

SOME HOMOLOGICAL PROPERTIES OF CATEGORY \mathcal{O} . III

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ABSTRACT. We prove that thick category \mathcal{O} associated to a semi-simple complex finite dimensional Lie algebra is extension full in the category of all modules. We also prove weak Alexandru conjecture both for regular blocks of thick category \mathcal{O} and the associated categories of Harish-Chandra bimodules.

MSC 2010 : 16E30, 17B10

Keywords : Extension fullness, BGG category \mathcal{O} , Harish-Chandra bimodules, homological dimension, Yoneda extensions

1. INTRODUCTION

This paper is motivated by the so-called (*weak*) *Alexandru conjectures* as stated in the PhD Thesis [Fu1] of Alain Fuser and further popularized in the series [Fu2, Fu3, Fu4, Fu5] of preprints and manuscript by the same author and in the two arXiv preprints [Ga1, Ga2] by Pierre-Yves Gaillard (some further similar manuscript were available online at different times). The conjectures concern certain homological properties of various categories of Harish-Chandra modules over real and complex Lie algebras modeled on the classical properties of the BGG category \mathcal{O} from [BGG].

Given an abelian category \mathcal{A} , Yoneda defined the extension groups $\text{Ext}_{\mathcal{A}}^d(M, N)$ for any $M, N \in \mathcal{A}$ and $d \geq 0$ using equivalence classes of exact sequences of length $d + 2$. For any abelian subcategory \mathcal{B} of \mathcal{A} with exact inclusion, the definition gives rise to a canonical map $\text{Ext}_{\mathcal{B}}^d(M, N) \rightarrow \text{Ext}_{\mathcal{A}}^d(M, N)$ which is neither injective nor surjective in general. We say that \mathcal{B} is *extension full* in \mathcal{A} if these canonical maps are isomorphisms for any M, N and d . Weak Alexandru conjecture could be roughly simplified to the conjecture that certain subcategories of categories of Harish-Chandra modules are extension full.

The property of being extension full in this context is motivated by a famous theorem of Cline, Parshall and Scott from [CPS1], which asserts that the Serre subcategory associated with a coideal of the partially ordered set indexing simple objects of some highest weight category \mathcal{C} is extension full in \mathcal{C} . All definitions are designed so that this result

of [CPS1], combined with well-known consequences of the Kazhdan-Lusztig conjecture (see [Hu]), automatically implies that weak Alexandru conjecture is true for the principal block \mathcal{O}_0 , we prove this in detail in Theorem 26 below. We also prove weak Alexandru conjecture for thick category \mathcal{O}_0 , but disprove it for a singular block in category \mathcal{O} . To the best of our knowledge, the general case of (weak) Alexandru conjectures is still open. The result on singular blocks in category \mathcal{O} shows that the properties required by weak Alexandru conjecture are less natural than and not equivalent to the extension fullness result in [CPS1].

Extension fullness, the key notion behind weak Alexandru conjectures, seems to be an interesting and non-trivial property. The aim of this paper is to investigate extension fullness for various pairs of categories of modules over complex semi-simple Lie algebras and basic classical Lie superalgebras, which appear in the context of Alexandru conjectures. Here is a short list of our main results:

- Category \mathcal{O} is extension full in the category of weight modules.
- Thick category \mathcal{O} is extension full in the category of all modules.
- The category of generalized weight modules is extension full in the category of all modules.
- Computation of projective dimension, inside the thick category \mathcal{O} , of structural modules from the usual category \mathcal{O} .
- Confirmation of weak Alexandru conjecture for the principal block of thick category \mathcal{O} and the associated category of Harish-Chandra bimodules.
- Disproof of weak Alexandru conjecture for a singular block in (thick) category \mathcal{O} .

The paper is organized as follows. Section 2 provides necessary background from homological algebra. Section 3 gives several effective criteria to check extension fullness for abelian categories in an abstract situation. In Section 4 we prove that category \mathcal{O} is extension full in the category of weight modules and that thick category \mathcal{O} is extension full in the category of generalized weight modules. In Section 5 we show that thick category \mathcal{O} is extension full in the category of all modules and even reduce computation of projective dimension for objects in the thick category \mathcal{O} to computation of projective dimension in the usual category \mathcal{O} . In Section 6 we focus on some basic homological properties in singular blocks of category \mathcal{O} . Section 7 proves weak Alexandru conjecture for regular blocks of (thick) category \mathcal{O} and disproves it for some singular blocks of (thick) \mathcal{O} based on an examples described in

Section 6. Finally, in Section 8 we extend our results to the category of Harish-Chandra bimodules.

Despite of the fact that we do use some results from the first two papers [Ma1, Ma2] in the series, the present paper is rather a complement to than a continuation of [Ma1, Ma2].

2. PRELIMINARIES

We denote by \mathbb{N} the set of all non-negative integers. All subcategories are assumed to be full. We abbreviate $\otimes_{\mathbb{C}}$ by \otimes .

2.1. Extensions. We start with recalling the classical approach of Yoneda, see [Bu] or [We94, Vista 3.4.6], to the definition of extension groups in arbitrary abelian categories. For any abelian category \mathcal{A} , two fixed objects $M, N \in \mathcal{A}$ and $d \in \mathbb{N}$, the set $\text{Ext}_{\mathcal{A}}^d(M, N)$ is defined as follows. Consider the set of all exact sequences of length $d + 2$,

$$(1) \quad \mathcal{X} : \quad 0 \rightarrow N \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_d \rightarrow M \rightarrow 0,$$

with $X_1, \dots, X_d \in \mathcal{A}$. Take two exact sequences \mathcal{X} and \mathcal{Y} of the above form. If there are morphisms $X_i \rightarrow Y_i$, for $1 \leq i \leq d$, such that the following diagram commutes:

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & N & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X_d & \longrightarrow & M & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & N & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_d & \longrightarrow & M & \longrightarrow & 0, \end{array}$$

we set $\mathcal{X} \sim \mathcal{Y}$. Then $\text{Ext}_{\mathcal{A}}^d(M, N)$ is the set of equivalence classes of such exact sequences with respect to the equivalence relation generated by \sim . This set has the natural structure of an abelian group, see [Bu].

By [Bu, Theorem 3.1], for any short exact sequence $X \hookrightarrow Y \rightarrow Z$ in \mathcal{A} and any $K \in \mathcal{A}$, there is the familiar long exact sequence

$$(2) \quad \begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}_{\mathcal{A}}(Z, K) & \rightarrow & \text{Hom}_{\mathcal{A}}(Y, K) & \rightarrow & \text{Hom}_{\mathcal{A}}(X, K) & \rightarrow \\ & & \text{Ext}_{\mathcal{A}}^1(Z, K) & \rightarrow & \text{Ext}_{\mathcal{A}}^1(Y, K) & \rightarrow & \text{Ext}_{\mathcal{A}}^1(X, K) & \rightarrow \\ & & \text{Ext}_{\mathcal{A}}^2(Z, K) & \rightarrow & \text{Ext}_{\mathcal{A}}^2(Y, K) & \rightarrow & \text{Ext}_{\mathcal{A}}^2(X, K) & \rightarrow \dots \end{array}$$

and similarly with K being the first argument.

2.2. Extension full subcategories. Consider an abelian category \mathcal{A} and an abelian full subcategory \mathcal{B} . Assume that the inclusion functor $\iota : \mathcal{B} \rightarrow \mathcal{A}$ is exact. By definition, the inclusion of \mathcal{B} into \mathcal{A} induces the canonical morphism of extension groups,

$$\varphi_{M,N}^d : \text{Ext}_{\mathcal{B}}^d(M, N) \rightarrow \text{Ext}_{\mathcal{A}}^d(M, N),$$

for any two objects $M, N \in \mathcal{B}$ and any $d \in \mathbb{N}$. For convenience we will leave out the reference to M, N and when we say that a property holds for φ^d , it is understood that it holds for any $\varphi_{M,N}^d$. In general the morphisms $\varphi_{M,N}^d$ are neither injective nor surjective.

We say that \mathcal{B} is **extension full** in \mathcal{A} if and only if φ^d is an isomorphism for every $d \in \mathbb{N}$. Note that φ^0 is always an isomorphism since \mathcal{B} is a full subcategory of \mathcal{A} , while φ^1 is an isomorphism if and only if \mathcal{B} is a closed subcategory (a Serre subcategory) of \mathcal{A} . For convenience we will slightly abuse notation and often write $\text{Ext}_{\mathcal{B}}^d(M, N) \cong \text{Ext}_{\mathcal{A}}^d(M, N)$ to state the more specific property that $\varphi_{M,N}^d$ is an isomorphism (in other words, we always assume that if $\text{Ext}_{\mathcal{B}}^d(M, N)$ and $\text{Ext}_{\mathcal{A}}^d(M, N)$ are isomorphic, the isomorphism is induced by $\varphi_{M,N}^d$).

We will often use the following easy observation which follows directly from the definitions using [Bu, Theorem 3.1] and [Mc, Lemma III.1.4].

Remark 1. The maps φ^d give rise to a morphism (i.e. a chain map) between the corresponding long exact sequences of the form (2) with respect to categories \mathcal{B} and \mathcal{A} .

2.3. Projective and global dimension. For $M \in \mathcal{A}$ the *projective dimension* $\text{pd}_{\mathcal{A}}M$ of M is the supremum of the set of all $k \in \mathbb{N}$ for which there exists an $N \in \mathcal{A}$ such that $\text{Ext}_{\mathcal{A}}^k(M, N) \neq 0$. If the category \mathcal{A} contains enough projective objects, the projective dimension of $M \in \mathcal{A}$ coincides with the minimal length of a projective resolution of M in \mathcal{A} . The supremum of all the projective dimensions over all objects in \mathcal{A} is called the *global dimension* of \mathcal{A} and is denoted by $\text{gl.dim } \mathcal{A}$.

Given a short exact sequence $A \hookrightarrow B \twoheadrightarrow C$ with $A, B, C \in \mathcal{A}$, the long exact sequence (2) implies the following inequalities:

$$(3) \quad \text{pd}_{\mathcal{A}}A \leq \max\{\text{pd}_{\mathcal{A}}B, \text{pd}_{\mathcal{A}}C - 1\};$$

$$(4) \quad \text{pd}_{\mathcal{A}}C \leq \max\{\text{pd}_{\mathcal{A}}A + 1, \text{pd}_{\mathcal{A}}B\}.$$

2.4. Guichardet categories. Consider an abelian category \mathcal{A} of finite global dimension and let $S_{\mathcal{A}}$ denote the class of simple objects in \mathcal{A} . An *initial segment* in \mathcal{A} is the Serre subcategory \mathcal{I} of \mathcal{A} generated by a subset $S_{\mathcal{I}} \subset S_{\mathcal{A}}$, for which the following condition is satisfied: for any $L, L' \in S_{\mathcal{A}}$ such that $\text{pd}_{\mathcal{A}}L' = \text{pd}_{\mathcal{A}}L - 1$, $L \in S_{\mathcal{I}}$ and $\text{Ext}_{\mathcal{A}}^1(L, L') \neq 0$, we have $L' \in S_{\mathcal{I}}$.

An abelian category \mathcal{A} of finite global dimension is called a *Guichardet category* if every initial segment \mathcal{I} is extension full in \mathcal{A} .

2.5. Various categories of Lie algebra modules. Let \mathfrak{g} be a finite dimensional semisimple complex Lie algebra and $U(\mathfrak{g})$ be its universal enveloping algebra. Denote by \mathfrak{b} a Borel subalgebra of \mathfrak{g} with Cartan

subalgebra \mathfrak{h} and nilradical \mathfrak{n} . Denote by I' an ideal of finite codimension in the local ring $S(\mathfrak{h})_{(0)}$. The corresponding ideal in $S(\mathfrak{h}) = U(\mathfrak{h})$ is denoted by $I = S(\mathfrak{h}) \cap I'$. Consider the following categories of \mathfrak{g} -modules, see e.g. [BGG, Hu, So1, So3]:

- \mathfrak{g} -mod: The category of finitely generated $U(\mathfrak{g})$ -modules.
- \mathcal{W}^∞ : The subcategory of \mathfrak{g} -mod consisting of *generalized weight modules*; that is modules on which the action of \mathfrak{h} is locally finite.
- \mathcal{W}^I : The subcategory of \mathcal{W}^∞ -mod consisting of modules for which the nilpotent part of the \mathfrak{h} -action factors over $S(\mathfrak{h})/I$ (note that this requires adjustment of weights for each generalized weight space).
- \mathcal{O}^∞ : The subcategory in \mathcal{W}^∞ of locally $U(\mathfrak{b})$ -finite modules.
- \mathcal{O}^I : The subcategory in \mathcal{W}^I of locally $U(\mathfrak{b})$ -finite modules.
- \mathcal{H} : The category of finitely generated \mathfrak{g} -bimodules which are locally finite for the adjoint action of \mathfrak{g} .
- ${}^k_{\chi}\mathcal{H}^l_{\theta}$: The subcategory in \mathcal{H} of bimodules which are annihilated by $(\ker \chi)^k$ on the left and by $(\ker \theta)^l$ on the right for two central characters χ and θ .
- ${}^\infty_{\chi}\mathcal{H}^l_{\theta} = \cup_{k \in \mathbb{N}} {}^k_{\chi}\mathcal{H}^l_{\theta}$; ${}^k_{\chi}\mathcal{H}^\infty_{\theta} = \cup_{l \in \mathbb{N}} {}^k_{\chi}\mathcal{H}^l_{\theta}$; ${}^\infty_{\chi}\mathcal{H}^\infty_{\theta} = \cup_{k \in \mathbb{N}} {}^k_{\chi}\mathcal{H}^\infty_{\theta}$.

In particular, we have $\mathcal{O}^\infty = \cup_I \mathcal{O}^I$ and $\mathcal{W}^\infty = \cup_I \mathcal{W}^I$. If I is chosen to be the maximal ideal \mathfrak{m} in $S(\mathfrak{h})_{(0)}$, we have $\mathcal{O}^{\mathfrak{m}} = \mathcal{O}$, the BGG category from [BGG]. Similarly, $\mathcal{W}^{\mathfrak{m}} = \mathcal{W}$ is the category of finitely generated weight modules. Simple objects in \mathcal{O}^∞ coincide with simple objects in \mathcal{O} . Objects of \mathcal{O}^∞ (and of \mathcal{O}^I) have finite length, so these categories are both, artinian and noetherian. The categories \mathcal{O}^∞ and \mathcal{W}^∞ have neither injective nor projective modules.

For each central character χ and every category \mathcal{X} of \mathfrak{g} -modules defined above we denote by \mathcal{X}_χ the full subcategory of \mathcal{X} consisting of all modules with generalized central character χ .

For $\lambda \in \mathfrak{h}^*$ we denote by $L(\lambda) \in \mathcal{O}$ the simple highest weight module with highest weight λ and by χ_λ the central character of $L(\lambda)$.

2.6. Restricted duality. We conclude by recalling the usual construction of *duality* (i.e. a contravariant exact involutive equivalence) on category \mathcal{O}^I . We use the transpose map τ on \mathfrak{g} described in [Hu, Section 0.5]. The map τ fixes \mathfrak{h} pointwise and sends the root space \mathfrak{g}_α to $\mathfrak{g}_{-\alpha}$ for each root α . The \mathfrak{g} -action on the classical dual module

$M^* = \text{Hom}_{\mathbb{C}}(M, \mathbb{C})$ is given by $(g\alpha)(v) = \alpha(\tau(g)v)$ for $g \in \mathfrak{g}$, $v \in M$ and $\alpha \in M^*$.

For each $\mu \in \mathfrak{h}^*$ and $M \in \mathcal{O}^I$ denote by M^μ the \mathfrak{h} -submodule consisting of all generalized weight vectors for weight μ . The dual module (through $\tau|_{\mathfrak{h}} = id$) is denoted by $(M^\mu)^*$. The nilpotent part of the action of $S(\mathfrak{h})$ on $(M^\mu)^*$ clearly factors over I and, moreover, we have $(M^*)^\mu \cong (M^\mu)^*$. Then define

$$M^* = \bigoplus_{\mu} (M^\mu)^* \in \mathcal{O}^I,$$

canonically as a submodule of M^* . By definition, this leads to a duality $\star : \mathcal{O}^\infty \rightarrow \mathcal{O}^\infty$, which also fixes every \mathcal{O}^I . This duality also induces the usual duality on \mathcal{O} as in [Hu, Section 3.2]. Similarly to [Hu, Theorem 3.2(e)] we have

$$(5) \quad \text{Ext}_{\mathcal{O}^\infty}^\bullet(M^*, N^*) \cong \text{Ext}_{\mathcal{O}^\infty}^\bullet(N, M)$$

for any two $M, N \in \mathcal{O}^\infty$.

2.7. Lie algebra cohomology. For a finite dimensional Lie algebra \mathfrak{a} , the algebra cohomology of \mathfrak{a} with values in $M \in \mathfrak{a}\text{-mod}$ satisfies

$$H^d(\mathfrak{a}, M) \cong \text{Ext}_{\mathfrak{a}}^d(\mathbb{C}, M) \quad \text{for } d \in \mathbb{N},$$

see Corollary 7.3.6 in [We94]. We will need the following simple lemma.

Lemma 2. *For any module $V \in \mathfrak{a}\text{-mod}$, for which there is a morphism $V \rightarrow \mathbb{C}$, with \mathbb{C} the trivial \mathfrak{a} -module, we have*

$$H^{\dim \mathfrak{a}}(\mathfrak{a}, V) \neq 0.$$

Proof. From the Chevalley-Eilenberg complex in [We94, Corollary 7.7.3], we know that $H^{\dim \mathfrak{a}}(\mathfrak{a}, \mathbb{C}) \neq 0$ and $H^d(\mathfrak{a}, M) = 0$, for $d > \dim \mathfrak{a}$ and any $M \in \mathfrak{a}\text{-mod}$. The analogue of equation (2), with $K = \mathbb{C}$ in the first argument, then yields

$$H^{\dim \mathfrak{a}}(\mathfrak{a}, V) \rightarrow H^{\dim \mathfrak{a}}(\mathfrak{a}, \mathbb{C}),$$

which concludes the proof. \square

3. CRITERIA FOR EXTENSION FULLNESS

In this section we will derive some useful criteria for extension fullness. Our setup consists of an abelian category \mathcal{A} and a full abelian subcategory \mathcal{B} . Further, we always assume that the inclusion functor $\iota : \mathcal{B} \rightarrow \mathcal{A}$ is exact.

Lemma 3. *Let \mathcal{A} and \mathcal{B} be as above. If all objects of \mathcal{B} have finite length, then \mathcal{B} is extension full in \mathcal{A} if and only if*

$$\varphi_{L,L'}^d : \text{Ext}_{\mathcal{B}}^d(L, L') \rightarrow \text{Ext}_{\mathcal{A}}^d(L, L')$$

is an isomorphism for any two simple objects $L, L' \in \mathcal{B}$ and any $d \in \mathbb{N}$.

Proof. The “only if” statement is clear. We prove the “if” statement by induction on the length of an object in \mathcal{B} . Assume that we have $\text{Ext}_{\mathcal{B}}^d(M, L') \cong \text{Ext}_{\mathcal{A}}^d(M, L')$ for all $d \in \mathbb{N}$ and for any simple $L' \in \mathcal{B}$ and $M \in \mathcal{B}$ of length smaller than or equal to $i - 1$. The module M admits a short exact sequence $N \hookrightarrow M \twoheadrightarrow K$ where $N, K \in \mathcal{B}$ have length smaller than i .

Consider the chain map induced by φ^d between the long exact sequences of the form (2) constructed with respect to both of the categories \mathcal{A} and \mathcal{B} (see Remark 1). Now the isomorphism

$$\text{Ext}_{\mathcal{B}}^d(M, L') \rightarrow \text{Ext}_{\mathcal{A}}^d(M, L')$$

follows from the Five Lemma (see e.g. [Mc, Lemma I.3.3]).

Now the proof that L' can also be replaced by an arbitrary object of \mathcal{B} is similar. \square

Lemma 4. *Let \mathcal{A} and \mathcal{B} be as above. Assume that \mathcal{B} has a full subcategory \mathcal{B}^0 with the following properties*

- \mathcal{B} is the Serre subcategory of \mathcal{A} generated by the objects of \mathcal{B}^0
- \mathcal{B}^0 has enough projective objects.

Then \mathcal{B} is extension full in \mathcal{A} if and only if, for $d \in \mathbb{N}$, the map

$$\text{Ext}_{\mathcal{B}}^d(P, K) \rightarrow \text{Ext}_{\mathcal{A}}^d(P, K)$$

is an isomorphism for every projective P in \mathcal{B}^0 and every $K \in \mathcal{B}^0$.

Proof. The “only if” statement is clear, so we prove the “if” statement.

We start by proving, by induction on d , that $\varphi_{M,K}^d$ is always a monomorphism for arbitrary $M, K \in \mathcal{B}^0$. Since \mathcal{B} is a Serre subcategory of \mathcal{A} , φ^1 is an isomorphism. Now we assume that φ^i (restricted to \mathcal{B}^0) is an monomorphism for $i < d$. Take arbitrary $C, K \in \mathcal{B}^0$, then there is a P , projective in \mathcal{B}^0 , such that there is a short exact sequence $X \hookrightarrow P \twoheadrightarrow C$ for some $X \in \mathcal{B}^0$. From (2) and Remark 1 we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} \text{Ext}_{\mathcal{B}}^{d-1}(P, K) & \longrightarrow & \text{Ext}_{\mathcal{B}}^{d-1}(X, K) & \longrightarrow & \text{Ext}_{\mathcal{B}}^d(C, K) & \longrightarrow & \text{Ext}_{\mathcal{B}}^d(P, K) \\ \varphi_{P,K}^{d-1} \downarrow & & \varphi_{X,K}^{d-1} \downarrow & & \varphi_{C,K}^d \downarrow & & \varphi_{P,K}^d \downarrow \\ \text{Ext}_{\mathcal{A}}^{d-1}(P, K) & \longrightarrow & \text{Ext}_{\mathcal{A}}^{d-1}(X, K) & \longrightarrow & \text{Ext}_{\mathcal{A}}^d(C, K) & \longrightarrow & \text{Ext}_{\mathcal{A}}^d(P, K) \end{array}$$

Now, by assumption, $\varphi_{P,K}^{d-1}$ and $\varphi_{P,K}^d$ are isomorphisms and from the induction step $\varphi_{X,K}^{d-1}$ is a monomorphism. The Four Lemma (see e.g [Mc, Lemma I.3.3(i)]) therefore implies that $\varphi_{C,K}^d$ is injective, for arbitrary $C, K \in \mathcal{B}^0$.

Now we prove, by induction on d , that $\varphi_{C,K}^d$ is actually an isomorphism. Assume φ^i is an isomorphism for $i < d$ and consider P and X as in the paragraph above. From (2) and Remark 1 we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} \mathrm{Ext}_{\mathcal{B}}^{d-1}(X, K) & \longrightarrow & \mathrm{Ext}_{\mathcal{B}}^d(C, K) & \longrightarrow & \mathrm{Ext}_{\mathcal{B}}^d(P, K) & \longrightarrow & \mathrm{Ext}_{\mathcal{B}}^d(X, K) \\ \varphi_{X,K}^{d-1} \downarrow & & \varphi_{C,K}^d \downarrow & & \varphi_{P,K}^d \downarrow & & \varphi_{X,K}^d \downarrow \\ \mathrm{Ext}_{\mathcal{A}}^{d-1}(X, K) & \longrightarrow & \mathrm{Ext}_{\mathcal{A}}^d(C, K) & \longrightarrow & \mathrm{Ext}_{\mathcal{A}}^d(P, K) & \longrightarrow & \mathrm{Ext}_{\mathcal{A}}^d(X, K) \end{array}$$

As $\varphi_{X,K}^{d-1}$ is a bijection by the induction step, $\varphi_{P,K}^d$ is a bijection by assumptions, and $\varphi_{X,K}^d$ is a monomorphism by the previous paragraph, the Four Lemma implies that $\varphi_{C,K}^d$ is an epimorphism.

By assumptions, any module in \mathcal{B} has a finite filtration with quotients in \mathcal{B}^0 . The claim of the lemma now follows using the same argument as in the proof of Lemma 3. \square

The following result is a special case of Lemma 4, but we provide an alternative proof, which is of interest in its own right.

Corollary 5. *Let \mathcal{A} and \mathcal{B} be as above and assume that they both have enough projective objects. If every projective object in \mathcal{B} is acyclic for the functor $\mathrm{Hom}_{\mathcal{A}}(-, K)$ for any $K \in \mathcal{B}$, then \mathcal{B} is extension full in \mathcal{A} .*

Proof. Consider $N \in \mathcal{B}$ fixed. We need to prove that the functor $\mathrm{Ext}_{\mathcal{A}}^j(-, N)$, restricted to category \mathcal{B} , is isomorphic to $\mathrm{Ext}_{\mathcal{B}}^j(-, N)$. We have the obvious isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(-, N) \cong \mathrm{Hom}_{\mathcal{A}}(-, N) \circ \iota,$$

of functors from the category \mathcal{B} to the category **Sets**.

By assumption, the exact functor ι maps projective modules in \mathcal{B} to acyclic modules for the functor $\mathrm{Hom}_{\mathcal{A}}(-, N)$. The classical Grothendieck spectral sequence, see [We94, Section 5.8], therefore implies the theorem. \square

Now we consider an extra abelian category \mathcal{C} , for which \mathcal{A} (and therefore also \mathcal{B}) is a full subcategory with exact inclusion. Denote by \mathcal{A}^∞ the Serre subcategory of \mathcal{C} generated by objects of \mathcal{A} . Furthermore we denote by \mathcal{A}^k , with $k \in \mathbb{N}$, the subcategory of \mathcal{A}^∞ of objects which have a filtration of length k with quotients inside \mathcal{A} , then $\mathcal{A}^\infty = \bigcup_{k \in \mathbb{N}} \mathcal{A}^k$.

We define similarly \mathcal{B}^k and \mathcal{B}^∞ as subcategories in \mathcal{C} , which are automatically subcategories of \mathcal{A}^k and \mathcal{A}^∞ , respectively.

We show how the Yoneda extension groups in \mathcal{A}^∞ , see Subsection 2.1, can be seen as a direct limit of the corresponding extension groups for \mathcal{A}^k . For $M, N \in \mathcal{A}^\infty$, we argue that the extensions $\text{Ext}_{\mathcal{A}^\infty}^d(M, N)$ correspond to exact sequences

$$0 \rightarrow N \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_d \rightarrow M \rightarrow 0,$$

where all modules M, N, X_1, \dots, X_d are contained in \mathcal{A}^k for some k and where two such exact sequences are equivalent in \mathcal{A}^∞ if and only if they are equivalent in (that is represent the same extension in) some \mathcal{A}^l with $l \geq k$. Indeed, an equivalence between two exact sequences in \mathcal{A}^∞ , as defined in Subsection 2.1, involves only a finite amount of other exact sequences, so, in particular, a finite amount of modules. Therefore all relevant exact sequences are contained in one particular \mathcal{A}^l . This implies the following description.

Proposition 6. *The extension groups $\text{Ext}_{\mathcal{A}^\infty}^d(M, N)$ where M and N are in $\mathcal{A}^{k_0} \subset \mathcal{A}^\infty$ correspond to the limit of the directed system*

$$\text{Ext}_{\mathcal{A}^k}^d(M, N) \rightarrow \text{Ext}_{\mathcal{A}^{k+1}}^d(M, N), \quad k \geq k_0,$$

where these morphism are in general neither injective nor surjective.

Corollary 7. *Consider abelian categories $\mathcal{B} \subset \mathcal{A} \subset \mathcal{C}$, with \mathcal{B}^k and \mathcal{A}^k as defined above. If \mathcal{B}^k is extension full in \mathcal{A}^k for each k , then \mathcal{B}^∞ is extension full in \mathcal{A}^∞ .*

Proof. To prove the isomorphism

$$\text{Ext}_{\mathcal{B}^\infty}^d(M, N) \cong \text{Ext}_{\mathcal{A}^\infty}^d(M, N)$$

for every $M, N \in \mathcal{B}^\infty$ and $d \in \mathbb{N}$, we need to prove two statements according to Proposition 6.

Statement I. Every exact sequence of the form (1), where all modules are contained in some \mathcal{A}^k and which is not a trivial extension in any of the categories \mathcal{A}^l for $l \geq k$, is equivalent to an extension in \mathcal{B}^∞ .

Statement II. Every exact sequence of the form (1), where all modules are contained in some \mathcal{B}^k and which is not a trivial extension in any of the categories \mathcal{B}^l for $l \geq k$, does not become a trivial extension in \mathcal{A}^∞ .

We prove Statement I. By assumption, the extension given by (1) is equivalent to one in \mathcal{B}^k . Since the same extension is not trivial in \mathcal{A}^l for an arbitrary $l \geq k$, it is also a non-trivial extension in \mathcal{B}^l . This proves that this extension is equivalent to a non-trivial extension in \mathcal{B}^∞ . Statement II is proved similarly. \square

4. CATEGORY \mathcal{O} AND WEIGHT MODULES

The main result of this section is stated in the following theorem.

Theorem 8. *Let \mathfrak{g} be a semisimple finite dimensional complex Lie algebra and I' an ideal of finite codimension in $S(\mathfrak{h})_{(0)}$.*

- (i) *The category $\mathcal{O}^{I'}$ is extension full in $\mathcal{W}^{I'}$.*
- (ii) *The category \mathcal{O}^∞ is extension full in \mathcal{W}^∞ .*

First we note that $\mathcal{O}^{I'}$ is a Serre subcategory of $\mathcal{W}^{I'}$ and hence for $M, N \in \mathcal{O}^{I'}$ we have

$$(6) \quad \text{Ext}_{\mathcal{O}^{I'}}^1(M, N) \cong \text{Ext}_{\mathcal{W}^{I'}}^1(M, N).$$

For each $k \in \mathbb{N}$ we can consider the ideal $I_k = \mathfrak{h}^k S(\mathfrak{h})$, for which we use the short-hand notation $\mathcal{O}^k = \mathcal{O}^{I_k}$ and $\mathcal{W}^k = \mathcal{W}^{I_k}$. Then we arrive in the situation described at the end of Section 3, with $\mathcal{W}^\infty = \cup_{k \in \mathbb{N}} \mathcal{W}^k$ and $\mathcal{O}^\infty = \cup_{k \in \mathbb{N}} \mathcal{O}^k$. For notational convenience we will work with the ideals I_k even though the results hold generally.

For every $k > 0$ and $\lambda \in \mathfrak{h}^*$ we define the \mathfrak{h}^k -module $V_{\lambda, k}$ as $U(\mathfrak{h})/J_{\lambda, k}$ where $J_{\lambda, k}$ is the ideal of $U(\mathfrak{h})$ generated by all elements of the form $(h_1 - \lambda(h_1))(h_2 - \lambda(h_2)) \cdots (h_k - \lambda(h_k))$ where $h_1, h_2, \dots, h_k \in \mathfrak{h}$. Further, for $n > 0$ we define the \mathfrak{g} -module

$$M_{n, k}(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} U(\mathfrak{b})/U(\mathfrak{b})\mathfrak{n}^n \otimes_{U(\mathfrak{h})} V_{\lambda, k}.$$

The following lemma is immediate from the definition.

Lemma 9. (i) *We have $M_{n, k}(\lambda) \in \mathcal{O}^k$.*

- (ii) *Both $M_{n+1, k}(\lambda)$ and $M_{n, k+1}(\lambda)$ surject onto $M_{n, k}(\lambda)$. Furthermore, the module $M_{1, 1}(\lambda)$ is isomorphic to the classical Verma module $M(\lambda)$ with highest weight λ .*

(iii) *There is the following short exact sequence:*

$$U(\mathfrak{g})\mathfrak{n}^n \otimes_{U(\mathfrak{h})} V_{\lambda, k} \hookrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V_{\lambda, k} \twoheadrightarrow M_{n, k}(\lambda).$$

We denote the maximal direct summand of $M_{n, k}(\lambda)$ belonging to the subcategory $\mathcal{O}_{\chi_\lambda}^k$, by $\widetilde{M}_{n, k}(\lambda)$.

Lemma 10. *For each $d > 0$ we have $\text{Ext}_{\mathcal{W}^k}^d(\widetilde{M}_{n, k}(\lambda), L(\nu)) = 0$ for all $\nu \in \mathfrak{h}^*$ and all $n \gg 0$.*

Proof. The result is trivial unless $\chi_\nu = \chi_\lambda$, so we assume that ν is in the Weyl group orbit of λ . This leaves only a finite amount of choices for ν .

The module $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V_{\lambda,k}$ is projective in \mathcal{W}^k . Lemma 9(iii) and equation (2) therefore imply that we have

$$(7) \quad \text{Ext}_{\mathcal{W}^k}^{d-1}(U(\mathfrak{g})\mathfrak{n}^n \otimes_{U(\mathfrak{h})} V_{\lambda,k}, L(\nu)) \rightarrow \text{Ext}_{\mathcal{W}^k}^d(M_{n,k}(\lambda), L(\nu)),$$

moreover, this is an isomorphism if $d > 1$.

The \mathfrak{b} -module $U(\mathfrak{b})\mathfrak{n}^n \otimes_{U(\mathfrak{h})} V_{\lambda,k}$ has a resolution in terms of modules $U(\mathfrak{b}) \otimes_{U(\mathfrak{h})} V_{\mu,k}$, where each $\mu \in \mathfrak{h}^*$ is of the form

$$\mu = \lambda + \alpha_1 + \cdots + \alpha_p,$$

where $p \geq n$ and α_i 's are positive roots. This implies that the module

$$U(\mathfrak{g})\mathfrak{n}^n \otimes_{U(\mathfrak{h})} V_{\lambda,k} \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} U(\mathfrak{b})\mathfrak{n}^n \otimes_{U(\mathfrak{h})} V_{\lambda,k}$$

has a projective resolution in \mathcal{W}^k , by modules $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V_{\mu,k}$, with the same condition on μ .

According to the above, in order to prove that the left-hand side of equation (7) is zero for $d > 0$, it suffices to show that the space

$$\text{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V_{\mu,k}, L(\nu)) \cong \text{Hom}_{\mathfrak{h}}(V_{\mu,k}, L(\nu))$$

(where the isomorphism is given by adjunction) is zero, for μ as above. For each ν , we can find an n large enough, such that all of weights μ of the above form do not appear in $L(\nu)$. Taking the maximum over this finite set of numbers yields the lemma. \square

For every $\mu \in \mathfrak{h}^*$, denote the projective cover of $L(\mu)$ in \mathcal{O}^k by $P^k(\mu)$. Let W be the Weyl group of \mathfrak{g} .

Proposition 11. *For n large enough, we have*

$$\widetilde{M}_{n,k}(\lambda) = \bigoplus_{w \in W} \dim(L(w \cdot \lambda)^\lambda) P^k(w \cdot \lambda).$$

Proof. Lemma 10 for $d = 1$ and the isomorphism in (6) imply that $\widetilde{M}_{n,k}(\lambda)$ is projective in \mathcal{O}^k , for n large enough. Furthermore, from Lemma 9(iii) and the computation in the proof of Lemma 10 we get

$$\text{Hom}_{\mathfrak{g}}(\widetilde{M}_{n,k}(\lambda), L(w \cdot \lambda)) \cong \text{Hom}_{\mathfrak{h}}(V_{\lambda,k}, L(w \cdot \lambda)),$$

which concludes the proof. \square

Corollary 12. *Consider $M \in \mathcal{O}^I$ and P projective in \mathcal{O}^I . Then for $d > 0$ we have*

$$\text{Ext}_{\mathcal{W}^I}^d(P, M) = 0.$$

Proof. If M is simple, this is an immediate consequence of the combination of Lemma 10 and Proposition 11. The general statement therefore follows, using the usual arguments with long exact sequences, from the fact that each module in \mathcal{O}^I has finite length. \square

Proof of Theorem 8. Claim (i) follows combining Corollaries 5 and 12. Claim (ii) follows from claim (i) and Corollary 7. \square

For $I = \mathfrak{m}$ the category \mathcal{O}^I is the usual BGG category \mathcal{O} . In this case Theorem 8(i) states that category \mathcal{O} is extension full in the category of weight modules, which recovers an old result of Delorme, see [De]. An important consequence is the following connection between \mathfrak{n} -cohomology and extensions with Verma modules in category \mathcal{O} , see [Hu, Theorem 6.15(b)].

Corollary 13. *For $\mu \in \mathfrak{h}^*$ and $N \in \mathcal{O}$ we have*

$$\mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_{\mu}, H^k(\mathfrak{n}, N)) = \mathrm{Ext}_{\mathcal{O}}^k(M(\mu), N).$$

Proof. The equality $\mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_{\mu}, H^k(\mathfrak{n}, N)) = \mathrm{Ext}_{\mathcal{W}}^k(M(\mu), N)$ follows immediately from the Frobenius reciprocity. The claim thus follows from Theorem 8(i) for the case $I = \mathfrak{m}$. \square

We would like to record the following observation.

Proposition 14. *We have*

$$\mathrm{gl.dim} \mathcal{O} = \mathrm{gl.dim} \mathcal{W} = \dim \mathfrak{g} - \dim \mathfrak{h},$$

whereas the global dimensions of \mathcal{O}^I and \mathcal{W}^I are infinite if $I \neq \mathfrak{m}$.

Proof. The global dimension of \mathcal{O} is well-known, see e.g. [Ma1, Proposition 2], [Hu, Section 6.9] or [BGG]. The global dimension of \mathcal{W} follows from the fact that a projective resolution in \mathcal{W} is a projective resolution for the relative $(\mathfrak{g}, \mathfrak{h})$ -cohomology.

The infinite global dimensions of \mathcal{O}^I and \mathcal{W}^I follow immediately from considering a projective resolution in \mathcal{O}^I (respectively \mathcal{W}^I) of a projective module in \mathcal{O} (respectively \mathcal{W}). \square

5. CATEGORY \mathcal{O}^{∞}

5.1. Category \mathcal{O}^{∞} is extension full in \mathfrak{g} -mod.

Theorem 15. *Let \mathfrak{g} be a complex semisimple finite dimensional Lie algebra. Then both categories \mathcal{O}^{∞} and \mathcal{W}^{∞} are extension full in \mathfrak{g} -mod.*

Before proving this, we note the following corollary.

Corollary 16. *We have $\mathrm{gl.dim} \mathcal{O}^{\infty} = \mathrm{gl.dim} \mathcal{W}^{\infty} = \dim \mathfrak{g}$.*

Proof. Theorem 15 implies that the global dimension of \mathcal{O}^{∞} and \mathcal{W}^{∞} are smaller than or equal to $\dim \mathfrak{g}$, the global dimension of \mathfrak{g} -mod. The classical fact

$$\mathrm{Ext}_{\mathfrak{g}}^{\dim \mathfrak{g}}(\mathbb{C}, \mathbb{C}) = H^{\dim \mathfrak{g}}(\mathfrak{g}, \mathbb{C}) \neq 0,$$

see Lemma 2, then shows that both global dimensions in question are equal to this value. \square

The remainder of this section is devoted to proving Theorem 15.

Lemma 17. *Consider a finite dimensional abelian Lie algebra \mathfrak{h} and the category $\mathcal{F} := \cup_{k \in \mathbb{N}} \mathcal{F}^k$ where \mathcal{F}^k is the category of all finite dimensional \mathfrak{h} -modules which are $S(\mathfrak{h})/\mathfrak{h}^k S(\mathfrak{h})$ -modules. Then the category \mathcal{F} is extension full in \mathfrak{h} -mod.*

Proof. The commutative algebra $S(\mathfrak{h})$ is positively graded in the natural way with \mathfrak{h} being of degree one. The algebra $S(\mathfrak{h})$, being isomorphic to the polynomial algebra in finitely many variables, is Koszul. Consider the graded Koszul resolution \mathcal{P}^\bullet of the trivial $S(\mathfrak{h})$ -module \mathbb{C} (the module structure on \mathbb{C} is given by $\mathfrak{h}\mathbb{C} = 0$). Then the $-i$ -th component \mathcal{P}^{-i} of this resolution is generated in degree i and $\mathcal{P}^{-i} = 0$ for $i > \dim \mathfrak{h}$.

For $k \in \mathbb{N}$ consider the algebra $A_k := S(\mathfrak{h})/\mathfrak{h}^k S(\mathfrak{h})$ together with the functor $F_k : S(\mathfrak{h})\text{-mod} \rightarrow A_k\text{-mod}$ given by $M \mapsto M/\mathfrak{h}^k S(\mathfrak{h})M$. Applying F_k to \mathcal{P}^\bullet gives a complex of projective A_k -modules which still has homology \mathbb{C} in the homological position zero and a lot of other homologies in negative homological positions. However, all those homologies are concentrated in degrees $\geq k$ of our grading. Resolving those homologies in $A_k\text{-mod}$ we obtain that for $k \gg 0$ the graded spaces $\text{Ext}_{\mathfrak{h}}^d(\mathbb{C}, \mathbb{C})$ and $\text{Ext}_{A_k}^d(\mathbb{C}, \mathbb{C})$ agree in all degrees up to $k - 1$ for all d . Hence, taking the limit for $k \rightarrow \infty$, yields isomorphism $\text{Ext}_{\mathcal{F}}^d(\mathbb{C}, \mathbb{C}) \cong \text{Ext}_{\mathfrak{h}}^d(\mathbb{C}, \mathbb{C})$.

Since all modules in \mathcal{F} have finite length and \mathbb{C} is the only simple module in \mathcal{F} , the result follows from Lemma 3. \square

Frobenius reciprocity for extensions follows from adjunction between derived functors. Since the category \mathcal{W}^∞ does not have projective modules, we need the following lemma. We introduce the notation $\mathcal{C}(\mathfrak{h})^I$ for the category of finite dimensional \mathfrak{h} -modules for which the nilpotent part of the $S(\mathfrak{h})$ -action factors over I , with I an ideal as in Section 2. Furthermore, we set $\mathcal{C}(\mathfrak{h})^\infty = \cup_I \mathcal{C}(\mathfrak{h})^I$.

Lemma 18. *For $K \in \mathcal{C}(\mathfrak{h})^\infty$, $M \in \mathcal{W}^\infty$ and $d > 0$ we have*

$$(8) \quad \text{Ext}_{\mathcal{W}^\infty}^d(\text{Ind}_{\mathfrak{h}}^{\mathfrak{g}} K, M) \cong \text{Ext}_{\mathcal{C}(\mathfrak{h})^\infty}^d(K, \text{Res}_{\mathfrak{h}}^{\mathfrak{g}} M).$$

Proof. There is an ideal J big enough such that both M and $\text{Ind}_{\mathfrak{h}}^{\mathfrak{g}} K$ belong to \mathcal{W}^J . By Proposition 6, both the left-hand side and the right-hand side of (8) are respectively given as limits of

$$\text{Ext}_{\mathcal{W}^I}^d(\text{Ind}_{\mathfrak{h}}^{\mathfrak{g}} K, M) \quad \text{and} \quad \text{Ext}_{\mathcal{C}(\mathfrak{h})^I}^d(K, \text{Res}_{\mathfrak{h}}^{\mathfrak{g}} M)$$

over $I \supset J$. The isomorphism between these two extension groups for every J follows from the usual Frobenius reciprocity. \square

Lemma 19. *Let I' be an ideal in $S(\mathfrak{h})_{(0)}$ of finite codimension. If $P \in \mathcal{W}^{I'}$ is projective and $M \in \mathcal{W}^\infty$ is arbitrary, then the morphism*

$$\mathrm{Ext}_{\mathcal{W}^\infty}^d(P, M) \rightarrow \mathrm{Ext}_{\mathfrak{g}}^d(P, M),$$

is an isomorphism for all $d \geq 0$.

Proof. Without loss of generality we may take

$$P \cong U(\mathfrak{g}) \otimes_{S(\mathfrak{h})} (S(\mathfrak{h})/I \otimes \mathbb{C}_\lambda),$$

where $\mathbb{C}_\lambda = V_{\lambda,1}$ is the simple 1-dimensional \mathfrak{h} -module corresponding to λ . By Lemma 18, the proposed statement then reduces to

$$(9) \quad \mathrm{Ext}_{\mathcal{C}(\mathfrak{h})^\infty}^d(S(\mathfrak{h})/I \otimes \mathbb{C}_\lambda, \mathrm{Res}_{\mathfrak{h}}^{\mathfrak{g}} M) \cong \mathrm{Ext}_{\mathfrak{h}}^d(S(\mathfrak{h})/I \otimes \mathbb{C}_\lambda, \mathrm{Res}_{\mathfrak{h}}^{\mathfrak{g}} M).$$

All modules in $\mathcal{C}(\mathfrak{h})^\infty$ decompose into generalized weight spaces and the category decomposes into equivalent blocks corresponding to different eigenvalues. It suffices to consider one block. The block corresponding to 0 is exactly \mathcal{F} from Lemma 17 and equation (9) is thus a consequence of that lemma. \square

Proof of Theorem 15. We apply Lemma 4, with \mathcal{A} , \mathcal{B} and \mathcal{B}^0 given by, respectively, $\mathfrak{g}\text{-mod}$, \mathcal{W}^∞ and \mathcal{W} . The fact that \mathcal{W}^∞ is extension full in $\mathfrak{g}\text{-mod}$ is therefore a consequence of Lemma 19 for the special case $I = \mathfrak{m}$.

The fact that \mathcal{O}^∞ is extension full in $\mathfrak{g}\text{-mod}$ is then an immediate consequence of the result for \mathcal{W}^∞ and Theorem 8(ii). \square

5.2. Projective dimensions in \mathcal{O}^∞ . In this section we calculate projective dimensions inside category \mathcal{O}^∞ for modules in \mathcal{O}^I .

Theorem 20. *(i) Consider $M \in \mathcal{O}^I$ for some ideal I' in $S(\mathfrak{h})_{(0)}$ of finite codimension, with $\mathrm{pd}_{\mathcal{O}^I} M < \infty$, then*

$$\mathrm{pd}_{\mathcal{O}^\infty} M = \dim \mathfrak{h} + \mathrm{pd}_{\mathcal{O}^I} M.$$

(ii) The minimal projective dimension of a module in \mathcal{O}^∞ is $\dim \mathfrak{h}$.

Before proving this, we note that this result yields the projective dimension of all structural modules from \mathcal{O} (that is simple, standard, costandard, tilting, injective, projective modules) inside the category \mathcal{O}^∞ by using the results in [Ma1, Ma2]. In particular, we have the following corollary.

Corollary 21. *Consider $\lambda \in \mathfrak{h}^*$ to be integral dominant. Then for $w \in W$ we have:*

- (a) $\text{pd}_{\mathcal{O}^\infty} L(w \cdot \lambda) = \dim \mathfrak{g} - l(w)$,
 (b) $\text{pd}_{\mathcal{O}^\infty} M(w \cdot \lambda) = \dim \mathfrak{h} + l(w)$.

Proof. This is a consequence of the combination of Theorem 20 with either [Hu, Theorem 6.9] or [Ma1, Propositions 3 and 6]. \square

The remainder of this subsection is devoted to proving Theorem 20.

Lemma 22. *For a projective $P \in \mathcal{O}^I$ we have $\text{pd}_{\mathcal{O}^\infty} P \leq \dim \mathfrak{h}$.*

Proof. For notational convenience we only consider the ideals I^k in this proof. According to Proposition 11, every projective module in \mathcal{O}^k is a direct summand of some $\widetilde{M}_{n,k}(\lambda) = (M_{n,k}(\lambda))_{\chi_\lambda}$, where $\lambda \in \mathfrak{h}^*$ and $n \gg 0$. We prove that $\text{pd}_{\mathcal{O}^\infty} \widetilde{M}_{n,k}(\lambda) \leq \dim \mathfrak{h}$.

Take $\nu \in \mathfrak{h}^*$ with $\chi_\nu = \chi_\lambda$. For $d > 0$, applying (2) inside the category $\mathfrak{g}\text{-mod}$ to the sequence from Lemma 9(iii) yields the exact sequence

$$\begin{aligned} \text{Ext}_{\mathfrak{g}}^{d-1}(U(\mathfrak{g})\mathfrak{n}^n \otimes_{U(\mathfrak{h})} V_{\lambda,k}, L(\nu)) &\rightarrow \text{Ext}_{\mathfrak{g}}^d(\widetilde{M}_{n,k}(\lambda), L(\nu)) \rightarrow \\ &\rightarrow \text{Ext}_{\mathfrak{g}}^d(U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V_{\lambda,k}, L(\nu)) \rightarrow \text{Ext}_{\mathfrak{g}}^d(U(\mathfrak{g})\mathfrak{n}^n \otimes_{U(\mathfrak{h})} V_{\lambda,k}, L(\nu)). \end{aligned}$$

We take the projective resolution of $U(\mathfrak{g})\mathfrak{n}^n \otimes_{U(\mathfrak{h})} V_{\lambda,k}$ in \mathcal{W}^k described in the proof of Lemma 10. This resolution is given in terms of modules of the form $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V_{\mu,k}$ with all $\mu \not\leq \nu$. As for such μ and for $p > 0$ we have

$$(10) \quad \text{Ext}_{\mathfrak{g}}^p(U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V_{\mu,k}, L(\nu)) = \text{Ext}_{\mathfrak{h}}^p(V_{\mu,k}, L(\nu)) = 0,$$

our resolution is an acyclic resolution for the functor $\text{Hom}_{\mathfrak{g}}(-, L(\nu))$, which can be used to compute $\text{Ext}_{\mathfrak{g}}^i(-, L(\nu))$. Since (10) is also true for $p = 0$, we obtain that

$$\text{Ext}_{\mathfrak{g}}^i(U(\mathfrak{g})\mathfrak{n}^n \otimes_{U(\mathfrak{h})} V_{\lambda,k}, L(\nu)) = 0 \quad \text{for } i \in \{d-1, d\}.$$

By Frobenius reciprocity we have

$$\text{Ext}_{\mathfrak{g}}^d(U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V_{\lambda,k}, L(\mu)) \cong \text{Ext}_{\mathfrak{h}}^d(V_{\lambda,k}, L(\mu)).$$

By Theorem 15 we have

$$\text{Ext}_{\mathfrak{g}}^d(\widetilde{M}_{n,k}(\lambda), L(\mu)) \cong \text{Ext}_{\mathcal{O}^\infty}^d(\widetilde{M}_{n,k}(\lambda), L(\mu)).$$

The above now implies that, for $d > 0$,

$$(11) \quad \text{Ext}_{\mathcal{O}^\infty}^d(\widetilde{M}_{n,k}(\lambda), L(\mu)) \cong \text{Ext}_{\mathfrak{h}}^d(V_{\lambda,k}, L(\mu)).$$

Since $\text{gl.dim } \mathfrak{h}\text{-mod} = \dim \mathfrak{h}$, we get $\text{pd}_{\mathcal{O}^\infty} \widetilde{M}_{n,k} \leq \dim \mathfrak{h}$. \square

Using (11) one can even show that $\text{pd}_{\mathcal{O}^\infty} \widetilde{M}_{n,k} = \dim \mathfrak{h}$, by considering the corresponding \mathfrak{h} -homology. However, this does not prove the corresponding result for arbitrary projective modules because of the nontrivial decomposition in Proposition 11.

Proof of Theorem 20. First we prove claim (ii). Consider $M \in \mathcal{O}^\infty$, it is a module of finite length with all simple subquotients from \mathcal{O} , therefore it has a weight λ such that $M^\lambda \neq 0$ and M contains no vectors of higher weights. Similarly to the proof of Lemma 22 (as in formula (11)) we have

$$\text{Ext}_{\mathcal{O}^\infty}^{\dim \mathfrak{h}}(\widetilde{M}_{n,1}(\lambda), M^*) \cong \text{Ext}_{\mathfrak{h}}^{\dim \mathfrak{h}}(\mathbb{C}_\lambda, M^*) \cong H^{\dim \mathfrak{h}}(\mathfrak{h}, \mathbb{C}_{-\lambda} \otimes M^*).$$

This is non-zero by Lemma 2. Equation (5) therefore implies that M has projective dimension at least $\dim \mathfrak{h}$.

Now we prove claim (i) by induction on the projective dimension of M inside \mathcal{O}^I . If the projective dimension is zero, the result follows from Lemma 22 and the previous paragraph. We assume the result holds up to projective dimension $p - 1$. For $M \in \mathcal{O}^I$, with $\text{pd}_{\mathcal{O}^I} M = p$, there is a P , projective in \mathcal{O}^I , and an $N \in \mathcal{O}^I$, with $\text{pd}_{\mathcal{O}^I} N = p - 1$, such that $N \hookrightarrow P \twoheadrightarrow M$. Formulae (3) and (4) yield

$$\begin{aligned} p + \dim \mathfrak{h} - 1 &\leq \max\{\dim \mathfrak{h}, \text{pd}_{\mathcal{O}^\infty} M - 1\} \\ \text{pd}_{\mathcal{O}^\infty} M &\leq \max\{p + \dim \mathfrak{h}, \dim \mathfrak{h}\}, \end{aligned}$$

which implies $\text{pd}_{\mathcal{O}^\infty} M = p + \dim \mathfrak{h}$. \square

5.3. Projective dimensions in \mathcal{W}^∞ .

Theorem 23. *Let I' be an ideal in $S(\mathfrak{h})_{(0)}$ of finite codimension. If $M \in \mathcal{W}^{I'}$ satisfies $\text{pd}_{\mathcal{W}^{I'}} M < \infty$, then*

$$\text{pd}_{\mathcal{W}^\infty} M = \dim \mathfrak{h} + \text{pd}_{\mathcal{W}^{I'}} M.$$

Proof. From Lemma 18 and the algebra cohomology of \mathfrak{h} it follows quickly that the projective dimension of projective modules in $\mathcal{W}^{I'}$ is equal to $\dim \mathfrak{h}$. The result then follows identically as in the proof of Theorem 20. \square

5.4. Basic classical Lie superalgebras. In this subsection we consider basic classical Lie superalgebras, we refer to [Mu] for definitions. We will denote a basic classical Lie superalgebra by $\widetilde{\mathfrak{g}}$ and the underlying Lie algebra of $\widetilde{\mathfrak{g}}$ by \mathfrak{g} . An important property of these Lie superalgebras is that the Cartan subalgebra of $\widetilde{\mathfrak{g}}$ is equal to the one of \mathfrak{g} . Therefore we have natural analogues of the categories introduced in Subsection 2.5 and we denote the corresponding categories by $\widetilde{\mathcal{O}}^{I'}$, $\widetilde{\mathcal{W}}^{I'}$ etc.

Theorem 24. *For a basic classical Lie superalgebra $\tilde{\mathfrak{g}}$ we have:*

- (i) *The BGG category $\tilde{\mathcal{O}}$ is extension full in $\tilde{\mathcal{W}}$.*
- (ii) *The categories $\tilde{\mathcal{O}}^\infty$ and $\tilde{\mathcal{W}}^\infty$ are extension full in $\tilde{\mathfrak{g}}\text{-mod}$.*

Proof. Consider a projective module \tilde{P} in $\tilde{\mathcal{O}}$. It is a direct summand of $\text{Ind}_{\tilde{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} P$ for a projective module P in \mathcal{O} . Using the Frobenius reciprocity, we have

$$\text{Ext}_{\tilde{\mathcal{W}}}^d(\text{Ind}_{\tilde{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} P, M) = \text{Ext}_{\mathcal{W}}^d(P, \text{Res}_{\tilde{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} M),$$

which is zero for $d > 0$ by Corollary 12. Claim (i) now follows from Corollary 5.

The same reasoning can be used to obtain the extension fullness of $\tilde{\mathcal{O}}^I$ into $\tilde{\mathcal{W}}^I$, for every ideal I' in $S(\mathfrak{h})_{(0)}$ of finite codimension. Therefore, Corollary 7 implies that $\tilde{\mathcal{O}}^\infty$ is extension full in $\tilde{\mathcal{W}}^\infty$.

Lemma 19 can be generalized immediately to basic classical Lie superalgebras, since their Cartan subalgebra coincides with the one of the underlying Lie algebra (alternatively, one can use the fact that projective modules in $\tilde{\mathcal{W}}^I$ are induced from projective modules in \mathcal{W}^I and Lemma 19). The fact that $\tilde{\mathcal{W}}^\infty$ is extension full in $\tilde{\mathfrak{g}}\text{-mod}$ then follows from Lemma 4. \square

6. SINGULAR BLOCKS IN CATEGORY \mathcal{O}

6.1. Singular blocks in category \mathcal{O} . Let λ be a dominant integral weight for \mathfrak{g} and W_λ denote the stabilizer of λ in W with respect to the dot action. Let w_0 be the longest element in W and w_0^λ be the longest element in W_λ . We also denote by $\mathfrak{a} : W \rightarrow \mathbb{N}$ Lusztig's \mathfrak{a} -function, see [Lu1, Lu2].

Consider the corresponding singular block $\mathcal{O}_\lambda = \mathcal{O}_{\chi_\lambda}$. Then we have the usual exact functors of translation out of and onto the W_λ -wall:

$$\theta^{\text{out}} : \mathcal{O}_\lambda \rightarrow \mathcal{O}_0 \quad \text{and} \quad \theta^{\text{on}} : \mathcal{O}_0 \rightarrow \mathcal{O}_\lambda,$$

see [BG] for details. These functors satisfy

$$(12) \quad \theta^{\text{on}} \theta^{\text{out}} \cong \text{Id}_{\mathcal{O}_\lambda}^{\oplus |W_\lambda|}.$$

Furthermore, the functor $\theta^{\text{out}} \theta^{\text{on}}$ is the unique indecomposable projective endofunctor of \mathcal{O}_0 sending $M(0)$ to the projective cover of $L(w_0^\lambda \cdot 0)$. This functor is usually denoted $\theta_{w_0^\lambda}$. The main result of this section is the following observation.

Theorem 25. (i) *The projective dimension of the simple Verma module in \mathcal{O}_λ equals $\mathfrak{a}(w_0 w_0^\lambda)$.*

- (ii) We have $\text{gl.dim } \mathcal{O}_\lambda = 2\mathbf{a}(w_0w_0^\lambda)$.
- (iii) The projective dimension of the dominant simple module in \mathcal{O}_λ equals $2\mathbf{a}(w_0w_0^\lambda)$.

Proof. Let L be the simple Verma module in \mathcal{O}_λ . Then $L \cong \theta^{\text{on}}L(w_0 \cdot 0)$ and hence

$$\theta^{\text{out}}L \cong \theta^{\text{out}}\theta^{\text{on}}L(w_0 \cdot 0) = \theta_{w_0^\lambda}L(w_0 \cdot 0)$$

is the indecomposable tilting module $T(w_0w_0^\lambda \cdot 0)$ in \mathcal{O}_0 with highest weight $w_0w_0^\lambda \cdot 0$. By [Ma2, Theorem 17], the projective dimension of $T(w_0w_0^\lambda \cdot 0)$ equals $\mathbf{a}(w_0w_0^\lambda)$. On the other hand, from (12) it follows that $\theta^{\text{on}}T(w_0w_0^\lambda \cdot 0)$ is a direct sum of copies of L . As projective functors are exact and send projective modules to projective modules, it follows that the projective dimensions of L and $T(w_0w_0^\lambda \cdot 0)$ coincide, proving claim (i).

The parabolic-singular Koszul duality from [BGS] asserts that the Koszul dual of \mathcal{O}_λ is the parabolic subcategory $\mathcal{O}_0^{W_\lambda}$ of \mathcal{O}_0 associated to W_λ . We use the normalization of Koszul duality which maps simple objects to indecomposable injective objects. By the graded length of a module we mean the number of non-zero graded components with respect to Koszul grading. Then Koszul duality maps a simple module of projective dimension p to an indecomposable injective module of graded length $p + 1$ and reverses the quasi-hereditary order. Therefore Koszul duality maps L to the dominant costandard module in $\mathcal{O}_0^{W_\lambda}$. We denote the latter module by M , which thus has graded length $\mathbf{a}(w_0w_0^\lambda) + 1$, by claim (i). The injective envelope I of the dual module M^* (the dominant standard module) is known to be projective-injective and hence tilting, see e.g. [Ma2, Section 3]. This is the only tilting module which contains the dominant simple as a subquotient. Therefore I is the tilting module associated to the standard module M^* , i.e. we have a (unique up to a nonzero scalar) injection $M^* \hookrightarrow I$ and a (unique up to a nonzero scalar) surjection $I \twoheadrightarrow M$ and the image of the composition of these two maps coincides with the simple socle of M . As the socles of I and M^* agree and, at the same time, the heads of I and M agree, it follows that

$$\text{graded length}(I) = \text{graded length}(M) + \text{graded length}(M^*) - 1.$$

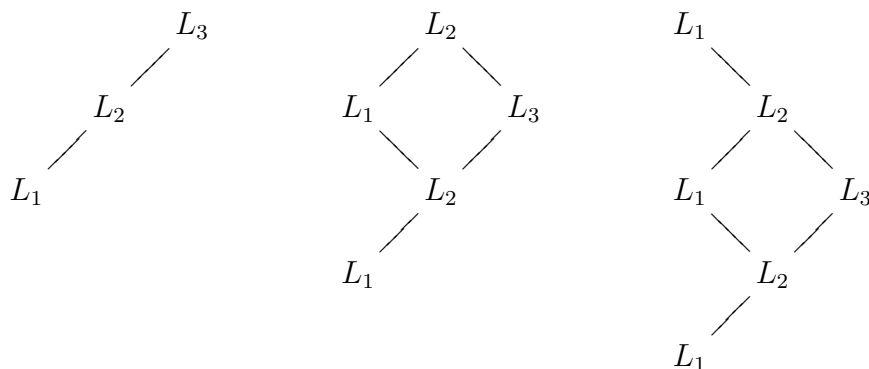
Clearly, the graded lengths of M and M^* coincide. In [Ma2, Section 3] it is shown that all projective-injective modules in $\mathcal{O}_0^{W_\lambda}$ have the same graded length and that each projective module is a submodule of a projective-injective module. This implies that the maximal graded length of an indecomposable injective module in $\mathcal{O}_0^{W_\lambda}$ is $2\mathbf{a}(w_0w_0^\lambda) + 1$ which implies claim (ii).

Claim (iii) follows from the fact that Koszul duality maps the dominant simple module in \mathcal{O}_λ to the antidominant injective in the parabolic category and the latter module is automatically projective and hence has maximal graded length, as mentioned in the previous paragraph. This completes the proof. \square

6.2. The \mathfrak{sl}_3 -examples. As described in [St, Section 5.2.1], any non-trivial singular integral block of the category \mathcal{O} for \mathfrak{sl}_3 is equivalent to the category of modules over the following quiver with relations:

$$(13) \quad \begin{array}{ccccc} & & a & & c \\ & & \curvearrowright & & \curvearrowright \\ 1 & \longleftarrow & 2 & \longleftarrow & 3 \\ & & b & & d \end{array} \quad cd = 0, \quad ab = dc.$$

Let L_i for $i = 1, 2, 3$ be simple modules corresponding to vertices in this quiver and P_i be their projective covers. Then P_3, P_2 and P_1 have the following Loewy structure, respectively:



A direct computation thus implies

$$\text{pd } L_1 = 1, \quad \text{pd } L_2 = \text{pd } L_3 = 2.$$

In particular, the global dimension of this module category equals 2. All this fully agrees with Theorem 25 and with [MaO].

Further, it is straightforward to check that the minimal projective resolution of L_3 has the following form:

$$0 \rightarrow P_3 \rightarrow P_2 \rightarrow P_3 \rightarrow L_3 \rightarrow 0.$$

This implies that we have

$$(14) \quad \text{Ext}^2(L_3, L_3) \neq 0$$

in this module category.

6.3. Singular speculations. It would be interesting to generalize the explicit description of homological invariants for structural modules in the block \mathcal{O}_0 described in [Ma1, Ma2], including projective dimension of simple, standard, indecomposable tilting and indecomposable injective

modules, to the singular case. Theorem 25 makes some steps in this direction.

Using the same arguments as in the proof of Theorem 25 one shows that computation of projective dimension of indecomposable tilting modules in \mathcal{O}_λ is equivalent to computation of projective dimension in \mathcal{O}_0 of the modules $\theta_{w_0\lambda}L(w \cdot 0)$ where w is a longest coset representative in W/W_λ . This is a special case of [Ma2, Problem 24].

7. REGULAR BLOCKS OF \mathcal{O} AND \mathcal{O}^∞ ARE GUICHARDET

Our main result in this section is the following.

Theorem 26. *Let \mathfrak{g} be a semisimple complex Lie algebra.*

- (i) *For χ a regular central character, both categories \mathcal{O}_χ and \mathcal{O}_χ^∞ are Guichardet.*
- (ii) *For θ a singular central character, the categories \mathcal{O}_θ and $\mathcal{O}_\theta^\infty$ are not always Guichardet.*

The first step in proving Theorem 26 is determining initial segments in the categories \mathcal{O} and \mathcal{O}^∞ .

A *coideal* Γ_W in the Weyl group W , with respect to the Bruhat order \geq , is a subset of W such that $w' \geq w$ and $w \in \Gamma_W$ imply $w' \in \Gamma_W$. We use the same conventions for the Bruhat order as in [Hu, Section 0.4].

An *ideal* Γ in \mathfrak{h}^* is a subset such that $\lambda' \leq \lambda$ and $\lambda \in \Gamma$ imply $\lambda' \in \Gamma$. For an integral dominant $\lambda \in \mathfrak{h}^*$, we have $w \cdot \lambda \geq w' \cdot \lambda$ if and only if $w \leq w'$, so there is a one to one correspondence between coideals in W and ideals in \mathfrak{h}^* contained in the orbit of a fixed integral regular weight.

Lemma 27. (i) *Consider a regular central character χ . The initial segments in \mathcal{O}_χ are the full Serre subcategories generated by a set of modules of the form $\{L(\lambda) \mid \lambda \in \Gamma\}$, for Γ some ideal in $\{\lambda \in \mathfrak{h}^* \mid \chi_\lambda = \chi\}$.*

- (ii) *The initial segments in \mathcal{O}^∞ are the Serre subcategories generated by the initial segments in \mathcal{O} .*

Proof. We start with the principal block \mathcal{O}_0 and show that the initial segments are the full Serre subcategories generated by a set of modules of the form $\{L(w \cdot 0) \mid w \in \Gamma_W\}$, for Γ_W some coideal in W .

By [Ma1, Proposition 6] we have $\text{pd}_{\mathcal{O}}L(w) = 2l(w_0) - l(w)$. The Ext^1 -quiver of \mathcal{O}_0 is known as a consequence of the Kazhdan-Lusztig conjecture, see e.g. [AS, Section 7]. In particular, we have

- if $w' \geq w$ and $l(w') = l(w) + 1$, then $\text{Ext}_{\mathcal{O}}^1(L(w), L(w')) \neq 0$;
- if $l(w') = l(w) + 1$ and w and w' are not comparable, then $\text{Ext}_{\mathcal{O}}^1(L(w), L(w')) = 0$.

The first property implies that every initial segment in \mathcal{O}_0 corresponds to an coideal in W . The second property implies that every coideal in W corresponds to an initial segment. This proves claim (i) for \mathcal{O}_0 .

Next, we consider an indecomposable block inside \mathcal{O}_χ , for χ a non-integral regular central character. By [So2], such a block is equivalent to some regular integral indecomposable block in category \mathcal{O} (possibly for a different Lie algebra), where the equivalence preserves the highest weight structure. Since an equivalence of categories maps initial segments to initial segments, claim (i) follows.

Now we turn to \mathcal{O}^∞ for arbitrary central characters. We argue that the Ext^1 -quiver of \mathcal{O}^∞ is the same one as for \mathcal{O} , up to loops (that is self-extensions of simple modules). Imagine there is a module $M \notin \mathcal{O}$ satisfying

$$L(\lambda) \hookrightarrow M \twoheadrightarrow L(\lambda')$$

for $\lambda, \lambda' \in \mathfrak{h}^*$. This module is clearly in \mathcal{O}^2 . If $\lambda' \not\leq \lambda$, then M is a quotient of $M(\lambda')$ by the universal property of Verma modules. If $\lambda' < \lambda$ we can use the duality \star on \mathcal{O}^2 , which preserves \mathcal{O} , to return to the previous situation. Therefore $\lambda = \lambda'$.

Going from \mathcal{O} to \mathcal{O}^∞ , we therefore have that the extension quivers coincide up to self-extensions and the projective dimensions of simple modules coincide up to a shift by $\dim \mathfrak{h}$, see Theorem 20. Therefore claim (ii) follows. \square

Lemma 27 allows us to apply a result on stratified algebras by Cline, Parshall and Scott in [CPS2], or the special case of quasi-hereditary algebras in [CPS1].

Proof of Theorem 26. Lemma 27(i) implies that, in every regular block, the initial segments correspond to ideals in the poset of weights. The property for \mathcal{O}_χ with χ regular therefore follows from [CPS1, Theorem 3.9(i)].

Indecomposable blocks of category \mathcal{O}^I are stratified (in the sense of [CPS2, Definition 2.2.1]) with respect to the order \leq on the weights, see [So3, Lemma 8] and [So3, Theorem 7] or [KKM, Corollary 9(a)]. Take Γ an ideal in the poset (of a regular block) and let \mathcal{O}_Γ^I be the Serre subcategory of \mathcal{O}^I generated by $\{L(\lambda) \mid \lambda \in \Gamma\}$. Then

$$\text{Ext}_{\mathcal{O}_\Gamma^I}^j(M, N) \rightarrow \text{Ext}_{\mathcal{O}^I}^j(M, N)$$

is an isomorphism for any $M, N \in \mathcal{O}_\Gamma^I$, see e.g. [CPS2, Equation 2.1.2.1]. Corollary 7 then implies that

$$\mathrm{Ext}_{\mathcal{O}_\Gamma^\infty}^d(M, N) \rightarrow \mathrm{Ext}_{\mathcal{O}^\infty}^d(M, N)$$

is an isomorphism for any $M, N \in \mathcal{O}_\Gamma^\infty$ and $d \geq 0$, so \mathcal{O}_χ^∞ is Guichardet by Lemma 27(ii), which proves claim (i).

For claim (ii), we consider the singular example for $\mathfrak{sl}(3)$ described in Subsection 6.2. In this example, the Serre subcategory generated by the simple module L_3 is an initial segment. This follows from the calculation of the projective dimensions and

$$\mathrm{Ext}^1(L_1, L_3) = \mathrm{Ext}^1(L_3, L_1) = 0,$$

because the vertices 1 and 3 in the quiver (13) are not connected. Since $\mathrm{Ext}^1(L_3, L_3) = 0$, this initial segment is semi-simple. However, it is not extension full by (14). \square

Remark 28. After Theorem 26(ii) it is natural to relax weak Alexandru conjecture for \mathcal{O} as follows: Let \mathcal{C} be a singular block of \mathcal{O} . For $d \in \mathbb{N}$ let \mathcal{C}_d denote the Serre subcategory of \mathcal{C} generated by all simple modules of projective dimension at most d . Is \mathcal{C}_d extension full in \mathcal{C} ?

8. HARISH-CHANDRA BIMODULES

Let χ be a regular central character and θ be a central character in the same weight lattice as χ . The equivalences of categories in [BG, Theorem 5.9] and [So1, Theorem 1] imply that for $k \geq 1$ the category \mathcal{O}_θ^k is equivalent to both ${}^k_\chi \mathcal{H}_\theta^\infty$ and ${}^\infty_\theta \mathcal{H}_\chi^k$ and that $\mathcal{O}_\theta^\infty$ is equivalent to both ${}^\infty_\theta \mathcal{H}_\chi^\infty$ and ${}^\infty_\chi \mathcal{H}_\theta^\infty$. As a consequence, the claims of the following result follow from Corollary 16, Theorem 20 and Theorem 26, respectively.

Theorem 29. *Let χ be a regular central character and θ be a central character in the same weight lattice as χ .*

(i) *The global dimension of the category ${}^\infty_\chi \mathcal{H}_\theta^\infty$ is finite. If χ is integral, then the global dimension of ${}^\infty_\chi \mathcal{H}_\chi^\infty$ is $\dim \mathfrak{g}$.*

(ii) *Consider $M \in {}^k_\chi \mathcal{H}_\theta^\infty$ with $\mathrm{pd}_{{}^k_\chi \mathcal{H}_\theta^\infty} M < \infty$, then*

$$\mathrm{pd}_{{}^\infty_\chi \mathcal{H}_\theta^\infty} M = \dim \mathfrak{h} + \mathrm{pd}_{{}^k_\chi \mathcal{H}_\theta^\infty} M.$$

(iii) *If θ is also regular, the categories ${}^\infty_\theta \mathcal{H}_\chi^\infty$, ${}^\infty_\chi \mathcal{H}_\theta^\infty$, ${}^1_\chi \mathcal{H}_\theta^\infty$ and ${}^\infty_\theta \mathcal{H}_\chi^1$ are Guichardet.*

We conclude with the result that, in spite of Theorem 15, the category ${}^\infty_\chi \mathcal{H}_\chi^\infty$ is not extension full in the category of bimodules.

Proposition 30. *The category ${}_{\chi}^{\infty}\mathcal{H}_{\chi}^{\infty}$ for χ a regular integral central character is not extension full in the category of \mathfrak{g} -bimodules.*

Proof. Without loss of generality we may assume that χ is the central character of the trivial \mathfrak{g} -module \mathbb{C} . As noted in Theorem 29, the global dimension of the category ${}_{\chi}^{\infty}\mathcal{H}_{\chi}^{\infty}$ is $\dim \mathfrak{g}$. The trivial \mathfrak{g} -bimodule \mathbb{C} is an object of ${}_{\chi}^{\infty}\mathcal{H}_{\chi}^{\infty}$. Identifying \mathfrak{g} -bimodules with $\mathfrak{g} \oplus \mathfrak{g}$ -modules, we find that

$$\mathrm{Ext}_{\mathfrak{g}\text{-mod-}\mathfrak{g}}^{2\dim \mathfrak{g}}(\mathbb{C}, \mathbb{C}) \cong H^{2\dim \mathfrak{g}}(\mathfrak{g} \oplus \mathfrak{g}, \mathbb{C}) \neq 0,$$

by Lemma 2. This implies that

$$0 = \mathrm{Ext}_{{}_{\chi}^{\infty}\mathcal{H}_{\chi}^{\infty}}^{2\dim \mathfrak{g}}(\mathbb{C}, \mathbb{C}) \rightarrow \mathrm{Ext}_{\mathfrak{g}\text{-mod-}\mathfrak{g}}^{2\dim \mathfrak{g}}(\mathbb{C}, \mathbb{C})$$

can not be an isomorphism. □

Acknowledgment. KC is a Postdoctoral Fellow of the Research Foundation - Flanders (FWO). VM is partially supported by the Swedish Research Council. We thank Vera Serganova and Catharina Stroppel for useful discussions.

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