F\otimes M(\lambda',p')$, where F is a finite-dimensional $\mathfrak{G}$-module, such that each subquotient of this filtration is isomorphic to some $M(\mu',q')$ (a Verma flag of $F\otimes M(\lambda',p')$). Moreover, we can precisely determine all GVM subquotients of this filtration (see also Chapter 12, which is independent of the present Section), and see, that the picture for their parameters in Ω is quite analogous to the situation with parameters of Verma modules in $\mathfrak{H}^*$. Having this knowledge, we just rewrite the proof of [D, Theorem 7.5.23] (this was done, with all necessary notation and explanations in [FM1, Section 7]). ☐

From Theorem 6.6.1 one can easily derive a simplicity criterion for $M(\lambda,p)$.

Corollary 6.6.1. *The module $M(\lambda,p)$ is a simple object in $\mathcal{K}^\alpha$ if and only if $(\beta,(\lambda,p))_r \notin \mathbb{N}$ for all $\beta\in\Delta^+_{\alpha,r}\setminus\{\eta(\alpha)\}$.*

6.7 G_2 case

Theorem 6.6.1 generalizes the BGG Theorem to the case of arbitrary simple complex finite-dimensional Lie algebras not of type G_2. The reason is that the definition of our action of W on Ω is based on the computational results for A_2 and B_2 algebras, obtained in Theorems 6.3.1, 6.4.1 and 6.4.2. We are not brave enough to try to do analogous calculations for the G_2 case, so we will try to guess the necessary action of W and then prove an analogue of the BGG Theorem, following the proof of Theorem 6.6.1. The hint, why it seems to be possible, is that we really did not use Theorems 6.3.1, 6.4.1 and 6.4.2 in the proof of Theorem 6.6.1 directly. In the first part of the proof we succeed to obtain enough information from highest weight submodules of non α-stratified modules $M(\lambda, p)$. The content of this Section is essentially the content of [KM2]. We also refer the reader to [KM2] for all technical details.

Retain the notation from Section 6.5. Let $\mathfrak{G}$ be a Lie algebra of type G_2, $\pi = \{\alpha, \beta\}$, $\pi(\mathfrak{A}) = \{\alpha\}$ (α can be a long root as well as a short one). Denote by $(\cdot,\cdot)^\circ$ the invariant bilinear form, which corresponds to Δ°. For $(\lambda, p) \in \Omega$ set

$$N_{(\lambda,p)} := \frac{\lambda(\alpha(H_\alpha)H_\beta - \alpha(H_\beta)H_\alpha) + \alpha(H_\beta)p}{\alpha(H_\alpha)}. \tag{5}$$

For all $\gamma^\circ \in \pi^\circ$ we define a map $\eta_r^\circ : \pi^\circ \to \Omega_r$ as follows

$$\eta_r^\circ(\gamma^\circ) := \begin{cases} (r\alpha, (\alpha^\circ, \alpha^\circ)^\circ), & \gamma^\circ = \alpha^\circ \\ \left(\frac{\alpha(H_\alpha)(\gamma^\circ,\gamma^\circ)^\circ - \alpha(H_\beta)(\alpha^\circ,\gamma^\circ)^\circ}{\alpha(H_\alpha)\beta(H_\beta) - \alpha(H_\beta)\beta(H_\alpha)}\beta + r\alpha, (\alpha^\circ, \gamma^\circ)^\circ\right), & \gamma^\circ \neq \alpha^\circ. \end{cases} \tag{6}$$

The map η_r° can be continued by linearity on Δ°. Set $\eta = \eta_r^\circ \circ \eta'$ and let $\Delta_{\alpha,r} = \eta(\Delta)$ with the fixed base $\eta(\pi)$. For all $(\lambda, p) \in \Omega, \gamma \in \pi$ we let

$$(\eta(\gamma), (\lambda, p))_r := \begin{cases} p & , \gamma = \alpha \\ N_{(\lambda,p)} & , \gamma \neq \alpha. \end{cases} \tag{7}$$

One can continue $(\cdot,\cdot)_r$ on Ω_r by linearity on the first argument and obtain a bilinear form $(\cdot,\cdot)_r : \Omega_r \times \Omega \to \mathbb{C}$ at the same way as it was done in Section 6.5.

We note that the formulae above are valid for any algebra of rank two, not only for type G_2. In the cases of algebras of type A_2 and B_2 they will coincide with the corresponding formulae from Section 6.5. It is easy to see that $(\cdot,\cdot)_r$ is non-degenerate on Ω_r and $\Delta_{\alpha,r}$ is a root system in Ω_r, $(\cdot,\cdot)_r$ of the same type, as Δ°. Now one can define an action of W on Ω via $s_{\eta(\gamma)}$, $\gamma \in \pi$. Retaining the notation from Section 6.6 and following all the proofs we obtain the following.

Theorem 6.7.1. *The statements of Theorem 6.5.1, Corollary 6.5.1, Theorem 6.6.1 and Corollary 6.6.1 extend to the case when $\mathfrak{G}$ is of type G_2.*

Finally, we remark that the structure of the poset of GVMs (with respect to inclusions), in this case, is the most difficult among all cases of algebras of rank two. In the case when α is a short root, the corresponding poset is a non-ordered union of linearly ordered sets and of quadruples occuring in B_2 case. In the case when α is a long root, to the posets descibed above one should add the poset of the Bruhat order on on a Weyl group of type A_2.

7 Gelfand-Zetlin modules

In the previous Chapter we generalized the BGG Theorem on GVMs, induced from a dense $\mathfrak{sl}(2,\mathbb{C})$-module. We crucially used the classification of such modules (classification of modules $V(a,b)$) and the fact that the modules $V(a,b)$ are, from some points of view, similar to finite-dimensional modules. In our case, this means that all the modules $V(a,b)$ have a natural basis, in which the action of the generators of $\mathfrak{A}$ is written using the same polynomial formulae, which occur in finite-dimensional modules. We really used this similarity in the proof of Theorem 6.6.1, where we determine all subquotients in a composition series of the tensor product of $V(a,b)$ with a simple finite-dimensional module. One of the most direct ways to generalize this machinery on bigger algebras, is to find a reasonable family of modules, similar to finite-dimensional modules in the above sense. Fortunately, this can be done, using the celebrated Gelfand-Zetlin basis for finite-dimensional $\mathfrak{A}$-modules ([GZ1, GZ2], see also [BR, Z1]). The classical result by Gelfand and Zetlin presents a nice basis for finite-dimensional modules over simple Lie algebras of type A_n, B_n and D_n. Jimbo extended the same construction on the case of quantum algebra $U_q(\mathfrak{gl}_n)$ ([Ji]).

In this Chapter we present a construction of a family of modules "similar to finite-dimensional modules" for all mentioned algebras. We have to note, that there are also some recent results for the C_n case and for certain (non-standard) quantum algebras (see, for example, [Mo1, Mo2, GI]), but we are not going to discuss them. These modules were first constructed by Drozd, Futorny and Ovsienko for the A_n case. The best explanation of their construction can be found in [DOF1], although this is not the first paper on this subject. One can also consult [DOF2, DOF4, DOF5]. This construction was extended to orthogonal Lie algebras in [M5] and to $U_q(\mathfrak{gl}_n)$ in [MT].

In this Chapter we will consider three cases (A_n, orthogonal algebras, $U_q(\mathfrak{gl}_n)$) separately. For each case we recall the classical rersult concerning finite-dimensional modules and present a construction of the new family of modules, which are called *generic Gelfand-Zetlin modules*. Regretfully, the only complete proof for the classical situation, available in the litarature ([Z1, Chapter 10]) covers only the A_n case. Some analogous calculations can also be found in [O1, O2]. In the last Section we give a unified proof for the construction of generic modules in all cases.

7.1 $\mathfrak{gl}(n,\mathbb{C})$ ($\mathfrak{sl}(n,\mathbb{C})$) case

In this Section we recall the results from [GZ1]. For our convenience we will work with a reductive Lie algebra $\mathfrak{A}'$, isomorphic to $\mathfrak{gl}(n,\mathbb{C})$. We fix a standard basis in $\mathfrak{A}'$, consisting of matrix units $e_{i,j}$, $i,j=1,2,\dots,n$. By a tableau we will mean a doubly indexed vector $[\,l\,]=[\,l_{i,j}\,]$ from $\mathbb{C}^{n(n+1)/2}$, where $i=1,2,\dots,n$ and $j=1,2,\dots,i$. Denote by $[\,\delta^{k,j}\,]$ the Kronecker tableau, i.e. one with $\delta^{k,j}_{k,j}=1$ and $\delta^{k,j}_{a,b}=0$ if $k\neq a$ or $j\neq b$.

Theorem 7.1.1. *Let $V(m)$ be a simple $\mathfrak{A}'$-module with the highest weight $m=(m_{n,1}, m_{n,2},\dots, m_{n,n})$, $m_{n,i}\geqslant m_{n,i+1}$, $m_{n,i}\in\mathbb{N}$. Then $V(m)$ possesses a basis consisting of all tableau $[\,s\,]$ such that $s_{n,j}=m_{n,j}$, $j=1,2,\dots,n$ and $s_{i+1,j}\geqslant s_{i,j}\geqslant s_{i+1,j+1}$, $i=1,2,\dots,n,$*

$j=1,2,\dots,i$ and the action of generators of $\mathfrak{A}'$ is given by the following formulae:

$$e_{k,k}[\,s\,]=\left(\sum_{i=1}^{k}s_{k,i}-\sum_{i=1}^{k-1}s_{k-1,i}+k\right)[\,s\,],\quad k=1,2,\dots,n,$$

$$e_{k,k+1}[\,s\,]=\sum_{j=1}^{k}a^{+}_{k,j}([\,s\,])([\,s\,]+[\,\delta^{k,j}\,]),\quad e_{k+1,k}[\,s\,]=\sum_{j=1}^{k}a^{-}_{k,j}([\,s\,])([\,s\,]-[\,\delta^{k,j}\,]),$$

where $[\,\delta^{k,j}\,]$ is the Kronecker tableau and for $l_{r,t}=s_{r,t}-t$ we have

$$a^{\pm}_{k,j}([\,s\,])=\mp\frac{\prod_{i}(l_{k\pm 1,i}-l_{k,j})}{\prod_{i\neq j}(l_{k,i}-l_{k,j})}.$$

The proof of this Theorem can be found in [Z1, Chapter 10]. We note, that not all simple $\mathfrak{A}$-modules occur as $V(m)$ in the above formulation. To cover all simple finite-dimensional $\mathfrak{A}$-modules one has to consider the modules $V(m,h)$, $h\in\mathbb{C}$ in which in all basis elements $[\,s\,]$ the entries $s_{i,j}$ are replaced by $s_{i,j}+h$. In this notation $V(m)=V(m,0)$. Nevertheless, if we consider the restrictions of all $V(m)$ from the above formulation on $\mathfrak{sl}(n,\mathbb{C})$ we will obtain all simple finite-dimensional $\mathfrak{sl}(n,\mathbb{C})$-modules (but this correspondence is not bijective).

Now we can see an analogue with $\mathfrak{sl}(2,\mathbb{C})$-case. Theorem 7.1.1 presents a basis of the finite-dimensional $\mathfrak{A}'$-module $V(m)$, and the action of generators of $\mathfrak{A}'$ on this basis is written by some polynomial (rational) formulae. The idea to construct a new family of $\mathfrak{A}'$-modules, is to take an analogous basis, in which all formulae are well-defined. More precisely, fix a tableau $[\,l\,]$ such that $l_{i,j}-l_{i,k}\notin\mathbb{Z}$ for all possible i and $j\neq k$. Consider the set $B([\,l\,])$, consisting of all tableaux $[\,l\,]$, such that

- $t_{n,j}=l_{n,j}$ for all j;
- $t_{i,j}-l_{i,j}\in\mathbb{Z}$ for all $1\leqslant i\leqslant n-1$ and all j.

Let $V([\,l\,])$ be a vector space with a basis $B([\,l\,])$. For $[\,t\,]\in B([\,l\,])$ set

$$e_{k,k}[\,t\,]=\left(\sum_{i=1}^{k}l_{k,i}-\sum_{i=1}^{k-1}l_{k-1,i}\right)[\,t\,],\quad k=1,2,\dots,n,$$

$$e_{k,k+1}[\,t\,]=\sum_{j=1}^{k}a^{+}_{kj}([\,t\,])([\,t\,]+[\,\delta^{k,j}\,]),\quad e_{k+1,k}[\,t\,]=\sum_{j=1}^{k}a^{-}_{kj}([\,t\,])([\,t\,]-[\,\delta^{k,j}\,]),$$

where

$$a^{\pm}_{kj}([\,t\,])=\mp\frac{\prod_{i}(t_{k\pm 1,i}-t_{k,j})}{\prod_{i\neq j}(t_{k,i}-t_{k,j})}.$$

We will call the formulae above the *Gelfand-Zetlin (GZ) formulae.*

Theorem 7.1.2. *1. The GZ-formulae define on $V([\,l\,])$ a structure of a $\mathfrak{A}'$-module of finite length.*

2. $V([\,l\,])$ is simple if and only if $l_{i,j} - l_{i-1,k} \notin \mathbb{Z}$ for all i, j, k.

The proof will be given in Sectrion 7.4. Some more explanation and properties of $V([\,l\,])$ will be presented in Section 7.5. The modules $V([\,l\,])$ will be called *generic* Gelfand-Zetlin modules.

7.2 $\mathfrak{O}(n, \mathbb{C})$ case

In this Section we recall the Gelfan-Zetlin construction of simple finite dimensional modules over the Lie algebra $\mathfrak{A} = \mathfrak{O}(n, \mathbb{C})$ (see [GZ2] or [BR, Section 10.1.B]). The results presented here were obtained in [M5]. Let $X_{i+1,i} = e_{i+1,i} - e_{i,i+1}$, $1 \leqslant i \leqslant n-1$ denote the standard generators of $\mathfrak{A}$. Assume that $n = 2k$ or $n = 2k+1$. In this Section, by a tableau we will mean the vector $[\,l\,]$ with complex entries, considered as a double indexed family $[\,l_{i,j}\,]$, where $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant s$ for $i = 2s-1$ or $i = 2s$. As usually, $[\,\delta^{i,j}\,]$ will denote the Kronecker tableau.

Fix a vector $m = (m_1, m_2, \dots, m_k)$ with integer or half-integer entries satisfying the following conditions:

1. For $n = 2k$: $m_1 \geqslant m_2 \geqslant \dots \geqslant m_{k-1} \geqslant |m_k|$.

2. For $n = 2k+1$: $m_1 \geqslant m_2 \geqslant \dots \geqslant m_{k-1} \geqslant m_k \geqslant 0$.

Consider the set $B(m)$ consisting of all tableaux with all integer or all half integer entries satisfying the following conditions:

$$\left.\begin{array}{ll} l_{n-1,i} = m_i, & \\ l_{2p+1,i} \geqslant l_{2p,i} \geqslant l_{2p+1,i+1}, & i = 1, 2, \dots, p-1, \\ l_{2p+1,p} \geqslant l_{2p,p} \geqslant |l_{2p+1,p+1}|, & \\ l_{2p,i} \geqslant l_{2p-1,i} \geqslant l_{2p,i+1}, & i = 1, 2, \dots, p-1, \\ l_{2p,p} \geqslant l_{2p-1,p} \geqslant -l_{2p,p} & \end{array}\right\}.$$

Consider the vectorspace $V(m)$ having $B(m)$ as basis (clearly $V(m)$ is finite-dimensional) and define an action of the generators $X_{i+1,i}$ on the basis elements as follows:

$$X_{2p+1,2p}[\,l\,] = \sum_{j=1}^{p} A(l_{2p-1,j})([\,l\,] + [\,\delta^{2p-1,j}\,]) - \sum_{j=1}^{p} A(l_{2p-1,j} - 1)([\,l\,] - [\,\delta^{2p-1,j}\,]),$$
$$X_{2p+2,2p+1}[\,l\,] = \sum_{j=1}^{p} B(l_{2p,j})([\,l\,] + [\,\delta^{2p,j}\,]) - \sum_{j=1}^{p} B(l_{2p,j} - 1)([\,l\,] - [\,\delta^{2p,j}\,]) + iC_{2p}[\,l\,],$$

where the functions A, B and C are defined in the following way: first we substitute $l_{2p-1,j}$ by $s_{2p-1,j}=l_{2p-1,j}+p-j$ and $l_{2p,j}$ by $s_{2p,j}=l_{2p,j}+p-j+1$ for all possible p, then we define

$$A(l_{2p-1,j})=\frac{1}{2}\left(\prod_{r=1}^{p-1}(s_{2p-2,r}-s_{2p-1,j}-1)(s_{2p-2,r}+s_{2p-1\,j})\right)^{1/2}\times$$
$$\times\left(\prod_{r=1}^{p}(s_{2p,r}-s_{2p-1,j}-1)(s_{2p,r}+s_{2p-1,j})\right)^{1/2}\times$$
$$\times\left(\prod_{r\neq j}(s_{2p-1,r}^2-s_{2p-1,j}^2)(s_{2p-1,r}^2-(s_{2p-1\,j}+1)^2)\right)^{-1/2},$$

$$B(l_{2p,j})=\left(\prod_{r=1}^{p}(s_{2p-1,r}^2-s_{2p,j}^2)\prod_{r=1}^{p+1}(s_{2p+1,r}^2-s_{2p,j}^2)\right)^{1/2}\times$$
$$\times\left(s_{2p,j}^2(4s_{2p,j}^2-1)\prod_{r\neq j}(s_{2p,r}^2-s_{2p,j}^2)((s_{2p,r}-1)^2-s_{2p,j}^2)\right)^{-1/2},$$

$$C_{2p}=\prod_{r=1}^{p}s_{2p-1,r}\prod_{r=1}^{p+1}s_{2p+1,r}\left(\prod_{r=1}^{p}s_{2p,r}(s_{2p,r}-1)\right)^{-1}.$$

Theorem 7.2.1. *The formulae above define on $V(m)$ the structure of a simple $\mathfrak{A}$-module. Moreover, any simple finite-dimensional $\mathfrak{A}$-module is isomorphic to $V(m)$ for some m as above.*

We will call the formulae above the *Gelfand-Zetlin (GZ) formulae* (for the orthogonal case). Regretfully, there is no complete proof of this theorem (which is again a long straightforward calculation) available in the literature. The reader can find some related calculations in [O1, O2]. Using Theorem 7.2.1, we define a new family of $\mathfrak{A}$-modules (which we will call *generic* Gelfan-Zetlin modules) as follows:

Fix a tableau $[\,l\,]$ with complex entries $l_{i,j}$, $1\leqslant i\leqslant n-1$ and $1\leqslant j\leqslant k$ if $i=2k-1$ or $i=2k$ satisfying the following defining conditions:

- all $l_{i,j}$ are not integers and half-integers for $i<n-1$;
- $l_{i,j}\pm l_{i,k}$ is not an integer for all $1\leqslant i\leqslant n-2$ and all $j\neq k$.

Consider a set $B([\,l\,])$ consisting of all tableaux $[\,t\,]$ such that

- $t_{n-1,j}=l_{n-1,j}$ for all j;

- $t_{i,j} - l_{i,j}$ is an integer for all $1 \leqslant i \leqslant n-2$ and all j.

Let $V([\,l\,])$ be a vector space with basis $B([\,l\,])$. For $[\,t\,] \in B([\,l\,])$ set

$$X_{2p+1,2p}[\,t\,] = \sum_{j=1}^{p} A(t_{2p-1,j})([\,t\,] + [\,\delta^{2p-1,j}\,]) - \sum_{j=1}^{p} A(t_{2p-1,j}-1)([\,t\,] - [\,\delta^{2p-1,j}\,]),$$
$$X_{2p+2,2p+1}[\,t\,] = \sum_{j=1}^{p} B(t_{2p,j})([\,t\,] + [\,\delta^{2p,j}\,]) - \sum_{j=1}^{p} B(t_{2p,j}-1)([\,t\,] - [\,\delta^{2p,j}\,]) + iC_{2p}[\,t\,],$$

where the functions A, B and C are taken from the GZ formulae for the orthogonal case. This action can be easily extended to $V([\,l\,])$ by linearity.

Theorem 7.2.2. 1. *The formulae above define on $V([\,l\,])$ the structure of a completely reducible $\mathfrak{A}$-module of finite length.*

2. *$V([\,l\,])$ is simple if and only if $l_{i,j} - l_{i-1,k} \notin \mathbb{Z}$ for all i, j, k.*

The proof is postponed till Section 7.4.

7.3 $U_q(\mathfrak{gl}_n)$ case

In this Section we apply the same construction to the quantum analogue $U_q(\mathfrak{gl}_n)$ of the algebra $\mathfrak{gl}(n, \mathbb{C})$. The original generalization of the Gelfan-Zetlin construction to this case was obtained in [Ji], but we will follow [KlSc], because [KlSc] contains a more convenient version of the Gelfand-Zetlin formulae. The reader may also consult [UTS1, UTS2], where the same results were obtained by different methods. The results of this Section were obtained in [MT]. In this Section by a tableau we will mean a tableau as in Section 7.1. We begin with the definition of $U_q(\mathfrak{gl}_n)$ we will use.

Let q be a non-zero complex non-root of unity. For any complex x we set $[x]_q = (q^x - q^{-x})/(q - q^{-1}) = (e^{xh} - e^{-xh})/(e^h - e^{-h})$, where $q = \exp(h)$. We define $U_q(\mathfrak{gl}_n)$ as a unital associative complex algebra generated by E_i, F_i, $i = 1, 2, \ldots, n-1$, K_j, K_j^{-1}, $j = 1, 2, \ldots, n$ subject to the relations

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$
$$K_iE_jK_i^{-1} = q^{\delta_{ij}/2}q^{-\delta_{i,j+1}/2}E_j,$$
$$K_iF_jK_i^{-1} = q^{-\delta_{ij}/2}q^{\delta_{i,j+1}/2}F_j,$$
$$[E_i, F_j] = \delta_{ij}\frac{K_i^2K_{i+1}^{-2} - K_i^{-2}K_{i+1}^2}{q - q^{-1}},$$
$$[E_i, E_j] = [F_i, F_j] = 0, \quad |i-j| \geqslant 2,$$
$$E_i^2E_{i\pm1} - (q + q^{-1})E_iE_{i\pm1}E_i + E_{i\pm1}E_i^2 = 0,$$
$$F_i^2F_{i\pm1} - (q + q^{-1})F_iF_{i\pm1}F_i + F_{i\pm1}F_i^2 = 0$$

(see, for example [KlSc, UTS1, UTS2]).

Theorem 7.3.1. *Let $V(m)$ be a simple $U_q(\mathfrak{gl}_n)$-module with a highest weight $m = (m_{n,1}, m_{n,2}, \ldots, m_{n,n})$, $m_{n,i} \geqslant m_{n,i+1}$, $m_{n,j} \in \mathbb{N}$. Then $V(m)$ possesses a basis consisting of all tableau $[\,s\,] = (s_{i,j})_{i=1,2,\ldots,n}^{j=1,2,\ldots,i}$ such that $s_{n,j} = m_{n,j}$, $j = 1, 2, \ldots, n$ and $s_{i+1,j} \geqslant s_{i,j} \geqslant s_{i+1,j+1}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, i$ and the action of generators of $U_q(\mathfrak{gl}_n)$ are given by the following formulae:*

$$K_k[\,s\,] = q^{a_k/2}[\,s\,], \quad a_k = \sum_{i=1}^{k} s_{k,i} - \sum_{i=1}^{k-1} s_{k-1,i}, \quad k = 1, 2, \ldots, n,$$

$$E_k[\,s\,] = \sum_{j=1}^{k} a_{k,j}^{+}([\,s\,])([\,s\,] + [\,\delta^{k,j}\,]), \quad F_k[\,s\,] = \sum_{j=1}^{k} a_{k,j}^{-}([\,s\,])([\,s\,] - [\,\delta^{k,j}\,]),$$

where for $l_{r,t} = s_{r,t} - t$ we have

$$a_{k,j}^{\pm}([\,s\,]) = \mp \frac{\prod_i [l_{k\pm1,i} - l_{k,j}]_q}{\prod_{i\neq j} [l_{k,i} - l_{k,j}]_q}.$$

The proof of this theorem can be found in [UTS1]. Analogously to Section 7.1 we construct a family of *generic* Gelfand-Zetlin modules over $U_q(\mathfrak{gl}_n)$ as follows.

Let $1(q)$ be the set of all complex x such that $q^x = 1$. Fix a tableau $[\,l\,]$ with complex entries $l_{i\,j}$, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant i$ satisfying the following condition:

- $2(l_{i,j} - l_{i,k}) \notin 1(q) + 2\mathbb{Z}$ for all $1 \leqslant i \leqslant n-1$ and all $j \neq k$.

Consider a set $B([\,l\,])$ consisting of all tableaux $[\,t\,]$ such that

- $t_{n,j} = l_{n,j}$ for all j;
- $t_{i,j} - l_{i,j}$ is an integer for all $1 \leqslant i \leqslant n-1$ and all j.

Let $V([\,l\,])$ be a vector space with a basis $B([\,l\,])$. For $[\,t\,] \in B([\,l\,])$ set

$$K_k[\,t\,] = q^{a_k/2}[\,t\,], \quad a_k = \sum_{i=1}^{k} l_{k,i} - \sum_{i=1}^{k-1} l_{k-1,i} + k, \quad k = 1, 2, \ldots, n,$$

$$E_k[\,t\,] = \sum_{j=1}^{k} a_{kj}^{+}([\,t\,])([\,t\,] + [\,\delta^{k,j}\,]), \quad F_k[\,t\,] = \sum_{j=1}^{k} a_{kj}^{-}([\,t\,])([\,t\,] - [\,\delta^{k,j}\,]),$$

where

$$a_{kj}^{\pm}([\,t\,]) = \mp \frac{\prod_i [t_{k\pm1,i} - t_{k,j}]_q}{\prod_{i\neq j} [t_{k,i} - t_{k,j}]_q}.$$

We will call the formulae above the *Gelfand-Zetlin (GZ) formulae.*

Theorem 7.3.2. *1. The GZ formulae define on $V([\,l\,])$ the structure of a $U_q(\mathfrak{gl}_n)$-module of finite length.*

2. The module $V([\,l\,])$ is simple if and only if $l_{i,j} - l_{i-1,k} \notin \mathbb{Z}$ for all i, j, k.

The proof will be given in the next Section.

7.4 Proof of Theorem 7.1.2, Theorem 7.2.2 and Theorem 7.3.2

Here we present a unified proof of Theorem 7.1.2, Theorem 7.2.2 and Theorem 7.3.2. For the unification we set $U(\mathfrak{A}) = U_q(\mathfrak{gl}_n)$ in Theorem 7.3.2.

First we show that the GZ formulae really define on $V([\,l\,])$ the structure of a $U(\mathfrak{A})$-module. Consider a relation $u = 0$ in $U(\mathfrak{A})$. Clearly, it is enough to show that u (as an element of the tensor algebra) acts trivially on $V([\,l\,])$, thus we have only to prove that $u[\,t\,] = 0$ for any $[\,t\,] \in B([\,l\,])$. Using the GZ formulae, we can write $u[\,t\,] = \sum_{[\,s\,] \in I(u,[\,t\,])} f([\,s\,])[\,s\,]$, where the set $I(u, [\,t\,]) - [\,t\,]$ depends only on u and for any fixed u each $f([\,t\,])$ is a rational function in $t_{i,j}$ (resp. $q^{t_{i,j}}$ in the quantum case). Thus, it is enough to show that each $f([\,s\,])$ is identically zero. Hence, we have to show that some polynomials in $t_{i,j}$ (resp. $q^{t_{i,j}}$) are zero. Let p be such a polynomial, k be its degree and s be the degree of u. Clearly, there exist a tableau $[\,\tilde{t}\,]$ such that all $\tilde{t}_{i,j}$ are positive integers and for any integers $-k-s \leqslant \tilde{s}_{i,j} \leqslant k+s$ the tableau $[\,\tilde{t}_{i,j} + \tilde{s}_{i,j}\,]$ occurs as a basis element in a finite-dimensional $U(\mathfrak{A})$-module. Taking into account in the quantum case, that $q^a \neq q^b$, if $a \neq b$ are positive integers, we conclude that p is identically zero, since GZ formulae really define the structure of an $\mathfrak{A}$-module on the finite-dimensional module as above. This complete the proof of the first step.

Now we are going to show that there exists a natural commutative subalgebra Γ in $U(\mathfrak{A})$, which distinguishes the basis elements $[\,t\,]$ from $B([\,l\,])$, i.e. for any different $[\,t\,]$ and $[\,s\,]$ from $B([\,l\,])$ there exists $z \in \Gamma$ such that $z[\,t\,] = a[\,t\,]$, $z[\,s\,] = b[\,s\,]$ and $a \neq b$. Consider an increasing chain of algebras $U(\mathfrak{gl}(1,\mathbb{C})) \subset U(\mathfrak{gl}(2,\mathbb{C})) \subset \cdots \subset U(\mathfrak{gl}(n,\mathbb{C}))$ (resp. $U(\mathfrak{O}(1,\mathbb{C})) \subset U(\mathfrak{O}(2,\mathbb{C})) \subset \cdots \subset U(\mathfrak{O}(n,\mathbb{C}))$, resp. $U_q(\mathfrak{gl}_1) \subset U_q(\mathfrak{gl}_2) \subset \cdots \subset U_q(\mathfrak{gl}_n)$) with natural inclusions with respect to the first basis elements. Denote by Γ the subalgebra of $U(\mathfrak{A})$ generated by all $Z(\hat{\mathfrak{A}})$, for $U(\hat{\mathfrak{A}})$ occuring in the above filtration. Set $U(\mathfrak{A}_k) = U(\mathfrak{gl}(k,\mathbb{C}))$ (resp. $U(\mathfrak{O}(k,\mathbb{C}))$, resp. $U_q(\mathfrak{gl}_k)$) from the above filtration. Then any generator of $U(\mathfrak{A}_k)$, acting on $[\,t\,]$ changes only $t_{i,j}$ with $j < k$. From this it follows that the action of any $z \in Z(\mathfrak{A}_k)$ on $[\,t\,]$ is a rational function in $t_{k,j}$ (resp. $q^{t_{k,j}}$), $j = 1, 2, \ldots, k$. Reducing our consideration to $U(\mathfrak{A}_k)$ we obtain, that it is enough to prove that the central characters (if they exist) of $V([\,l(1)\,])$ and $V([\,l(2)\,])$ with $l(1)_{n,j} - l(2)_{n,j} \in \mathbb{Z}$ for all j and $l(a)_{n,j} - l(a)_{n,i} \notin \mathbb{Z}$ for all a and $j \neq i$ do not coincide. Consider a finite-dimensional $U(\mathfrak{A})$-module $V(m)$, as in Theorem 7.1.1 (resp. Theorem 7.2.1, resp. Theorem 7.3.1). The upper row of any tableau $[\,s\,]$ which indexes some GZ-basis element of $V(m)$ determines the highest weight of $V(m)$. Hence, any $z \in Z(\mathfrak{A})$ acts on $V(m)$ via a scalar, which is a polynomial in $m_{n,j}$, $j = 1, 2, \ldots, n$, that can be computed via the Harish-Chandra homomorphism. Denote this scalar by $f_z(m_{n,j})$. Since we have infinitely many finite-dimensional modules with a Gelfand-Zetlin basis, we obtain that $z[\,t\,] = f_z(t_{n,j})[\,t\,]$ for any

$[t] \in B([l])$. In particular, $V([l])$ has a central character. Using a trivial calculation with the Weyl group, we see that $(l(1)_{n,j})$ and $(l(2)_{n,j})$ do not belong to the same orbit, since $l(1)_{n,j} - l(2)_{n,j} \in \mathbb{Z}$ for all j and $l(a)_{n,j} - l(a)_{n,i} \notin \mathbb{Z}$ for all a and $j \neq i$. Now the desired statement follows immediately from the Harish-Chandra Theorem (Theorem 3.2.2, resp. [J, Claim 6.26,Section 8.30]).

From the previous paragraph it follows that any submodule of $V([l])$ is determined by the set of basis elements, $[t] \in B([l])$, which it contains. Consider a graph with vertices $[t] \in B([l])$, in which $[t]$ and $[t] + [\delta^{i,j}]$ are connected by an edge if there are generators X, Y of $U(\mathfrak{A})$, such that $X[t]$ has a non-zero coefficient in $[t] + [\delta^{i,j}]$ and $Y([t] + [\delta^{i,j}])$ has a non-zero coefficient in $[t]$, $i < n$, $j \leqslant i$. From the GZ formulae it follows easily, that this graph contains only finite many connected components. Hence $V([l])$ has a finite length. All other statements follow immediately from the GZ formulae.

7.5 General theory of Gelfand-Zetlin modules

In Section 7.4 we introduced an important technical tool in the study of $V([t])$ – the subalgebra Γ of $U(\mathfrak{A})$. Using it we can abstractly define a general class of $U(\mathfrak{A})$-modules, which contains all the modules $V([l])$. We say that a $U(\mathfrak{A})$-module M is a *Gelfand-Zetlin module*, if it decomposes into a direct sum of non-isomorphic finite-dimensional Γ-modules. Since Γ is commutative, an equivalent condition is that M is a Γ-root module (i.e. a direct sum of Γ-root subspaces) with finite-dimensional root subspaces (a Γ-root subspace M^{χ} of M, which corresponds to $\chi \in \Gamma^*$ is the set of all $v \in M$ such that $(z - \chi(z))^N v = 0$ for all $z \in \Gamma$ and for big enough $N \in \mathbb{N}$). From Section 7.4 it follows easily, that each $V([l])$ is a Gelfand-Zetlin module. The algebra Γ is usually called the Gelfand-Zetlin subalgebra of $U(\mathfrak{A})$.

In Section 7.4 we saw, that there is a natural (not bijective) parametrization of $\chi \in \Gamma^*$ by tableaux. Really, if $[l]$ is a tableau, we will say that it parametrizes a GZ weight $\chi \in \Gamma^*$ if for any $1 \leqslant k \leqslant n$ the highest weioght $U(\mathfrak{A}_k)$-module with the highest weight $(l_{k,j})_{j=1,2,\dots,k}$ has the central character $\chi|_{Z(\mathfrak{A}_k)}$. It follows from the Harish-Chandra Theorem, that two tableaux parametrize the same chracter of Γ if and only if they belong to the same orbit of the natural action of the product $\hat{W}$ of all Weyl groups of $U(\mathfrak{A}_k)$ on the set of all tableaux.

The category of Gelfand-Zetlin modules is quite interesting. It was studied in different cases in [DOF1, DOF2, DOF4, DOF5, M4, M5, MO, MT]. Some recent results were also obtained in [Ov]. In particular, the situation of the Gelfand-Zetlin subalgebra Γ in $U(\mathfrak{A})$ is a partial case of an abstract Harish-Chandra situation, defined and studied in [DOF1]. Thus, Γ is a Harish-Chandra subalgebra of $U(\mathfrak{A})$ in the sense of [DOF1] (see [DOF1, M5, MT]). This means (for a commutative algebra Γ), that $\Gamma u \Gamma$ is a finitely generated bimodule for any $u \in U(\mathfrak{A})$. An abstract machinery, developed in [DOF1], allows one to obtain many interesting properties of the category of Gelfand-Zetlin modules. For example, it follows easily, that any simple $V([l])$ has only trivial extensions with any other (non-isomorphic) simple Gelfand-Zetlin module. We will not need these properties, so we will not discuss them. The reader can consult [DOF1] directly.

In what follows we will use constructed generic Gelfand-Zetlin modules $V([l])$ only over $\mathfrak{gl}(n,\mathbb{C})$ ($\mathfrak{sl}(n,\mathbb{C})$) algebra. We have already mentioned, that the main advantage of modules $V([\,l\,])$, is their similarity to finite-dimensional modules. Using this, we will obtain an analogue of the BGG Theorem for modules $M_{\mathcal{P}}(V([\,l\,]))$ by the same methods as used in Chapter 6. For any simple $\mathfrak{A}$, bigger than $\mathfrak{sl}(2,\mathbb{C})$, this will not be recovered using the general machinery of Shapovalov form in Chapter 9.

8 An analogue of the BGG Theorem, II

In this Chapter we give a generalization of the BGG Theorem for GVMs $M_{\mathcal{P}}(V)$, induced from a simple generic GZ module $V \simeq V([\,l\,])$. The results, presented here were obtained in [M4, MO]. The content in [MO] is a generalization of the BGG Theorem, but in [M4] the main content is different and there only a slight generalization of the results from [MO] mentioned. This is why [MO] is much more detailed than [M4], so we refer the reader to [MO] for all technical details.

8.1 GVMs induced from $V([\,l\,])$

In this Chapter we fix the Lie algebra $\mathfrak{G} = \mathfrak{gl}(n, \mathbb{C})$ (or $\mathfrak{G} = \mathfrak{sl}(n, \mathbb{C})$ just by restiction) with the standard Cartan subalgebra and the standard root basis. Then the algebra $\mathfrak{A}$ is a direct sum of some $\mathfrak{gl}(n_i, \mathbb{C})$ ($\mathfrak{sl}(k_i, \mathbb{C})$), $i = 1, 2, \dots, k$. In a natural way we extend the notion of a generic GZ module to the algebra $\mathfrak{A}$. Thus, any generic GZ module V over $\mathfrak{A}$ is defined by a sequence of tableaux $[\,l\,] = ([\,l(i)\,])_{i=1,2,\dots,k}$. Let $V([\,l\,])$ be a simple $\mathfrak{A}'$-module, whose restricion on $\mathfrak{A}$ is a simple generic GZ $\mathfrak{A}$-module. Consider the corresponding GVM $M_{\mathcal{P}}(V([\,l\,]))$. It follows directly from the definition of GZ modules and from the generalized Harish-Chandra homomorphism, that $M_{\mathcal{P}}(V([\,l\,]))$ is a GZ module over $\mathfrak{G}$. We can parameterize $M_{\mathcal{P}}(V([\,l\,]))$ in a regular way by $[\,l\,]$ and by the $\mathfrak{H}_{\mathfrak{A}}$-highest weight $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$ of $M_{\mathcal{P}}(V([\,l\,]))$. Since $M_{\mathcal{P}}(V([\,l\,]))$ is a GZ module, this parameterization can be changed by a bit more convenient one. If we fix a Γ-weight vector $v \in M_{\mathcal{P}}(V([\,l\,]))_\lambda$, whose GZ weight is parameterized by some tableau $[\,t\,]$, then $[\,t\,]$ uniquely defines $V([\,l\,])$ and λ. Hence $[\,t\,]$ uniquely defines $M_{\mathcal{P}}(V([\,t\,]))$. For us it will be much more convenient to parameterize GVMs by such tableaux $[\,t\,]$, so we need to know, which $[\,t\,]$ can occur as parameters of GVMs. In the situation described above we set $M([\,t\,]) = M_{\mathcal{P}}(V([\,l\,]))$. Correspondingly, we set $L([\,t\,])$ to be the unique simple quotient of $M([\,t\,])$. To formulate such a condition, we assume that π is ordered in a natural way, $\pi = \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$. From Section 7.5 we know that the tableaux parameterize the Γ weights up to the action of $\hat{W}$. This means that all our results should be also formulated up to this action. For a subset $A \in \mathbb{N}$ we will denote by $I(A)$ its convex hull in $\mathbb{N}$.

Lemma 8.1.1. *Up to the action of $\hat{W}$, the tableau $[\,t\,]$ parametrizes the GVM induced from a generic GZ module over $\mathfrak{A}$ if and only if the following conditions are satisfied:*

- $t_{i,j} - t_{i,k} \notin \mathbb{Z}$, $j \neq k$ *if* $\alpha_s \in \pi(\mathfrak{A})$ *for any* $s \in I(\{i, j, k\})$;
- $t_{i+1,j} = t_{i,j}$ *if* $\alpha_s \notin \pi(\mathfrak{A})$ *for some* $s \in I(\{i, j\})$.

Proof. Follows from the facts, that $l_{i,j} - l_{i,k} \notin \mathbb{Z}$, $i < n$, $j \neq k$ for a generic $V([\,l\,])$ and $l_{i+1,j} = l_{i,j}$ for all i, j if $[\,l\,]$ paramitrizes the highest weight of a highest weight module. □

Let Ω denote the set of all tableaux, satisfying the conditions of Lemma 8.1.1. From the disscusion above it follows, that we can consider our GVMs as $M([\,t\,])$, $[\,t\,] \in \Omega$. Fix $[\,t\,] \in \Omega$, such that $M([\,t\,])$ is the GVM induced from a simple generic GZ module. Consider the vectorspace T, consisting of all tableaux, which satisfy the following conditions:

- $t_{i,j} = 0$ if $\alpha_s \in \pi(\mathfrak{A})$ for all $s \in I(\{i,j\})$;
- $t_{i+1,j} = t_{i,j}$ if $\alpha_s \notin \pi(\mathfrak{A})$ for some $s \in I(\{i,j\})$.

Denote by $\Omega_{[t]}$ the affine hyperplane $[t]+T$. In the next Section we will define an action of W on $\Omega_{[t]}$ and in Section 8.3 we will give a generalization of the BGG Theorem in terms of this action.

8.2 An action of the Weyl group and the Harish-Chandra Theorem

There is a natural action of the Weyl group W on the set of all tableaux: it just permutes the elements of the upper row of a tableau. From the definition of $\Omega_{[t]}$ is is clear, that we can not just restrict the natural action on $\Omega_{[t]}$, because $\Omega_{[t]}$ is not invariant under it. Nevertheless, it is easy to extend the natural action of W on the upper row of $[s] \in \Omega_{[t]}$ to an action of W on $\Omega_{[t]}$. Suppose that W acts naturaly on the upper row of any $[s] \in \Omega_{[t]}$. For $[s] \in \Omega_{[t]}$ and $w \in W$ set $w[s] = [m]$, where $m_{i,j} = s_{i,j}$ if $s_{i,j} \neq s_{n,j}$ and $m_{i,j} = s_{n,w(j)}$ if $s_{i,j} = s_{n,j}$. Clearly, this defines an action of W on $\Omega_{[t]}$.

Lemma 8.2.1. *For $[s] \in \Omega_{[t]}$ and $w \in W(\mathfrak{A})$ we have $M([s]) \simeq M(w[s])$.*

Proof. Follows from the fact that the action of $W(\mathfrak{A})$ does not change $[s]$ up to the action of $\hat{W}$. $\square$

For $[s] \in \Omega_{[t]}$ let $\chi_{[s]}$ denote the central character of the module $M([s])$.

Lemma 8.2.2. *For $[s]$, $[m]$ from $\Omega_{[t]}$ holds $\chi_{[s]} = \chi_{[m]}$ if and only if $[s] \in W[m]$.*

Proof. $\chi_{[s]}$ (resp. $\chi_{[m]}$) coincides with $\chi_{(s_{n,j})_{j=1,2,\dots,n}}$ (resp. $\chi_{(n_{n,j})_{j=1,2,\dots,n}}$). Since the action of W on the upper row of $[s]$ (resp. $[m]$) coincides with the action of W on $\mathfrak{H}^*$, the statement follows from Theorem 3.2.2. $\square$

This result can now be easily extended to a more general statement, which is a generalization of the Harish-Chandra isomorphism Theorem. Denote by $\mathcal{A}$ the vectorspace of all tableau $[t]$, satisfying the following condition: $t_{i+1,j} = t_{i,j}$ if $\alpha_s \notin \pi(\mathfrak{A})$ for some $s \in I(\{i,j\})$. Then the action of W on each $\Omega_{[t]}$ can be easily extended to an action of W on $\mathcal{A}$. Since any element of $\mathcal{A}$ parameterize some element of Γ^*, we have a natural map from Γ to $S(\mathcal{A}^*)$. Let p denote the composition of this map with the natural inclusion of $Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}})$ in Γ.

Consider the following commutative diagram:

$$\begin{array}{ccccc} & & S(\mathcal{A}^*) & \xrightarrow{j} & S(\mathfrak{H}) \\ & & {\scriptstyle p}\uparrow & & \uparrow \\ Z(\mathfrak{G}) & \xrightarrow{(1\otimes\gamma^{\mathfrak{A}})\circ\varphi_{\mathfrak{A}}} & Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}}) & \xrightarrow{(\gamma_{\mathfrak{A}}\circ\varphi)\otimes 1} & S(\mathfrak{H}(\mathfrak{A}))^{W(\mathfrak{A})} \otimes S(\mathfrak{H}_{\mathfrak{A}}) \end{array}$$

Here j is the natural projection with respect to the upper row of the tableau. We can extend our commutative diagram by the row of the invariant algebras:

$$\begin{tikzcd}
 & S(\mathcal{A}^*) \arrow[r, "j"] & S(\mathfrak{H}) \\
Z(\mathfrak{G}) \arrow[r, "(1\otimes\gamma^{\mathfrak{A}})\circ\varphi_{\mathfrak{A}}"] & Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}}) \arrow[u, "p"] \arrow[r, "(\gamma_{\mathfrak{A}}\circ\varphi)\otimes 1"] & S(\mathfrak{H}(\mathfrak{A}))^{W(\mathfrak{A})} \otimes S(\mathfrak{H}_{\mathfrak{A}}) \arrow[u] \\
 & S(\mathcal{A}^*)^W \arrow[u, "q"] \arrow[r, "j_{\mathfrak{A}}"] & S(\mathfrak{H})^W, \arrow[u]
\end{tikzcd}$$

where q is the canonical inclusion and $J_{\mathfrak{A}}$ is an isomorphism. Now we can consider a composition $\psi_{\mathfrak{A}} = q^{-1} \circ (1 \otimes \gamma^{\mathfrak{A}}) \circ \varphi_{\mathfrak{A}}$. From Lemma 8.2.2 and the commutativity of the diagram above we obtain.

Theorem 8.2.1. $Z(\mathfrak{G}) \overset{\psi_S}{\simeq} S(\mathcal{A}^*)^W$.

Proof. Follows from the discussion above. $\square$

Note, that one can also prove Theorem 8.2.1 independently from Lemma 8.2.2 by methods, analogous to that used in the proof of Theorem 6.5.1. Theorem 8.2.1 is a generalization of the Harish-Chandra isomorphism Theorem.

8.3 The BGG Theorem and a simplicity criterion

Now we are ready to state and prove an analogue of the BGG Theorem for modules $M([\,s\,])$, $[\,s\,] \in \Omega_{[\,t\,]}$ in terms of the new action of W on $\Omega_{[\,t\,]}$. To formulate the Theorem we have to introduce a partial order $\prec$ on $\Omega_{[\,t\,]}$. For any transposition (ij), $i < j$ in W we will write $(ij)[\,s\,] \prec [\,s\,]$ provided $s_{n,i} - s_{n,j} \in \mathbb{Z}_+$. We also denote by $\prec$ the transitive closure of $\prec$.

Theorem 8.3.1. *For $[\,s\,], [\,m\,] \in \Omega_{[\,t\,]}$ the following conditions are equivalent:*

1. $M([\,s\,]) \subset M([\,m\,])$.
2. $L([\,s\,])$ *is a subquotient of* $M([\,m\,])$.
3. $[\,s\,] \prec [\,m\,]$

The complete proof of this theorem can be found in [MO]. Here we present only a scheme of the proof. The original proof from [MO] is, in fact, analogous to that of Theorem 6.6.1, or, more generally, to the original proof ([D, Theorem 7.6.23]) of the BGG Theorem. From Theorem 8.3.1 we immediately obtain the following criterion of simplicity for $M([\,s\,])$.

Corollary 8.3.1. *For $[\,s\,], [\,m\,] \in \Omega_{[\,t\,]}$ the following conditions are equivalent:*

1. *$M([\,s\,])$ is simple.*

2. *For any $\alpha \in \Delta_+ \setminus \Delta(\mathfrak{A})$ holds $s_\alpha[\,s\,] \not\prec [\,s\,]$.*

3. *For any $1 \leqslant i < j \leqslant n$ such that $\{\alpha_i, \alpha_{i+1}, \dots, \alpha_{j-1}\} \not\subset \pi(\mathfrak{A})$ holds $s_{n,i} - s_{n,j} \notin \mathbb{N}$.*

As we already mentioned in Section 6.6, in order to be able to follow the original proof of BGG, we have to know the subquotients of the tensor product $M([\,s\,]) \otimes F$ with a finite dimensional $\mathfrak{G}$-module F. Of course, for this it is sufficient to decompose $V([\,l\,]) \otimes F$, where $V([\,l\,])$ is a simple generic GZ module over $\mathfrak{A}$ and F is a simple finite-dimensional module over $\mathfrak{A}$. Clearly, we can assume that $\mathfrak{A} \simeq \mathfrak{gl}(n,\mathbb{C})$, $(\mathfrak{sl}(n,\mathbb{C}))$. Denote by $[\,\hat{p}\,]$ the natural inclusion of $\mathfrak{H}(\mathfrak{A})^*$ in the space of all tableaux, which sends $\lambda \in \mathfrak{H}(\mathfrak{A})^*$ to the tableau $[\,\hat{p}(\lambda)\,]$, such that $\hat{p}(\lambda)_{n,j} = \lambda_j$ and $\hat{p}(\lambda)_{i,j} = 0$ otherwise. Denote also by $\hat{p}^*$ the canonical inverse map, which sends any tableau to the weight defined by its upper row.

Lemma 8.3.1. *Let $V([\,l\,])$ be a simple generic GZ module over $\mathfrak{A}$ and F be a simple finite-dimensional $\mathfrak{A}$-module. Then any subquotient of $V([\,l\,]) \otimes F$ is of the form $V([\,t\,] + [\,\hat{p}(\lambda)\,])$ for $\lambda \in \mathrm{supp}(F)$. Moreover, the multiplicity of $V([\,t\,] + [\,\hat{p}(\lambda)\,])$ as a subquotient of $V([\,l\,]) \otimes F$ equals $\sum \dim(F_\mu)$, where the sum is taken over all $\mu \in \mathrm{supp}(F)$ such that $\hat{p}^*([\,t\,] + [\,\hat{p}(\mu)\,]) \in W(\hat{p}^*([\,t\,] + [\,\hat{p}(\mu)\,]))$.*

Proof. First we note that, according to [GZ1], F decomposes into a direct sum of non-isomorphic one-dimensional Γ-modules. As we have seen in Section 7.4, the same is true for $V([\,l\,])$. Further, from the construction of $V([\,l\,])$ and Section 7.5 it follows that two modules $V([\,l(1)\,])$ and $V([\,l(2)\,])$, such that $l(1)_{i,j} = l(2)_{i,j}$ for all $1 \leqslant i \leqslant n-1$ and all j are isomorphic if and only if their central characters coincide.

Now the statement, that any subquotient of $V([\,l\,]) \otimes F$ is of the form $V([\,t\,] + [\,\hat{p}(\lambda)\,])$ for some $\lambda \in \mathrm{supp}(F)$, follows directly from Kostant's Theorem.

Denote by $\widetilde{\mathrm{supp}}(F)$ the multi-support of F, i.e. the support in which all weights are counted with their multiplicities. Denote by Γ' a natural subalgebra of Γ, which is the Gelfand-Zetlin subalgebra of $U(\mathfrak{gl}(n-1))$ $(U(\mathfrak{sl}(n-1)))$. By the construction, the module $V([\,l\,])$ is dense with respect to Γ' (i.e. its Γ'-support coincides with a weight lattice) and all Γ'-weight subspaces are one-dimensional. Since tensoring with a finite-dimensional module preserves the weight lattice, we conclude that $V([\,l\,]) \otimes F$ is a dense module and all non-trivial Γ'-weight subspaces of it are of dimension $\dim F$. Hence, applying [DOF1, Corollary 33], we obtain that the length of $V([\,l\,]) \otimes F$ equals $\dim(F)$.

Now we want to substitute $V([\,l\,])$ by a finite-dimensional module E and apply a similarity of $V([\,l\,])$ with E, mentioned in Chapter 7. Suppose that E lies far enough from the walls, i.e. the length of $E \otimes F$ equals $\dim(F)$. We will call a Γ'-weight subspace of E *generic* provided the dimension of this weight subspace in $E \otimes F$ equals $\dim(F)$. Clearly, any E lying far from the walls has a generic Γ'-weight subspace. Fix E lying far from the walls and a generic Γ'-weight ν (i.e. E_ν is a generic Γ'-weight subspace). Fix $z \in Z(\mathfrak{A})$. Choose a basis in $(E \otimes F)_\nu$ and write the characteristic polynomial $f_z(X)$ of z in this basis.

Let λ be a highest weight of E. From the Littlewood-Richardson rule we obtain

$$f_z(X) = \pm\left(\prod_{\lambda'\in\widetilde{\mathrm{supp}}(F)} (X - \chi_{\lambda+\lambda'}(z)).\right) \tag{8}$$

Note that from the GZ formulae, which define the action of the generators of $\mathfrak{G}$ on finite-dimensional modules, it follows that the coefficients of $f_z(X)$ are just the rational functions in the entries of the tableau, parametrizing ν. Since we can find sufficiently many modules E lying far from the walls and sufficiently many generic Γ'-weight subspaces in E we conclude for any generic Γ'-weight ν in any simple module M, defined using Gelfand-Zetlin formulae, the polynomial $f_z(X)$ has also the form (8).

To complete the proof we only have to recall that the module $V([\,l\,])$ was constructed using the GZ formulae and, as it was mentioned above, any Γ'-weight subspace of $V([\,l\,])$ is generic. □

From Lemma 8.3.1, we see that generic GZ modules behave even more regular with respect to tensoring with finite-dimensional modules, than finite-dimensional modules themselves. One notes that Lemma 8.3.1 is a precise analogue of the corrsponding statement for modules $V(a,b)$ from Section 6.6. In fact, it also includes the mentioned statement in the case of simple $V(a,b)$.

In order to feel free while working with $M([\,s\,])$, $[\,s\,]\in\Omega_{[\,t\,]}$, one needs also some classical results about GVMs. More precisely, one needs an analogue of Proposition 6.2.1.

Proposition 8.3.1. *For* $[\,s\,],[\,m\,]\in\Omega_{[\,t\,]}$ *holds* $\dim\mathrm{Hom}(M([\,s\,]),M([\,m\,]))\leqslant 1$ *and any non-zero homomorphism from this space is injective.*

Proof. Analogous to the proof of the analogous statement for Verma modules, but uses some technical innovations, like relative Gelfand-Kirillov dimension. These are purely technical calculations so we omit them. The reader can consult [MO, Section 6]. □

Now the strategy to prove Theorem 8.3.1 is quite transparent.

Proof of Theorem 8.3.1. Again the first statement clearly implies the second one. So, first we prove that $[\,s\,]\prec[\,m\,]$ implies $M([\,s\,])\subset M([\,m\,])$. For this it is enough to consider $\alpha\in\Delta_+\setminus\Delta(\mathfrak{A})$ and $[\,s\,]=s_\alpha[\,m\,]\prec[\,m\,]$. First assume that $\pi=\pi(\mathfrak{A})\cup\{\alpha_{n-1}\}$ and that s_α transposes $m_{n,i}$ with $m_{n,n}$. If we assume that $m_{n,i}-m_{n,i'}\notin\mathbb{Z}$ for all $i'\neq i,n$, the module $M([\,m\,])$ can be constructed via GZ formulae and the statement follows by direct calculation. Now for the general $\alpha\in\Delta_+\setminus\Delta(\mathfrak{A})$ the proof repeats the proof of the analogous statement in Theorem 6.6.1.

Now we have to prove that the second statement implies the third one. Having Lemma 8.3.1 and Proposition 8.3.1, this is a direct translation of the original proof of BGG ([D, Theorem 7.6.23]). □

8.4 Tableaux realization of Verma modules and GVMs

In the last two Chapters we saw, that it is quite convenient to work with a GZ $\mathfrak{G}$-module M, constructed directly from the GZ formulae. For this the set of tableaux, parametrizing the GZ weights of M, should be chosen such that the GZ formulae are well-defined on all these tableaux, i.e. all denominators of the GZ formulae are non-zero. This leads us to a useful notion of a $\mathfrak{G}$-module having a tableaux realization. In this section we define a family of $\mathfrak{G}$-modules having a tableaux realization and construct a tableaux realization for some Verma modules and some GVMs. The content of this section is taken from [M4], where the notion of a module having a tableaux realization was introduced.

We fix $\mathfrak{G} = \mathfrak{gl}(n, \mathbb{C})$ (or $\mathfrak{sl}(n, \mathbb{C})$), but we note that analogous constructions can be done in other cases (see, for example, [M5]). A tableau $[\, l\,]$ will be called good, provided $l_{i,j} - l_{i,k} \notin \mathbb{Z}$ for all $i < n$ and $j \neq k$. A GZ module M over $\mathfrak{G}$ is said to have a tableaux realization, if it decomposes into a direct sum of one-dimensional Γ-modules, parameterized by good tableaux. From Theorem 7.1.1 it follows that if an indecomposable $\mathfrak{G}$-module M has a tableaux parametrization, then there exists a set $B(M)$ of tableaux, which can be taken as a basis for M, and action of $\mathfrak{G}$ on this basis is defined by the GZ formulae. The simplest examples of modules having a tableaux realization are finite-dimensional $\mathfrak{G}$-modules or generic GZ modules $V([\, l\,])$. In this Section we also construct a tableaux realization for some Verma modules and some GVMs. Tha main advantage of such modules is that they are given by precise formulae, so they are quite easy for calculations.

Let $a = (a_1, \dots, a_n) \in \mathfrak{H}^*$ be such that $a_k - a_j \notin \mathbb{Z}$ and $a_j > a_k$ for all $1 \leqslant j < k \leqslant n$. Consider the tableau $[\, l\,] = [\, l\,](a)$ defined as follows: $l_{i,j} = a_j$ for all $1 \leqslant j \leqslant i \leqslant n$. Let $B([\, l\,])$ denote the set of all tableaux $[\, t\,]$ satisfying the following conditions:

1. $l_{n,j} = t_{n,j}$, $j = 1, \dots, n$;
2. $l_{i,j} - t_{i,j} \in \mathbb{Z}_+$ for all $1 \leqslant j \leqslant i < n$;
3. $t_{i,j} \geqslant t_{i-1,j}$ for all $1 < i \leqslant n$, $1 \leqslant j \leqslant i$.

It follows directly from the GZ formulae, that $[\, t\,] \pm [\, \delta^{i,j}\,] \in B([\, l\,])$ for $[\, t\,] \in B([\, l\,])$, if $a^{\pm}_{ij}([\, t\,]) \neq 0$. Let $M = M([\, l\,])$ denote the $\mathfrak{G}$-module with a basis $B([\, l\,])$ and the action of generators of $\mathfrak{G}$, defined by the GZ formulae (here we have an abuse of notation, but we will see that $M([\, l\,])$ is a Verma module with the highest weight $a - \rho$, so everything is correct). It follows directly from Section 7.4, that $M([\, l\,])$ is really a $\mathfrak{G}$-module.

Theorem 8.4.1. *1. $M([\, l\,]) \simeq M(a)$.*

2. $M([\, l\,])$ is simple.

3. $M([\, l\,]) \otimes F$ is completely reducible for any finite-dimensional $\mathfrak{G}$-module F.

4. $\mathrm{Ext}^1(M([\, l\,](a)), M([\, l\,](a'))) = \mathrm{Ext}^1(M([\, l\,](a')), M([\, l\,](a))) = 0$ for any a, a' as above.

Proof. It follows directly from the GZ formulae, that the GZ weight space of $M([\,l\,])$, corresponding to $[\,l\,]$ is a highest weight and generates $M([\,l\,])$. Thus $M([\,l\,])$ is a quotient of $M(a)$. Comparing the dimensions of the weight spaces we obtain that they coincide, hence $M([\,l\,]) \simeq M(a)$. Direct application of the GZ formulae also shows that $[\,l\,]$ is the only highest weight of $M([\,l\,])$. Hence $M([\,l\,])$ is simple. The third statement follows from Corollary 5.4.1. The last one follows from the third one and Theorem 3.7.3. $\square$

Now, as in Section 8.2, assume that $\pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and fix $\pi(\mathfrak{A}) \subset \pi$. Consider a tableau $[\,t\,]$, satisfying the following conditions:

- $t_{i,j} - t_{i,k} \notin \mathbb{Z}$ for any $i < n$ and any j, k;
- $t_{i+1,j} = t_{i,j}$ if $a_s \notin \pi(\mathfrak{A})$ for some $s \in I(\{i,j\})$.

Let $B([\,t\,])$ denote the set of all tableaux $[\,s\,]$, satisfying the following conditions:

1. $s_{n,j} = t_{n,j}$, $j = 1, \dots, n$;
2. $s_{i,j} - t_{i,j} \in \mathbb{Z}_+$ for all $1 \leqslant j \leqslant i < n$ such that $a_s \notin \pi(\mathfrak{A})$ for some $s \in I(\{i,j\})$;
3. $s_{i,j} \geqslant s_{i-1,j}$ for all $1 < i \leqslant n$, $1 \leqslant j \leqslant i$ such that $a_s \notin \pi(\mathfrak{A})$ for some $s \in I(\{i,j\})$;
4. $s_{i,j} - t_{i,j} \in \mathbb{Z}$ if $a_s \in \pi(\mathfrak{A})$ for any $s \in I(\{i,j\})$.

It follows directly from the GZ formulae, that $[\,s\,] \pm [\,\delta^{i,j}\,] \in B([\,t\,])$ for $[\,s\,] \in B([\,t\,])$, if $a^{\pm}_{ij}([\,s\,]) \neq 0$. Let $M = M([\,t\,])$ denote the $\mathfrak{G}$-module with a basis $B([\,t\,])$ and the action of generators of $\mathfrak{G}$, defined by the GZ formulae (here we have an abuse of notation, but we will see that $M([\,t\,])$ is a GVM, so again everything is correct). It follows directly from Section 7.4 that $M([\,t\,])$ is really a $\mathfrak{G}$-module.

Theorem 8.4.2. *$M([\,l\,])$ is a simple GVM, induced from a simple generic GZ module over $\mathfrak{A}$. Moreover, $M([\,l\,]) \otimes F$ is completely reducible for any finite-dimensional $\mathfrak{G}$-module F.*

Proof. Folows directly from Section 8.2 and Corollary 5.4.1. $\square$

Theorem 8.4.2 differs from the description in Section 8.2, since here we present a precise construction of $M([\,t\,])$, which was not done in Section 8.2.

9 Generalized Shapovalov form

In this Chapter we generalize a modern technique of studying of Verma modules using the Shapovalov form ([KK]) on GVMs. This machinery was worked out in [KM3], so we refer the reader to this paper for all missing technical details. In particular, we define a generalization of the Shapovalov form on a GVM $M(\lambda, p)$, see Chapter 6. We calculate the determinant of this form. Using the determinant formula we generalize the BGG Theorem about the embeddings of GVMs. In particular, these results covers all known generalizations of the BGG Theorem for α-stratified modules, obtained in Chapter 6 and in [F7] in the case of an affine algebra of type $A_1^{(1)}$. Since the proposed machinery works in a more general situation than just simple finite-dimensional algebras, we begin with the setup.

9.1 Definition of the form

Let $\mathfrak{G}$ be a complex contragradient Lie algebra associated with a complex $(n\times n)$-matrix $A=(a_{ij})$. We fix the standard triangular decomposition $(\mathfrak{G}_+, \mathfrak{H}, \Delta_+, \sigma)$ of $\mathfrak{G}$. Let Δ be the set of roots of the algebra $\mathfrak{G}$ i.e. $\Delta=\Delta_+\cup -\Delta_+$ ([MP]). For the rest of this Chapter we fix a basis π of Δ_+ and an element $\alpha\in\pi$ satisfying the following conditions: the subalgebra $\mathfrak{A}$ of $\mathfrak{G}$ generated by $\mathfrak{G}_{\pm\alpha}$ should be isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{G}$ should be an integrable (i.e. direct sum of finite-dimensional modules) $\mathfrak{A}$-module under the adjoint action. This generalizes the situation, described in Chapter 6, where α was a basis of the root system $\Delta(\mathfrak{A})$ and $\mathfrak{A}$ was the semisimple part of the Levi factor of $\mathcal{P}$. We can just let $\mathcal{P}$ be generated by $\mathfrak{G}_+$, $\mathfrak{H}$ and $\mathfrak{G}_{-\alpha}$. So, we keep all the notation from Chapter 4. We fix the dual elements $H_\beta\in\mathfrak{H}$ for all $\beta\in\Delta_+$.

Under the above choice of α the simple reflection s_α on $\mathfrak{H}^*$ is correctly defined and satisfies all the standard properties of a simple reflection. Let P denote the standard Kostant partition function with respect to π and $\hat{P}$ denote the standard Kostant partition function with respect to $s_\alpha(\pi)$. By a *quasiroot* we will mean any element $q\alpha\in\mathfrak{H}^*$, where $\alpha\in\Delta_+$ and q is a positive rational number.

For a contragradient Lie algebra with a symmetrizable Cartan matrix let $(\cdot,\cdot)$ denote the bilinear form on $\mathfrak{G}$ ([Ka, MP]). The corresponding bilinear form on $\mathfrak{H}^*$ will also be denoted by $(\cdot,\cdot)$. For a restricted weight $\mathfrak{G}$-module V we introduce the action of the Kac-Casimir operator Ω ([KK]) on V as follows: for $v\in V_\mu$, $\mu\in\mathfrak{H}^*$ let

$$\Omega v=(\mu+2\rho,\mu)v+2\sum_{\beta\in\mathbb{Q}_+}\sum_i e_{-\beta}^{(i)}e_\beta^{(i)}v,$$

where ρ is an element in $\mathfrak{H}^*$ such that $(\rho,\gamma)=1$ for all $\gamma\in\pi$, $e_\beta^{(i)}$ form a basis of $\mathfrak{G}_\beta$ and $e_{-\beta}^{(i)}$ form the dual basis of $\mathfrak{G}_{-\beta}$. One can easily check that the form $(\cdot,\cdot)$ on $\mathfrak{H}^*$ is invariant under s_α.

Set $\mathfrak{H}(\alpha)=U(\mathfrak{H})\otimes\mathbb{C}[c]$. Consider the following decomposition of $U(\mathfrak{G})$ ([F7, page 88]):

$$U(\mathfrak{G})=(\mathfrak{N}(\mathfrak{A})U(\mathfrak{G})+U(\mathfrak{G})\mathfrak{N})\oplus\mathfrak{H}(\alpha)\mathbb{C}[X_\alpha]X_\alpha\oplus\mathfrak{H}(\alpha)\mathbb{C}[X_{-\alpha}]X_{-\alpha}\oplus\mathfrak{H}(\alpha).$$

Let p be the projection of $U(\mathfrak{G})$ on $\mathfrak{H}(\alpha)$ with respect to the above decomposition. We define an α-Shapovalov form (or a generalized Shapovalov form) F_α on $U(\mathfrak{G})$ as a symmetric bilinear form with values in $\mathfrak{H}(\alpha)$ as follows (see also [F7, KK, MP, Sh]):

$$F_\alpha(x, y) = p(\sigma(x)y), \quad x, y \in U(\mathfrak{G}).$$

It is straightforward that the graded components $U(\mathfrak{G})_\xi$, $\xi \in \mathbb{Z}\Delta$, are orthogonal with respect to F_α. Moreover, F_α is contravariant, i.e. $F_\alpha(zx, y) = F_\alpha(x, \sigma(z)y)$ for all $x, y, z \in U(\mathfrak{G})$.

Consider a vector subspace

$$\mathcal{M} = U(\mathfrak{N}(\mathfrak{A}) \oplus \mathfrak{G}_\alpha) + U(\mathfrak{N}(\mathfrak{A}) \oplus \mathfrak{G}_{-\alpha})$$

in $U(\mathfrak{G})$. For $\xi \in \mathbb{Z}\Delta$ we set $\mathcal{M}_\xi = \mathcal{M} \cap U(\mathfrak{G})_\xi$. Clearly, each $\mathcal{M}_\xi$ is finite-dimensional. To calculate the dimension of $\mathcal{M}_\xi$ we have to introduce the notion of the Kostant α-function P_α (see also [MO]).

For $\gamma = \sum_{\beta\in\pi} a_\beta \beta \in \Delta$ set $\psi_\alpha(\gamma) = \sum_{\beta\in(\pi\setminus\{\alpha\})} a_\beta\beta$. Define the Kostant α-function P_α : $\mathfrak{H}^* \to \mathbb{N} \cup \{0\}$ as follows: for $\lambda \in \mathfrak{H}^*$ set $P_\alpha(\lambda)$ to be the maximum number of the decompositions

$$\lambda + n\alpha = \sum_{\beta\in\Delta_+\setminus\{\alpha\}} n_\beta \psi_\alpha(\beta)$$

with non-negative integer coefficients, where n runs through all integers. It follows easily from the definition of P_α that $\dim \mathcal{M}_{-\xi} = P_\alpha(\xi)$.

For $\eta \in \mathbb{Z}\Delta$ we denote by F_α^η the restriction of F_α to $\mathcal{M}_{-\eta}$.

Fix $a, b \in \mathbb{C}$ and denote by $\hat{V}(a, b)$ the $\mathfrak{A}$-module uniquely defined by the following properties: $\hat{V}(a, b)$ has the same subquotients as $V(a, b)$ and $\hat{V}(a, b)$ is generated by $\hat{V}(a, b)_a$ as a $\mathfrak{A}$-module. Clearly, $\hat{V}(a, b) \simeq V(a, b)$ if the module $V(a, b)$ is α-stratified (and hence simple by Proposition 5.3.1). We will denote by $M(\lambda, b)$, $\lambda(H_\alpha) = a$, the corresponding GVM $M_\mathcal{P}(\hat{V}(a, b))$. We note that $M_\mathcal{P}(\hat{V}(a, b)) \simeq M_\mathcal{P}(V(a, b))$ if and only if $V(a, b)$ is α-stratified, but these two modules always have the same subquotients. Let $0 \neq v_{(\lambda,b)} \in M(\lambda, b)_\lambda$ be a canonical generator of $M(\lambda, b)$. It is well-known (see for example [CF]) that $\mathcal{M}v_{(\lambda,b)} = M(\lambda, b)$. We can naturally identify $\mathfrak{H}(\alpha)$ with the ring of polynomials on the $\mathbb{C}$-space $\{(\lambda, b) \mid \lambda \in \mathfrak{H}^*, b \in \mathbb{C}\}$ by setting $c^* = (0, 1)$. Thus we can define the value $F_\alpha^\eta((\lambda, b))$ of F_α^η at the point (λ, b).

Now we can define a bilinear $\mathbb{C}$-valued form, $\hat{F}_\alpha$, on $M(\lambda, b)$ by setting

$$\hat{F}_\alpha(u_1 v_{(\lambda,b)}, u_2 v_{(\lambda,b)}) = F_\alpha(u_1, u_2)((\lambda, b)), \quad u_1, u_2 \in \mathcal{M}.$$

The following Lemma presents the standard properties of $\hat{F}_\alpha$:

Lemma 9.1.1. *1. The kernel of $\hat{F}_\alpha$ coincides with the unique maximal submodule of the module $M(\lambda, b)$.*

2. *$\hat{F}_\alpha$ is non-degenerate on $M(\lambda, b)$ if and only if $M(\lambda, b)$ is simple.*

3. *All weight subspaces of $M(\lambda, b)$ are orthogonal with respect to $\hat{F}_\alpha$.*

Proof. Proof is analogous to that for the classical Shapovalov form (see for example [MP]). □

9.2 Determinant formula

The main advantage of the generalized Shapovalov form is that it is possible to calculate the determinant of its restriction to a weight space of $M(\lambda, v)$. This is given by the following *determinant formula.*

Theorem 9.2.1. *Let $\mathfrak{G}$ be a contragradient Lie algebra with a symmetrisable Cartan matrix. Then for any $\eta \in \mathfrak{H}^*$*

$$\begin{aligned}
\det F_\alpha^\eta = &\prod_{k=1}^{\infty} (X_{-a}X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))^{P(\eta - k\alpha)} \times \\
&\times \prod_{k=1}^{\infty} (X_{-a}X_\alpha + (1-k)(H_\alpha + \rho(H_\alpha) - (1-k)))^{\hat{P}(\eta - k\alpha)} \times \\
&\times \prod_{\substack{\beta \in \Delta_+ \setminus \{\alpha\},\\ s_\alpha(\beta) = \beta}} \prod_{k=1}^{\infty} \left(H_\beta + \rho(H_\beta) - k\frac{(\beta,\beta)}{2} \right)^{P_\alpha(\eta - k\beta)} \times \\
&\times \prod_{\substack{\{\beta, s_\alpha(\beta) | \\ \beta \in \Delta_+ \setminus \{\alpha\},\\ s_\alpha(\beta) \neq \beta\}}} \prod_{k=1}^{\infty} \left(\left(H_\beta + \rho(H_\beta) - k\frac{(\beta,\beta)}{2} \right) \cdot \right. \\
&\left. \cdot \left(H_{s_\alpha(\beta)} + \rho(H_{s_\alpha(\beta)}) - k\frac{(\beta,\beta)}{2} \right) + \alpha(H_\beta)\alpha(H_{s_\alpha(\beta)})X_{-\alpha}X_\alpha \right)^{P_\alpha(\eta - k\beta)}.
\end{aligned}$$

up to a non-zero constant factor, where all the roots β are taken with their multiplicities.

We note that the product in the last factor of the above formula runs through all non-orderd pairs $\{\beta, s_\alpha(\beta)\}$ such that $\beta \neq s_\alpha(\beta)$.

The formula from Theorem 9.2.1 looks quite unattractive, however we will later see (during the proof), that the division into four factors as above is quite natural. In this Section we present a proof of this formula, a more detailed version of which can be found in [KM3]. We begin with two necessary lemmas.

Lemma 9.2.1. *Up to a non-zero constant factor,* $\det F_\alpha^\eta$ *is a product of factors having one of the following forms:*

1. $(X_{-a}X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))$;

2. $(X_{-a}X_\alpha + (1-k)(H_\alpha + \rho(H_\alpha) - (1-k)))$;

3. $\left(H_\beta + \rho(H_\beta) - k\frac{(\beta,\beta)}{2}\right)$, *where* β *is a quasiroot such that* $s_\alpha(\beta) = \beta$.

4. $\left(\left(H_\beta + \rho(H_\beta) - k\frac{(\beta,\beta)}{2}\right) \cdot \left(H_{s_\alpha(\beta)} + \rho(H_{s_\alpha(\beta)}) - k\frac{(\beta,\beta)}{2}\right) + \alpha(H_\beta)\alpha(H_{s_\alpha(\beta)})X_{-\alpha}X_\alpha\right)$, *where* β *is a quasiroot such that* $s_\alpha(\beta) \neq \beta$.

Proof. Consider a GVM $M(\lambda, b)$ generated by a non-zero element $v_{(\lambda,b)} \in M(\lambda,b)_\lambda$. Clearly, $M(\lambda, b)$ is restricted, hence the action of Ω is well-defined on it. Applying Ω to $v_{(\lambda,b)}$ one obtains $\Omega v_{(\lambda,b)} = ((\lambda+2\rho,\lambda) + (b - ((\lambda,\alpha)+1)^2)/2)v_{(\lambda,b)}$ and thus Ω acts as $((\lambda+2\rho,\lambda) + (b-(\lambda+\rho,\alpha)^2)/2)id$ on $M(\lambda,b)$.

There are two general possibilities for $M(\lambda, b)$ to be reducible. The first one: $M(\lambda,b)$ is reducible if $\hat{V}(a,b)$ is. The second one: $M(\lambda,b)$ is reducible if there exists an α-highest weight vector in some $M(\lambda,b)_\mu$ with $\mu - \lambda \notin \mathbb{Z}\alpha$. First assume that $\hat{V}(a,b)$ is reducible. This is possible if and only if for some $m \in \mathbb{N}$, $X_\alpha^m X_{-\alpha}^m v_{(\lambda,b)} = 0$ or $X_{-\alpha}^m X_\alpha^m v_{(\lambda,b)} = 0$ holds. By direct calculations with $U(\mathfrak{G}_\alpha)$ we obtain

$$\prod_{k=1}^{m} (X_{-a}X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))\, v_{(\lambda,b)} = 0$$

or

$$\prod_{k=1}^{m} (X_{-a}X_\alpha + (1-k)(H_\alpha + \rho(H_\alpha) - (1-k)))\, v_{(\lambda,b)} = 0.$$

Further, suppose that there exists an α-highest weight vector w in $M(\lambda,b)_\mu$ for some $\mu \in \mathfrak{H}^*$ such that $\mu - \lambda \notin \mathbb{Z}\alpha$. Then the eigenvalues of Ω on $v_{(\lambda,b)}$ and w coincide and we obtain

$$(\lambda+2\rho,\lambda) + (b - ((\lambda,\alpha)+1)^2)/2 = (\mu+2\rho,\mu) + (b' - ((\mu,\alpha)+1)^2)/2 \tag{9}$$

for some $b' \in \mathbb{C}$. Clearly, the difference $b'-b$ polynomialy depends on $\sqrt{b}$ after fixing λ^α and $\mu-\lambda$ (see [FM1]). Thus the formula above can be applied to the case $(a+1+2n)^2 = b$, $n \in \mathbb{Z}$. For such $N(a,b)$ we get that $M(\lambda,b)$ is an extension of two Verma modules (with respect to different bases in Δ). Now, using the fact that the action of Ω on a Verma module can be calculated at the highest weight vector, we obtain that $b' = b + 2\sqrt{b}(\mu-\lambda,\alpha) + (\mu-\lambda,\alpha)^2$ (here $\sqrt{b}$ is the complex square root function which has two different values as soon as $b \neq 0$).

If $(\mu - \lambda, \alpha) = 0$ the equality (9) reduces to $(\lambda+2\rho,\lambda) = (\mu+2\rho,\mu)$ and we can use the same arguments as in proof of [KK, Lemma 3.2] obtaining the factors $(H_\beta - \rho(H_\beta) - (\beta,\beta)/2)$ (here β is not necessary a quasiroot).

If $(\mu-\lambda,\alpha)\neq 0$ we can take two equalities of the form (9) corresponding to different values b_1 and b_2 of $\sqrt{b}$, transfer everything in the left-hand side and multiply them. We obtain the following (here $\beta=\lambda-\mu$):

$$(2(\lambda+\rho,\beta)-(\beta,\beta)-(\lambda+\rho,\alpha)(\beta,\alpha))^2-b(\beta,\alpha)^2=0.$$

The last equality can be rewritten in the form

$$(2(\lambda+\rho,\beta)-(\beta,\beta)-(\lambda+\rho,\alpha)(\beta,\alpha))^2-(\beta,\alpha)^2(\lambda+\rho,\alpha)^2-(\beta,\alpha)^2(b-(\lambda+\rho,\alpha)^2)=0.$$

We note that

$$(\lambda+\rho,\beta)-(\lambda+\rho,\alpha)(\beta,\alpha)=(\lambda+\rho,\beta-(\beta,\alpha)\alpha)=(\lambda+\rho,\beta-\frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha)=(\lambda+\rho,s_\alpha(\beta)).$$

From this it follows that

$$\begin{aligned}(2(\lambda+\rho,\beta)-(\beta,\beta))\,(2(\lambda+\rho,s_\alpha(\beta))-(s_\alpha(\beta),s_\alpha(\beta)))&+\\ +4(\frac{1}{4}(b-(\lambda+\rho,\alpha)^2))(\alpha,\beta)(\alpha,s_\alpha(\beta))&=0.\end{aligned}$$

Taking into account that $\frac{1}{4}(b-(\lambda+\rho,\alpha)^2)$ is an eigenvalue of the operator $X_{-\alpha}X_\alpha$, we obtain the factor of the form

$$\big((H_\beta+\rho(H_\beta)-(\beta,\beta)/2)\,\big(H_{s_\alpha(\beta)}+\rho(H_{s_\alpha(\beta)})-(\beta,\beta)/2\big)+\alpha(H_\beta)\alpha(H_{s_\alpha(\beta)})X_{-\alpha}X_\alpha\big)$$

with the same arguments as in [KK, Lemma 3.2].

Now we only need to show that all β appearing above are quasiroots. Suppose not. Thus we will have some factor of the determinant of F_α corresponding to a non-quasiroot β. Calculating F_α on a Verma submodule for some reducible $N(a,b)$ we obtain a contradiction with [KK, Theorem 1]. Lemma is proved. □

To proceed we define a new *α-gradation* on $U(\mathfrak{G})$ by setting the degree of $X_{\pm\alpha}$ and the degree of H_α to be 0 and all the degrees of other base elements in $\mathfrak{G}$ to be 1.

Lemma 9.2.2. *Up to a factor of grade zero the leading term of* $\det F_\alpha^\eta$ *with respect to the α-gradation is equal to*

$$\prod_{\beta\in\Delta_+\setminus\{\alpha\}}\prod_{k=1}^{\infty}H_\beta^{P_\alpha(\eta-k\beta)}.$$

Proof. From the classical Shapovalov determinant formula ([KK]) it follows that the above formula is correct for $\det F_\alpha^{\eta-l\alpha}$, where $l\in\mathbb{N}$ is big enough. To complete the proof it is sufficient to show that the leading term of $\det F_\alpha^\eta$ in the α-gradation does not depend on the shift on α.

Choose some PBW monomial base $v_1, \dots, v_t$ in $\mathcal{M}_{-\eta}$ and suppose that as soon as some v_i contains $X_{-\alpha}$ this monomial should start with this $X_{-\alpha}$. Consider the elements $X_\alpha v_1, \dots, X_\alpha v_t$ and let W be a linear span of these elements. For $1 \leqslant i \leqslant t$ set $\hat{v}_i = X_\alpha v_i$ if v_i does not contain $X_{-\alpha}$ and $\hat{v}_i = w_i$ if $v_i = X_{-\alpha} w_i$. Clearly, elements $\hat{v}_1, \dots, \hat{v}_t$ form a basis of $\mathcal{M}_{-\eta+\alpha}$. Moreover, it follows from the definition of $\hat{v}_i$ that up to a factor of zero degree the leading term of $\det F_\alpha^{\eta-\alpha}$ coincides with the leading term of the determinant of the form F_α restricted to W (we will denote it by $F_\alpha(W)$). Since the base change from $v_1, \dots, v_t$ to $X_\alpha v_1, \dots, X_\alpha v_t$ is defined by the elements of zero grade it follows that $\det F_\alpha^\eta$ differs from $\det F_\alpha(W)$ by a factor of grade zero. This implies that the leading term of $\det F_\alpha^\eta$ in the α-gradation does not depend on the shift on α. $\square$

The proof of the classical determinant formula for Shapovalov form ([KK]) is based on the so-called Jantzen filtration of a Verma module. In order to prove Theorem 9.2.1 we have to generalize this notion for GVMs. Choose $z \in \mathfrak{H}^*$ such that $(z, \beta) \neq 0$ for all $\beta \in \mathbb{Z}\Delta_+ \setminus 0$. Let t be an indeterminate. By standard techniques, we can extend $M(\lambda, b)$ to the module $\widetilde{M}(\widetilde{\lambda, b})$ over the algebra $\widetilde{U(\mathfrak{G})} = U(\mathfrak{G}) \otimes \mathbb{C}[t]$, where $\widetilde{(\lambda, b)} = (\lambda, b) + t(z, 1) \in \widetilde{\mathfrak{H}(\alpha)}^* = \mathfrak{H}(\alpha)^* \otimes \mathbb{C}[t]$. Further we can trivially extend σ on $\widetilde{U(\mathfrak{G})}$ and construct a bilinear form $\widetilde{F_\alpha}$. Using $\widetilde{F_\alpha}$ one can define a bilinear $\mathbb{C}[t]$-valued form $\widetilde{F_\alpha^t}$ on $\widetilde{M}(\widetilde{\lambda, b})$. Setting $\widetilde{M^i}$ to be equal to the set of all elements v in $\widetilde{M}(\widetilde{\lambda, b})$ such that $\widetilde{F_\alpha^t}(v, w)$ is divisible by t^i for all $w \in \widetilde{M}(\widetilde{\lambda, b})$ we define a Jantzen filtration

$$\widetilde{M}(\widetilde{\lambda, b}) = \widetilde{M^0} \supset \widetilde{M^1} \supset \dots$$

on $\widetilde{M}(\widetilde{\lambda, b})$. The canonical epimorphism $\varphi : \widetilde{M}(\widetilde{\lambda, b}) \to M(\lambda, b)$ $(t \to 0)$ induces a filtration

$$M(\lambda, b) = M^0 \supset M^1 \supset \dots$$

of $M(\lambda, b)$ which we will also call a *Jantzen filtration.*

Proof of Theorem 9.2.1. We have only to calculate the degrees in $\det F_\alpha^\eta$ of the factors described in Lemma 9.2.1. For a quasiroot β, which is not proportional to α, the proof of this fact is exactly the same as in [KK, Proof of Theorem 1] because of Lemma 9.2.2 and the remark that the functions $P_\alpha(x - y)$ $y \in \alpha^\perp$ are linearly independent (here $\alpha^\perp$ is taken with respect to $(\cdot, \cdot)$).

Thus we have only to calculate the degrees of the factors of the form

- $(X_{-a} X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))$;
- $(X_{-a} X_\alpha + (1 - k)(H_\alpha + \rho(H_\alpha) - (1 - k)))$.

We will do it for the first kind of factors. One can apply analogous arguments for the second case. Consider a factor $(X_{-a} X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))$ for some fixed $k \in \mathbb{N}$. Let $N(a, b)$ be such that it has the unique submodule starting at the highest weight $a - k\alpha$. We note that in this case $a \notin \mathbb{Z}$. One can easily choose $\lambda \in \mathfrak{H}^*$ $(\lambda(H_\alpha) = a)$ such that

GVM $M(\lambda, b)$ has the unique non-trivial submodule N. Clearly, in the described case N is isomorphic to the Verma module $M(\lambda - k\alpha)$. From the definition of Jantzen filtration we have $M^0 = M(\lambda, b)$ and $M^1 = N$. Our goal is to prove that $M^2 = 0$. Since N is irreducible it follows that either $M^2 = N$ or $M^2 = 0$. Consider $\widetilde{U(\mathfrak{G})}$-modules $\widetilde{M(\lambda, b)}$ and $\widetilde{N}$ and let w be a canonical generator of $\widetilde{N}$. Use the definition of $\widetilde{F_\alpha}$ to calculate $\widetilde{F_\alpha^t}(w, w)$. By the direct application of $\mathfrak{sl}(2)$-theory we obtain that

$$\widetilde{F_\alpha^t}(w, w) = \prod_{i=1}^{k} f_k(t),$$

where $f_k(t) \in \mathbb{C}[t]$ such that $f'_k(0) \neq 0$ satisfy the following condition: the difference between constant terms in f_{k+1} and f_k is equal to $a - 2k$. Since a is not integer it follows that the product in the formula above is divisible by at most t. But it is divisible by t since $\widetilde{N}$ is a submodule. Thus the canonical generator of N belongs to M^1 and does not belong to M^2. Hence $M^2 = 0$. Now we can claim that from the construction of the Jantzen filtration, it follows immediately that $\det F_\alpha^\eta$ is divisible exactly by the $P(\eta - k\alpha)$-th power of $(X_{-a}X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))$ (see [KK, Proof of Theorem 1] and [MP, Section 6.6]). This completes our proof. $\square$

9.3 Generalization of the BGG Theorem

As in the classical case, the determinant formula for F_α enables one to prove a generalization of the BGG Theorem (see [KK, Theorem 2] and [MP, Section 6.7]). In this Section we will formulate an analogous result for our GVMs induced from $\mathfrak{A}$.

For $\lambda, \mu \in \mathfrak{H}^*$ and $b_1, b_2 \in \mathbb{C}$ we set $(\lambda, b_1) \rightharpoonup (\mu, b_2)$ in one of the following cases:

1. $b_1 = b_2$ and $\lambda = \mu - k\alpha$ for some $k \in \mathbb{Z}$;

2. $b_1 = b_2 \pm 2\sqrt{b_2}(k\beta, \alpha) + (k\beta, \alpha)^2$ for $k \in \mathbb{N}$ and $\beta \in \Delta_+ \setminus \{\alpha\}$ such that $\lambda = \mu - k\beta$ and

$$2(\lambda + \rho)(H_\beta) - k(\beta, \beta) - (\lambda + \rho)(H_\alpha)(\beta, \alpha) = \pm\sqrt{b_2}(\beta, \alpha).$$

   (here an analytic branch of $\sqrt{z}$ function is fixed).

Denote by $\prec$ the transitive closure of the relation $\rightharpoonup$ on $\mathfrak{H}^* \times \mathbb{C}$.

For each pair $\beta \neq s_\alpha(\beta)$ of roots in Δ_+ we fix some bijective map

$$\mathrm{sign} : \{\beta, s_\alpha(\beta)\} \to \{\pm 1\}.$$

We also set $\mathrm{sign}(\beta) = 0$ if $(\alpha, \beta) = 0$ and fix some analytic branch of $\sqrt{z}$ function. For $\beta \in \Delta_+ \setminus \{\alpha\}$, $k \in \mathbb{N}$ and $b \in \mathbb{C}$ set $f_{\beta,k}(b) = b + 2\,\mathrm{sign}(\beta)\sqrt{b_2}(k\beta, \alpha) + (k\beta, \alpha)^2$.

First of all one can formulate the following criterion of simplicity for the module $M(\lambda, b)$, which follows immediately from Theorem 9.2.1 and Lemma 9.1.1.

Theorem 9.3.1. *$M(\lambda, b)$ is simple if and only if the two following conditions are satisfied:*

1. $((\lambda+\rho, \alpha)+2k)^2 \neq b$ *for all* $k \in \mathbb{Z}$.

2. $((2(\lambda+\rho, \beta) - k(\beta, \beta))\,(2(\lambda+\rho, s_\alpha(\beta)) - k(s_\alpha(\beta), s_\alpha(\beta))) + (\alpha, \beta)(\alpha, s_\alpha(\beta)) \cdot$ $\cdot\, (b - (\lambda+\rho, \alpha)^2)) \neq 0$ *for all* $\beta \in \Delta_+ \setminus \{\alpha\}$ *and for all* $k \in \mathbb{N}$.

We remark that the first condition of the Theorem 9.3.1 is equivalent to the condition that the module $N(a, b)$ (see definition of $M(\lambda, b)$) and thus the module $M(\lambda, b)$ is α-stratified. Hence for α-stratified modules one needs to check only the second condition. The following theorem is a generalization of the BGG Theorem and [KK, Theorem 2].

Theorem 9.3.2. *The following statements are equivalent:*

1. $L(\lambda, b_1)$ *is a subquotient of* $M(\mu, b_2)$.

2. $M(\lambda, b_1) \subset M(\mu, b_2)$.

3. $(\lambda, b_1) \prec (\mu, b_2)$.

The proof is standard and analogous to the proof of the corresponding result for Verma modules (see, for example, [MP, KK]). One can also consult [KM3] for some technical details.

10 The BGG resolution

In this Chapter we begin the study of simple (as objects in $\mathcal{K}^\alpha$) quotients $L(\lambda, p)$ of the GVMs $M(\lambda, p)$ considered in Chapter 6. In fact, we construct an analogue of the BGG resolution for such modules. We also present a BGG resolution for some simple quotients of GVMs, induced from generic GZ modules. The results from this Chapter were obtained in [FM2, FM3, M2, M4], where the reader can find missing technical details. We retain the notation from Chapter 6.

10.1 General case – semidominant parameters

For this Section we assume that $\mathfrak{G}$ is an arbitrary simple finite-dimensional complex Lie algebra. We begin with a cohomological construction of an exact sequence of GVMs. Denote by W^+ the subgroup of W, generated by all s_β, $\beta \in \pi \setminus \pi(\mathfrak{A})$ (this means $\beta \in \pi$, $\beta \neq \alpha$). Recall the set Ω of parameters of GVMs, defined in Section 6.5, and the action of W on Ω constructed in the same Section. We will call an element $(\lambda, p) \in \Omega$ *minimal*, if $(\lambda, p) - s_\beta(p) = (\beta, p_\beta)$ for all $\beta \in \pi \setminus \pi(\mathfrak{A})$. For some time we fix a minimal element $(\lambda, p) \in \Omega$.

Let $\hat{\Delta}$ be the set of all positive roots, generated by $\pi \setminus \{\alpha\}$. Denote by $B = B(\alpha)$ the subalgebra of $\mathfrak{G}$ generated by all root subspaces $\mathfrak{G}_{-\beta}$, $\beta \in \hat{\Delta}$. Consider B as a module over the subalgebra $A = \mathfrak{H} \oplus \mathfrak{N}$ under the action $h \cdot a = [h, a] + \lambda(h)a$ for any $h \in \mathfrak{H}$ and $a \in B$, and

$$b \cdot a = \begin{cases} [b, a], & [b, a] \in B; \\ 0, & [b, a] \notin B. \end{cases}$$

for all $b \in \mathfrak{N}$ and $a \in B$. Clearly, this action can be naturally extended to an action on the exterior powers $\bigwedge^k B$, for all $k \in \mathbb{N}$.

Let ε be the unique eigenvalue on $M(\lambda, p)$ of the quadratic Casimir operator

$$C = h_0 + \sum_{\alpha \in \Delta_+} X_{-\alpha} X_\alpha,$$

where h_0 is a certain fixed element in $S(\mathfrak{H})$. Note that this eigenvalue is determined uniquely by (λ, p) via the generalized Harish-Chandra homomorphism.

Define $U_\varepsilon = U(\mathfrak{G})/(C - \varepsilon)$ and consider the following $\mathfrak{G}$-modules:

$$D_k = U_\varepsilon \bigotimes_{U(A)} \bigwedge^k B,$$

where $k \in \mathbb{Z}_+$.

Following [BGG1], for $k \in \mathbb{N}$, define the homomorphisms $d_k : D_k \to D_{k-1}$ as follows:

$$d_k(X \otimes X_1 \wedge X_2 \wedge \cdots \wedge X_k) = \sum_{i=1}^{k} (-1)^{i+1} X X_i \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k + \sum_{1 \leqslant i < j \leqslant k} (-1)^{i-j} X \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k.$$

Since $d_k \circ d_{k+1} = 0$, we immediately obtain that the sequence

$$0 \leftarrow D_0 / \operatorname{Im} d_1 \xleftarrow{\eta} D_0 \xleftarrow{d_1} D_1 \xleftarrow{d_2} D_2 \xleftarrow{d_3} \ldots,$$

where η is a natural projection, is a complex. We will denote this complex by $\mathfrak{V}_\alpha(\lambda, \varepsilon)$.

Theorem 10.1.1. *The complex $\mathfrak{V}_\alpha(\lambda, \varepsilon)$ is exact.*

Proof. The algebra U_ε inherits the natural gradation on $U(\mathfrak{G})$ by the degree of the monomials. This leads to a gradation on D_k. For $l \geqslant k$, let $D_k^{(l)}$ be a subspace spanned by the elements $x \otimes y$ where x is an element in U_ε of degree less than or equal to $l - k$ and $y \in \bigwedge^k B$. It is clear that $d_k(D_k^{(l)}) \subset D_{k-1}^{(l)}$ and thus d_k induces a homomorphism $d_k^{(l)} : D_k^{(l)}/D_k^{(l-1)} \to D_{k-1}^{(l)}/D_{k-1}^{(l-1)}$. Denote $\hat{D}_k^{(l)} = D_k^{(l)}/D_k^{(l-1)}$. Also set $M^{(l)} = \hat{D}_0^{(l)}/\operatorname{Im} d_1^{(l)}$ and let $\eta^{(l)}$ be the corresponding induced homomorphism.

It is sufficient to show that, for every l, the complex

$$0 \leftarrow M^{(l)} \xleftarrow{\eta^{(l)}} \hat{D}_0^{(l)} \xleftarrow{d_1^{(l)}} \hat{D}_1^{(l)} \xleftarrow{d_2^{(l)}} \hat{D}_2^{(l)} \xleftarrow{d_3^{(l)}} \ldots \tag{10}$$

is exact.

By the PBW Theorem, for every $k \in \mathbb{Z}_+$, one can write:

$$D_k = \left(U(\mathfrak{N}_-) \bigotimes \bigwedge^k B \right) \bigoplus \left(\sum_{m \geqslant 1} X_\alpha^m U(\mathfrak{N}(\mathfrak{A})) \bigotimes \bigwedge^k B \right)$$

and hence

$$\hat{D}_k^{(l)} \simeq \left(U(\mathfrak{N}_-)^{(l-k)} \bigotimes \bigwedge^k B \right) \bigoplus \left(\sum_{m=1}^{l-k} X_\alpha^m U(\mathfrak{N}(\mathfrak{A}))^{(l-k-m)} \bigotimes \bigwedge^k B \right).$$

We will denote by $s_\alpha \mathfrak{N}_-$ the subalgebra generated by $\mathfrak{N}(\mathfrak{A})$ and X_α. Let $\mathfrak{N}_-^B$ (resp. $s_\alpha \mathfrak{N}_-^B$) be the subalgebra generated by $X_{-\beta}$, $\beta \in \Delta_+$, $\beta \notin \hat{\Delta}$ (resp. $\beta \in s_\alpha \Delta_+$, $\beta \notin s_\alpha(\hat{\Delta})$) and let $S_j(B)$ be the set of all homogeneous elements of degree j in the symmetric algebra of B. Then

$$\hat{D}_k^{(l)} \simeq \left(\sum_{j=0}^{l-k} U(\mathfrak{N}_-^B)^{(l-j-k)} S_j(B) \bigotimes \bigwedge^k B \right) \bigoplus \left(\sum_{j=0}^{l-k} U(s_\alpha \mathfrak{N}_-^B)^{(l-j-k)} S_j(B) \bigotimes \bigwedge^k B \right).$$

For any homogeneous element $u \in U(\mathfrak{N}_-^B)$ (resp. $u \in U(s_\alpha \mathfrak{N}_-^B)$) of degree $l-j-k$ we have that $d_k^{(l)}(uS_j(B) \otimes \bigwedge^k B) \subset uS_{j+1}(B) \otimes \bigwedge^{k-1} B$. Therefore $d_k^{(l)}$ induces a complex which is in fact the Koszul complex, and hence is exact. Using the PBW theorem we conclude that the complex (10) decomposes into a direct sum of exact complexes and therefore is exact. The theorem is proved. $\square$

From Theorem 10.1.1 one immediately obtains the following formula for the formal character of $D_0/\operatorname{Im} d_1$, which we will need later on in Section 11.1:

Corollary 10.1.1.

$$\operatorname{ch}(D_0/\operatorname{Im} d_1) = \sum_{i \geqslant 0} (-1)^i \operatorname{ch} D_i.$$

Denote by $P(\alpha)^{++}$ the set of all $(\lambda, p) \in \Omega$, such that $w(\lambda, p) \prec (\lambda, p)$ for all $w \in W$. Let w_0 be the longest element in W and w_0^α be the longest element in W^+. Denote by $P(\alpha)^+$ the set of all $w_0^\alpha w_0(\lambda, p)$, $(\lambda, p) \in P(\alpha)^{++}$. We will call the elements from $P(\alpha)^{++}$ (resp. $P(\alpha)^+$) dominant (resp. semidominant). In this section we will construct the BGG resolution of the module $L(\lambda, p)$, $(\lambda, p) \in P(\alpha)^+$. Let W_k^+ denote the set of all elements of W of length $k \in \mathbb{Z}_+$.

Theorem 10.1.2. *Let $(\lambda, p) \in P(\alpha)^+$. Denote by C_k the direct sum of all $M(w(\lambda, p))$, $w \in W_k^+$. Then there exists an exact sequence*

$$0 \leftarrow L(\lambda, p) \xleftarrow{\eta} C_0 \xleftarrow{\delta_1} C_1 \xleftarrow{\delta_2} \dots \xleftarrow{\delta_m} C_m \leftarrow 0,$$

where η is a natural projection and δ_i are the standard homomorphisms, defined with respect to the Bruhat order on W^+, see [BGG1, Lemma10.4].

Proof. It follows directly from Theorem 6.6.1 that all the homomorphisms are well-defined and that this sequence is a complex. First suppose that the statement is true for all minimal (λ, p). Then using the translation functors (i.e tensoring the corresponding sequences with finite-dimensional $\mathfrak{G}$-modules and taking the summand which corresponds to a central character) we produce the resolutions for all $(\lambda, p) \in P(\alpha)^+$. So, it is enough to prove the statement for minimal (λ, p). Fix such (λ, p) and consider the corresponding exact complex $\mathfrak{V}_\alpha(\lambda, \varepsilon)$. Assume that we know that $D_0/\operatorname{Im}(d_1) \simeq L(\lambda, p)$. Then, following [BGG1] or [RC], one constructs a sequence of homomorphisms $\nu^i : B_i \to C_i$ which makes the following diagram commutative:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & B_2(\lambda, p) & \xrightarrow{d_2} & B_1(\lambda, p) & \xrightarrow{d_1} & B_0(\lambda, p) & \xrightarrow{\eta} & L(\lambda, p) & \longrightarrow & 0 \\
 & & {\scriptstyle \nu^2}\downarrow & & {\scriptstyle \nu^1}\downarrow & & {\scriptstyle \nu^0}\downarrow & & {\scriptstyle 1}\downarrow & & \\
\dots & \longrightarrow & C_2(\lambda, p) & \xrightarrow{\delta_2} & C_1(\lambda, p) & \xrightarrow{\delta_1} & C_0(\lambda, p) & \xrightarrow{\eta} & L(\lambda, p) & \longrightarrow & 0.
\end{array}$$

To complete the proof we have only to show that $D_0/\operatorname{Im}(d_1) \simeq L(\lambda, p)$, or in other words, that

$$M = M(\lambda, p)/\sum_{\beta\in(\pi\setminus\{\alpha\})} M(s_\beta(\lambda, p))$$

is a simple object in $\mathcal{K}^\alpha$. From Theorem 6.6.1, it follows that the only simple subquotients that can occur in M are $L(w(\lambda, p))$ for $w \in W^+$. Let $w \in W^+$, $w \neq 1$. Then there exists $\beta \in (\pi \setminus \{\alpha\})$ such that the length of $s_\beta w$ is less than the length of w. Hence, the growth of the dimensions of the weight spaces of $L(w(\lambda, p))_{\mu-k\beta}$ is strictly bigger the the growth of the dimensions of $M_{\mu-k\beta}$. This means that $L(w(\lambda, p))$ is not a subquotient of M. Hence, the only subquotient of M is $L(\lambda, p)$, which has multiplicity not bigger than 1 and we obtain $M \simeq L(\lambda, p)$. This completes the proof. □

It is natural to call the sequence, constructed in Theorem 10.1.2, the BGG resolution of $L(\lambda, p)$.

10.2 Simply laced case – dominant parameters

It is clear, that almost all the arguments used in Section 10.1, except those about the simplicity of $D_0/\operatorname{Im}(d_1)$, can be extended to the case $(\lambda, p) \in P(\alpha)^{++}$. In fact, we can state that, from Section 10.1, the following statement follows easily.

Proposition 10.2.1. *Let $(\lambda, p) \in P(\alpha)^{++}$ be a minimal element. Then there exists an exact sequence*

$$0 \leftarrow D_0/\operatorname{Im}(d_1) \xleftarrow{\eta} C_0 \xleftarrow{\delta_1} C_1 \xleftarrow{\delta_2} \dots \xleftarrow{\delta_m} C_m \leftarrow 0,$$

where η and δ_i are as in Theorem 10.1.2.

To prove, that $D_0/\operatorname{Im}(d_1) \simeq L(\lambda, p)$ for a minimal $(\lambda, p) \in P(\alpha)^{++}$, we have to assume that $\mathfrak{G}$ has a simply-laced Dynkin diagram (note we can not prove the corresponding result for a non-simply laced case; we do not know if it is true there).

Lemma 10.2.1. *Assume, that $\mathfrak{G}$ has a simply-laced Dynkin diagram and $(\lambda, p) \in P(\alpha)^{++}$ is a minimal element. Then $D_0/\operatorname{Im}(d_1) \simeq L(\lambda, p)$.*

Proof. The proof is essentially the content of [FM2, Section 5], so we skip some technical details which can be found there. Let $K(\mathfrak{G})$ denote the subalgebra of $\mathfrak{G}$, generated by all $X_{-\beta}$, $\beta \in (\Delta_+ \setminus \hat{\Delta})$.

First we consider arbitrary $M(\mu, q)$ with a weight generator v. Call an element $u \in M(\mu, q)$ quasi-primitive, if there exists a submodule $M \subset M(\mu, q)$, $M \neq M(\mu, q)$, such that $\mathfrak{N}u = 0$ in the quotient $M(\mu, q)/M$. It is easy to see, that as soon as $K(\mathfrak{G})v$ intersects any proper submodule M of $M(\mu, q)$ then $K(\mathfrak{G})v$ contains a quasi-primitive element of F. The main observation, we are going to prove is the claim.

Lemma 10.2.2. *The only quasi-primitive elements of $K(\mathfrak{G})v$ are $\mathbb{C}X_{-\alpha}^k v$, $k \in \mathbb{Z}_+$.*

Proof. A direct calculation shows that for any $\tau \in \mathfrak{H}^*$ the existence of a non-zero α-primitive element in $K(\mathfrak{G})v$ of weight $\mu - \tau$ is equivalent to a system of linear equations on $\mu(H_\beta)$, $\beta \in \pi$, and does not depend on q. But this contradicts Theorem 6.6.1. Thus the only α-primitive elements in $K(\mathfrak{G})v$ are $\mathbb{C}X_{-\alpha}^k v$, $k \in \mathbb{Z}_+$.

Now suppose that $v' \in (K(\mathfrak{G})v)_\nu$ is quasi-primitive and $(K(\mathfrak{G})v)_\xi$ has no quasi-primitive elements if $p_{\mathfrak{A}}(\nu) \leqslant p_{\mathfrak{A}}(\xi)$. Consider a basis T in $\Delta_+ \setminus \{\alpha\}$ containing $\pi \setminus \{\alpha\}$. Then $X_\gamma v' = 0$ for all $\gamma \in \pi \setminus \{\alpha\}$. If $\gamma \in T \setminus \pi$, we have $(\gamma, \alpha) \neq 0$ since $\mathfrak{G}$ has a simply laced Dynkin diagram. Let $Q \simeq \mathfrak{sl}(2, \mathbb{C})$ be a subalgebra generated by $X_{\pm\gamma}$ and F be a Q-module generated by v'. Suppose that $X_\gamma v' \neq 0$. Since v' is quasi-primitive it implies that $v' \notin F'$, where F' is a Q-module generated by $X_\gamma v'$. Then F'_ν contains a non-zero element v'' such that $X_\gamma v'' = 0$ and hence F' has a finite-dimensional quotient. Since $M(\mu, q)$ is α-stratified then $v_k = X_{-\alpha}^k v'$ is quasi-primitive for all $k > 0$. Note that $X_\gamma v_k = 0$ for all k. Indeed, if $X_\gamma v_k \neq 0$ for some $k > 0$ then we can apply to v_k the same arguments as above and conclude that a Q-module generated by $X_\gamma v_k$ also has a finite-dimensional quotient of the same dimension. But $(\alpha, \gamma) \neq 0$ and hence these finite-dimensional modules have different highest weights which is a contradiction from the $\mathfrak{sl}(2, \mathbb{C})$-theory. Therefore, $X_\gamma v_k = 0$ for all $k > 0$. Using the fact that the root system Δ is finite we find $m \geqslant 0$ such that $X_\beta v_m = 0$ for all $\beta \in T$. Hence, v_m is α-primitive and thus belongs to $\mathbb{C}X_{-\alpha}^k v$ for some $k \in \mathbb{Z}_+$. We conclude that v' is α-primitive and belongs to $\mathbb{C}X_{-\alpha}^k v$ for some $k \in \mathbb{Z}_+$. ☐

Now it is clear, that for any quotient V of $M(\mu, q)$ holds $\dim V_\nu \geqslant \dim(K(\mathfrak{G})v)_\nu$. Consider the module $M(\lambda, p)$. Form Corollary 10.1.1 we deduce that $\dim(D_0/\operatorname{Im}(d_1))_\nu = \dim(K(\mathfrak{G})v)_\nu$ for all ν such that $(K(\mathfrak{G})v)_\nu$ is non-zero. This implies that $D_0/\operatorname{Im}(d_1)$ is a simple module, and hence $D_0/\operatorname{Im}(d_1) \simeq L(\lambda, p)$. ☐

From Lemma 10.2.1 and Proposition 10.2.1 we immediately obtain the BGG resolution of $L(\lambda, p)$, $(\lambda, p) \in P(\alpha)^{++}$ for $\mathfrak{G}$ having a simply-laced Dynkin diagram.

Corollary 10.2.1. *Assume that $\mathfrak{G}$ has a simply-laced Dynkin diagram and $(\lambda, p) \in P(\alpha)^{++}$. Then there exists an exact sequence*

$$0 \leftarrow L(\lambda, p) \xleftarrow{\eta} C_0 \xleftarrow{\delta_1} C_1 \xleftarrow{\delta_2} \dots \xleftarrow{\delta_m} C_m \leftarrow 0,$$

where η and δ_i are as in Theorem 10.1.2.

Proof. Analogous to that of Theorem 10.1.2. ☐

10.3 Corank one case of GZ modules

There is also a transparent way to construct an analogue for the BGG resolution in some cases considered in Chapter 8. For this Section we retain the notation from Chqpter 8. Our results are quite easy and far from being general, so we omit all technical details and refer the reader to [M4].

Let $\mathfrak{G}$ be the $\mathfrak{gl}(\mathfrak{N}, \mathbb{C})$ (or $\mathfrak{sl}(n, \mathbb{C})$) algebra with $\pi = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ and $\pi \setminus \pi(\mathfrak{A}) = \{\alpha_1\}$. Let $[\,l\,]$ be a tableau, such that

- $l_{i,1} = l_{n,1}$ for all i;
- $l_{n,i} - l_{n,i+1} \in \mathbb{Z}$ for all i;
- $l_{k,i} - l_{k,j} \notin \mathbb{Z}$ for all $k < n$ and all i, j;
- $l_{k,i} - lk + 1, j \notin \mathbb{Z}$ for all $k < n$ and all $i, j > 1$.

We will denote by P the set of all tableaux satisfying the conditions above. Let $M([\,l\,])$ be the GVM constructed in the Section 8.4. Recall that, in Section 8.2, we defined an action of W on P. The BGG resolution of $M([\,l\,])$ for some special $[\,l\,]$ has the following form:

Theorem 10.3.1. *Assume that $l_{n,i} - l_{n,i+1} \in \mathbb{N}$ for all i. Then the following natural sequence is exact:*

$$0 \to M(s_{\alpha_1}([\,l\,])) \to M([\,l\,]) \to L([\,l\,]) \to 0.$$

Proof. In fact we have to prove that $M = M([\,l\,])/M(s_{\alpha_1}([\,l\,]))$ is a simple module. From the construction of both $M([\,l\,])$ and $M(s_{\alpha_1}([\,l\,]))$ from Section 8.4, it follows that M has no non-trivial α-primitive elements, and hence is simple. $\square$

We have called this situation a corank one case, since the difference in ranks between $\mathfrak{G}$ and $\mathfrak{A}$ equals one. We also note that the corresponding result for another corank one situation, when $\pi \setminus \pi(\mathfrak{A}) = \{\alpha_{n-1}\}$ is still a conjecture for $n > 3$. It is also natural to call the sequence from Theorem 10.3.1 the BGG resolution of $L([\,l\,])$. We also note, that, using the notation from Section 10.1, W^+ is generated by s_{α_1} in our case, so this sequence has the same form, as ones constructed in Theorem 10.1.2 and Corollary 10.2.1.

11 Character formulae and Schubert filtration

In this Chapter we apply the results obtained in Chapter 10 to derive two chracter formulae for simple subquotients of α-stratified GVMs and to construct an analogue of the Schubert filtration. These results were obtained in [FM2, KM5].

11.1 Weyl character formula

Using Theorem 10.1.2 and Corollary 10.2.1 it is easy to write down the formal character of $L(\lambda, p)$, $(\lambda, p) \in P(\alpha)^{++} \cup P(\alpha)^{+}$. Let ρ' denote the halfsum of all roots in $\hat{\Delta}$ and $K = \Delta_{+} \setminus \hat{\Delta}$.

Theorem 11.1.1. *Let $(\lambda, p) \in P(\alpha)^{++} \cup P(\alpha)^{+}$. Then there exists an element $a(\lambda, p) \in \mathfrak{H}^*$ such that*

$$\mathrm{ch}(L(\lambda, p)) = \left(\sum_{i=-\infty}^{+\infty} e^{i\alpha}\right) \left(\prod_{\beta \in K \setminus \{\alpha\}} (1 - e^{-\beta})^{-1}\right) \times \\ \times \left(\sum_{w \in W^{+}} (-1)^{l(w)} e^{w(\lambda + a(\lambda,p) + \rho') - a(\lambda,p)}\right) \left(\sum_{w \in W^{+}} (-1)^{l(w)} e^{w(\rho')}\right)^{-1}$$

Proof. We note that the character of $L(\lambda, p)$ is invariant under a shift by α. This gives us the first factor. Now using Theorem 10.1.2 and Corollary 10.2.1, one reduces the rest of the calculation to the Weyl character formula for the semisimple Lie algebra which corresponds to $\hat{\Delta}$. The reader can consult [FM2, Section 7] for more technical details. $\square$

11.2 Demazure formula

In this Section we extend the Weyl character formula, obtained in Section 11.1 to the Demazure character formula (see [A, De, Z2] for the classical case). Here we keep the notation from Chapter 10.

For $\beta \in \hat{\Delta}$ set

$$d_{\beta} = (1 - e^{-\beta})^{-1}(1 - e^{-\beta} s_{\beta}).$$

Let

$$T = \left(\sum_{i=-\infty}^{+\infty} e^{i\alpha}\right) \left(\prod_{\beta \in -K \setminus \{\alpha\}} (1 - e^{-\beta})^{-1}\right).$$

Let W^{α} be the Weyl group generated by s_{β}, $\beta \in \pi \setminus \{\alpha\}$. We recall that with respect to π $(\pi \setminus \{\alpha\})$ there is a standard length function defined on W (W^{α}).

Theorem 11.2.1. *Let $(\lambda, p) \in P(\alpha)^{++} \cup P(\alpha)^+$. Then there exists an element $a(\lambda, p) \in \mathfrak{H}^*$ such that for any reduced decomposition, $w = s_{\beta_1} s_{\beta_2} \dots s_{\beta_k}$, of the longest element $w \in W^\alpha$ the following equality holds:*

$$\mathrm{ch}(L(\lambda, p)) = Te^{a(\lambda,p)} \left(d_{\beta_1} d_{\beta_2} \dots d_{\beta_k} e^{\lambda-\rho-a(\lambda,p)}\right).$$

Proof. By virtue of Theorem 11.1.1, there exists $a(\lambda, p) \in \mathfrak{H}^*$ such that $\mathrm{ch}(L(\lambda, p)) = Te^{a(\lambda,p)}Q$, where

$$Q = \left(\sum_{w \in W(\alpha)} (-1)^{l(w)} e^{w(\lambda-\rho-a(\lambda,p)+\rho_\alpha)}\right) \left(\sum_{w \in W(\alpha)} (-1)^{l(w)} e^{w(\rho_\alpha)}\right)^{-1}.$$

On the other hand, by the Weyl charcter formula ([D, Theorem 7.5.9]), Q can be considered as the character of a simple finite-dimensional module over the Lie algebra associated with $\hat{\Delta}$. From the Demazure character formula [Z2, Theorem 2.5.3], we obtain that

$$Q = d_{\beta_1} d_{\beta_2} \dots d_{\beta_k} e^{\lambda-\rho-a(\lambda,p)}$$

for any reduced decomposition, $w = s_{\beta_1} s_{\beta_2} \dots s_{\beta_k}$, of the longest element $w \in W^\alpha$. This completes the proof. □

11.3 Schubert filtrations

The classical Schubert filtration of a simple finite-dimensional module is associated with the Demazure character formula ([Z2]). In this Section we construct a filtration of $L(\lambda, p)$, $(\lambda, p) \in P(\alpha)^{++} \cup P(\alpha)^+$ by $U(\mathcal{P})$-modules, associated with the formula obtained in Theorem 11.2.1.

Theorem 11.3.1. *Let $(\lambda, p) \in P(\alpha)^{++} \cup P(\alpha)^+$. There is an element $a(\lambda, p) \in \mathfrak{H}^*$ such that for any reduced decomposition, $w = s_{\beta_1} s_{\beta_2} \dots s_{\beta_k}$, of the longest element $w \in W^\alpha$ there exists a (canonical) filtration*

$$L(\lambda, p) = L_1 \supset L_2 \supset \dots \supset L_k \supset L_{k+1} = 0$$

of $L(\lambda, p)$ (viewed as a $\mathcal{P}$-module) by the $\mathcal{P}$-modules L_j, $j = 1, 2, \dots, k$ such that

$$\mathrm{ch}(L_j) = Te^{a(\lambda,p)} \left(d_{\beta_j} \dots d_{\beta_k} e^{\lambda-\rho-a(\lambda,p)}\right).$$

Proof. **Step 1.** To simplify our notation, we set $L = L(\lambda, p)$. First we use Mathieu's localization to shift our module $L(\lambda, p)$ to $\mathcal{O}(\mathcal{P}, \mathcal{L}(V(\mathfrak{l}, p^2)))$. We fix $x \in \mathbb{C}$ such that the $\mathfrak{G}$-module $L' = \theta_x(L)$ is not α-stratified and hence is a (non-split) extension of a highest weight module and a module which is a highest weight module with respect to the basis $s_\alpha(\pi)$. Set $L(1)$ to be the set of all element of L on which X_α acts locally nilpotent and $L(2) = L/L(1)$. Then $L(1)$ is a highest weight module and $L(2)$ is a highest weight module with respect to $s_\alpha(\pi)$.

Let μ be the highest weight of $L(1)$. Then $\mu = \lambda - \rho + x\alpha$ by Lemma 5.5.1. Since $(\lambda, p) \in P(\alpha)^{++} \cup P(\alpha)^{+}$, we obtain that $p \in \mathbb{Z}$ and $p \neq 0$, hence $\mu(H_\alpha) = |p| - 1$. From this, it follows that the unique simple quotient of $L(1)$ is finite-dimensional (or, equivalently, μ is integral dominant).

Step 2. Let $\mathfrak{G}^\alpha = \mathfrak{N}_-^\alpha \oplus \mathfrak{H}_\alpha \oplus \mathfrak{N}_+^\alpha$ denote the Lie subalgebra of $\mathfrak{G}$, generated by $\mathfrak{G}_\beta$, $\beta \in \pm\hat{\Delta}$ with the inherited triangular decomposition. Consider a $\mathfrak{G}^\alpha$-module $M = U(\mathfrak{G}^\alpha)L(1)_\mu$. This is a highest weight $\mathfrak{G}^\alpha$-module. The support of any non-trivial $\mathfrak{G}^\alpha$-submodule of M does not intersects the highest weight of M and thus this submodule generates in $L(1)$ a non-trivial $\mathfrak{G}$-submodule, whose support does not intersect the intersection of $L(1)$ with $N(\lambda', p)$. Inducing to $U(\alpha)$, this gives a non-trivial $\mathfrak{G}$-submodule in L' on which $X_{-\alpha}$ acts bijectively. And after the return twist with θ_{-x} we get that L is not simple, which contradicts our assumptions. Hence, M does not contain any non-trivial $\mathfrak{G}^\alpha$-submodule, therefore M is a simple $\mathfrak{G}^\alpha$-module. As μ is integral doiminant we also get that M is finite-dimensional.

Consider a reduced decomposition $\hat{w}_0 = s_{\beta_1} s_{\beta_2} \dots s_{\beta_k}$. According to the classical Demazure character formula and the classical Schubert filtration one can consider a Schubert filtration

$$M = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0$$

of M by $\mathfrak{N}_+^\alpha$-module, corresponding to the above decomposition of $\hat{w}_0$, with $\operatorname{ch} M_i$ given by the Demazure formula. Let v_j denote a (canonical) generator of M_j for $j \in \{1, 2, \dots, k\}$.

Step 3. By Section 10.2, the maximal submodule of $M(\lambda, p)$ is generated by the images of all $M(s_\beta(\lambda, p))$ in $M(\lambda, p)$, where β runs through the set of all simple roots different form α. Lifting this to $L(1)$ we get that $L(1)$ is the quotient of the Verma module $M(\mu + \rho)$ over the submodule generated by the images of all $M(s_\beta(\mu + \rho))$ in $M(\mu + \rho)$, where $\beta \neq \alpha$ is simple.

By PBW Theorem, $U(\mathfrak{N}_-)$ is free over $\hat{K}$ with the basis $U(\mathfrak{N}_-^\alpha)$. We can identify each Verma module with $U(\mathfrak{N}_-)$ as an $\mathfrak{N}_-$-module. We have that $M(\mu + \rho)$ is free over $\hat{K}$ with the basis $U(\mathfrak{N}_-^\alpha)M(\mu + \rho)_\mu$. Denote by $\hat{M}$ the submodule of $U(\mathfrak{N}_-^\alpha)M(\mu + \rho)_\mu$ generated by the intersection of $U(\mathfrak{N}_-^\alpha)M(\mu + \rho)_\mu$ with all images of $M(s_\beta(\mu + \rho))$ in $M(\mu + \rho)$, where $\beta \neq \alpha$ is simple. We have $L(1) \simeq M(\mu + \rho)/U(\mathfrak{G})\hat{M}$ and $U(\mathfrak{G})\hat{M} = U(\mathfrak{N}_-)\hat{M} = \hat{K}\hat{M}$. Since $\hat{M}$ is a subset of $U(\mathfrak{N}_-^\alpha)M(\mu + \rho)_\mu$ and the last one is a $\hat{K}$-free basis of $M(\mu + \rho)$, we get that $L(1)$ is $\hat{K}$-free with a basis $U(\mathfrak{N}_-^\alpha)M(\mu + \rho)_\mu/\hat{M} \simeq M$ (here we mean that any $\mathbb{C}$-basis of $U(\mathfrak{N}_-^\alpha)M(\mu + \rho)_\mu$, resp. $L(1)$ is a $\hat{K}$-free basis).

Step 4. Let N be arbitrary $\mathfrak{N}_-^\alpha$-submodule of M. Then $\hat{K}N$ is a $\mathcal{P}^*$-module (here and on $*$ is with respect to the Chevalley involution), moreover $\operatorname{ch} \hat{K}N = \operatorname{ch} \hat{K} \times \operatorname{ch} N$. Indeed, the character formula follows from the free action of $\hat{K}$. Clearly, $\hat{K}N$ is a $\hat{K}$-module and an $\mathfrak{H}$-module. As $[\mathfrak{N}_-^\alpha, \hat{K}] \subset \hat{K}$, $\hat{K}N$ is a $\mathfrak{N}_-^\alpha$-module. So, it is enough to show that $\hat{K}N$ is closed under the action of X_α. This now follows from the fact that $[X_\alpha, \hat{K}] \subset U(\mathfrak{N}_- \oplus \mathfrak{H})$ and $X_\alpha N = X_\alpha M = 0$.

Step 5. For each $1 \leqslant i \leqslant k$ there exists a $\mathfrak{N}_-^\alpha$-submodule, N_i, of M such that $\operatorname{ch} M/N_i = \operatorname{ch} M_i$. Moreover, we can choose N_i such that $N_i \subset N_{i+1}$. Indeed, as M_i is a $\mathfrak{N}_+^\alpha$-submodule of $M \simeq M^*$, we can apply $*$ and obtain a $\mathfrak{N}_-^\alpha = (\mathfrak{N}_+^\alpha)^*$-submodule of $M \simeq M^*$, whose quotient is isomorphic to M_i^*. As $*$ preserves the character this is exactly N_i we need. The statment about inclusions for N_i's follows from the opposite inclusions of M_i's and contravariantness of $*$.

Step 6. Set $I_i = \hat{K}N_i$. By Steps 4 and 5 we have $\operatorname{ch} I_i = \operatorname{ch} \hat{K} \times (\operatorname{ch} M - \operatorname{ch} M_i)$. Clearly, $X_{-\alpha}$ acts injectively on I_i. Inducing I_i up to $U(\alpha)$ and shifting by θ_x we obtain a filtration, $\hat{L}_i$, of L by D^*-modules. Moreover, as $X_{-\alpha}$ acts injectively on $\hat{K}$, $\operatorname{ch} \hat{K}$ changes to T during the induction process, therefore we obtain that $\operatorname{ch} \hat{L}_i = T \times (\operatorname{ch} M - \operatorname{ch} M_i)$. Note that $N(\lambda, p)^* \simeq N(\lambda, p)$ as $N(\lambda, p)$ is α-stratified and hence $L^* \simeq L$ since $*$ preserves the character of the module and L is completely determined by (λ, p). Applying the duality one more time we get that there exists a filtration of $L \simeq L^*$ by $U(\mathcal{P})$-modules L_i such that $\operatorname{ch} L_i = \operatorname{ch} L - \operatorname{ch} \hat{L}_i$. Now the desired result for this filtration L_i follows from the fact that $\operatorname{ch} L = T \times \operatorname{ch} M$ (Theorem 11.1.1). $\square$

12 Categories $\mathcal{O}(\mathcal{P}, \mathcal{L})$, I: Definition and basic properties

In the next four Chapters we present some recent results concerning the structure of certain categories connected with GVMs. These results were obtained in [FM4, FKM1, FKM2, FKM3, KlMa], where the reader can find all necessary technical details. We also use some classical results from [BGG3, RC, RCW2, CF, FP]. We start in this Chapter with a definition of our categories in a general situation and fix all the notations from Chapter 4. Our goal here is to study basic properties analogous to those of the category $\mathcal{O}$, in particular, decomposition into a direct sum of module categories over associative finite-dimensional algebras and different analogues of the BGG reciprocity.

12.1 Admissible categories

Let $\mathcal{L}$ be a full abelian subcategory of the category of all finitely generated $\mathfrak{A}'$-modules. The category $\mathcal{L}$ will be called *admissible* if the following conditions are satisfied:

1. Any $M \in \mathcal{L}$ is finitely generated.
2. Any $M \in \mathcal{L}$ is weight with respect to the center of $\mathfrak{A}'$.
3. For any finite-dimensional simple $\mathfrak{A}'$-module F the functor $F \otimes \_$ preserves $\mathcal{L}$ and exact.

In this Chapter we will always assume that $\mathcal{L}$ is an admissible category and will often consider the objects in $\mathcal{L}$ only as $\mathfrak{A}$-modules.

Let $\mathcal{L}$ be an admissible category of $\mathfrak{A}'$-modules. Denote by $\mathcal{O}(\mathcal{P}, \mathcal{L})$ the full subcategory of the category of $\mathfrak{G}$-modules consisting of modules which are

1. finitely generated;
2. $\mathfrak{N}$-finite;
3. a direct sum of modules from $\mathcal{L}$, when viewed as $\mathfrak{A}'$-modules.

Proposition 12.1.1. *1. The category $\mathcal{O}(\mathcal{P}, \mathcal{L})$ is closed under the operations of taking submodules, quotients, finite direct sums and under tensoring with finite-dimensional $\mathfrak{G}$-modules.*

2. The modules $M_{\mathcal{P}}(V)$ and $L_{\mathcal{P}}(V)$ are objects of $\mathcal{O}(\mathcal{P}, \mathcal{L})$ for any simple $V \in \mathcal{L}$.

3. If V is a simple module in $\mathcal{O}(\mathcal{P}, \mathcal{L})$, then $V \simeq L_{\mathcal{P}}(\hat{V})$ for some simple $\hat{V} \in \mathcal{L}$.

Proof. Statement (1) is obvious. To prove (2), it is enough to show that $M_{\mathcal{P}}(V)$ as an $\mathfrak{A}$-module decomposes into a direct sum of $\mathfrak{A}$-submodules in $\mathcal{L}$. This follows from the fact that $M_{\mathcal{P}}(W) \simeq U(\mathfrak{N}_{-}) \otimes W$ as a vector space by Proposition 4.2.1 and this isomorphism carries over the decomposition of $U(\mathfrak{N}_-)$ as a direct sum of finite-dimensional α-modules with respect to the adjoint action. We conclude that $M_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ and also $L_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P}, \mathcal{L})$.

Let V be a simple module in $\mathcal{O}(\mathcal{P}, \mathcal{L})$. Since V is $\mathfrak{N}$-finite and $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable, there exists a non-zero element $v \in V$ such that $\mathfrak{N}v = 0$ and $hv = \lambda(h)v$ for all $h \in \mathfrak{H}_\alpha$ and some $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$. Put $V_\lambda = U(\alpha)v$. Then $\mathfrak{N}w = 0$ for any $w \in V_\lambda$ implying that V_λ is a simple α-module and $V \simeq L_{\mathcal{P}}(V_\lambda)$ by Proposition 4.2.1. This completes the proof. $\square$

12.2 Projectives in $\mathcal{O}(\mathcal{P}, \mathcal{L})$

Proposition 12.2.1. *Let $\mathcal{L}$ be admissible and let V be a projective module in $\mathcal{L}$. Fix a non-negative integer k. Then the module*

$$P(V, k) = U(\mathfrak{G}) \bigotimes_{U(\mathcal{P})} \big((U(\mathfrak{N})/(U(\mathfrak{N})\mathfrak{N}^k)) \otimes V\big)$$

admits a (possibly infinite) filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_i \subset \cdots \subset V_s = P(V, k)$$

with components indexed by a linearly ordered set I, such that for any $0 < i \in I$ the module

$$V_i / \sum_{k<i} V_k$$

is isomorphic to $M_{\mathcal{P}}(W_i)$ for some projective $W_i \in \mathcal{L}$.

Proof. Since V is a $\mathfrak{H}_{\mathfrak{A}}$-weight module, so is $P(V, k)$. Moreover, since k is finite, among all the weights of $P(V, k)$ there exists a maximal, say λ, with respect to the natural order. Consider a $\mathfrak{A}$-module $P(V, k)_\lambda$. The PBW theorem guarantees that the $U(\mathfrak{G})$–submodule generated by $P(V, k)_\lambda$ in $P(V, k)$ is $U(\mathfrak{N}(\mathfrak{A}))$-free. Since $U(\mathfrak{G})$ is a direct sum of finite-dimensional $\mathfrak{A}$-modules under the adjoint action, it follows that $P(V, k)_\lambda$ is isomorphic to $V \otimes F$ as a $\mathfrak{A}$-module for some finite-dimensional module F. Since $\mathcal{L}$ is admissible we can decompose $V \otimes F$ into a direct sum of modules in $\mathcal{L}$. Further, since tensoring with a finite-dimensional module is an exact functor we conclude that $V \otimes F = \oplus_t X(t)$ and each $X(t)$ is projective in $\mathcal{L}$. Since λ is a maximal weight, it follows that all $M_{\mathcal{P}}(X(t))$ are submodules in $P(V, k)$ and so we can construct the first step of our filtration. Now one just proceeds by induction completing the proof. $\square$

The filtration obtained in Proposition 12.2.1 will be called the *standard filtration*. For a given standard filtration of V we will denote by $[V : M_{\mathcal{P}}(V)]$ the number (finite or infinite) of i such that $M_{\mathcal{P}}(W_i)$ is isomorphic to $M_{\mathcal{P}}(V)$. The module $P(V, k)$ constructed above need not to be projective in $\mathcal{O}(\mathcal{P}, \mathcal{L})$ in general. In order to construct some projective modules in $\mathcal{O}(\mathcal{P}, \mathcal{L})$ we have to assume that $\mathcal{O}(\mathcal{P}, \mathcal{L})$ has certain properties.

A direct summand (or block) $\mathcal{O}_i$ of $\mathcal{O}(\mathcal{P}, \mathcal{L})$ is said to be *quasi-finite* (resp. *finite*) if the set of highest weights (with respect to $\mathfrak{H}_{\mathfrak{A}}$) of all simple modules in $\mathcal{O}_i$ is finite (resp. $\mathcal{O}_i$

contains only finitely many simple objects up to isomorphism). We will say that $\mathcal{O}(\mathcal{P},\mathcal{L})$ has a *quasi block decomposition* if $\mathcal{O}(\mathcal{P},\mathcal{L})$ decomposes into a direct sum of quasi-finite full subcategories. For example, from Corollary 4.3.3 it follows, that $\mathcal{O}(\mathcal{P},\mathcal{L})$ has a quasi block decomposition if $\mathcal{L} = \oplus_{\chi \in Z^*(\alpha)} \mathcal{L}_\chi$, where $\mathcal{L}_\theta$ consists of those $M \in \mathcal{L}$ such that $(z - \chi(z))^k m = 0$ fo all $z \in Z(\mathfrak{G})$, $m \in M$ and some $k \in \mathbb{N}$. By the same argument, if each $\mathcal{L}_\theta$ has only finitely many non-isomorphic simple modules, then $\mathcal{O}(\mathcal{P},\mathcal{L})$ has a block decomposition.

Theorem 12.2.1. *Suppose that $\mathcal{O}_i$ is a quasi-finite block of $\mathcal{O}(\mathcal{P},\mathcal{L})$ and V is an indecomposable projective in $\mathcal{L}$ such that $M_{\mathcal{P}}(V) \in \mathcal{O}_i$. Then for k big enough, the $\mathcal{O}_i$–projection of the module $P(V,k)$ is projective in $\mathcal{O}(\mathcal{P},\mathcal{L})$.*

Proof. Let λ be an $\mathfrak{H}_{\mathfrak{A}}$-weight of V. Since $\mathcal{O}_i$ is quasi-finite, there exist a positive integer N such that for any $M \in \mathcal{O}_i$ holds $\mathfrak{N}^{(N)} M_\lambda = 0$. Let $k > N$. From the construction of $P(V,k)$ it follows that there is a canonical isomorphism between $\mathrm{Hom}_{\mathfrak{G}}(P(V,k), M)$ and $\mathrm{Hom}_{\mathfrak{A}'}(V, M_i)$ for any $M \in \mathcal{O}(\mathcal{P},\mathcal{L})$ (here M_i denotes the direct summand of M lying in $\mathcal{O}_i$). Since V is projective in $\mathcal{L}$ we conclude that the direct summand of $P(V,k)$ lying in $\mathcal{O}_i$ is projective in $\mathcal{O}(\mathcal{P},\mathcal{L})$, as stated. □

Corollary 12.2.1. *Suppose that $\mathcal{L}$ has enough projective modules (i.e. any module is a quotient of a projective module) and $\mathcal{O}_i$ is a quasi-finite block of $\mathcal{O}(\mathcal{P},\mathcal{L})$, then*

1. *$\mathcal{O}_i$ has enough projective modules;*
2. *Every projective in $\mathcal{O}_i$ has a standard filtration;*
3. *There is a one-to-one correspondence between the simple objects in $\mathcal{O}_i$ and the indecomposable projective objects in $\mathcal{O}_i$.*

Proof. The second statement follows form Proposition 12.2.1. The first and the third ones follow from Theorem 12.2.1 using standard arguments. □

Corollary 12.2.2. *Suppose that $\mathcal{L}$ has enough projective modules and $\mathcal{O}(\mathcal{P},\mathcal{L})$ has a quasi block decomposition, then*

1. *$\mathcal{O}(\mathcal{P},\mathcal{L})$ has enough projective modules;*
2. *Every projective in $\mathcal{O}(\mathcal{P},\mathcal{L})$ has a standard filtration;*
3. *There is a one-to-one correspondence between the simple objects in $\mathcal{O}(\mathcal{P},\mathcal{L})$ and the indecomposable projective modules in $\mathcal{O}(\mathcal{P},\mathcal{L})$.*

12.3 Blocks of $\mathcal{O}(\mathcal{P},\mathcal{L})$

Theorem 12.3.1. *Suppose that $\mathcal{L}$ has enough projective modules, $\mathcal{O}_i$ is a finite block of $\mathcal{O}(\mathcal{P},\mathcal{L})$ and any object in $\mathcal{O}_i$ has finite length. Then $\mathcal{O}_i$ is isomorphic to the module category of a finite-dimensional algebra.*

Proof. Consider the endomorphism algebra of the sum of projective covers of all simple modules in $\mathcal{O}_i$. $\square$

Corollary 12.3.1. *Suppose that $\mathcal{L}$ has enough projective modules, $\mathcal{O}(\mathcal{P}, \mathcal{L})$ has a block decomposition and any object in $\mathcal{O}(\mathcal{P}, \mathcal{L})$ has finite length. Then each block of $\mathcal{O}(\mathcal{P}, \mathcal{L})$ is isomorphic to the module category of a finite-dimensional algebra.*

Now we discuss which finite-dimensional algebras can appear in Corollary 12.3.1. For this we need some abstract notation. Let A be a finite dimensional algebra. A two-sided ideal J in A is called *projectively stratifying* if J is generated (as a two-sided ideal) by a primitive idempotent and J is projective as a left A-module. The algebra A is called *projectively stratified* if there exists an ordering $e_1, \dots, e_n$ of the equivalence classes of primitive idempotents of A such that for each l the idempotent e_l generates a projectively stratifying ideal in the quotient algebra $A/ < e_1, \dots, e_{l-1} >$.

This is equivalent to requiring that each projective module P has a filtration of the following form: $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = P$ where M_{l+1}/M_l is a direct sum of copies of the module $(A/ < e_1, \dots, e_l >) \cdot e_{l+1}$ which is projective over the quotient algebra $A/ < e_1, \dots, e_l >$. Any projectively stratified algebra is a stratifying endomorphism algebra in the sense of Cline, Parshall and Scott [CPS2]. Any quasi-hereditary algebra is projectively stratified. A projectively stratified algebra A is quasi-hereditary if and only if all the rings E_l are semisimple if and only if A has finite global dimension (see [CPS2]). We will disscuss projectively stratified algebras in general, especially those related to the example of $\mathcal{O}(\mathcal{P}, \mathcal{L})$ to be presented in Chapter 13, in Chapter 15.

Theorem 12.3.2. *Assume that Λ is a sum of module categories of projectively stratified algebras. Then any finite block of $\mathcal{O}(\mathcal{P}, \mathcal{L})$ (if it exists), such that all its objects have finite length, also is the module category of a projectively stratified algebra.*

Proof. By Proposition 12.1.1 there is a natural bijection between simple objects in $\mathcal{L}$ and $\mathcal{O}(\mathcal{P}, \mathcal{L})$. The induction process can glue several blocks of $\mathcal{L}$ together into one block of $\mathcal{O}(\mathcal{P}, \mathcal{L})$. Assume that a finite block of $\mathcal{O}(\mathcal{P}, \mathcal{L})$ is given and call it $\mathcal{O}_i$. Fix the direct summand $\mathcal{L}_i$ of $\mathcal{L}$ (in general, this is a product of several blocks) such that the above bijection restricts to a bijection between $\mathcal{L}_i$-simples and $\mathcal{O}_i$-simples.

The functors occuring in the construction of projective objects in $\mathcal{O}(\mathcal{P}, \mathcal{L})$ are exact and hence transport filtrations from $\mathcal{L}$ to $\mathcal{O}(\mathcal{P}, \mathcal{L})$. Start with a module V which contains (up to an isomorphism) at least one copy of each isomorphism class of each non-isomorphic indecomposable projective in $\mathcal{L}_i$. Then the tensor product $(U(\mathfrak{N})/(U(\mathfrak{N})\mathfrak{N}^k)) \otimes V$ is again projective (in $\mathcal{L}$) and maps onto all projectives in $\mathcal{L}_i$. Hence it contains at least one copy of each isomorphism class of indecomposable projective modules in $\mathcal{L}_j$. A filtration of this module, as in the definition of projectively stratified algebra yields a similar filtration of the induced $U(\mathfrak{G})$-module. Since the number of isomorphism classes of indecomposable projectives in the block $\mathcal{O}_i$ equals the number of indecomposable projectives in $\mathcal{L}_j$, the resulting filtration has the correct length. $\square$

The following Corollary from Theorem 12.3.2 is obvious.

Corollary 12.3.2. *Under the conditions of Theorem 12.3.2 the following are equivalent for a finite block $\mathcal{O}_i$ of $\mathcal{O}(\mathcal{P},\mathcal{L})$:*

1. *The block $\mathcal{O}_i$ is equivalent to the module category of a quasi-hereditary algebra.*

2. *The block $\mathcal{O}_i$ has finite global dimension.*

3. *For any simple $L(V) \in \mathcal{O}_i$ the module V is projective in $\mathcal{L}$.*

12.4 Reciprocities in $\mathcal{O}(\mathcal{P},\mathcal{L})$

In this Section we give two different generalizations of BGG reciprocity for $\mathcal{O}(\mathcal{P},\mathcal{L})$. The first is based on dualities on projectively stratified algebras and the second is based on some numerical properties of simple modules in $\mathcal{L}$. We also disscuss a connection between them. The first analogue of BGG reciprocity is an abstract property of a projectively stratified algebra.

Theorem 12.4.1. *Let A be a projectively stratified algebra over an algebraically closed field k. Assume that A has a duality (i.e. a contravariant exact equivalence, which preserves simple objects). Assume also that each projective A–module has a filtration by "Verma modules" $M(i)$ (indexed by i in I, the set of isomorphism classes of indecomposable projective A–modules) satisfying $(M(i):L(i))=1$ and $(M(i):L(j))\neq 0$ implies $j\leqslant i$. Denote by $l(i)$ the k–dimension of $End_A(\Delta(i))$, wher $\Delta(i)$ is the i-th standard module (see also Chapter 15), which equals $[P(i):M(i)]$. Then for all $i,j\in I$ there is a BGG-reciprocity*

$$[P(i):M(j)] = l(j)(M(j):L(i)).$$

Note that the only properties of Verma modules needed here are the ones mentioned in the assumptions. No universality is needed.

Proof. We proceed by induction along the filtration of A which makes it a projectively stratified algebra. Let j be a maximal index. Write $P(i)=Ae$ and $P(j)=Af$ for some primitive idempotents e and f. By the choice of j the trace ideal AfA is projective as a left module. We have $Ae\cap AfA=(Af)^l$ for some l which can be computed as $l = dim_k \operatorname{Hom}_A(Af,Ae)/l(j)$. By the condition on Verma modules, all occurences of $M(j)$ in a filtration of A are inside the ideal AfA. Hence $[P(i):M(j)] = l\cdot l(j) = dim_k \operatorname{Hom}_A(Af,Ae) = \dim_k(fAe)$. Applying the duality on A we get $\dim_k(fAe) = \dim_k(eAf) = (P(j):L(i))$. Again by the defining condition on Verma modules we have $(P(j):L(i)) = l(j)\cdot(M(j):L(i))$. □

We note, that if all $l(i)=1$ we will obtain a quasi-hereditary algebra and the classical BGG-reciprocity. We remark that all the results above are also true if the notion of a simple $\mathfrak{G}$-module and a simple object in $\mathcal{L}$ do not coincide.

The second analogue of the BGG reciprocity is the following Theorem.

Theorem 12.4.2. *Suppose that $\mathcal{L}$ has a block decomposition, with each block being the module category over a local finite-dimensional associative algebra. Suppose also that for any simple modules X and Y in $\mathcal{L}$ there exists a constant $\mathfrak{i}(X,Y)$ such that for any finite-dimensional α-module F holds $((F\otimes X):Y)=\mathfrak{i}(X,Y)((F\otimes Y):X)$. (We will call this condition the* duality condition.*) Then for any two simple modules V and W in $\mathcal{L}$ holds*

$$[P(L(V)):M_{\mathcal{P}}(W)]=\mathfrak{i}(V,W)\mathfrak{l}(\tilde{V})(M_{\mathcal{P}}(W):L(V)),$$

where $P(L(X))$ is the projective cover of $L(X)$ in $\mathcal{O}(\mathcal{P},\mathcal{L})$ and $\mathfrak{l}(\tilde{X})$ is the length of the projective cover $\tilde{X}$ of X in $\mathcal{L}$.

Proof. By Theorem 12.3.2 each block of $\mathcal{O}(\mathcal{P},\mathcal{L})$ corresponds to a projectively stratified finite-dimensional algebra. First we note that $\dim \mathrm{Hom}(P(L(V)),M)=(M:L(V))$ for any module $M\in\mathcal{O}(\mathcal{P},\mathcal{L})$. Thus we have only to show that

$$[P(L(V)):M_{\mathcal{P}}(W)]=\mathfrak{i}(V,W)\mathfrak{l}(\tilde{V})\dim \mathrm{Hom}(P(L(V)),M_{\mathcal{P}}(W))$$

for any two simple modules V and W in $\mathcal{L}$. Fix a block $\mathcal{O}_j$. Clearly, we need to check our equality inside $\mathcal{O}_j$ only, so we can assume that $L(V)$ and $L(W)$ belong to $\mathcal{O}_j$. Let k be big enough. Let $P(V,k)_j$ be the direct summand of $P(V,k)$ in $\mathcal{O}_j$. Then

$$P(V,k)_j=\sum_{L(K)\in \mathrm{Irr}(\mathcal{O}_j)} n_K(V)P(L(K))$$

and

$$n_K(V)=\dim \mathrm{Hom}_{\mathfrak{G}}(P(V,k)_j,L(K))=\dim \mathrm{Hom}_{\mathfrak{A}'}(\tilde{V},L(K)).$$

In particular $n_K(V)=0$ if $V\not<K$ with respect to the order induced from $\mathfrak{H}_\alpha$ and $n_V(V)=1$. This allows us to proceed by induction. From the linearity of our formula (in the induction step) we obtain that it is enough to prove that $[P(V,k)_j:M_{\mathcal{P}}(W)]=\mathfrak{i}(V,W)\mathfrak{l}(\tilde{V})\dim \mathrm{Hom}(P(V,k)_j,M_{\mathcal{P}}(W))$. Further it is clear that we only have to check that $[P(V,k):M_{\mathcal{P}}(W)]=\mathfrak{i}(V,W)\mathfrak{l}(\tilde{V})\dim \mathrm{Hom}(P(V,k),M_{\mathcal{P}}(W))$. Clearly, from the construction of $P(V,k)$ it follows that there exists a finite-dimensional α-module F such that $[P(V,k):M_{\mathcal{P}}(W)]=((F\otimes\tilde{V}):W)=\mathfrak{l}(\tilde{V})((F\otimes V):W)$. On the other hand $\dim \mathrm{Hom}(P(V,k),M_{\mathcal{P}}(W))=\dim \mathrm{Hom}_{\alpha\oplus\mathfrak{H}_\alpha}(\tilde{V},M_{\mathcal{P}}(W))=\dim \mathrm{Hom}_{\mathfrak{A}}(\tilde{V},F\otimes W)=((F\otimes W):V)$ by the projectivity of $\tilde{V}$. Application of the duality condition for $\mathcal{L}$ completes the proof. □

Comparing Theorem 12.4.1 with Theorem 12.4.2 one obtains the following result characterizing the behavior of simple modules.

Corollary 12.4.1. *Assume that $\mathcal{L}$ has a block decomposition with enough projective modules and a unique simple module in each block. Assume also that there is only finitely many simples in $\mathcal{L}$ having the same central character and $\mathcal{O}(\mathcal{P},\mathcal{L})$ has a duality (for some $\mathfrak{G}$ and $\mathcal{P}$). If for any two simple X and Y in $\mathcal{L}$ with projective covers $P(X)$ and $P(Y)$ respectively there exists a constant $\mathfrak{i}(X,Y)$ such that for any finite-dimensional α-module F holds $((F\otimes X):Y)=\mathfrak{i}(X,Y)((F\otimes Y):X)$ then $\mathfrak{i}(X,Y)=\mathfrak{l}(P(Y))/\mathfrak{l}(P(X))$.*

Proof. It is easy to see that all the conditions of both Theorem 12.4.1 and Theorem 12.4.2 are satisfied, hence the statement follows by comparing the two reciprocity formulae. $\square$

12.5 Gelfand-Zetlin example

In this Section we present an example of an admissible category $\mathcal{L}$ consisting of generic GZ-modules. Assume that $\mathfrak{A}$ is $\mathfrak{gl}(n, \mathbb{C})$ (or $\mathfrak{sl}(n, \mathbb{C})$). We retain the notation from Chapter 7. Let $[\,l\,]$ be a tableau satisfying the conditions

- $l_{k,i} - l_{k,j} \notin \mathbb{Z}$ for all $k < n$ and all $i \neq j$;
- $l_{k,i} - l_{k+1,j} \notin \mathbb{Z}$ for all $k < n$ and all i, j,

and let $V([\,l\,])$ be the corresponding generic GZ module. It follows directly from Lemma 8.3.1 that the category $\mathcal{L} = \mathcal{L}([\,l\,])$ of all subquotients of $V([\,l\,]) \otimes F$, where F runs through the set of all finite-dimensional $\mathfrak{A}$-modules, is admissible. From the arguments, presented in Section 12.2, we easily obtain that the corrsponding $\mathcal{O}(\mathcal{P}, \mathcal{L})$ has a block decomposition and, from Theorem 12.3.2, it follows that each finite block of $\mathcal{O}(\mathcal{P}, \mathcal{L})$ is a module category over a projectively stratified algebra.

There is a natural duality on $\mathcal{O}(\mathcal{P}, \mathcal{L})$, which corrspondts to σ and defined in a standard way (see, for example, [FM4] or [FKM1]). Hence, we have an abstract BGG reciprocity on $\mathcal{O}(\mathcal{P}, \mathcal{L})$:

$$\begin{aligned}
&[P(L(V([\,l(1)\,]))) : M(V([\,l(2)\,]))] = \\
&\qquad [P(L(V([\,l(2)\,]))) : M(V([\,l(2)\,]))] \cdot (M(V([\,l(2)\,])) : L(V([\,l(1)\,]))).
\end{aligned}$$

We note that, from Section 8.3, it follows that each indecomposable block of $\mathcal{L}$ has a unique simple module. In particular, this means that $[P(L(V([\,l(2)\,]))) : M(V([\,l(2)\,]))]$ equals the length $\mathfrak{l}(\hat{V}([\,l(2)\,]))$ of the projective cover $\hat{V}([\,l(2)\,])$ of $V([\,l(2)\,])$ in $\mathcal{L}$. Hence, we have

$$[P(L(V([\,l(1)\,]))) : M(V([\,l(2)\,]))] = \mathfrak{l}(\hat{V}([\,l(2)\,]))(M(V([\,l(2)\,])) : L(V([\,l(1)\,]))).$$

To prove the second BGG reciprocity we need the following result.

Lemma 12.5.1. *Let $\mathfrak{G}$ be a simple finite-dimensional complex Lie algebra, $\mathfrak{H}$ be its Cartan subalgebra and W be the Weyl group. For $\lambda, \mu \in \mathfrak{H}^*$ set $\lambda \sim \mu$ if and only if $\lambda \in W \cdot \mu$ (here $w \cdot \mu = w(\mu + \rho) - \rho$ is the standard dot-action of W). Then for any $\lambda, \mu \in \mathfrak{H}^*$ and any simple finite-dimensional $\mathfrak{G}$-module F holds*

$$|W\lambda| \sum_{\nu : \nu + \lambda \sim \mu} \dim F_\nu = |W\mu| \sum_{\nu : \nu + \mu \sim \lambda} \dim F_\nu.$$

Proof. Let W_λ (resp. W_μ) be the subgroup of W stabilizing λ (resp. μ). Then we can rewrite our equality in the form

$$|W_\mu| \sum_{\nu:\nu+\lambda\sim\mu} \dim F_\nu = |W_\lambda| \sum_{\nu:\nu+\mu\sim\lambda} \dim F_\nu.$$

Let $w_1, w_2, \ldots, w_k$ be all the elements of W and define ν_i by $\nu_i + \lambda = w_i \cdot \mu$. We can rewrite the last equality as $w_i^{-1} \cdot \lambda = \mu - w_i^{-1}(\nu_i)$. Since $\dim F_\xi = \dim F_{w(\xi)}$ for all $\xi \in \mathfrak{H}^*$ and $w \in W$ we conclude that both the left and the right hand sides of the desired equality coincide with

$$\sum_{i=1}^{n} \dim F_{\nu_i}$$

and the lemma follows. $\square$

Combining Lemma 12.5.1 and Lemma 8.3.1, we obtain that for two simple modules $V([\,l(1)\,])$ and $V([\,l(2)\,])$ from $\mathcal{L}$ having central characters χ_{μ_1} and χ_{μ_2} respectively holds

$$\mathfrak{i}(V([\,l(1)\,]), V([\,l(2)\,])) = |W(\mu_2)|/|W(\mu_1)|.$$

Thus all conditions of Theorem 12.4.2 are also satisfied and the corresponding analogue of BGG reciprocity holds. Since we have both reciprocities, we can apply Corollary 12.4.1 and obtain for $[\,l(1)\,]$, $[\,l(2)\,]$, μ_1, μ_2 as above

$$\mathfrak{i}(V([\,l(1)\,]), V([\,l(2)\,])) = \mathfrak{l}(P(V([\,l(2)\,])))/\mathfrak{l}(P(V([\,l(1)\,]))) = |W(\mu_2)|/|W(\mu_1)|.$$

If we fix $[\,l(1)\,]$ such that $l(1)_{n,i} = 0$ for all i, we will have $|W(\mu_1)| = 1$ and $\mathfrak{l}(P(V([\,l(1)\,])))$. Hence $\mathfrak{l}(P(V([\,l(2)\,]))) = |W(\mu_2)|$. We can now combine this in the following statement:

Theorem 12.5.1. *1. For any simple generic $V([\,l\,])$, having the central character χ_μ, the length of the projective cover of $V([\,l\,])$ in $\mathcal{L} = \mathcal{L}([\,l\,])$ equals $|W(\mu)|$.*

2. The BGG reciprocity in $\mathcal{O}(\mathcal{P}, \mathcal{L})$, $\mathcal{L} = \mathcal{L}([\,l\,])$ has the following form:

$$[P(L(V([\,l(1)\,]))) : M(V([\,l(2)\,]))] = |W(\mu)|(M(V([\,l(2)\,])) : L(V([\,l(1)\,]))),$$

where χ_μ is the central character of $L([\,l(2)\,])$.

13 Categories $\mathcal{O}(\mathcal{P},\mathcal{L})$, II: Induction from $\mathfrak{sl}(2,\mathbb{C})$

In this Chapter we specialize the situation studied in the previous Chapter to the case $\mathfrak{A}\simeq\mathfrak{sl}(2,\mathbb{C})$. Our goal is a more detailed study of the corresponding categories. We will show that corresponding $\mathcal{O}(\mathcal{P},\mathcal{L})$ are equivalent to full sabcategories of classical $\mathcal{O}$, which possess an analogue of Soergel's combinatorial description and have reach in content theory of tilting modules.

13.1 Functor E

During Sections 13.1 – 13.3 we study, in detail, the category $\mathcal{O}(\mathcal{P},\mathcal{L})$ in the case $\mathfrak{A}\simeq\mathfrak{sl}(2,\mathbb{C})$ and $\mathcal{L}$ is associated with a module $V(a,b)$, $a,b\in\mathbb{C}$ (see Section 5.3). This is exactly the case which corresponds to GVMs $M(\lambda,p)$, studied in Chapter 6. We begin by describing the category $\mathcal{O}(\mathcal{P},\mathcal{L})$ more precisely. We retain the notation from Chapter 6.

Call a weight $\mathfrak{A}$-module V with finite-dimensional weight spaces *admissible*, provided f acts bijectively on V. By definition, any $V(a,b)$ is admissible. Let $\tilde{\mathcal{L}}=\tilde{\mathcal{L}}(V(a,b))$ denote the full subcategory of the category of $\mathfrak{A}$-modules, which consists of all admissible submodules and all admissible quotients of all modules having the form $V(a,b)\otimes F$, where F is a finite-dimensional $\mathfrak{A}$-module. One can easily see that $\tilde{\mathcal{L}}$ inherits the abelian structure from the category of all $\mathfrak{A}$-modules, i.e. $\tilde{\mathcal{L}}$ is closed under taking kernels and cokernels of homomorphisms. Moreover, $\tilde{\mathcal{L}}$ is closed under taking finite direct sums. One can extend $\tilde{\mathcal{L}}$ to the corresponding admissible category $\mathcal{L}=\mathcal{L}(V(a,b))$ of $\mathfrak{A}'$-modules. Set $\mathcal{O}(\mathcal{P},\mathcal{L})=\mathcal{O}(\mathcal{P},\mathcal{L}(V(a,b)))$.

Since any $V(a,b)$ is a generic GZ module over $\mathfrak{A}$, we know from Section 12.5 that $\mathcal{L}$ has a block decomposition with a unique simple in each block and that the length of any projective in $\mathcal{L}$ is either 1 or 2. It is easy to see that this length equals 2 if and only if b is the square of a positive integer. Moreover, it follows from Section 5.4 that $V(a,(\sqrt{b}+i)^2)$, $i\in\mathbb{Z}$ is a complete list of simple objects in $\mathcal{L}(V(a,b))$. First of all, we establish an equivalence of certain categories $\mathcal{O}(\mathcal{P},\mathcal{L})$.

Theorem 13.1.1. *The categories $\mathcal{O}(\mathcal{P},\mathcal{L}(V(a_1,b)))$ and $\mathcal{O}(\mathcal{P},\mathcal{L}(V(a_2,b)))$ are (blockwise) equivalent (i.e. they are independent on a).*

Proof. To prove this we will use Mathieu's localization (see Section 5.5). We can assume that $a_1\neq a_2$. Since $\mathbb{C}$ is one-dimensional over itself, there exists $x\in\mathbb{C}$ such that $a_1=a_2+x\alpha$. Moreover, $x\notin\mathbb{Z}$ according to our assumption. By the definition of $\mathcal{O}(\mathcal{P},\mathcal{L})$, f acts bijectively on any module $V\in\mathcal{O}(\mathcal{P},\mathcal{L})$. Thus any V can be trivially extended to a U_f-module.

Now suppose that M is a U_f-module and $0\neq v\in M$ such that $H_\alpha v=av$ for some $a\in\mathbb{C}$. We know, from Lemma 5.5.1, that $\theta_y(H_\alpha)v=(a+2y)v$ for any $y\in\mathbb{C}$. From this, it follows immediately that the twist by θ_{-x} (resp. θ_x) is a well-defined functor from $\mathcal{O}(\mathcal{P},\mathcal{L}(V(a_1,b)))$ to $\mathcal{O}(\mathcal{P},\mathcal{L}(V(a_2,b)))$ (resp. from $\mathcal{O}(P,\mathcal{L}(V(a_2,b)))$ to $\mathcal{O}(\mathcal{P},\mathcal{L}(V(a_1,b))))$. Since the composition of θ_x and θ_{-x} is an identity, we easily conclude that these fuctors are mutually inverse. The block version follows immediately. This completes the proof. $\square$

According to Theorem 13.1.1, the properties of the category $\mathcal{O}(\mathcal{P}, \mathcal{L}(V(a,b)))$ do not depend on a. Recall that for a fixed b there exists at least one $V(a,b)$ which is not simple. More precisely, if b is a square of an integer, such $V(a,b)$ is unique and if b is not a square of an integer, there are precisely two non-isomorphic non-simple modules $V(a',b)$ and $V(a'',b)$. Let $V(\mathfrak{l},b)$ be a non-simple module. The aim of this Section is to define and investigate a functor from $\mathcal{O}(\mathcal{P}, \mathcal{L}(V(\mathfrak{l},b)))$ to $\mathcal{O}$.

For $M \in \mathcal{O}(\mathcal{P}, \mathcal{L}) = \mathcal{O}(\mathcal{P}, \mathcal{L}(V(\mathfrak{l},b)))$ denote by $E(M)$ the space of locally e-finite elements of M. Since e is locally ad-nilpotent, $E(M)$ is a $\mathfrak{G}$-submodule of M. We know, that any module in $\mathcal{O}(\mathcal{P}, \mathcal{L})$ has finite length as a $\mathfrak{G}$-module. From this one deduces, that $E(M) \in \mathcal{O}$. We define E by restriction on homomorphism $\varphi : M \to N$ to $E(\varphi) : E(M) \to E(N)$. Thus E is a well-defined functor from $\mathcal{O}(\mathcal{P}, \mathcal{L}) = \mathcal{O}(\mathcal{P}, \mathcal{L}(V(\mathfrak{l},b)))$ to $\mathcal{O}$.

Theorem 13.1.2. *Let $M \in \mathcal{O}(\mathcal{P}, \mathcal{L})$.*

1. *$E(M) = 0$ if and only if $M = 0$.*
2. *$\dim(M_{\mu-k\alpha}) = \dim(E(M)_{\mu-k\alpha})$ holds for any $\mu \in \mathrm{supp}(M)$ and for any $k \in \mathbb{N}$ big enough.*
3. *For any finite-dimensional $\mathfrak{G}$-module F holds $E(M \otimes F) = E(M) \otimes F$.*
4. *E sends projectives from $\mathcal{O}(\mathcal{P}, \mathcal{L})$ to projectives in $\mathcal{O}$.*
5. *E sends indecomposable modules to indecomposable modules.*
6. *For $f : M \to N$ holds $E(f) = 0$ if and only if $f = 0$.*
7. *If M is a simple object in $\mathcal{O}(\mathcal{P}, \mathcal{L})$, then $E(M)$ has a unique simple highest weight submodule, say $L(\hat{E}(M))$, $\hat{E}(L) \in \mathfrak{H}^*$. Moreover, for any $M \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ and any simple $L \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ holds $(M : L) = (E(M) : L(\hat{E}(L)))$.*
8. *Let $P(L)$ be the projective cover of a simple $L \in \mathcal{O}(\mathcal{P}, \mathcal{L})$. Then $E(P(L)) = P(\hat{E}(L))$.*
9. *E is a full functor.*

Proof. We will give the scheme for the proof. The reader can consult [FKM2] for more details.

The functor E is defined on $\mathfrak{A}$-modules, so statements (1), (2), (3) and (7) follow from trivial $\mathfrak{sl}(2,\mathbb{C})$ computation. Now (4) follows from (3) and Section 12.2. (6) is obvious.

Prove (5). Assume that $M \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ is indecomposable, but $E(M) = N_1 \oplus N_2$ is a non-trivial decomposition in $\mathcal{O}$. Let $M_i = v \in M \mid X_{-\alpha}^k v \in N_i,$ for some $k \in \mathbb{N}$, $i = 1, 2$. It is easy to see that $M = M_1 \oplus M_2$; moreover, both M_i, $i = 1, 2$ are $\mathfrak{G}$-submodules in M. This contradicts our assumption. This proves (5). Now (8) follows from (4), (5) and (7). To complete our proof we have to prove only (9).

Clearly, it is enough to show that, for any $M, N \in \mathcal{O}(\mathcal{P}, \mathcal{L})$, holds

$$\dim \operatorname{Hom}_{\mathcal{O}(\mathcal{P},\mathcal{L})}(M, N) = \dim \operatorname{Hom}_{\mathcal{O}}(E(M), E(N)).$$

From (7) and (8) it follows that this is true if M is a projective module. Assume that M is indecomposable. Let $P(M)$ be a projective cover of M. It is enough to prove that for any $f : E(M) \to E(N)$ there is ψ in $\mathcal{O}(\mathcal{P}, \mathcal{L})$ such that $f = E(\psi)$. We have that $E(P(M))$ is a projective cover of $E(M)$. Let $z : E(P(M)) \to E(M)$ be a canonical epimorphism. Since $P(M)$ is projective, there exists an epimorphism $x : P(M) \to M$ and a homomorphism $y : P(M) \to N$ such that $z = E(x)$ and $f \circ z = E(y)$. For $m \in M$ set $\psi(m) = y \circ x^{-1}(m)$. We have to show that this is a well-defined map. But $\ker z \subset \ker f \circ z$, hence $\ker x \subset \ker y$ since $X_{-\alpha}$ acts bijectively on $P(M)$ and E acts on homomorphisms by restriction. This means that ψ is well-defined. Since both x and y are $\mathfrak{G}$-morphisms we deduce that ψ is also a $\mathfrak{G}$-morphism. Clearly, $E(\psi) = f$, since E is just restriction. This completes the proof of our theorem. □

Corollary 13.1.1. *$\mathcal{O}(\mathcal{P}, \mathcal{L})$ is equivalent to a full subcategory of $\mathcal{O}$. Moreover, the image of a block of $\mathcal{O}(\mathcal{P}, \mathcal{L})$ is contained in a block of $\mathcal{O}$.*

Theorem 13.1.2 also answers the question about multiplicities of simple subquotients in GVMs (analogue of Kazhdan-Lusztig Theorem), reducing this question to the solved one in $\mathcal{O}$.

Corollary 13.1.2. *Let V_1, V_2 be simple in $\mathcal{L}$. Then $(M(V_1) : L(V_2)) = (E(M(V_1)) : L(\hat{E}(L(V_2))))$.*

13.2 Soergel's Theorems

Using Theorem 13.1.2 it is possible to obtain an analogue of Soergel's combinatorial description of algebras, arising from $\mathcal{O}(\mathcal{P}, \mathcal{L})$. Here we retain the notation from Section 3.7. Consider the principal block, $\mathcal{O}_{triv} = \mathcal{O}(\chi_{\lambda_0})$, of $\mathcal{O}$ and the big projective module $P(w_0(\lambda_0))$ in it. Suppose that b is the square of an integer. Denote by $\mathcal{O}(\mathcal{P}, \mathcal{L})_{triv}$ the direct summand of $\mathcal{O}(\mathcal{P}, \mathcal{L}(V(\mathfrak{l}, \gamma)))$ which has a non-trivial image in $\mathcal{O}_{triv}$ under E. Since E acts blockwise and $\mathcal{O}_{triv}$ is indecomposable, such indecomposable $\mathcal{O}(\mathcal{P}, \mathcal{L})_{triv}$ is unique. Let L be a simple object in $\mathcal{O}(\mathcal{P}, \mathcal{L})$ such that $\hat{E}(L) = w_0(\lambda_0)$. L exists, since f acts injectively on $L(w_0(\lambda_0)) = M(w_0(\lambda_0))$. Call the projective module $P(L)$ in $\mathcal{O}(\mathcal{P}, \mathcal{L})_{triv}$ the *big projective module*. We remark that $P(L)$ can be characterized as the unique indecomposable projective in $\mathcal{O}(\mathcal{P}, \mathcal{L})_{triv}$, such that any GVM from this principal block occurs as a subquotient in a standard filtration of $P(L)$. The following Theorem is a direct analogue of the first part of Theorem 3.7.4.

Theorem 13.2.1. $\operatorname{End}_{\mathcal{O}(\mathcal{P},\mathcal{L})}(P(L)) \simeq \operatorname{End}_{\mathcal{O}}(P(w_0 \cdot 0))$. *In fact,* $\operatorname{End}_{\mathcal{O}(\mathcal{P},\mathcal{L})}(P(L))$ *is the coinvariant algebra.*

Proof. Follows from Theorem 13.1.2 and [S1, Endomorphismensatz 3]. □

Using Theorem 13.1.1, one can generalize this result to arbitrary $\mathcal{O}(\mathcal{P}, \mathcal{L}(V(a,b)))$, where b is the square of an integer. The next result generalizes Soergel's double centralizer Theorem (second part of Theorem 3.7.4).

Theorem 13.2.2. *Let B denote the (projectively stratified finite-dimensional) algebra associated with $\mathcal{O}(\mathcal{P}, \mathcal{L})_{triv}$. Then B is isomorphic to the endomorphism algebra of the big projective module, viewed as a module over its endomorphism ring.*

It is more convenient to prove this theorem in an "abstract" setting which we are going to introduce now. Let A (resp. B) denote the algebra associated with the principal block of $\mathcal{O}$ (resp. $\mathcal{O}(\mathcal{P}, \mathcal{L})$). We recall, that according to Theorem 13.1.2 and Corollary 13.1.1, B is a (matrix) subalgebra of A. Let e be the primitive idempotent of A such that Ae is the big projective module in $\mathcal{O}_{triv}$. Then Be is the big projective module in $\mathcal{O}(\mathcal{P}, \mathcal{L})_{triv}$ and $C = eAe = eBe$ is the coinvariant algebra, which is the endomorphism algebra of Ae and Be. Let $T = \mathrm{Hom}_A(Ae, \_)$ denote Soergel's functor ([S1]). Recall, that by Soergel's Theorem ([S1, Struktursatz 2]), for any $M \in \mathcal{O}_{triv}$ and any projective $Q \in \mathcal{O}_{triv}$, holds

$$\mathrm{Hom}_A(M, Q) \simeq \mathrm{Hom}_{C=eAe}(T(M), T(Q)).$$

Proof of Theorem 13.2.2. We start from $B = \mathrm{Hom}_B(B, B)$. Applying Theorem 13.1.2, we have $\mathrm{Hom}_B(B, B) \simeq \mathrm{Hom}_A(E(B), E(B))$. Now applying the mentioned result by Soergel we obtain $\mathrm{Hom}_A(E(B), E(B)) \simeq \mathrm{Hom}_{eAe}(T(E(B)), T(E(B)))$. We know from Theorem 13.2.1, that $eAe = eBe$. Recall that $E(Be) = Ae$, hence $T(E(B)) = \mathrm{Hom}_A(Ae, E(B)) = \mathrm{Hom}_A(E(Be), E(B)) \simeq \mathrm{Hom}_B(Be, B) = eB$. Finally,

$$\mathrm{Hom}_{eAe}(T(E(B)), T(E(B))) \simeq \mathrm{Hom}_{eBe}(eB, eB).$$

Now we note that B is a matrix subalgebra of A and we can apply duality on A to the last endomorphism ring, obtaining $\mathrm{Hom}_{eBe}(eB, eB) \simeq \mathrm{Hom}_{eBe}(Be, Be)$, which completes the proof. $\square$

We note that in the proof above, we can also apply the canonical duality on B, mentioned in [FP, FKM1].

13.3 Tilting modules

In this Section we develop the theory of tilting module for $\mathcal{O}(\mathcal{P}, \mathcal{L}) = \mathcal{O}(\mathcal{P}, \mathcal{L}(V(\mathfrak{l}, b)))$. As a preliminary result we have to investigate a family of tilting modules in $\mathcal{O}$. There is a general result about tilting modules for stratified algebras ([AHLU]), but we are going to use a more symmetric definition of tilting modules for $\mathcal{O}(\mathcal{P}, \mathcal{L})$; we will have to do some preliminary work. Retain the notation from Section 3.7.

Assume that $\lambda \in \mathfrak{H}^*$ is such that $s_\alpha(\lambda) = \lambda + k\alpha$ for some $k \in \mathbb{N}$. Consider the indecomposable projective module $P(\lambda) \in \mathcal{O}$. Clearly, there exists a Verma flag $P(\lambda) = P_0 \supset P_1 \supset P_2 \supset \dots$ of $P(\lambda)$ such that $P_0/P_1 \simeq M(\lambda)$ and $P_1/P_2 \simeq M(s_\alpha(\lambda))$. Set $\hat{P}(\lambda) = P(\lambda)/P_2$. Then $\hat{P}(\lambda)$ has a Verma flag with $M(\lambda)$ on the top and $M(s_\alpha(\lambda))$ on the bottom. Define a class $K(\alpha)$ of modules in $\mathcal{O}$ as follows: if $\lambda \in \mathfrak{H}^*$ is such that $s_\alpha(\lambda) - \lambda \notin (\mathbb{Z}\alpha \setminus \{0\})$ then $K(\alpha)$ contains $M(\lambda)$; in the other case, $K(\alpha)$ contains $\hat{P}(\lambda)$ if $s_\alpha(\lambda) - \lambda \in \mathbb{N}\alpha$ and $K(\alpha)$ contains $\hat{P}(s_\alpha(\lambda))$ if $\lambda - s_\alpha(\lambda) \in \mathbb{N}\alpha$.

Denote by $O_1(\alpha)$ (resp. $O_2(\alpha)$) the full subcategory of $\mathcal{O}$, containing all modules which admit a filtration with subquotients from $K(\alpha)$ (resp. with subquotients of the form M^*, $M \in K(\alpha)$). Since any module in $K(\alpha)$ has a Verma flag, we have $O_1(\alpha) \subset O_1$, $O_2(\alpha) \subset O_2$ and $\mathcal{O}(K(\alpha)) = O_1(\alpha) \cap O_2(\alpha) \subset O_1 \cap O_2$. Hence any module in $\mathcal{O}(K(\alpha))$ (if there is any) is a tilting module. So to determine $\mathcal{O}(K(\alpha))$ we have to find out which indecomposable tilting modules belong to it.

Lemma 13.3.1. *For any $M \in K(\alpha)$ (resp. M such that $M^* \in K(\alpha)$) and any finite-dimensional $\mathfrak{G}$-module F the module $F \otimes M$ belongs to $O_1(\alpha)$ (resp. $O_2(\alpha)$).*

Proof. Follows from the exactness of $F \otimes \_$ by standard arguments combined with the observation that we are considering objects which are induced from projective objects. $\square$

Proposition 13.3.1. *$T(\lambda) \in \mathcal{O}(K(\alpha))$ if and only if either $s_\alpha(\lambda) - \lambda \notin \mathbb{Z}\alpha \setminus \{0\}$ or $\lambda - s_\alpha(\lambda) \in \mathbb{N}\alpha$.*

Proof. If $s_\alpha(\lambda) - \lambda \notin (\mathbb{Z}\alpha \setminus \{0\})$ then, according to the definition of $K(\alpha)$, any Verma module (resp. dual Verma module), occuring in the Verma flag (resp. dual Verma flag) of $T(\lambda)$ belongs to $K(\alpha)$ (resp. is of the form M^* for some $M \in K(\alpha)$). Hence $T(\lambda) \in \mathcal{O}(K(\alpha))$.

Recall that any increasing Verma flag of $T(\lambda)$ starts with $M(\lambda)$. From the definition of $K(\alpha)$ it follows that for any $\lambda \in \mathfrak{H}^*$ such that $s_\alpha(\lambda) - \lambda \in \mathbb{N}\alpha$ there are no modules in $K(\alpha)$ such that their increasing Verma flag starts with $M(\lambda)$. Since any filtration with quotients from $K(\alpha)$ can be extended to a Verma flag, we obtain that in the case $s_\alpha(\lambda) - \lambda \in \mathbb{N}\alpha$ the module $T(\lambda)$ cannot belong to $K(\alpha)$.

So we only have to prove that $T(\lambda) \in K(\alpha)$ in case $\lambda - s_\alpha(\lambda) \in \mathbb{N}\alpha$. This will follow easily if we recall the inductive construction of tilting modules via tensoring with finite-dimensional modules. Suppose that λ is such that $\lambda - s_\alpha(\lambda) \in \mathbb{N}\alpha$ and $M(s_\alpha(\lambda))$ is simple. Then $T(\lambda) \simeq \hat{P}(s_\alpha(\lambda))$, by the construction of $\hat{P}(s_\alpha(\lambda))$ and hence $T(\lambda) \in O_1(\alpha)$. But $T(\lambda)$ is also self-dual as a tilting module in the category $\mathcal{O}$, hence $T(\lambda) \in O_2(\alpha)$. Finally, $T(\lambda) \in \mathcal{O}(K(\alpha))$.

Now we note, that from Lemma 13.3.1 it follows that $\mathcal{O}(K(\alpha))$ is stable under tensoring with finite-dimensional modules. In particular, it means that if we fix λ as in the previous paragraph, then $T(\lambda) \otimes F$ belongs to $\mathcal{O}(K(\alpha))$ for any finite-dimensional $\mathfrak{G}$-module F. To complete the proof we only need to recall that any $T(\mu)$ with $\mu - s_\alpha(\mu) \in \mathbb{N}\alpha$ occurs as a direct summand in $T(\lambda) \otimes F$ for some λ as in the previous paragraph and some finite-dimensional F ([CI]). $\square$

The modules in $\mathcal{O}(K(\alpha))$ will be called *strong tilting modules*. Later we will see that they are closely related to tilting modules in $\mathcal{O}(\mathcal{P}, \mathcal{L})$. Since λ_0 satisfies the conditions of Proposition 13.3.1, we have.

Corollary 13.3.1. *The big projective module is a strong tilting module.*

Let $\mathcal{L} = \mathcal{L}(V(\mathfrak{l}, b))$, b the square of an integer. In order to introduce the notion of a tilting module in $\mathcal{O}(\mathcal{P}, \mathcal{L})$, we need a natural duality on $\mathcal{O}(\mathcal{P}, \mathcal{L})$. This can be easily done, using σ for $\mathcal{O}(\mathcal{P}, \mathcal{L}(V(a, b)))$, in the case when $V(a, b)$ is a simple α-module. The same direct construction for the case $\mathcal{O}(\mathcal{P}, \mathcal{L}(V(\mathfrak{l}, b)))$ does not work, because dualization does not preserve the bijectivity of the action of f. In fact, e acts bijectively on the dual module. There are two ways to solve this problem. The first way is to fix a non-integer x and to define a duality * on $\mathcal{O}(\mathcal{P}, \mathcal{L})$ as the composition of θ_x, the natural duality on $\theta_x(\mathcal{O}(\mathcal{P}, \mathcal{L}))$, which can be constructed via σ (here everything works since both e and f act bijectively on $\theta_x(\mathcal{O}(\mathcal{P}, \mathcal{L}))$), and θ_{-x}. The second way is to compose σ with the natural automorphism of $\mathfrak{G}$ corresponding to the simple reflection s_α. We choose the second way and from now on for $M \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ we will denote by M^* the corresponding dual module.

Let $\mathcal{G}(\Delta)$ (resp. $\mathcal{G}(\bigtriangledown)$) denote the full subcategory of $\mathcal{O}(\mathcal{P}, \mathcal{L})$ which consists of all modules having a standard filtration, i.e. a filtration, whose subquotients are isomorphic to $M_{\mathcal{P}}(W)$, with W being a projective in $\mathcal{L}$ (resp. a dual standard filtration, i.e. a filtration whose subquotients are isomorphic to $M_{\mathcal{P}}(W)^*$, W projective in $\mathcal{L}$). A module, $M \in \mathcal{O}(\mathcal{P}, \mathcal{L})$, will be called a *tilting module* if $M \in \mathcal{G}(\Delta) \cap \mathcal{G}(\bigtriangledown)$.

So far, we do not know if there is any tilting module in $\mathcal{O}(\mathcal{P}, \mathcal{L})$. The aim of this section is to describe all tilting module in $\mathcal{O}(\mathcal{P}, \mathcal{L})$. We recall that our definition of tilting module does not coincide with the general definition in [AHLU]. The difference is in the definition of $\mathcal{G}(\bigtriangledown)$. In [AHLU], the existence of a filtration is required, whose subquotients are isomorphic to $M_{\mathcal{P}}(W)^*$, where W is simple in $\mathcal{L}$. Our condition is more restrictive. Taking into account the uniqueness of characteristic tilting modules for standardly stratified algebras (this class includes, in particular, projectively stratified algebras) in [AHLU], we only have to show that for any simple $L = L_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ there exists an indecomposable tilting module $T(L) \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ such that the standard filtration of $T(L)$ starts with $M_{\mathcal{P}}(W)$.

Lemma 13.3.2. *For any $M \in \mathcal{G}(\Delta)$ (resp. $M \in \mathcal{G}(\bigtriangledown)$) and any submodule N occuring in a standard filtration (resp. dual standard filtration) of M holds $E(N) \subset E(M)$ and $E(M/N) \simeq E(M)/E(N)$.*

Proof. Follows from the definition of E and the fact that $M \simeq N \oplus (M/N)$ as a $\mathfrak{A}$-module. □

Lemma 13.3.3. *Let T be a tilting module in $\mathcal{O}(\mathcal{P}, \mathcal{L})$. Then $E(T)$ is a strong tilting module in $\mathcal{O}$.*

Proof. From the definition of $K(\alpha)$, it follows immediately that for any projective $W \in \mathcal{L}$ holds $E(M_{\mathcal{P}}(W)) \in K(\alpha)$. Now, by Lemma 13.3.2, the standard (resp. dual standard) filtration of T is sent by E to a filtration whose subquotients are elements of $K(\alpha)$ (resp. with subquotients, dual to modules in $K(\alpha)$). This completes the proof. □

Lemma 13.3.4. *For any strong tilting module $T' \in \mathcal{O}$, there exists a tilting module $T \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ such that $E(T) \simeq T'$.*

Proof. Clearly, it is enough to prove this statement for indecomposable T', so we can suppose that $T' = T(\lambda)$. First, assume that $M(s_\alpha(\lambda))$ is a simple $\mathfrak{G}$-module. Clearly, $M(\lambda)$ belongs to the image of E, hence $M(\lambda) = E(M_{\mathcal{P}}(W))$ for some simple object $W \in \mathcal{L}$. Let W' be the projective cover of W. From the definition of $K(\alpha)$ one immediately obtains $T(\lambda) = E(M_{\mathcal{P}}(W'))$. Now the statement follows from Theorem 13.1.2, the inductive construction of strong tilting modules, and the remark that tilting modules in $\mathcal{O}(\mathcal{P},\mathcal{L})$ are self-dual. □

Theorem 13.3.1. *For any simple object $L = L_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P},\mathcal{L})$ there exists exactly one indecomposable tilting module $T(L) \in \mathcal{O}(\mathcal{P},\mathcal{L})$ such that the standard filtration of $T(L)$ starts with $M_{\mathcal{P}}(W')$, where W' is a projective cover of W in $\mathcal{L}$. The set $T(L)$, where L runs through simple modules in $\mathcal{O}(\mathcal{P},\mathcal{L})$ is a complete set of indecomposable tilting modules in $\mathcal{O}(\mathcal{P},\mathcal{L})$. Any tilting module is a finite direct sum of indecomposable tilting modules.*

Proof. Existence follows from Lemma 13.3.4. The rest follows from [AHLU, 2.1 and 2.2]. □

We have already proved the existence of tilting modules in $\mathcal{O}(\mathcal{P},\mathcal{L})$. Now we are going to determine the formal character of a tilting module. Clearly, it is sufficient to do this for an indecomposable module $T(L)$, where L is a simple module in $\mathcal{O}(\mathcal{P},\mathcal{L})$. Further, by the definition, $T(\lambda)$ has a standard filtration, hence it has a filtration by $M_{\mathcal{P}}(W)$, where $W \in \mathcal{L}$ is a simple object. Since $M_{\mathcal{P}}(W)$ is an extension of two Verma modules (with respect to a different basis in $\mathfrak{G}$), its character is known. So the problem is to determine the multiplicities $[T(L) : M_{\mathcal{P}}(W)]$. We solve this problem by reducing it to the recently solved analogous problem for $\mathcal{O}$ (see [S4]).

Theorem 13.3.2. *Let W_1 and W_2 be simple objects in $\mathcal{L}$. Denote by $l(W_2')$ the length of the projective cover W_2' of W_2 in $\mathcal{L}$. Then*

$$[T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2)] = l(W_2')[E(T(L_{\mathcal{P}}(W_1))) : E(M_{\mathcal{P}}(W_2))].$$

Proof. Set $m = [T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2)]$. Then $m = l(W_2')[T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2')]$ and by Lemma 13.3.2, $[T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2')] = [E(T(L_{\mathcal{P}}(W_1))) : E(M_{\mathcal{P}}(W_2'))]$. If $l(W_2') = 1$ then $W_2 = W_2'$ and we are done. Otherwise, it follows, from the definition of $K(\alpha)$, that the number of Verma modules in a Verma flag of $E(M_{\mathcal{P}}(W_2'))$ equals 2; moreover, $[E(M_{\mathcal{P}}(W_2')) : E(M_{\mathcal{P}}(W_2))] = 1$. This completes the proof. □

According to Lemma 4.3.1, $E(T(L_{\mathcal{P}}(W_1)))$ is a strong tilting module. In particular, it is a tilting module in $\mathcal{O}$. Furthermore, $E(M_{\mathcal{P}}(W_2))$ is a Verma module in $\mathcal{O}$, hence, the multiplicity $[E(T(L_{\mathcal{P}}(W_1))) : E(M_{\mathcal{P}}(W_2))]$ can be computed by Soergel's Theorem, [S4, Theorem 5.12 and Theorem 6.7]. We also note, that again, applying the Mathieu's functors θ_x, one extends the above results to an arbitrary category $\mathcal{O}(\mathcal{P},\mathcal{L}(V(l,\gamma)))$.

Finally, if one looks at the proof of [S4, Theorem 2.1], one sees that it implies another interesting result for the principal block $\mathcal{O}(\mathcal{P},\mathcal{L})_{triv}$ of $\mathcal{O}(\mathcal{P},\mathcal{L})$. We keep the notation from [S4]. Let $S = S_\delta$ denote the semi-regular bimodule, associated with a semi-infinite character

δ. As it was shown in [S4], the functor $S \otimes_{U(\mathfrak{G})} \_$ maps an indecomposable projective $P(w(\lambda))$, $\lambda \in P^{++}$, $w \in W$, into an indecomposable tilting module $T(ww_0(\lambda)) \in \mathcal{O}$, where w_0 is the longest element of the Weyl group. Comparing Theorem 13.1.2, (8) with the definition of strong tilting module we see that for any indecomposable projective module $P(L) \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ the module $S \otimes_{U(\mathfrak{G})} E(P(L))$ is an indecomposable strong tilting module. If we recall Lemmas 13.3.3 and 13.3.4 and the fact that $S \otimes_{U(\mathfrak{G})} \_$ is an equivalence of certain categories ([S4, Section 2]) preserving short exact sequences, we obtain the following result:

Theorem 13.3.3. *The projectively stratified algebra of $\mathcal{O}(\mathcal{P}, \mathcal{L})_{triv}$ is its own Ringel dual (see [R1, KlKo] for detail).*

14 Catogories $\mathcal{O}(\mathcal{P}, \mathcal{L})$, III: Case of arbitrary $\mathfrak{A}$

The aim of this Chapter is to generalize the machinary worked out in Chapter 13 in order to study categories $\mathcal{O}(\mathcal{P}, \mathcal{L})$ in the case of arbitrary $\mathfrak{A}$, in particular, we want to study $\mathcal{O}(\mathcal{P}, \mathcal{L})$ from the Gelfand-Zetlin example (Section 12.5). We will proceed by defining and analyzing an appropriate candidate for the image of "virtual" functor E. Only in the last Section we will construct an analogue of E for the Gelfand-Zetlin example.

14.1 Complete modules having a quasi Verma flag in $\mathcal{O}$

Let $\mathcal{O} = \mathcal{O}(\mathfrak{A})$ be the classical BGG-category $\mathcal{O}$ associated with the standard triangular decomposition of $\mathfrak{A}$. A module, M, from $\mathcal{O}$ is said to have a *quasi Verma flag* if there is a filtration, $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$, such that each M_i/M_{i-1} is a (non-zero) submodule in a Verma module. As each Verma module has a simple socle, which is itself a Verma module, the length of a quasi Verma flag does not depend on the choice of a flag and equals the number of simple Verma subquotients of M.

Fix a Weyl-Chevalley basis in $\mathfrak{A}$. With any simple root α we can associate an *elementary Enright completion*, r_α, defined as the composition of the following three functors ([E, Dh, Ma]). The first one is an induction from $U = U(\mathfrak{A})$ to the Mathieu's localization U_S of U with respect to the powers of $X_{-\alpha}$. The second one is the restriction back to U and the last one is taking the locally X_α-finite part. Clearly $r_\alpha : \mathcal{O} \to \mathcal{O}$ and $r_\alpha \circ r_\alpha = r_\alpha$. It is known that (on objects which are torsion-free over $\mathfrak{N}_-$) the functors r_α satisfy the braid relations (main result in [De]), hence for any element w of the Weyl group W we can define the corresponding composition r_w. Now by *Enright completion* we will mean $r = r_{w_0}$, where w_0 is the longest element in W.

A module, M, from $\mathcal{O}$ is said to be *complete* if $r(M) = M$. Our main object of interest here is the full subcategory $\mathcal{K}$ of $\mathcal{O}$ which consists of all complete modules having a quasi Verma flag and integral support.

Theorem 14.1.1. *$M \in \mathcal{K}$ if and only if it is a kernel of homomorphisms between two modules, each of which is a direct sum of projective covers of simple Verma modules (with integral parameters).*

Proof. Assume that $0 \to M \to P_1 \to P_2$ is exact with P_1, P_2 being direct sums of projective covers of simple Verma modules. As P_1 is projective, it has a Verma flag and hence each its submodule has a quasi-Verma flag. Moreover, with respect to any $\mathfrak{sl}(2) - subalgebra$ associated with a simple root, both P_1 and P_2 are direct sums of projective, therefore complete modules. Direct verification in $\mathfrak{sl}(2)$ case shows that the kernel of any homomorphism between complete modules is complete. We obtain that $r_\alpha(M) = M$ for any simple root α and hence M is complete.

Conversely, since M has a quasi Verma flag and socle of any Verma module is a simple Verma module, the injective envelope P_1 of M is a direct sum of injective envelops of simple Verma modules, which are isomorphic to the corresponding projective covers. Now, as both M and P_1 are complete, each $X_{-\alpha}$ acts injectively on the socle of P_1/M and hence this socle is a direct sum of simple Verma modules. This means that the injective envelope of P_1/M is also a direct sum of injective envelops (=projective covers) of simple Verma modules. □

Corollary 14.1.1. *$\mathcal{K}$ is an abelian category and decomposes to a direct sum of module categoriers over local algebras.*

Proof. This follows from abstract nonsence for an endomorphism algebra of a projective-injective module in the corresponding block of $\mathcal{O}$. □

Corollary 14.1.2. *For any finite-dimensional $\mathfrak{A}$-module F, $F\otimes\_$ preserves $\mathcal{K}$ and is exact on $\mathcal{K}$.*

Proof. Follows from the fact that $F\otimes\_$ preserves category of projective-injective modules. □

14.2 Soergel's Theorems

Consider the category $\mathcal{O}(\mathcal{P},\mathcal{L})$ for $\mathcal{L}=\mathcal{K}$ as in the previous Section. It is clear that this $\mathcal{O}(\mathcal{P},\mathcal{L})$ coincides with the full subcategory of $\mathcal{O}(\mathfrak{G})$ consisting of all $\mathfrak{A}$-complete modules with integral $\mathfrak{A}$-weights. Decomposing $\mathcal{O}(\mathcal{P},\mathcal{L})$ with respect to $Z(\mathfrak{G})$ we get that $\mathfrak{A}$-complete projective modules from $\mathcal{O}$ are projective in $\mathcal{O}(\mathcal{P},\mathcal{L})$ and hence indecomposable projectives in $\mathcal{O}(\mathcal{P},\mathcal{L})$ are precisely $\mathfrak{A}$-complete indecomposable projectives in $\mathcal{O}$. In particular, all projective covers of simple Verma modules belong to $\mathcal{O}(\mathcal{P},\mathcal{L})$. As in Section 13.2 by a *principal block* of $\mathcal{O}(\mathcal{P},\mathcal{L})$ we will mean the direct summand lying in the principal block of $\mathcal{O}$.

Theorem 14.2.1. *The algebra associated with the principal block of $\mathcal{O}(\mathcal{P},\mathcal{L})$ is isomorphic to the endomorphism algebra of the big projective module from $\mathcal{O}(\mathcal{P},\mathcal{L})$ viewed as a module over its endomorphism algebra.*

Proof. Analogous to that of Theorem 13.2.2. □

14.3 Tilting modules

For an indecomposable projective, V, in $\mathcal{L}$ call $M_{\mathcal{P}}(V)$ a *standard module.* Since V is self-dual and projective in $\mathcal{L}$, $M_{\mathcal{P}}(V)$, as an $\mathfrak{A}$-module, is a direct sum of self-dual projective modules in $\mathcal{L}$.

Lemma 14.3.1. *Let $M_{\mathcal{P}}(V)\in\mathcal{O}(\mathcal{P},\mathcal{L})$ be a standard module. Then the dual modules to $M_{\mathcal{P}}(V)$ in $\mathcal{O}(\mathcal{P},\mathcal{L})$ and in $\mathcal{O}$ are isomorphic.*

Proof. We reduce our consideration to a block of $\mathcal{O}(\mathcal{P},\mathcal{L})$, which corresponds to a projectively stratified finite-dimensional algebra. Let S be the partially ordered set of simple modules. Then S also parametrizes the standard modules. From the description of projective modules in $\mathcal{O}(\mathcal{P},\mathcal{L})$ it follows that $M_{\mathcal{P}}(V)$ can be written as $P(V)/N$, where $P(V)$

is an indecomposable projective module, N has a standard filtration and all the standard subquotients of this filtration are bigger than $M_{\mathcal{P}}(V)$ with respect to S. We know that the dual modules for $P(V)$ in $\mathcal{O}(\mathcal{P},\mathcal{L})$ and $\mathcal{O}$ coincides. Now the statement follows by induction in S. $\square$

We will call $M_{\mathcal{P}}(V)^*$, V is an indecomposable projective in $\mathcal{L}$, *costandard modules.* Consider the full subcategory O_1 (resp. O_2) of $\mathcal{O}(\mathcal{P},\mathcal{L})$ which consists of all modules having a *standard filtration* (resp. *costandard filtration*), i.e. a filtration, whose quotients are standard (resp. costandard) modules.

Corollary 14.3.1. *Let $M \in O_1 \cup O_2$. Then the dual modules to M in $\mathcal{O}(\mathcal{P},\mathcal{L})$ and in $\mathcal{O}$ are isomorphic.*

Proof. Follows from Lemma 14.3.1 and exactness of the dualities. $\square$

A module $M \in O_1 \cap O_2$ will be called a *tilting* module. Hence, by virtue of Corollary 14.3.1, it should be a tilting module in $\mathcal{O}$. Now we have to determine those $T(\lambda)$ belonging to $\mathcal{O}(\mathcal{P},\mathcal{L})$.

Lemma 14.3.2. *$T(\lambda) \in \mathcal{O}(\mathcal{P},\mathcal{L})$ if and only if λ is $\mathfrak{A}$-integral and belongs to the closure of the $\mathfrak{A}$-dominant Weyl chamber.*

Proof. Let $M \in \mathcal{O}(\mathcal{P},\mathcal{L})$ be a module having a standard filtration. This filtration can be refined to a Verma flag in $\mathcal{O}$. Let $M(\lambda)$ be a Verma submodule in M occuring in this Verma flag. Then $M(\lambda)$ is complete in $\mathcal{L}$ and hence λ is $\mathfrak{A}$-integral and belongs to the closure of the $\mathfrak{A}$-dominant Weyl chamber. Therefore, the only candidates for being in $\mathcal{O}(\mathcal{P},\mathcal{L})$ are $T(\lambda)$, which satisfy the condition of our Lemma.

Let w_0 denote the longest element in the Weyl group of $\mathfrak{A}$. First consider $T(\mu)$, where μ is $\mathfrak{A}$-integral and belongs to the closure of the $\mathfrak{A}$-dominant Weyl chamber, such that $M(w_0(\mu))$ is simple. Then $T(\mu)$ is a self-dual standard module and hence $T(\mu) \in \mathcal{O}(\mathcal{P},\mathcal{L})$. To complete the proof we recall that $\mathcal{O}(\mathcal{P},\mathcal{L})$, O_1 and O_2 are closed under tensoring with finite-dimensional modules and any $T(\lambda)$ such that λ satisfies the condition of our Lemma can be obtained as a direct summand in $T(\mu) \otimes F$ for some finite-dimensional F and some $T(\mu)$ as above ([CI]). $\square$

Theorem 14.3.1. *Any tilting module in $\mathcal{O}(\mathcal{P},\mathcal{L})$ is a direct sum of indecomposable tilting modules of the form $T(\lambda)$, where λ is $\mathfrak{A}$-integral and belongs to the closure of the $\mathfrak{A}$-dominant Weyl chamber.*

Proof. We have already proved that all $T(\lambda)$, where λ is $\mathfrak{A}$-integral and belongs to the closure of the $\mathfrak{A}$-dominant Weyl chamber, are tilting modules in $\mathcal{O}(\mathcal{P},\mathcal{L})$. Recall that blocks of $\mathcal{O}(\mathcal{P},\mathcal{L})$ correspond to projectively stratified finite-dimensional algebras. Now the uniqueness of an indecomposable tilting module corresponding to a given simple module follows from an abstract result [AHLU, 2.1 and 2.2] on tilting modules over stratified algebras. $\square$

Consider again the algebra B, which corresponds to the principal block of $\mathcal{O}(\mathcal{P},\mathcal{L})$ and let T be the direct sum of all indecomposable tilting modules in $\mathcal{O}(\mathcal{P},\mathcal{L})_{triv}$.

Theorem 14.3.2. $B \simeq \mathrm{End}(T)$*, ie.* B *is Ringel self-dual.*

Proof. Let S denote the semi-regular $U(\mathfrak{G})$-bimodule ([S4]) and let λ_0 be the highest weight of the trivial $\mathfrak{G}$-module. Let w_0 be the longest element in the Weyl group W of $\mathfrak{G}$. Then the composition of $S\otimes\_$ with the graded duality maps $P(w(\lambda_0))$ to $T(ww_0(\lambda_0))$ for any $w \in W$ ([S4]). Note that if $w(\lambda_0)$ belongs to the closure of the $\mathfrak{A}$-antidominant Weyl chamber, then $ww_0(\lambda_0)$ belongs to the closure of the $\mathfrak{A}$-dominant Weyl chamber. Hence the composition of $S \otimes \_$ with the graded duality transfers projective modules from $\mathcal{O}(\mathcal{P},\mathcal{L})_{triv}$ to tilting modules in $\mathcal{O}(\mathcal{P},\mathcal{L})_{triv}$, producing an isomorphism between B and its Ringel dual. □

14.4 Equivalence with the Gelfand-Zetlin example

Assume that $\mathfrak{A} = \mathfrak{sl}(n,\mathbb{C})$ and let $\mathcal{F}$ denote the admissible category generated by a simple generic Gelfand-Zetlin module with integral central character, defined in Section 12.5.

Theorem 14.4.1. *There exists a canonical equivalence of categories between* $\mathcal{F}$ *and* $\mathcal{K}$*, which commutes with* $F \otimes \_$*,* F *finite dimensional. In particular, categories* $\mathcal{O}(\mathcal{P},\mathcal{F})$ *and* $\mathcal{O}(\mathcal{P},\mathcal{K})$ *are equivalent and hence Soergel's Theorems and tilting theory can be stated for* $\mathcal{O}(\mathcal{P},\mathcal{F})$

Lemma 14.4.1. *Fix* $\chi \in Z(\mathfrak{A})^*$*. Then the lengths of the indecomposable projectives in* $\mathcal{F}_\chi$ *and in* $\mathcal{K}_\chi$ *are the same and this common number coincides with the number of non-isomorphic Verma modules (over* $\mathfrak{A}$*) having central character* χ*.*

Proof. Let l be the number of non-isomorphic Verma modules over $\mathfrak{A}$ having central character χ. That the length of the indecomposable projective in $\mathcal{F}_\chi$ equals l is proved in Section 12.5. For $\mathcal{K}_\chi$ it can be shown as follows. Recall that the indecomposable projective in $\mathcal{K}_\chi$ is the big projective module $P(\lambda)$, where λ belongs to the closure of the antidominant Weyl chamber. Moreover, its length in K_χ coincides with the composition multiplicity $(P(\lambda) : L(\lambda))$. As $L(\lambda)$ is a simple socle of each Verma module in $\mathcal{O}_\chi$, the last number equals the length of any Verma flag of $P(\lambda)$. By BGG-reciprocity and the mentioned description of the socles of Verma modules, each Verma module from $\mathcal{O}_\chi$ occurs exactly once in any Verma flag of $P(\lambda)$. Hence, the length of $P(\lambda)$ in $\mathcal{K}_\chi$, which coincides with the length of a (quasi) Verma flag of $P(\lambda)$, equals l. □

Lemma 14.4.2. *Assume that we have already constructed an exact functor,* f*, from* $\mathcal{F}$ *to* $\mathcal{K}$*, which commutes with* $F \otimes \_$ *for any finite-dimensional* F*, faithful on morphisms and sends (for each* χ*) the simple from* $\mathcal{F}_\chi$ *to the simple from* $\mathcal{K}_\chi$ *inducing an isomorphism on the endomorphism rings. Then* f *is the desired equivalence, which proves Theorem 14.4.1.*

Proof. Denote by $\hat{\chi}$ the central integral character of a simple-projective Verma module. For this $\hat{\chi}$, the simple module in $\mathcal{F}_{\hat{\chi}}$ (or $\mathcal{K}_{\hat{\chi}}$) coincides with the corresponding indecomposable projective. As $F\otimes \_$ is exact on both $\mathcal{F}$ and $\mathcal{K}$ and f commutes with $F\otimes \_$, we get that f sends the indecomposable projective from $\mathcal{F}_\chi$ to the indecomposable projective in $\mathcal{K}_\chi$ for any χ. As f is exact, it sends simples to simples. All $\mathcal{F}_\chi$ and $\mathcal{K}_\chi$ are module categories over local algebras. Moreover, f acts blockwise, so it is enough to prove that $f:\mathcal{F}_\chi\to\mathcal{K}_\chi$ is an equivalence. But the lengths of the indecomposable projectives in $\mathcal{F}_\chi$ and $\mathcal{K}_\chi$ coincide by Lemma 14.4.1. Since f preserves the endomorphism ring of a simple and is exact, we derive that f is full on morphisms and the final statement follows from the exactness of f. □

By Lemma 14.4.2, to prove Theorem 14.4.1 we need only to construct an exact functor from $\mathcal{F}$ to $\mathcal{K}$, which commutes with all $F\otimes \_$, is faithful on morphisms and sends simples from $\mathcal{F}$ to simples in $\mathcal{K}$ preserving the central character and the endomorphism ring of any simple. We will construct this functor composing several m_α^x and l_α. Hence, as the next step we review some properties of these functors.

Fix a simple root α and denote by $\mathfrak{A}(\alpha)$ the $\mathfrak{sl}(2,\mathbb{C})$ subalgebra of $\mathfrak{A}$ associated with α. Let $\mathcal{L}_\alpha$ denote the full subcategory of the category of all finitely-generated $\mathfrak{A}(\alpha)$-modules, which consists of all direct summands of the modules $F\otimes M$, where F is finite-dimensional and M is a finitely generated weight module with one-dimensional weight spaces and such that $X_{-\alpha}$ acts bijectively on M. Let $\mathcal{M}_\alpha$ denote the full subcategory of the category of all finitely generated $\mathfrak{A}$-modules, which consists of all modules M, that can be decomposed into a direct sum of modules from $\mathcal{L}_\alpha$ when viewed as $\mathfrak{A}(\alpha)$-modules. It is easy to see that $\mathcal{L}_\alpha$ (resp. $\mathcal{M}_\alpha$) inherits an abelian structure from the category of all $\mathfrak{A}(\alpha)$ (resp. $\mathfrak{A}$) -modules .

Let $\mathcal{L}^\alpha$ denote the full subcategory of the category of all finitely generated $\mathfrak{A}(\alpha)$-modules, which consists of complete modules having a quasi-Verma flag from the corresponding category $\mathcal{O}$. Let $\mathcal{M}^\alpha$ denote the full subcategory of the category of all finitely-generated $\mathfrak{A}$-modules, which consists of all modules M, that can be decomposed into a direct sum of modules from $\mathcal{L}^\alpha$ when viewed as $\mathfrak{A}(\alpha)$-modules. $\mathcal{L}^\alpha$ has a natural abelian structure with usual kernels and cokernels defined for $f:M\to N$ as $r_\alpha(N/r_\alpha(M))$. In a natural way, this abelian structure can be extended to $\mathcal{M}^\alpha$. In the following Lemma we will refer to this abelian structure on $\mathcal{M}^\alpha$.

Lemma 14.4.3. *$l_\alpha:\mathcal{M}_\alpha\to\mathcal{M}^\alpha$ is exact, commutes with $F\otimes \_$ for any finite-dimensional F and faithful on morphisms.*

Proof. All properties can be checked on the $\mathfrak{A}(\alpha)$-level, where they are trivial. □

Lemma 14.4.4. *For any $x\in\mathbb{C}$ we have that $m_\alpha^x:\mathcal{M}_\alpha\to\mathcal{M}_\alpha$ is exact and faithful on morphisms.*

Proof. Follows directly from the definitions of m_α^x and $\mathcal{M}_\alpha$. □

Lemma 14.4.5. *Let $\mathcal{M}_\alpha^{GZ}$ denote the full subcategory of $\mathcal{M}_\alpha$ consisting of all Gelfand-Zetlin modules having strong tableaux realization. Then for any $x\in\mathbb{C}$ the functor m_α^x sends $\mathcal{M}_\alpha^{GZ}$ into itself and its restriction to this category commutes with $F\otimes \_$ for any finite-dimensional F.*

We note, that the category $\mathcal{M}_\alpha$ does not contain all Gelfand-Zetlin modules (even not those having a strong tableaux realization) if α is not the first simple root. We also note that the statement can be easily extended to $m^x_\alpha : (l_\alpha(\mathcal{M}^{GZ}_\alpha)) \to \mathcal{M}^{GZ}_\alpha$.

Proof. Let $X_\alpha = X_i$ for some $i \in \{1, 2, \ldots, n-1\}$. By exactness of m^x_α we have only to prove the statement for simple objects from $\mathcal{M}^{GZ}_\alpha$. Let M be a simple object in $\mathcal{M}^{GZ}_\alpha$. As Y_i acts bijectively on M, there is a finite number of Γ-weight generators v_j, $j \in J$ such that the corresponding tableaux $[t^j_{k,l}]$ satisfy the following condition: $t^j_{i-1,l} = t^j_{i,l}$ for all $l = 1, 2 \ldots, i-1$. Hence $X_{i-1}v_j = 0$ for all j. Moreover, $Xv_j = 0$ for any X corresponding to a positive root having X_{i-1} as a summand and all other summands of the form X_k, $k < i-1$. Y_i commutes with all $Z(\mathfrak{sl}(k, \mathbb{C}))$, $k \neq i$, and with $(H_\alpha)^\perp$. By polynomiality of Mathieu's twist, m^x_α sends an H_α-weight vector of weight y to an H_α-weight vector of weight $y + 2x$. Now let $c \in Z(\mathfrak{sl}(i, \mathbb{C}))$ and $cv_j = y_j v_j$. From $X_{i-1}v_i = 0$ we get $[X_{i-1}, X_{i-2}]V_i = 0$, $[[X_{i-1}, X_{i-2}], X_{i-3}]v_j = 0$ and so on. Thus we can apply the generalized Harish-Chandra homomorphism. We obtain $c_1 \in Z(\mathfrak{sl}(i-1, \mathbb{C}))$ and $H \in S(\mathfrak{H})$ such that $m^x_\alpha(c)v_j = m^x_\alpha(c_1 + H)v_j$. We conclude that the images of all v_j are Γ-weight vectors, thus implying $m^x_\alpha(M) \in \mathcal{M}^{GZ}_\alpha$. Moreover, one sees that $m^x_\alpha(M)$ is a simple object of $\mathcal{M}^{GZ}_\alpha$, which can be precisely computed in terms of $[t^j_{k,l}]$ and x. Now the statement about $F \otimes \_$ follows from Lemma 8.3.1. □

Lemma 14.4.6. *Both functors l_α and m^x_α respect the action of the center. In particular, they respect (generalized) central characters.*

Proof. Obvious. □

Lemma 14.4.7. *Both l_α and m^x_α commute with parabolic inductions.*

Proof. Trivial. □

Lemma 14.4.8. *Let $\mathcal{P}$ be a parabolic subalgebra of a semi-simple Lie algebra $\mathfrak{G}$ and V be a simple module over the Levi factor of $\mathcal{P}$ which is turned into a $\mathcal{P}$-module via the trivial action of the nilradical. Then any endomorphism of the module $U(\mathfrak{G}) \otimes_{U(\mathcal{P})} V$ (which usually is called a generalized Verma module associated with $\mathcal{P}$ and V) is scalar.*

Proof. $U(\mathfrak{G}) \otimes_{U(\mathcal{P})} V$ is generated by V and any endomorphism of $U(\mathfrak{G}) \otimes_{U(\mathcal{P})} V$ sends the unique copy of V in $U(\mathfrak{G}) \otimes_{U(\mathcal{P})} V$ into itself. Now the statement follows from [D, Proposition 2.6.5]. □

Proof of Theorem 14.4.1. By virtue of Lemmas above we need only to find a composition of different m^x_α and l_α which sends simples from $\mathcal{F}$ to simples in $\mathcal{K}$ preserving their endomorphism rings. These endomorphism rings equal $\mathbb{C}$ in the case of $\mathcal{F}$ by [D, Proposition 2.6.5]. We note, that simple objects in $\mathcal{K}$ are not simple $\mathfrak{A}$-modules but coincide with projective Verma modules (which are quite far from being simple $\mathfrak{A}$-modules) which also have $\mathbb{C}$ as endomorphism ring.

Using usual induction together with exactness of parabolic induction and Lemma 14.4.7, it is sufficient to construct a composition of different m^x_α and l_α which sends simples from

$\mathcal{F}$ to the generalized Verma modules over α induced from simple Gelfand-Zetlin modules having strong tableaux realization over the parabolic subalgebra with simple Levi part generated by X_i, Y_i, $i > 1$. In fact, on each step the endomorphism ring will be preserved by Lemma 14.4.8 and iterating this process inductively we will end up with a Verma module from $\mathcal{O}$. As both m_α^x and l_α respect the central character, the result will be in the correct block of $\mathcal{O}$. As we will also see later, on each step we will obtain a module, complete with respect to some r_α, where α is a simple root, so that the final module will be complete. This will prove our Theorem. Using the integer shift of tableaux one can also see that it is sufficient to construct such a composition for one fixed simple module from $\mathcal{F}$.

So, fix some $V([l]) \in \mathcal{F}$ such that the upper row of $[l]$ defines the projective Verma module in $\mathcal{O}$ (this means that the entries of the row decrease). The only Y_i acting bijectively on $V([l])$ is Y_1. Let $X_\alpha = X_1$. Then, clearly, $V([l]) \in \mathcal{M}_\alpha^{GZ}$ and from the proof of Lemma 14.4.5 it follows that $m_\alpha^x(V([l])) \simeq V([s])$, where $s_{i,j} = l_{i,j}$, $i > 1$ and $s_{1,1} = l_{1,1} + 2x$. Choose x such that $l_{1,1} + 2x = l_{2,1}$ and consider the module $M_1 = l_\alpha(m_\alpha^x(V([l])))$. It is generated by a Γ-weight vector corresponding to the tableau $[s]$ as above. Let us show that Y_2 acts bijectively on M_1. Indeed, any tableau $[p]$ appearing as a basis element in M_1 satisfies the condition $p_{2,1} - p_{1,1} \in \mathbb{Z}_+$ because of the local nilpotency of X_1. Assume that $p_{2,1} = p_{1,1}$ and consider the set P of all tableaux obtained from $[p]$ by integer shift of $p_{2,2}$. Applying Y_2 to any tableau from P and using Gelfand-Zetlin fomulae we see that we can reduce either $p_{2,1}$ or $p_{2,2}$ by 1, but $p_{2,1} - 1 < p_{1,1}$ and hence in fact we can only reduce $p_{2,2}$. This means that Y_2 sends any tableau from P into a (non-zero by Gelfand-Zetlin formulae) multiple of another tableau. From this we obtain that Y_2 acts bijectively on the subspace generated by P, moreover this subspace is a simple dense module over $\mathfrak{A}(\beta)$. Letting Y_1 act on all tableaux with $p_{2,1} = p_{1,1}$ we will obtain all basis elements of M_1. This means that M_1 is generated by a direct sum of simple dense $\mathfrak{A}(\beta)$-modules. From the fact that $U(\mathfrak{A})$ is a direct sum of finite-dimensional $\mathfrak{A}(\beta)$-modules under adjoint action we get that, as an $\mathfrak{A}(\beta)$-module, M_1 is a direct sum of subquotients of the modules $V \otimes F$, where V is simple dense and F is finite-dimensional. This means that Y_2 acts bijectively on M_1 and $M_1 \in \mathcal{M}_\beta^{GZ}$. Hence we are allowed to apply m_β^x.

Again from the proof of Lemma 14.4.5 one gets that this is equivalent to changing $s_{2,2}$, which can be choosen arbitrarily, for example equal to $s_{3,1}$. Now we can apply m_α^x and make $s_{1,1}$ equal to $s_{2,2} = s_{3,1}$. Again applying m_β^x we can achieve $s_{2,1} = s_{3,2}$. As our tableaux are defined up to permutations of the elements in each row, we can have $s_{1,1} = s_{2,1} = s_{3,1}$ and $s_{2,2} = s_{3,2}$. Now it is clear that proceeding with other simple roots as above we will be able to arrive at a module N generated by a Γ-weight element v corresponding to the tableau $[t]$ defined as follows: $t_{n,i} = l_{n,i}$ for all i, $t_{n-1,i} = l_{n-1,i}$ for all $i > 1$, $t_{i,1} = t_{n,1}$ for all i and $t_{k,i} = t_{n-1,i}$ for all $k < n$. By Lemma 14.4.3, this module will be automatically r_α-complete, i.e. $r_\alpha(N) = N$. (Hence, at the very end of the induction process we get a complete module.) But one also has $X_\gamma v = 0$ for any positive root γ containing α. From this we get that N is isomorphic to a generalized Verma module induced from a simple Gelfand-Zetlin module, $\hat{N}$, over the parabolic subalgebra with simple Levi part generated by X_i, Y_i, $i > 1$. From $t_{n-1,i} - t_{n-1,j} \notin \mathbb{Z}$ we have that $\hat{N}$ has a strong tableaux realization. Now induction completes our proof. $\square$

15 Categories $\mathcal{O}(\mathcal{P},\mathcal{L})$, IV: Projectively startified algebras

As it was already shown in Chapter 12, quite often the categories $\mathcal{O}(\mathcal{P},\mathcal{L})$ docompose into a direct sum of module categories over projectively stratified (associative finite-dimensional) algebras. The aim of this Chapter is to transfer the Lie-algebraic machinery of construction of GVMs (standard module) via induction from a parabolic subalgebra $\mathcal{P}$ of $\mathfrak{G}$ to the associative level. In other words, for a projectively statified algebra associated with a block of $\mathcal{O}(\mathcal{P},\mathcal{L})$, studied in Chapter 13, we are going to show that it's standard modules can be induced from a natural "parabolic" subalgebra. This Chapter is quite abstract and we will return to $\mathcal{O}(\mathcal{P},\mathcal{L})$ only in the last Section, where we will prove the above result. Our abstract setup requires some new definitions and notation. The results presented in this Chapter are the content of [KlMa] and generalize the corresponding results for quasi-hereditary algebras ([Koe1, Koe2, Koe3]). Since we will use a definition of projectively stratified algebra slightly different from those given in Section 12.3, we start with defining all main objects of this Chapter.

15.1 Abstract setup

We let A be a finite dimensional algebra over $\mathbf{k}$, an algebraically closed field. When we want to make clear over which algebra we are taking a module we will give an indication via subscripts. Let $J = AeA$ be a two-sided ideal in A, generated by a primitive idempotent e. J is called *left projectively stratifying (resp. projectively stratifying)* if it is a projective left (resp. two-sided) A-module. If we can order the equivalence classes $e_1,\dots,e_n$ of primitive idempotents of A such that for each l the idempotent e_l generates a left projectively stratifying (resp. projectively stratifying) ideal in the quotient algebra $A/<e_n,\dots,e_{l+1}>$, then A is called *left projectively stratified (resp. projectively stratified)* (compare with Section 12.3). We will indicate the (left) projectively stratified structure on an algebra, A, by the pair $(A,\leqslant)$ with $\leqslant$ the above order on the idempotents. Left stratified algebras have already been studied by Cline, Parshall and Scott ([CPS2]) under the name of *standardly stratified algebras.*

Two lemmas follow immediately from this definition.

Lemma 15.1.1. *Let $(A,\leqslant)$ be a projectively stratified algebra. Then $(A^{op},\leqslant)$ is projectively stratified (with the same order on the isoclasses of primitive idempotents).*

Lemma 15.1.2. *Let A be an algebra with an order $\leqslant$ on the isomorphism classes of primitive idempotents. Then $(A,\leqslant)$ is projectively stratified if and only if both $(A,\leqslant)$ and $(A^{op},\leqslant)$ are left projectively stratified.*

First, we remark that that the definition of projectively stratified algebra in Section 12.3 coressponds to the present definition of left projectively startified algebra. As an example of a left projectively stratified algebra which is not projectively stratified, consider the algebra of the quiver Γ with two vertices $\{a, b\}$, three arrows $\{\alpha : a \to b, \beta : b \to a, \gamma : a \to a\}$ and relations $\gamma^2 = 0$, $\gamma\alpha = 0$, $\beta\gamma = 0$, and $\beta\alpha\beta = 0$.

We also note that a left projectively stratified algebra is a stratifying endomorphism algebra in the sense of Cline, Parshall and Scott [CPS2] and all quasi-hereditary algebras are projectively stratified. It is also the case that a left projectively stratified algebra is quasi-hereditary if and only if it has finite global dimension [AHLU, CPS2].

Let $(A, \leqslant)$ be a (left) projectively stratified algebra. In what follows we will denote by $L(\lambda)$ the simple A-module which corresponds to e_λ, and will call λ a *weight*. We will also denote by $P(\lambda)$ (resp. $I(\lambda)$) the corresponding projective cover (resp. injective envelope). Following [CPS2], for a simple A-module L, corresponding to the idempotent e_λ, we define the *standard* module $\Delta(\lambda)$ as $A/_{\langle e_n, \ldots, e_{\lambda+1}\rangle} e_\lambda$ and *costandard* module $\nabla(\lambda)$ as the largest submodule of $I(\lambda)$ having factors $L(\mu)$ with $\mu \leqslant \lambda$. We note that each projective has a *standard flag*, i.e. a filtration whose quotients are standard modules.

Let $(B, \leqq)$ be a finite-dimensional algebra with $\leqq$ a partial order on the set of equivalence classes of simple modules. $(B, \leqq)$ is called *quasi-directed* if $\mathrm{Ext}^k_B(L, L') \neq 0$ for some k implies $L' \leqq L$. By a *quasi-local algebra* we will mean a direct sum of local algebras.

For a quasi-directed algebra B call an indecomposable module M *local* if all its simple composition factors are isomorphic. Call it *projectively local* if it is projective over the maximal quasi-local subalgebra of B (whose existence will be proved in Section 15.3).

Let S be a quasi-local subalgebra of an algebra B. We say B is *S-diagonalizable* if B is projective as left and right S-module. A quasi-directed algebra diagonalizable over its maximal quasi-local subalgebra will be called *pyramidal*.

Definition 15.1.1. *Let $(A, \leqslant)$ be a projectively stratified algebra and B and C subalgebras of A.*

1. *We will call B an* exact Borel subalgebra *of A if*
   - *there is a one-to-one correspondence between the simples of B and the simples of A;*
   - *$(B, \geqq)$ is pyramidal with the opposite order induced from the simples of A;*
   - *the tensor induction functor $A \otimes_B \_$ is exact;*
   - *$A \otimes_B \_$ sends the projectively local B-module V to the standard A-module of the same weight;*
   - *$[A \otimes_B L_B(i) : L_A(i)] = 1$.*
2. *We will call C a* Δ-subalgebra *of A if*
   - *there is a one-to-one correspondence between the simples of C and the simples of A;*
   - *$(C, \leqq)$ is pyramidal with the order induced from the simples of A;*
   - *for each weight i the indecomposable projective Ae_i occurs exactly once in the decomposition of the projective A-module $A \otimes_C Ce_i$;*

- *fixing epimorphisms $\kappa(i) : A \otimes_C Ce_i \to \Delta(i)$, one has isomorphisms $\kappa(i)|_{Ce_i}$: $1 \otimes Ce_i \xrightarrow{\simeq} \Delta(i)$.*

The importance of the last condition on exact Borel subalgebras can be seen from the example $A = \mathbf{k}[x]/(x^4)$ and $B = \mathbf{k}[x^2]/(x^4) \subset A$. In this example the unique standard objects are the algebras and induction clearly doubles the length of modules.

In the remainder of the paper, set $i \leqq j$ if and only if $i \leqslant j$. The first symbol will indicate quasi-directedness and the second will indicate a projectively stratified structure.

Definition 15.1.2. *Let A be an algebra and $\leqslant$ be a total order on the set of equivalence classes of simple A-modules. Let B and C be subalgebras of A such that $B \cap C = S$ is a quasi-local subalgebra of A containing at least one representative from the classes of isomorphisms of primitive idempotents, maximal in both B and C. Assume that $(B, \geqq)$ and $(C, \leqq)$ are pyramidal. Call (B, C) a* parabolic decomposition *of A provided that the multiplication in A induces isomorphisms $C \otimes_S B \simeq A$ as left C-modules and right B-modules.*

We have chosen this terminology to reflect the fact that the major example we know comes from tensor induction from a parabolic subalgebra of a Lie algebra.

15.2 Module-theoretic characterization of projectively stratified alghebras

A quasi-hereditary algebra $(A, \leqslant)$, can be characterized by the existence of standard modules $\Delta(i)$ having simple subquotients $L(k)$ with $k \leqslant i$ and $L(i)$ occuring once, and such that the projective $P(j)$ has a standard flag with sections $\Delta(i)$ with $j \leqslant i$ among which $\Delta(j)$ occurs exactly once. Left projectively stratified algebras ([CPS2, Section 2.2]) have the same module-theoretic characterization except that the multiplicity of $L(i)$ in $\Delta(i)$ may exceed one.

Theorem 15.2.1. *$(A, \leqslant)$ is left projectively stratified if and only if there exist modules $\Delta(i)$ such that*

(i) there is a surjection $\varphi_i : \Delta(i) \to L(i)$ whose kernel has composition factors $L(j)$, $j \leqslant i$.

(ii) $P(i)$ surjects onto $\Delta(i)$ and the kernel of this map has a standard flag with sections $\Delta(j)$, $j \nleqslant i$.

Proof. Let A be left projectively stratified and $\Delta(i)$ be the standard modules defined in Section 15.1. Then (i) is automatically satisfied since the standard module, $\Delta(i)$ is defined over the algebra $A/\langle e_n, \dots, e_{i+1}\rangle$ and this algebra only has simple modules of the required type. To prove (ii) we proceed by induction. Let $e = e_n$ be the maximal (with respect to

$\leqslant$) primitive idempotent of A. We have $P(n) = \Delta(n)$ and by [DR, Statement 7] we have a bijection

$$Ae \otimes_{eAe} eA \to AeA.$$

Now $Ae \otimes_{eAe} eA \simeq Ae \otimes_{eAe} \operatorname{Hom}(Ae, A)$ and so if Ae_i, $i \neq n$, appears as a direct summand of AeA then there is a surjection of Ae onto Ae_i and this contradicts the indecomposability of Ae. This completes the proof in one direction.

To prove the converse, suppose we have standard modules satisfying conditions (i) and (ii). Let $e = e_n$ be the maximal primitive idempotent. Then $Ae = P(n) = \Delta(n)$ and we have a surjection

$$Ae \otimes_{eAe} eA \simeq Ae \otimes_{eAe} \operatorname{Hom}(Ae, A) \to Ae_n A.$$

Now, $Ae \otimes_{eAe} \operatorname{Hom}(Ae, A) \simeq Ae \otimes_{eAe} \oplus_i \operatorname{Hom}(Ae, Ae_i)$. So, the image of $Ae \otimes_{eAe} eA$ is the direct sum of its images in each projective of A. We show that the image of $Ae \otimes_{eAe} \operatorname{Hom}(Ae, Ae_i)$ in $P(i)$ is a direct sum of copies of Ae. This will follow if we prove the existence of a short exact sequence,

$$0 \to M' = \oplus \Delta(n) \to P(i) \to M \to 0,$$

with M having a standard flag with sections $\Delta(j)$, $j < n$. From the projectivity of $\Delta(n)$ we have that any N with $0 \to N' \to N \to \Delta(n) \to 0$ is isomorphic to $N' \oplus \Delta(n)$ thus by induction on the length of the standard flag we have the result. The image of $Ae \otimes_{eAe} \operatorname{Hom}(Ae, Ae_i)$ in $P(i)$ certainly contains M'. Since there are no non-zero maps $\Delta(n) \to M$, M' is this image. So, AeA is a sum of projective modules and, hence, projective and left projectively stratifying. The projectives of A/AeA are the projectives of A modulo the trace of $P(n) = \Delta(n)$ and so have flags satisfying (ii) and with the exception of n all standard modules remain unchanged. By induction in the number of idempotents, we are then done. $\square$

Corollary 15.2.1. *Let $(A, \leqslant)$ be an algebra. Then $(A, \leqslant)$ is quasi-directed if and only if $(A, \leqslant)$ is left projectively stratified with projective standard modules.*

Proof. If $(A, \leqslant)$ is left projectively stratified with projective standard modules then $(A, \leqslant)$ is quasi-directed by definition. Now assume that $(A, \leqslant)$ is quasi-directed. Define $\Delta(i) = P(i)$. Condition (ii) is trivial and condition (i) follows by directedness. $\square$

One can clearly obtain the notion of *right* projectively stratified algebras by requiring right projectivity of the stratifying ideals.

Corollary 15.2.2. *$(A, \leqslant)$ is right projectively stratified if and only if there exist a choice of costandard modules $\nabla(i)$ such that*

(i) there is an injection $\varphi_i : L(i) \to \nabla(i)$ whose cokernel has composition factors $L(j)$, $j \leqslant i$.

(ii) $\nabla(i)$ *injects into* $I(i)$ *and the cokernel of this map has a costandard flag with sections* $\nabla(j)$, $j \nleqslant i$.

Proof. Pass to the opposite algebra. The functor $\mathrm{Hom}(\_, \mathbf{k})$ carries injectives into projectives and costandard modules into standard modules. Now apply Theorem 15.2.1. $\square$

Corollary 15.2.3. $(A, \leqslant)$ *is projectively stratified if and only if there exist a choice of standard modules* $\Delta(i)$ *and costandard modules* $\nabla(i)$ *satisfying necessary conditions from Theorem 15.2.1 and Corollary 15.2.2.*

15.3 Pyramidal algebras as projectively stratified algebras

From the theory of quasi-hereditary algebras we have that an algebra $(A, \leqslant)$ is *directed* (i.e. $\mathrm{Ext}(L, L') \neq 0$ implies $L \nleqslant L'$) if and only if it is quasi-hereditary with projective standard modules. We have already seen (Corollary 15.2.1) that the same relationship holds between quasi-directed and left projectively stratified algebras. In this Section, we examine pyramidal algebras; in fact, we prove that all pyramidal algebras are projectively stratified.

Lemma 15.3.1. *Let* $(B, \leqq)$ *be a quasi-directed algebra. Then the maximal quasi-local subalgebra* S *of* B *is isomorphic to* $\oplus_i e_i B e_i$.

Proof. Clearly, $S' = \oplus_i e_i B e_i$ is a subalgebra of B. Because of the directedness of B no endomorphisms of Be_i can factor through a non-isomorphic projective, so each $e_i B e_i$ is local, and hence S' is quasi-local. Now let P be an indecomposable summand of S. Then P equals Se_i is local and so $e_i S e_i = Se_i$ is a subalgebra of $e_i B e_i$ and hence $S \subset S'$. $\square$

Lemma 15.3.2. *Let* $(B, \leqq)$ *be quasi-directed. Then the projectively local module* $K(i)$ *is isomorphic to* Be_i/N*, where* N *is the trace of all* $P(j)$ *with* $j \leqq i$.

Proof. Because of the directedness of B we have

$$Be_i = e_i B e_i \oplus \sum_{j<i} e_j B e_i$$

as a vectorspace. For each element of $e_j B e_i$ there is a map for $P(j)$ to $P(i)$ covering it, so $\sum_{j<i} e_j B e_i \subset N$. But $[Be_i : L(i)] = [e_i B e_i : L(i)]$ by directedness of B. This completes the proof. $\square$

Proposition 15.3.1. *Let* $(B, \leqq)$ *be a pyramidal algebra. Then* $(B, \leqslant)$ *is projectively stratified.*

Proof. $(B, \leqslant)$ is left projectively stratified by Corollary 15.2.1. Now consider B^{op}. Then the algebra $(B^{op}, \geqq)$ is quasi-directed. This implies that a left projective B^{op}-module $P(i)$ has only $L(j)$ with $i \leqslant j$ as composition subquotients. Since B^{op} is pyramidal, each $P(i)$ has a projectively local flag, whose subquotients satisfy the same order condition. Let S be the maximal quasi-local subalgebra of B. Choose for B^{op}, $\Delta(i) = e_i S^{op} e_i$. By Lemma 15.3.2 these are the projectively local modules. Then these standards clearly satisfy conditions (i) and (ii) of Theorem 15.2.1, and so $(B^{op}, \leqslant)$ is left projectively stratified. And now, by Lemma 15.1.2, $(B, \leqslant)$ is projectively stratified. $\square$

Thus, a pyramidal algebra $(B, \leqq)$ has both $(B, \leqslant)$ and $(B, \geqslant)$ projectively stratified structures. To finish this Section we give necessary and sufficient conditions for a quasi-directed algebra to be projectively stratified. We begin with the following Lemma.

Lemma 15.3.3. *Let e be a primitive idempotent in a quasi-directed algebra $(B, \leqq)$ and X an eAe-module. Then $X \otimes_{eBe} eB$ is right B-projective if and only if X is right eBe-projective.*

Proof. If X_{eBe} is projective, then X is free over eBe and hence $X \otimes_{eBe} eB$ is right B-projective. On the other hand, suppose $X \otimes_{eBe} eB$ is right B-projective. Let L be the maximal local top of eB. X has top $(\hat{L})^n$, where $\hat{L}$ is the unique simple eBe-module. We have an exact sequence

$$0 \to T \xrightarrow{\varphi} (eBe)^n \to X \to 0.$$

Inducing to B we have

$$T \otimes_{eBe} eB \xrightarrow{\beta} (eBe)^n \otimes_{eBe} eB \xrightarrow{\alpha} X \otimes_{eBe} eB \to 0.$$

Since X surjects on $(\hat{L})^n$, we have the following chain of surjections

$$X \otimes_{eBe} eB \to (\hat{L} \otimes_{eBe} eB)^n \to (\hat{L} \otimes_{eBe} L)^n \to L^n.$$

$X \otimes_{eBe} eB$ is projective and must contain, as a direct summand, the projective cover, $(eB)^n$, of L^n. This implies that α is an isomorphism, $\beta = 0$ and, last, $\beta \cdot e = 0$. But $\beta \cdot e = \varphi : T \otimes_{eBe} eBe \to (eBe)^n \otimes_{eBe} eBe$ is non-zero if X is not projective. This contradiction proves the Lemma. $\square$

And this Lemma allows us to give the following characterization of when a quasi-directed algebra is projectively stratified.

Proposition 15.3.2. *Let $(B, \leqq)$ be quasi-directed. Then the following conditions are equivalent.*

(i) B is projectively stratified.

(ii) the maximal quasi-local subalgebra S of B is an exact Borel subalgebra.

(iii) Be is eBe-projective for every primitive idempotent e.

Proof. ($(i) \Leftrightarrow (iii)$) By [DR, Statment 7] we have $Be \otimes_{eSe} eB \simeq BeB$, where e is the maximal primitive idempotent. So BeB is right B-projective if and only if Be is right eSe-projective (Lemma 15.3.3).

($(ii) \Rightarrow (iii)$) Since $B \otimes_S \_$ is exact, B is a right flat, hence right projective ([Fa, 11.31]) S-module. Then Be is a projective S module and hence projective eSe-module.

($(iii) \Rightarrow (ii)$) We have $B \otimes_S \_ = \oplus_i Be_i \otimes_{e_i S e_i} \_$ and hence exact since each Be_i is e_iSe_i-projective. Now we just need to prove that this functor carries $\Delta_S(i) = e_iSe_i$ to $\Delta_B(i)$. Set $e = e_i$. We have $B \otimes_S eSe = B \otimes_{eSe} eSe = Be \otimes_{eSe} eSe = Be = \Delta_B(i)$. □

15.4 Brauer-Humphreys reciprocities

We will now prove an appropriate generalization of the classical Brauer-Humphreys reciprocities; this will be the refinement of [ADL, Theorem 2.5] in our situation. As is usual, $[M : \Delta(i)]$ (resp. $[M : \nabla(i)]$) is the number of occurrences of $\Delta(i)$ (resp. $\nabla(i)$) in a standard (resp. costandard) flag of M, should it exist. Similarly, for a simple L, $[M : L]$ means the corresponding composition multiplicity.

Theorem 15.4.1. *Let $(A, \leqslant)$ be a projectively stratified algebra. Then for any pair of weights i and j we have*

$$\dim_{\mathbf{k}}(\mathrm{End}(\Delta(j)))[P(i) : \Delta(j)] = [\nabla(j) : L(i)],$$
$$\dim_{\mathbf{k}}(\mathrm{End}(\nabla(j)))[I(i) : \nabla(j)] = [\Delta(j) : L(i)].$$

Proof. By duality it is sufficient to prove the first. By induction it is sufficient to prove it for maximal $j = n$. In this case $P(i) = Ae_i$, $P(n) = \Delta(n) = Ae_n$ and $I(n) = \nabla(n)$. Set $l = [P(i) : \Delta(n)]$. From the proof of Theorem 15.2.1, this is equal to

$$l = \frac{\dim_{\mathbf{k}} \mathrm{Hom}_A(P(n), P(i))}{\dim_{\mathbf{k}} \mathrm{End}_A(P(n))};$$

that is $Ae_i \cap Ae_nA = (Ae_n)^l$. So, it remains to show that $\dim_{\mathbf{k}} \mathrm{Hom}_A(P(n), P(i)) = [I(n) : L(i)]$. Passing to the opposite algebra we have

$$\begin{aligned}\dim_{\mathbf{k}} \mathrm{Hom}_A(P(n), P(i)) &= \dim_{\mathbf{k}}(e_nAe_i) = \dim_{\mathbf{k}}(e_iA^{op}e_n) \\ &= \dim_{\mathbf{k}} \mathrm{Hom}_A(A^{op}e_i, A^{op}e_n) = [A^{op}e_n : L(i)] = [I(n) : L(i)].\end{aligned}$$

□

And as corollary we get a weakened form of Theorem 12.4.1.

Corollary 15.4.1. *Assume that A is projectively stratified and has a duality (i.e. a contravariant exact equivalence on the category of A-modules, which preserves simples). Then*

$$\dim_{\mathbf{k}}(\mathrm{End}(\Delta(j)))[P(i) : \Delta(j)] = [\Delta(j) : L(i)].$$

Proof. The duality carries standards to costandards and we have $[\Delta(i) : L(j)] = [\nabla(i) : L(j)]$, which completes the proof. □

15.5 Duality between exact Borel and Δ-subalgebras

In this Section we explore the left-right symmetry of projectively stratified algebras and their exact Borel and Δ-subalgebras. For the remainder of the paper we will assume that the projectively stratified structures on exact Borel and Δ-subalgebras are given by the same order on the idempotents of the algebra. With this projectively stratified structure the fourth condition of an exact Borel subalgebra can be rephrased by: tensor induction carries standard modules to standard modules.

Lemma 15.5.1. *Let $(A, \leqslant)$ be a projectively stratified algebra. Then*

$$\mathrm{Ext}_A^k(\Delta(i), \nabla(j)) = 0$$

unless $k = 0$ and $i = j$.

Proof. Assume $k > 0$. Let m be the maximum of i and j. Put $e = \sum_{s=m+1}^{n} e_s$ and put $A' = A/AeA$. Over this algebra $\Delta(m)$ is projective and $\nabla(m)$ injective. Thus we have for $k \neq 0$ $\mathrm{Ext}_A^k(\Delta(i), \nabla(j)) = \mathrm{Ext}_{A'}^k(\Delta(i), \nabla(j)) = 0$ ([CPS2, 2.1.2]). If $k = 0$, $i \neq j$ then the image of any map $\varphi : \Delta(i) \to \nabla(j)$ must have $L(i)$ as top and other composition factors $L(j)$, $j \leqslant i$, since it is a quotient of $\Delta(i)$. If this map is non-zero then $L(i)$ is a composition factor of $\nabla(j)$ that is $i < j$. But then every submodule of $\nabla(j)$ has composition factor $L(j)$ which forces $j < i$. □

Let $(B, \geqq)$ be an exact Borel subalgebra of a projectively stratified algebra $(A, \leqslant)$. In the case that A is quasi-hereditary the standard objects are induced from simple B-modules. In the general case, this no longer holds; however, these modules, $\tilde{\Delta}_A(i) = A \otimes_B L_B(i)$, which we will call *substandard modules*, continue to play an important role (see also [AHLU]). In particular, $\Delta(i)$ has a $\tilde{\Delta}(i)$-flag.

Lemma 15.5.2. *Let $(A, \leqslant)$ be a projectively stratified algebra with an exact Borel subalgebra B. Then*

$$\mathrm{Ext}_A^k(\tilde{\Delta}(i), \nabla(j)) = 0,$$

unless $k = 0$ and $i = j$.

Proof. For $k = 0$, $\tilde{\Delta}(i)$ is an image of $\Delta(i)$ and the statement follows from Lemma 15.5.1. Consider $k \neq 0$. Let l be the maximal of i and j. Then $\tilde{\Delta}(i)$ and $\nabla(j)$ are modules over $A/\langle e_{k+1}, \ldots, e_n \rangle$ and so (by [CPS2, 2.1.2]) we may assume that $l = n$. If $j = n$, then $\nabla(j)$ is injective and the current Lemma clearly holds. So now assume $i = n$ and $j < i$ and consider the exact sequence

$$0 \to N \to \Delta(i) \to \tilde{\Delta}(i) \to 0.$$

Apply $\mathrm{Hom}_A(\_, \nabla(j))$ and pass to the long exact sequence. We get $\mathrm{Ext}_A^1(\tilde{\Delta}(i), \nabla(j)) = 0$ and $\mathrm{Ext}_A^{l+1}(\tilde{\Delta}(i), \nabla(j)) \simeq \mathrm{Ext}_A^l(N, \nabla(j))$. And the Lemma follows from the standard dimension shift arguments. □

Lemma 15.5.3. *Let $(B, \geqq)$ be an exact Borel subalgebra of a projectively stratified algebra, $(A, \leqslant)$. Then for all weights,* $\dim \mathrm{End}_A(\Delta_A(\lambda)) = \dim \mathrm{End}_B(\Delta_B(\lambda))$.

Proof. We prove the Lemma by induction. Let n be the maximal weight. We have

$$\dim \mathrm{End}_X(\Delta_X(n)) = \dim \mathrm{Hom}_X(P_X(n), \Delta_X(n)) = [\Delta_X(n) : L_X(n)]$$

for both $X = A$ or $X = B$. By the last condition for an exact Borel subalgebra we have $[\Delta_A(n) : L_A(n)] = [\Delta_B(n) : L_B(n)]$ and the statment follows for the maximal weight. Induction is clear. □

Proposition 15.5.1. *Let $(A, \leqslant)$ be a projectively stratified algebra and $(B, \geqq)$ a pyramidal subalgebra of A with the same poset of isoclasses of primitive idempotents. Then B is an exact Borel subalgebra of A if and only if for each weight i, restriction from A to B induces an isomorphism $\nabla_A(i) \simeq \nabla_B(i)$ as B-modules.*

Proof. Assume that B is an exact Borel subalgebra of A. We compare dimensions of $\nabla_A(i)$ and $\nabla_B(i)$. Using Theorem 15.4.1 we have

$$\dim_{\mathbf{k}} \nabla_X(i) = \sum_j \dim_{\mathbf{k}} L_X(j)[\nabla_X(i) : L_X(j)] =$$

$$= \sum_j \dim \mathrm{End}(\Delta_X(i)) \dim_{\mathbf{k}} L_X(j)[P_X(j) : \Delta_X(i)] = \dim \mathrm{End}(\Delta_X(i))[X : \Delta_X(i)]$$

for both $X = A$ and $X = B$. By the exactness of induction, $[A : \Delta_A(i)] = [B : \Delta_B(i)]$. By Lemma 15.5.3, $\dim \mathrm{End}(\Delta_A(i)) = \dim \mathrm{End}(\Delta_B(i))$ and so $\dim_{\mathbf{k}} \nabla_A(i) = \dim_{\mathbf{k}} \nabla_B(i)$.

From Lemma 15.5.2 it follows that the functor $\mathrm{Hom}_A({}_-, \nabla_A(j))$ is exact on A-modules having a substandard flag. Thus $\mathrm{Hom}_B({}_-, \nabla_A(j))$ is exact on the category of B-modules. So, $\nabla_A(j)$ containing $\nabla_B(j)$ is an injective B-module. The previous dimension count says that they are, in fact, equal.

Now assume that for each weight i, restriction from A to B induces an isomorphism $\nabla_A(i) \simeq \nabla_B(i)$ as B-modules. Since $(B, \geqq)$ is pyramidal, $(B, \leqslant)$ is projectively stratified by Lemma 15.3.1 with injective costandard modules (Corollary 15.2.1 and Corollary 15.2.3). We want to prove that A is right projective over B implying $A \otimes_B {}_-$ is exact. The right standard modules for A (and both right standard and projective over B) are $\nabla_A(i)^*$, so, as a right projective A-module, A has a $\nabla_A(i)^*$-flag, the last being a direct sum decomposition over B.

We are finished when we show that $A \otimes_B {}_-$ sends projectively local B-modules to standard A-modules. Let $K(i)$ be a projectively local B-module corresponding to the weight i. First we show $\dim_{\mathbf{k}}(A \otimes_B K(i)) = \dim_{\mathbf{k}} \Delta_A(i)$. We have $d := \dim_{\mathbf{k}}(A \otimes_B K(i)) = \dim_{\mathbf{k}}(\mathrm{Hom}_B(K(i), A^*))^*$. Since A is a right projective B-module, A^* is left injective and as such decomposes into a direct sum of injective B-modules, $\nabla_B(i)$. On each summand we have

$$\dim_{\mathbf{k}} \mathrm{Hom}_B(K(i), \nabla_B(j)) = \begin{cases} 0, & i \neq j; \\ \dim_{\mathbf{k}} \mathrm{End}_B(K(i)), & i = j; \end{cases}$$

this follows from the fact that $\nabla_B(i) = (e_iB)^*$ implies $[\nabla_B(i) : L_B(i)] = [K(i) : L_B(i)]$. Now $\dim \mathrm{End}_B(K(i)) = [K(i) : L(i)]$ because $K(i)$ is projective in the category of B-modules filtered by $L_B(i)$. Thus d equals $\dim_{\mathbf{k}} \mathrm{End}_B(K(i)) \cdot [A^* : \nabla_B(i)] = \dim_{\mathbf{k}} \mathrm{End}_B(K(i)) \cdot [A^* : \nabla_A(i)]$. Further, $\dim_{\mathbf{k}} \mathrm{End}_A(\nabla_A(i)) = \dim_{\mathbf{k}} \mathrm{End}_{A^{op}}(\Delta_{A^{op}}(i)) = \dim_{\mathbf{k}} \mathrm{End}_{B^{op}}(K(i)^*) = \dim_{\mathbf{k}} \mathrm{End}_B(K(i))$ by Lemma 15.5.3. Applying Brauer-Humphreys reciprocity we get

$$
\begin{aligned}
d = \dim_{\mathbf{k}} \mathrm{End}_B(K(i)) \cdot [A^* : \nabla_A(i)] = \\
&= \sum_j \dim_{\mathbf{k}} \mathrm{End}_A(\nabla_A(i)) \cdot [I_A(j) : \nabla_A(i)] \cdot \mathrm{mult}_{A^*}(I_A(j)) = \\
&\qquad = \sum_j [\Delta_A(i) : L_A(j)] \cdot \mathrm{mult}_A(P_A(j)) = \dim_{\mathbf{k}}(\Delta_A(i)).
\end{aligned}
$$

From the quasi-directedness of B and adjunction we have that $A \otimes_B P_B(i)$ has $P_A(i)$ as a direct summand exactly once and all other direct summands (if any) are of the form $P_A(j)$, $j > i$. So, for the largest weight we have $\Delta_A(n) = P_A(n) = A \otimes_B P_B(n) = A \otimes_B K(n)$. Now we proceed by induction. We have an exact sequence:

$$0 \to V \to P_B(i) \to K(i) \to 0$$

with V filtered by $K(j)$, $j > i$ because $\Delta_B(j) = K(j)$. By exactness of $A \otimes_B \_$ we obtain

$$0 \to A \otimes_B V \to A \otimes_B P_B(i) \to A \otimes_B K(i) \to 0,$$

and by the inductive hypothesis, $A \otimes_B V$ is filtered by $\Delta_A(j)$ with $j > i$. Now, since $P_A(i)$ occurs as a summand of $A \otimes_B P_B(i)$, there is a surjection of $A \otimes_B P_B(i)$ onto $\Delta_A(i)$ and the kernel V' is the the sum of the images of all possible maps from $P_A(j)$ for $j > i$. Hence $V \subset V'$ and thus $A \otimes_B K(i)$ surjects onto $\Delta_A(i)$ and the isomorphism follows from the dimension count. $\square$

Corollary 15.5.1. *B is an exact Borel subalgebra of a projectively stratified algebra $(A, \leqslant)$ if and only if B^{op} is a Δ-subalgebra of $(A^{op}, \leqslant)$.*

Proof. Follows from Proposition 15.5.1, its proof and the duality of the conditions for a Δ-subalgebra and the equivalent conditions for an exact Borel subalgebra. $\square$

15.6 Projectively stratified structure of algebras with parabolic decomposition

In this Section we prove a Theorem relating parabolic decomposition to projectively stratified algebras. It generalizes the corresponding result for quasi-hereditary algebras [Koe2, Theorem 4.1]. The proof closely follows the ideas of the proof there.

Theorem 15.6.1. *Let A be a finite-dimensional algebra and $\leqslant$ be a total order on the set of isomorphism classes of simple A-modules. Assume that $(B, \geqq)$ and $(C, \leqq)$ are pyramidal basic subalgebras, whose intersection $B \cap C = S$ is the maximal quasi-local subalgebra of both B and C. The following statments are equivalent.*

(i) The algebra A is projectively stratified with an exact Borel subalgebra B and a Δ-subalgebra C.

(ii) (B, C) is a parabolic decomposition of A.

Proof. Assume we have listed the idempotents in A (and hence in B and C) with respect to the natural total order.

($(i) \Rightarrow (ii)$) We proceed by induction on the number of direct summands in S (the number of weights). If S is local, then $A = B = C = S$ and we are done. Assume S is not local and $e = e_n$ is the maximal primitive idempotent in A. Since A is projectively stratified, AeA is a projectively stratifying ideal and hence is projective, as left A-module. In particular, by [DR, Statement 7], we have that the multiplication in A induces a bijection

$$Ae \otimes_{eAe} eA \to AeA.$$

We also have $eAe = eSe$, since e is the maximal primitive idempotent. We have the identifications

$$\Delta_A(n) \simeq \Delta_C(n) \simeq Ce$$

by the definition of Δ-subalgebra and Corollary 15.2.1, and

$$\nabla_A(n)^* \simeq \nabla_B(n)^* \simeq eB$$

by dual arguments (Corollary 15.5.1). We get a bijection

$$Ce \otimes_{eSe} eB \simeq Ae \otimes_{eAe} eA \simeq AeA,$$

compatible with left C and right B multiplication. Continuing by induction (see arguments in [Koe2, Theorem 4.1]) we see that $\mathbf{k}$-dimensions of both sides of

$$C \otimes_S B \to A$$

are equal. We are done.

($(ii) \Rightarrow (i)$) Let $e = e_n$ be the maximal primitive idempotent. We wish to show that AeA is projectively stratifying. Since B and C are quasi-directed, we have $eC = eCe = eSe$ and $Be = eBe = eSe$ are projectively local modules. First we show that $eSe = eAe$ (this says $\mathrm{End}_A(\Delta_A(n)) \simeq \mathrm{End}_C(\Delta_C(n))$). We have

$$eAe \simeq eC \otimes_S Be \simeq eSe \otimes_S eSe \simeq eSe \otimes_{eSe} eSe \simeq eSe.$$

Analogously,

$$Ae \simeq C \otimes_S Be \simeq C \otimes_S eSe \simeq C \otimes_{eSe} eSe \simeq Ce$$

and

$$eA \simeq eC \otimes_S B \simeq eSe \otimes_S B \simeq eSe \otimes_{eSe} B \simeq eB.$$

Now, since B and C are pyramidal they are projective over S both as left and right modules. So $Ae = Ce$ (resp. $eB = eA$) is a right (resp. left) projective eSe-module. Thus they are free over eSe and so $Ae \otimes_{eSe} eA$ is projective as left and right A-module. And the Theorem follows from standard induction. $\square$

15.7 Construction of algebras with given parabolic decomposition

In this Section we give a general construction of a projectively stratified algebra having given exact Borel and Δ-subalgebras. The central Theorem of this Section allows us to construct a projectively stratified algebra as an extension of a projectively stratified algebra with a pyramidal algebra. In the case that the projectively stratified algebra has a parabolic decomposition, the extension will as well.

To state the Theorem we assume the following set-up: Given a semi-local algebra S by an *S-algebra* we will mean an algebra $T_S(M)/I$, where M is an S-bimodule, finite-dimensional over $\mathbf{k}$. Let $(A, \leqslant)$ be a basic projectively stratified algebra with an exact Borel subalgebra B. Let $(D, \leqq)$ be a basic pyramidal algebra. Assume that B and D have isomorphic maximal quasi-local subalgebras, S (in particular, they have the same set of idempotents). To fix notation then $A \simeq T_S(M_A)/I_A$, $B \simeq T_S(M_B)/I_B$ and $D \simeq T_S(M_D)/I_D$. Let $A' = T_S(M_A \oplus M_D)/(I_A + I_D + \langle a \otimes_S d \,|\, a \in M_A, d \in M_D\rangle)$.

Theorem 15.7.1. *Let A, B, D and A' be defined as above. A' is projectively stratified and isomorphic to $D \otimes_S A$ as left D-module and right A-module. B is an exact Borel subalgebra of A' via the embedding $b \mapsto 1 \otimes b$. If $C \simeq T_S(M_C)$ is a Δ-subalgebra of A containing S, then A' has a Δ-subalgebra $C' \simeq T_S(M_C \oplus M_D)/(I_C + I_D + \langle c \otimes_S d \,|\, c \in M_C, d \in M_D\rangle)$. Last, $C' \simeq D \otimes_S C$ as left D-module and right C-module.*

Proof. By construction, we have appropriate module isomorphisms: $A' \simeq D \otimes_S A$ (and when relevant $C' \simeq D \otimes_S C$). Let e be the maximal primitive idempotent. Then $e = e \otimes_S e$ ($= 1 \otimes_S e$). So, $J = A'eA' = D \otimes_S C \cdot e \otimes_S e \cdot D \otimes_S C = D \otimes_S AeA$. Since D (resp. AeA) is a two-sided projective D (resp. A)-module, J is a two-sided projective A'-module and A' is projectively stratified by induction.

We prove B is an exact Borel subalgebra of A'. To begin, $D \otimes_S A \otimes_B \_$ is exact, since $A \otimes_B \_$ is exact and D is pyramidal and hence flat over S. By induction, $D \otimes_S A \otimes_B \_$ sends standard B-modules to standard A'-modules. Indeed, let e be the maximal primitive idempotent. $D \otimes_S A \otimes_B Be \simeq D \otimes_S Ae \simeq (D \otimes_S A)(1 \otimes_S e)$, which is A'-standard. It remains to show that $[A' \otimes_B L_B(i) : L_{A'}(i)] = 1$. Now $A' \otimes_B L_B(i) = D \otimes_S A \otimes_B L_B(i)$. Since B is an exact Borel subalgebra of A, we have $[A \otimes_B L_B(i) : L_A(i)] = 1$ and $[A \otimes_B L_B(i) : L_A(j)] \neq 0$ implies $j \leqslant i$. We are done, if we show $[D \otimes_S L_A(j) : L_{A'}(j)] = 1$ and $[D \otimes_S L_A(j) : L_{A'}(k)] \neq 0$ implies $k \leqslant j$, for this would clearly imply that $[A' \otimes_B L_B(i) : L_{A'}(i)] = 1$. But $D \otimes_S L_A(j) \simeq \sum_{k \leqslant j} e_k D e_j \otimes_S L_A(j)$ and $[\sum_{k \leqslant j} e_k D e_j \otimes_S L_A(j) : L_S(m)] \neq 0$, $m \leqslant j$ and so $[D \otimes_S L_A(j) : L_{A'}(k)] \neq 0$ implies $k \leqslant j$. Further, $[\sum_{k \leqslant j} e_k D e_j \otimes_S L_A(j) : L_S(j)] = 1$ implies $[D \otimes_S L_A(j) : L_{A'}(j)] = 1$. The remaining statements follow arguments already seen. □

Corollary 15.7.1. *In the case that $C = A$ we have that $A' \simeq C \otimes_S B$ is a parabolic decomposition.*

15.8 Parabolic decomposition of projectively stratified algebras attached to blocks of $\mathcal{O}(\mathcal{P}, \mathcal{L})$

We have mentioned that our motivation stems from categories $\mathcal{O}(\mathcal{P}, \mathcal{L})$. In this Section we give a parabolic decomposition for algebras of $\mathcal{O}(\mathcal{P}, \mathcal{L})$ studied in Chapter 13. So, throughout this Section we fix the notation from Chapter 13.

Theorem 15.8.1. *For every block of $\mathcal{O}(\mathcal{P}, \mathcal{L})$ there is an algebra, A, with parabolic decomposition, whose module category is equivalent to this block.*

We will prove this by explicit construction of A and its exact Borel and Δ-subalgebras. We require more terminology. For the rest of the section we fix a block, $\mathcal{O}_i$, of $\mathcal{O}(\mathcal{P}, \mathcal{L})$ assumed to have finitely many simples. For a weight λ and a weight $\mathfrak{G}$-module V set $V[\lambda] = \oplus_{k \in \mathbb{Z}} V_{\lambda + k\alpha}$, which is closed under the action of $\mathfrak{A}$.

Recall (Section 12.5), that indecomposable modules from $\mathcal{L}$ have the form $V(a, b)$ or $\tilde{V}(a, b)$, which is a self-extension of $V(a, b)$. For simple $V \in \mathcal{L}$ denote by $\hat{V}$ its projective cover (in $\mathcal{L}$), which is either V itself or its self-extension. Given $M \in \mathcal{O}_i$, a weight λ and a $b \in \mathbb{C}$ set

$$M_{\lambda,b} = \{v \in M_\lambda \mid \text{ there exists } k \in \mathbb{N} \text{ such that } (\mathfrak{c} - b)^k v = 0\}.$$

Since, as an $\mathfrak{A}'$-module, M decomposes into a direct sum of objects from $\mathcal{L}$, one has that $M_{\lambda,b} = \{v \in M_\lambda \mid (\mathfrak{c} - b)^2 v = 0\}$. For a simple module $L \in \mathcal{O}_i$ denote μ_L its α-highest weight. Then $L[\mu_L] \simeq V(a_L, b_L)$ for some a_L and b_L. Put $V_L = V(a_L, b_L)$.

Lemma 15.8.1. *Let L be a simple module in $\mathcal{O}_i$. There exists a projective module P^L such that $\mathrm{Hom}_{\mathfrak{G}}(P^L, M) \simeq M_{\mu_L, b_L}$ for any $M \in \mathcal{O}_i$.*

Proof. Set $\mu = \mu_L$, $a = a_L$ and $b = b_L$. We can pick $k \in \mathbb{N}$ big enough such that $\mathfrak{N}^k M_\mu = 0$ for any $M \in \mathcal{O}_i$ and consider the $\mathfrak{G}$-module

$$\hat{P}^L = U(\mathfrak{G}) \bigotimes_{U(\mathcal{P})} \left((U(\mathfrak{N}) / (U(\mathfrak{N}) \mathfrak{N}^k)) \otimes \hat{V}_L \right).$$

Now we can take P^L to be the $\mathcal{O}_i$-projection of $\hat{P}^L$. That $\mathrm{Hom}_{\mathfrak{G}}(P^L, M) \simeq M_{\mu,b}$ for any $M \in \mathcal{O}_i$ is identical to the classical argument in [BGG2, Theorem 2]. □

Since the α-highest weight of L is unique up to shifts by α, P^L is independent of the choice of this α-highest weight. Now we take

$$A = \mathrm{End}_{\mathfrak{G}}\left(\bigoplus_{L \text{ simple in } \mathcal{O}_i} P^L\right).$$

Clearly, the category of A-modules is equivalent to $\mathcal{O}_i$.

Consider an $\mathfrak{A}'$-submodule $V^L = 1 \otimes 1 \otimes \hat{V}_L$ in P^L and set $M_L = U(\mathfrak{N}_-)V^L$. Let $\Delta(L)$ denote the standard module associated with L. We have $\Delta(L) \simeq U(\mathfrak{G}) \otimes_{U(\mathcal{P})} \hat{V}_L$. For any simple $L \in \mathcal{O}_i$ fix the canonical generator, $p(L) = 1 \otimes 1 \otimes v$, of P^L, where v is a canonical generator of $\hat{V}_L$; then the map $\varphi \mapsto \varphi(p(L))$ is a $\mathbb{C}$-isomorphism between $\mathrm{Hom}_{\mathfrak{G}}(P^L, M)$ and M_{μ_L, b_L} for any $M \in \mathcal{O}_i$.

Lemma 15.8.2. *Any surjection $P^L \to \Delta(L)$ induces an $U(\mathfrak{N}_-)$-isomorphism $M_L \to \Delta(L)$.*

Proof. Let $\varphi : P^L \to \Delta(L)$ be a surjection. It carries $p(L)$ to a generator of $\Delta(L)$ and hence it induces an $X_{-\alpha}$-isomorphism $M_L[\mu_L] \to \Delta(L)[\mu_L]$. Now the statment follows from the fact that $U(\mathfrak{N}_-) = U(\mathfrak{N}^-) \otimes_{\mathbb{C}} U(X_{-\alpha})$ and that both M_L and $\Delta(L)$ are $U(\mathfrak{N}(\mathfrak{A}))$-free. □

Lemma 15.8.3. *M_L is an $\mathfrak{A}'$-module.*

Proof. Follows from the construction of P^L and the definition of M_L. □

Lemma 15.8.4. *Assume that $\varphi : P^{L_j} \to P^{L_k}$ is a homomorphism and $\varphi(p(L_j)) \in M_{L_k}$. Then $\varphi(M_{L_j}) \subset M_{L_k}$.*

Proof. By definition, $p(L_j)$ generates V^{L_j} as $\mathfrak{A}'$-module. Since φ is an $\mathfrak{A}'$-homomorphism, Lemma 15.8.3 says that $\varphi(V^{L_j}) \subset M^{L_k}$ and the statment follows from the fact that $M_{L_j} = U(\mathfrak{N}_-)V^{L_j}$ and the fact that M_{L_k} is stable under left $U(\mathfrak{N}_-)$-multiplication. □

Proof of Theorem 15.8.1. First we prove the existence of a Δ-subalgebra in A. Denote by I an indexing poset of simple modules in $\mathcal{O}_i$. Put

$$C = \bigoplus_{j,k \in I} \{\varphi \in \mathrm{Hom}_{\mathfrak{G}}(P^{L_j}, P^{L_k}) \mid \varphi(p(L_j)) \in M_{L_k}\}.$$

$C \subset A$ is a vectorspace, which is a subalgebra by Lemma 15.8.4. By Lemma 15.8.2, C has trivial intersection with the kernel of the projection $A \to \oplus_{j\in I}\Delta_A(i)$. Clearly, C is quasi-directed and contains a maximal quasi-local subalgebra which is isomorphic to $\oplus_{j\in I}\mathrm{End}(\Delta(j))$. Now, we have to prove that the vectorspace Ce_j is large enough, i.e. $\dim_{\mathbb{C}}(Ce_j) = \dim_{\mathbb{C}}(\Delta_A(j))$. Let $t = \dim_{\mathbb{C}}(\Delta_A(j))$. By the definition of A and by Lemma 15.8.1, we have

$$t = \sum_{k\in I} \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathfrak{G}}(P^{L_k}, \Delta(L_j))) = \sum_{k\in I} \dim_{\mathbb{C}}(\Delta(L_j)_{\mu_{L_k}, b_{L_k}}) =$$

$$= \sum_{k\in I} \dim_{\mathbb{C}}((M_{L_j})_{\mu_{L_k}, b_{L_k}}) = \sum_{k\in I} \dim_{\mathbb{C}}(e_k C e_j) = \dim_{\mathbb{C}} Ce_j.$$

So, we have only to show that C is pyramidal. The maximal quasi-local subalgebra of C is

$$S = \bigoplus_{j \in I} \{\varphi \in \mathrm{Hom}_{\mathfrak{G}}(P^{L_j}, P^{L_j}) \mid \varphi(p(L_j)) \in M_{L_j}\}.$$

We will show that C is right S-projective. Left projectivity can be proved analogously. In fact, we will show that for any $j, k \in I$

$$e_j C e_k = \{\varphi \in \mathrm{Hom}_{\mathfrak{G}}(P^{L_j}, P^{L_k}) \mid \varphi(p(L_j)) \in M_{L_k}\}$$

is a free right $e_k S e_k$-module. Recall that M_{L_k} maps bijectively onto $\Delta(L_k)$ for any surjection from P^{L_k} to $\Delta(L_k)$. Let $M_{\mathcal{P}}(V_{L_k})$ be the generalized Verma module associated with L_k. It follows from the description of $\mathcal{L}$ that either $M_{\mathcal{P}}(V_{L_k}) \simeq \Delta(L_k)$ or $\Delta(L_k)$ is a self-extension of $M_{\mathcal{P}}(V_{L_k})$. Let M^k denote a vectorsubspace of M_{L_k}, which maps bijectively on $M_{\mathcal{P}}(V_{L_k})$ under any composition $P^{L_k} \to \Delta(L_k) \to M_{\mathcal{P}}(V_{L_k})$. Such M^k clearly exists. Now, an $e_k S e_k$ basis of $e_j C e_k$ is given by any linear basis of the vectorspace of all maps $\varphi \in \mathrm{Hom}_{\mathfrak{G}}(P^{L_j}, P^{L_k})$, such that $\varphi(p(L_j)) \in M^k$. Hence, $e_j C e_k$ is $e_k S e_k$-free.

Since A has a Δ-subalgebra and there is a duality on $\mathcal{O}_i$, one has that A has an exact Borel subalgebra; the statement follows. $\square$

16 Extended $\mathfrak{sl}(3,\mathbb{C})$ example

In this Chapter we illustrate all the results obtained above for the case of the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$. In order to simplify the presentation we will recall all notation and adjust them to our fixed situation.

So we set $\mathfrak{G} = \mathfrak{sl}(3,\mathbb{C})$ – the Lie algebra of 3×3 complex matricies with zero trace. Denoting by $e_{i,j}$ the matrix units we have the standard basis $e_{1,2}$, $e_{2,3}$, $e_{1,3}$, $e_{2,1}$, $e_{3,2}$, $e_{3,1}$, $e_{1,1}-e_{2,2}$, $e_{2,2}-e_{3,3}$. We fix the Cartan subalgebra $\mathfrak{H}$, which consists of diagonal matrices. Then the root system Δ of $\mathfrak{G}$ is of type A_2, hence we can choose the basis $\pi = \{\alpha,\beta\}$ in Δ, corresponding to the roots $e_{1,2}$ and $e_{2,3}$ and then $\Delta_+ = \{\alpha,\beta,\alpha+\beta\}$. We fix the Weyl-Chevalley basis $X_\alpha = e_{1,2}$, $X_\beta = e_{2,3}$, $X_{\alpha+\beta} = e_{1,3}$, $X_{-\alpha} = e_{2,1}$, $X_{-\beta} = e_{3,2}$, $X_{-\alpha-\beta} = e_{3,1}$, $H_\alpha = e_{1,1}-e_{2,2}$, $H_\beta = e_{2,2}-e_{3,3}$. The Weyl grop W is generated by s_α and s_β and is isomorphic to the symmetric group S_3. We also have $\rho = \alpha+\beta$. For $\lambda\in\mathfrak{H}^*$ we will write $\lambda = (\lambda(H_\alpha),\lambda(H_\beta))$. Hence $\alpha = (2,-1)$, $\beta = (-1,2)$ and $\rho = \alpha+\beta = (1,1)$. Under this choice, $\mathfrak{N}_+$ (resp. $\mathfrak{N}_-$) is the subalgebra of strictly upper-triangular (resp. strictly lower-triangular) matrices and $\mathfrak{B}$ is the subalgebra of upper-triangular matrices.

Let $\mathcal{P}$ be a non-trivial parabolic subalgebra of $\mathfrak{G}$ containing $\mathfrak{B}$ (i.e $\mathcal{P}\neq\mathfrak{B}$ and $\mathcal{P}\neq\mathfrak{G}$). Using the symmetry of the Dynkin diagram we can assume, that $\mathcal{P} = \mathfrak{B}\oplus\mathfrak{G}_{-\alpha}$. Hence, we have $\mathfrak{A}\simeq\mathfrak{sl}(2,\mathbb{C})$ is generated by $\mathfrak{G}_{\pm\alpha}$ and has a basis $X_{-\alpha}$, X_α, H_α. Then $\mathfrak{A}' = \mathfrak{A}\oplus\mathfrak{H}(\mathfrak{A})$, where $\mathfrak{H}(\mathfrak{A})$ is one dimensional with generator $H_\alpha+2H_\beta$. Further $\pi(\mathfrak{A}) = \{\alpha\}$, $\Delta(\mathfrak{A})_+ = \{\alpha\}$, $W(\mathfrak{A})$ is generated by s_α and is isomorphic to the symmetric group S_2. Then $\mathfrak{N} = \mathfrak{G}_\beta\oplus\mathfrak{G}_{\alpha+\beta}$, $\mathfrak{N}(\mathfrak{A}) = \mathfrak{G}_{-\beta}\oplus\mathfrak{G}_{-\alpha-\beta}$, $\rho_\alpha = (1,-1/2)$, $\rho^\alpha = (0,3/2)$. We also have $\mathfrak{c} = (H_\alpha+1)+4X_{-\alpha}X_\alpha$.

16.1 Modules $M(\lambda,p)$ and $L(\lambda,p)$

Let $V(a,b)$, $a,b\in\mathbb{C}$, be the $\mathfrak{A}$-module with basis v_i, $i\in\mathbb{Z}$ and the following action of generators: $X_{-\alpha}v_i = v_{i-1}$, $H_\alpha v_i = (a+2i)v_i$, $X_\alpha v_i = (b-(a+2i+1)^2)/4v_{i+1}$. Choose $\lambda\in\mathfrak{H}^*$ such that $(\lambda-\rho)(H_\alpha) = a$, i.e $\lambda = (\lambda_1,\lambda_2)$ and $\lambda_1 = a+1$, $\lambda_2\in\mathbb{C}$. Set $p^2 = b$. Then the module $M(\lambda,p)$, which we will denote by $M(\lambda_1,\lambda_2,p)$, is, by definition,

$$M(\lambda,p) = U(\mathfrak{G})\bigotimes_{U(\mathcal{P})} V(a,b),$$

where $X_\beta V(a,b) = X_{\alpha+\beta}V(a,b) = 0$ and $(H_\alpha+2H_\beta)v = (\lambda_1+2\lambda_2-2)v$ for any $v\in V(a,b)$. Inside $\mathcal{K}^\alpha$ the module $M(\lambda_1,\lambda_2,p)$ has a unique simple quotient $L(\lambda_1,\lambda_2,p) = L(\lambda,p)$.

First of all we will obtain information about the support of $M(\lambda_1,\lambda_2,p)$. Clearly, $\lambda-\rho\in\mathrm{supp}(M(\lambda_1,\lambda_2,p))$. Since $X_{-\alpha}$ acts bijectively on $M(\lambda_1,\lambda_2,p)$, we obtain that, together with any μ, $\mathrm{supp}(M(\lambda_1,\lambda_2,p))$ contains also $\mu+k\alpha$ for all $k\in\mathbb{Z}$. Hence $\lambda-\rho+k\alpha\in\mathrm{supp}(M(\lambda_1,\lambda_2,p))$. Further, $M(\lambda_1,\lambda_2,p)$ is $U(\mathfrak{N}(\mathfrak{A})) = U(\mathfrak{G}_{-\beta}\oplus\mathfrak{G}_{-\alpha-\beta})$-free and we obtain

$$\mathrm{supp}(M(\lambda_1,\lambda_2,p)) = \{\lambda-\rho+k\alpha-m\beta \mid k\in\mathbb{Z}, m\in\mathbb{Z}_+\}.$$

In Section 10.2, we have seen that there exists $i \in \mathbb{Z}$ such that the $U(\langle X_{\pm\beta}\rangle)$-module $U(\langle X_{\pm\beta}\rangle)v_i$ is simple infinite-dimensional. Hence $\mathrm{supp}(L(\lambda_1, \lambda_2, p)) = \mathrm{supp}(M(\lambda_1, \lambda_2, p))$.

Finally, in this Section, we also calculte the dimensions of the weight spaces in $M(\lambda_1, \lambda_2, p)$. For $k \in \mathbb{Z}$ and $m \in \mathbb{Z}_+$, we have that $\dim(M(\lambda_1, \lambda_2, p)_{\lambda-\rho+k\alpha-m\beta})$ equals the number of decompositions of $m\beta$ as $x\alpha + y\beta + z(\alpha+\beta)$, where $x \in \mathbb{Z}$ and $y, z \in \mathbb{Z}_+$. Clearly, this last equals $m+1$. Hence

$$\dim(M(\lambda_1, \lambda_2, p)_{\lambda-\rho+k\alpha-m\beta}) = m+1.$$

16.2 Inclusions and multiplicities

We can view $(\lambda_1, \lambda_2, p)$ as an element of $\mathbb{C}^3 = \Omega$. According to Theorem 6.3.1, $M(\mu_1, \mu_2, q)$ is a submodule in $M(\lambda_1, \lambda_2, q)$ if and only if $\mu = \lambda - m\beta + k\alpha$, $m \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, and either $m = 0$ and $q = \pm p$ or $m = n^{\pm}(\lambda, p) = (\lambda_1 + 2\lambda_2 \pm p)/2$ and $q^2 = (p \mp n^{\pm}(\lambda, p))^2$. Hence, for $n \in \mathbb{N}$, we can consider the hyperplanes $H(n, \pm) = \{(\lambda_1, \lambda_2, p) \mid \lambda_1 + 2\lambda_2 \pm p - 2n = 0\}$. Let H be the union of all $H(n, \pm)$, $n \in \mathbb{N}$. Then we have, that $M(\lambda_1, \lambda_2, p)$ is a simple object in $\mathcal{K}^{\alpha}$ if and only if $(\lambda_1, \lambda_2, p) \in \Omega \setminus H$. In this case $L(\lambda_1, \lambda_2, p) = M(\lambda_1, \lambda_2, p)$.

From Theorem 6.3.1, it follows also that, for $(\lambda_1, \lambda_2, p) \in H(n, \pm)$, one has $M(\lambda_1 + n, \lambda_2 - 2n, p \mp n) \subset M(\lambda_1, \lambda_2, p)$. In particular, the maximal (in $\mathcal{K}^{\alpha}$) submodule in $M(\lambda_1, \lambda_2, p)$ coincides with some $M(\lambda_1 + n, \lambda_2 - 2n, p \mp n)$. Moreover, using Theorem 6.3.1, it is easy to describe this maximal submodule precisely. For $(\lambda_1, \lambda_2, p) \in H(n_1, +) \cap H(n_2, -)$, $p > 0$ we always have $n_1 > n_2$ and hence the maximal submodule in $M(\lambda_1, \lambda_2, p)$ coincides with $M(\lambda_1 + n_2, \lambda_2 - 2n_2, p + n_2)$. In other cases, it coincides with the unique Verma submodule in $M(\lambda_1 + n_2, \lambda_2 - 2n_2, p + n_2)$.

Now let $M(\lambda_1, \lambda_2, p)$ be a non-simple (in $\mathcal{K}^{\alpha}$) GVM and $M(\lambda_1 + n, \lambda_2 - 2n, p \mp n)$ be its maximal submodule. Then we have $L(\lambda_1, \lambda_2, p) = M(\lambda_1, \lambda_2, p)/M(\lambda_1 + n, \lambda_2 - 2n, p \mp n)$ and, hence, we can compute the dimensions of the weight spaces in $L(\lambda_1, \lambda_2, p)$:

$$\dim(L(\lambda_1, \lambda_2, p)_{\lambda-\rho+k\alpha-m\beta}) = \begin{cases} m+1, & m < n \\ n, & m \geqslant n \end{cases}.$$

In our situation the multiplicities of simple subquotients in a GVM can be also obtained directly from Theorem 6.3.1. In fact, we have that $(M(\lambda_1, \lambda_2, p) : L(\mu_1, \mu_2, q)) = 1$ if and only if $M(\mu_1, \mu_2, q) \subset M(\lambda_1, \lambda_2, p)$. In the other case, the corresponding multiplicity is just 0.

16.3 BGG resolution

According to Section 16.2, the maximal submodule (in $\mathcal{K}^{\alpha}$) of $M(\lambda_1, \lambda_2, p)$ is either 0 (in this case $M(\lambda_1, \lambda_2, p) = L(\lambda_1, \lambda_2, p)$) or coincides with $M(\mu_1, \mu_2, q)$ for some $(\mu_1, \mu_2, q) \in \Omega$. In the first case, we have a trivial resolution

$$0 \to M(\lambda_1, \lambda_2, p) \to L(\lambda_1, \lambda_2, p) \to 0.$$

In the second case, we also have a straightforward resolution of $L(\lambda_1, \lambda_2, p)$ by GVMs:

$$0 \to M(\mu_1, \mu_2, q) \to M(\lambda_1, \lambda_2, p) \to L(\lambda_1, \lambda_2, p) \to 0.$$

So, it is not necessary to apply Theorem 10.1.2 and Corollary 10.2.1. However, we have that W^+ is generated by s_β ad isomorphic to S_2. Further,

$$P(\alpha)^{++} = \bigcup_{m,n\in\mathbb{N}} \left(H(n,+) \cap H(m,-)\right), \quad P(\alpha)^{+} = s_\alpha s_\beta P(\alpha)^{++}.$$

And for $(\lambda_1, \lambda_2, p) \in P(\alpha)^{++} \cup P(\alpha)^{+}$ we have

$$0 \to M(s_\beta(\lambda_1, \lambda_2, p)) \to M(\lambda_1, \lambda_2, p) \to L(\lambda_1, \lambda_2, p) \to 0.$$

This is the BGG resolution of $L(\lambda_1, \lambda_2, p)$.

16.4 Weyl formula

The character formula for any $L(\lambda_1, \lambda_2, p)$, $(\lambda_1, \lambda_2, p) \in \Omega$, can be easily obtained from the resolutions given in the previous Section. Nevertheless, we will calculate $\mathrm{ch}(L(\lambda_1, \lambda_2, p))$ for $(\lambda_1, \lambda_2, p) \in P(\alpha)^{++} \cup P(\alpha)^{+}$ using Theorem 11.1.1. We have $K = \{-\alpha-\beta\}$, $\rho' = (-1/2, 1)$, and hence, for any element $(\lambda_1, \lambda_2, p) \in P(\alpha)^{++} \cup P(\alpha)^{+}$, there exists an element $a(\lambda_1, \lambda_2, p) \in \mathfrak{H}^*$ such that

$$\mathrm{ch}(L(\lambda_1, \lambda_2, p)) = \frac{1}{1-e^{-\alpha-\beta}} \times \frac{e^{\lambda+\rho'} - e^{s_\beta(\lambda+a(\lambda_1,\lambda_2,p)+\rho')-a(\lambda_1,\lambda_2,p)}}{e^{\rho'} - e^{s_\beta(\rho')}} \times \sum_{i=-\infty}^{+\infty} e^{i\alpha}.$$

We can rewrite this as

$$\mathrm{ch}(L(\lambda_1, \lambda_2, p)) = \left(\sum_{i=-\infty}^{+\infty} e^{i\alpha}\right) \times \left(\sum_{i=0}^{\infty} e^{i(-\alpha-\beta)}\right) \times \left(e^{\lambda-\rho} + e^{\lambda-\rho-\beta} + \cdots + e^{\lambda-\rho-(N-1)\beta}\right),$$

where $N = (\beta, (\lambda_1, \lambda_2, p))_r$ according to the choice of $a(\lambda_1, \lambda_2, p)$.

16.5 Tableaux realization

The module $M(\lambda_1, \lambda_2, p)$ does not always have a tableaux realization. Sometimes it is convenient to change the basis π of the root system, in order to obtain a tableaux realization of this module. In this Section, we construct two series of GVMs over $\mathfrak{sl}(3, \mathbb{C})$ ($\mathfrak{gl}(3, \mathbb{C})$) with their tableaux realization. We note that we will obtain GVMs with respect to different choices of π and $\pi(\mathfrak{A})$. We retain the notation from Section 7.1 and Section 8.4.

As the first case, consider a tableau $[\,l\,] \in \mathbb{C}^6$, which satisfies the following conditions:

- $l_{2,1} - l_{2,2} \notin \mathbb{Z}$;

- $l_{3,i} = l_{2,i}$, $i = 1, 2$.

Consider the set $B([\,l\,])$, which consists of all tableaux $[\,t\,]$ such that

- $t_{3,i} = l_{3,i}$, $i = 1, 2, 3$;
- $l_{2,i} - t_{2,i} \in \mathbb{Z}_+$, $i = 1, 2$;
- $l_{1,1} - t_{1,1} \in \mathbb{Z}$.

Then all the tableaux from $B([\,l\,])$ are good and the set $B([\,l\,])$ is closed under GZ formulae. So, the GZ formulae define on the vectorspace $M^1([\,l\,])$, with the basis $B([\,l\,])$, the structure of a $\mathfrak{G}$-module. It follows from the GZ fomulae that $M^1([\,l\,])$ is a GVM and is α-stratified if and only if $l_{1,1} - l_{2,i} \notin \mathbb{Z}$, $i = 1, 2$. In the α-stratified situation the space of all semiprimitive generators of $M^1([\,l\,])$ is generated by $[\,l\,] + i[\,\delta^{1,1}\,]$, $i \in \mathbb{Z}$.

As the second case, consider a tableau $[\,l\,] \in \mathbb{C}^6$, which satisfies the following conditions:

- $l_{2,1} - l_{2,2} \notin \mathbb{Z}$;
- $l_{3,1} = l_{2,1} = l_{1,1}$.

Consider the set $B([\,l\,])$, which contains all tableaux $[\,t\,]$ such that

- $t_{3,i} = l_{3,i}$, $i = 1, 2, 3$;
- $l_{2,2} - t_{2,2} \in \mathbb{Z}$;
- $l_{i,1} - t_{i,1} \in \mathbb{Z}_+$, $i = 1, 2$;
- $t_{2,1} \geqslant t_{1,1}$.

Then all the tableaux from $B([\,l\,])$ are good and the set $B([\,l\,])$ is closed under GZ formulae. So, the GZ formulae define on the vectorspace $M^2([\,l\,])$, with the basis $B([\,l\,])$, the structure of a $\mathfrak{G}$-module. It follows from the GZ fomulae that $M^2([\,l\,])$ is a GVM and is β-stratified if and only if $l_{3,i} - l_{2,2} \notin \mathbb{Z}$, $i = 2, 3$. In the β-stratified situation the space of all semiprimitive generators of $M^2([\,l\,])$ is generated by $[\,l\,] + i[\,\delta^{2,2}\,]$, $i \in \mathbb{Z}$.

16.6 Schubert filtration

Let $(\lambda_1, \lambda_2, p) \in P(\alpha)^{++} \cup P(\alpha)^{+}$. Since W^+ consists of two elements, the Schubert filtration of $L(\lambda_1, \lambda_2, p)$ will contain only two terms $L(\lambda_1, \lambda_2, p) = L_1 \supset L_0$, so we have only to construct the first term L_0. According to Sections 11.2 and 11.3, L_0 should be a $\mathcal{P}$-module, $\mathrm{supp}(L_0) = \mathrm{supp}(L(\lambda_1, \lambda_2, p))$, and all non-trivial weight spaces of L_0 should be one-dimensional.

The construction of L_0 is quite transparent. We will not use the tricks used in the proof of Theorem 11.3.1 and will construct L_0 directly. Let $M(\mu_1, \mu_2, q)$ be the maximal submodule in $M(\lambda_1, \lambda_2, p)$. Then either $q = p + n$ or $q = p - n$ for some $n \in \mathbb{N}$. Consider the $\mathfrak{A}$-module $V(a,p^2)$ from the definition of $M(\lambda_1,\lambda_2,p)$. Clearly, $1\otimes V(a,p^2)\subset L_0$. Now consider the second $\mathfrak{A}$-level, $M_1=\oplus_{k\in\mathbb{Z}}M(\lambda_1,\lambda_2,p)_{\lambda-\rho-\beta+k\alpha}$. As a $\mathfrak{A}$-module, M_1 is isomorphic to $F_2\otimes V(a,p^2)$, where F_2 is a simple two-dimensional $\mathfrak{A}$-module. Hence, the subquotients of M_2 are $V(a,(p\pm 1)^2)$, counted with their multiplicities. If $p=0$, then M_2 is indecomposable and we add to L_0 the submodule $V(a,1)$. Clearly, this gives us a $\mathcal{P}$-module. If $q=p+n$ (resp. $q=p-n$) and $p\neq 0$, then we can add both $V(a,(p\pm 1)^2)$ obtaining a new $\mathcal{P}$-module. We choose $p-1$ if $q=p+n$ and $p+1$ if $q=p-n$. Now we continue the procedure above, applying it to the chosen $V(a,(p\pm 1)^2)$. As a limit we will obtain the desired $\mathcal{P}$-module L_0 having the necessary dimensions of the weight spaces. It is also clear that the above conditions uniquely define L_0 in $M(\lambda_1,\lambda_2,p)/M(\mu_1,\mu_2,q)$.

16.7 Category $\mathcal{O}(\mathcal{P},\mathcal{L})$

To describe the category $\mathcal{O}(\mathcal{P},\mathcal{L})$ for our case, we first study the category $\mathcal{L}(V(a,b))$ of $\mathfrak{A}$-modules. According to the equivalence given by Mathieu's functor (Theorem 13.1.1), it is enough to consider the case of a non-simple $V(a,b)$, i.e the case $V(\mathfrak{l},b)$, $(\mathfrak{l}+1)^2=b$.

If b is not the square of an integer, the module $V(\mathfrak{l},b)\otimes F$ decomposes into a direct sum of $V(\mathfrak{l},b_i)$, $i=1,2,\dots,\dim(F)$ for any simple finite-dimesional $\mathfrak{A}$-module F. Hence all indecomposable object of $\mathcal{L}(V(a,b))$ are simple and have the form $V(\mathfrak{l},(\sqrt{b}+i)^2)$, $i\in\mathbb{Z}$. This last follows directly from Kostant's Theorem.

Now suppose that b is the square of an integer. Then the simple objects in $\mathcal{L}(V(\mathfrak{l},b))$ are $V(\mathfrak{l},i^2)$, $i\in\mathbb{Z}$. Hence $\mathcal{L}(V(\mathfrak{l},b))=\mathcal{L}(V(\mathfrak{l},0))$. Moreover, if $i>0$, then $\mathcal{L}(V(a,b))$ also contains a length two selfextension $\hat{V}(\mathfrak{l},i^2)$ of $V(\mathfrak{l},i^2)$, which occurs as a direct summand in $V(\mathfrak{l},0)\otimes F$ for any simple F, such that $\dim(F)>i$. Moreover, from Kostant's Theorem, it follows that $V(\mathfrak{l},i^2)$, $i\in\mathbb{Z}_+$ and $\hat{V}(\mathfrak{l},i^2)$, $i\in\mathbb{N}$ exhaust all indecomposable modules in $\mathcal{L}(V(\mathfrak{l},0))$. For $i\in\mathbb{N}$ we have a natural non-split extension

$$0\to V(\mathfrak{l},i^2)\to\hat{V}(\mathfrak{l},i^2)\to V(\mathfrak{l},i^2)\to 0.$$

Applying the functor E we obtain the following sequence.

$$0\to E(V(\mathfrak{l},i^2))\to E(\hat{V}(\mathfrak{l},i^2))\to E(V(\mathfrak{l},i^2))\to 0.$$

Clearly, $E(V(\mathfrak{l},i^2))$ is a Verma module with the central character i^2, hence $E(V(\mathfrak{l},i^2))=M(i)$, $i\in\mathbb{Z}_+$. Further, $\hat{V}(\mathfrak{l},i^2)$ is projective in $\mathcal{L}(V(\mathfrak{l},0))$, hence $E(\hat{V}(\mathfrak{l},i^2))$ is also projective and thus $E(\hat{V}(\mathfrak{l},i^2))=P(-i)$, $i\in\mathbb{N}$, is the big projective module. Now we can rewrite the above sequence as

$$0\to M(i)\to P(-i)\to M(i)\to 0.$$

This sequence is not exact in $\mathcal{O}$, but it is exact in the image of E. This shows that the image of E does not inherit the abelian structure from $\mathcal{O}$. We also recall that $M(i)$, $i\in\mathbb{N}$ is projective in $\mathcal{O}$. By virtue of the last exact sequence, it is no longer projective in the image of E. It's projective cover there coincides with $P(-i)$.

Now we are ready to turn to $\mathfrak{sl}(3,\mathbb{C})$. We will restrict our consideration to the principal block $\mathcal{O}(\mathcal{P},\mathcal{L})_{triv}$, where $\mathcal{L}=\mathcal{L}(V(1,0))$. We know that $\mathcal{O}(\mathcal{P},\mathcal{L})_{triv}$ is a full subcategory of $\mathcal{O}_{triv}$, so first we recall the stucture of $\mathcal{O}_{triv}$. The Verma modules in $\mathcal{O}_{triv}$ are $M(\alpha+\beta)$, $M(\alpha)$, $M(\beta)$, $M(-\alpha)$, $M(-\beta)$, $M(-\alpha-\beta)$. Thus the simple modules in $\mathcal{O}_{triv}$ are $L(\alpha+\beta)$, $L(\alpha)$, $L(\beta)$, $L(-\alpha)$, $L(-\beta)$, $L(-\alpha-\beta)$ and the projective modules in $\mathcal{O}_{triv}$ are $P(\alpha+\beta)$, $P(\alpha)$, $P(\beta)$, $P(-\alpha)$, $P(-\beta)$, $P(-\alpha-\beta)$. We know that $L(\alpha+\beta)$ is the trivial (one-dimensional) $\mathfrak{G}$-module, $M(\alpha+\beta)=P(\alpha+\beta)$ is the *big Verma module*, $L(-\alpha-\beta)=M(-\alpha-\beta)$ is the unique socle in any Verma module in $\mathcal{O}_{triv}$ and $P(-\alpha-\beta)$ is the *big projective module*. Set $I=\{\alpha+\beta,\alpha,\beta,-\alpha,-\beta,-\alpha-\beta\}$. Since $M(-\alpha-\beta)$ is the unique socle in any Verma module from $\mathcal{O}_{triv}$, from BGG reciprocity we obtain $[P(-\alpha-\beta):M(\lambda)]=1$, for any $\lambda\in I$. We also recall that $s_\alpha(\alpha+\beta)=\beta$, $s_\beta(\alpha+\beta)=\alpha$, $s_\beta s_\alpha(\alpha+\beta)=-\beta$, $s_\alpha s_\beta(\alpha+\beta)=-\alpha$, $s_\alpha s_\beta s_\alpha(\alpha+\beta)=s_\beta s_\alpha s_\beta(\alpha+\beta)=-\alpha-\beta$.

Now we can describe the image $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ of E. For this we have to recall our $\mathfrak{sl}(2,\mathbb{C})$ description of $\mathcal{L}$. According to this description, $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ contains only those Verma modules whose highest weight generators generate a non-simple $\mathfrak{A}$-module. Hence, $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ contains only $M(\alpha+\beta)$, $M(\alpha)$, $M(-\beta)$. The corresponding simple objects in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ are the unique quotients of $M(\alpha+\beta)$ (resp. $M(\alpha)$, resp. $M(-\beta)$), which are the extensions of $L(\alpha+\beta)$ and $L(\beta)$ (resp $L(\alpha)$ and $L(-\alpha)$, resp. $L(-\beta)$ and $L(-\alpha-\beta)$). We note, that the mentioned extension of $L(-\beta)$ and $L(-\alpha-\beta)$ is in fact $M(-\beta)$. According to our description of projective modules (Theorem 13.1.2), $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ contains $P(\beta)$, $P(-\alpha)$ and $P(-\alpha-\beta)$. As in the $\mathfrak{sl}(2,\mathbb{C})$ case, the abelian structure on $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ comes from $\mathcal{O}(\mathcal{P},\mathcal{L})$ and is different from that on $\mathcal{O}$. For example, in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ we have a natural exact sequence

$$0\to M(\alpha+\beta)\to P(\beta)\to M(\alpha+\beta)\to 0,$$

which is not exact in $\mathcal{O}$.

In $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ we also have the standard modules $S(\beta)$, $S(-\alpha)$ and $S(-\alpha-\beta)$, which are induced from the projective modules in $\mathcal{L}$. They can be presented as self-extensions of Verma modules in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$:

$$0\to M(\alpha+\beta)\to S(\beta)\to M(\alpha+\beta)\to 0,$$

$$0\to M(\alpha)\to S(-\alpha)\to M(\alpha)\to 0,$$

$$0\to M(-\beta)\to S(-\alpha-\beta)\to M(-\beta)\to 0.$$

Now we can say something about the structure of projective modules. We know (Proposition 12.2.1) that any projective in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ has a standard filtration. In fact, $P(\beta)=S(\beta)$, $P(-\alpha)\supset S(\beta)$ and $P(-\alpha)/S(\beta)\simeq S(-\alpha)$ and finally, $P(-\alpha-\beta)\supset P(-\alpha)$ and $P(-\alpha-\beta)/P(-\alpha)\simeq S(-\alpha-\beta)$.

It is easy to see, that the projectively stratified finite-dimensional algebra, which corresponds to $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$, has infinite global dimension. Really, the minimal projective resolution of $M(\alpha+\beta)$ in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ has the form

$$\dots P(\beta) \to P(\beta) \to P(\beta) \to M(\alpha+\beta) \to 0,$$

and thus, is infinite. Here we also note that the big Verma module $M(\alpha+\beta)$, which is projective in $\mathcal{O}$, is no longer projective in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$; moreover, as we have seen, it has an infinite minimal projective resolution. We also note, that in contrast with classical $\mathcal{O}$, simple modules in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ can have self-extensions. For example, the standard module $S(-\alpha-\beta)$ is a self-extension of $M(-\beta)$ and the last one is simple in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$.

Now a little bit about the tilting modules. The indecomposable tilting modules in $\mathcal{O}_{triv}$ are $T(\lambda)$, $\lambda \in I$. According to Lemma 13.3.4 and Proposition 13.3.1, $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ contains only $T(-\beta)$, $T(\alpha)$ and $T(\alpha+\beta)$. It is clear, that $T(-\beta) \simeq S(-\alpha-\beta)$ and $T(\alpha+\beta) \simeq P(-\alpha-\beta)$. According to Soergel's character fomula, $T(\alpha)$ has a standard filtration, $T(\alpha) \supset S(-\alpha)$ and $T(\alpha)/S(-\alpha) \simeq S(-\alpha-\beta)$.

Further, the functor $S \otimes_{U(\mathfrak{G})} \_$ sends $P(-\alpha-\beta)$ to itself $T(\alpha+\beta) \simeq P(-\alpha-\beta)$, $P(-\alpha)$ to $T(\alpha)$ and $P(\beta)$ to $T(-\beta)$. Hence, it produces the statement that the algebra of $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ coincides with its own Ringel dual.

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