On the relation between finitistic and good filtration dimensions

Volodymyr Mazorchuk and Alison E. Parker

Dedicated to the memory of Sheila Brenner

Abstract

In this paper we discuss generalizations of the concepts of good filtration dimension and Weyl filtration dimension, introduced by Friedlander and Parshall for algebraic groups to properly stratified algebras. We introduce the notion of the *finitistic Δ-filtration dimension* for such algebras and show that the finitistic dimension for such an algebra is bounded by the sum of the finitistic Δ-filtration dimension and the $\overline{\nabla}$-filtration dimension. In particular the finitistic dimension must be finite. We also conjecture that this bound is exact when the algebra has a simple preserving duality. We give several examples of well-known algebras where this is the case, including many of the Schur algebras, and blocks of category $\mathcal{O}$. We also give an explicit combinatorial formula for the global dimension in this case.

1 Introduction

The study of homological invariants like the global or finitistic dimension of an algebra is very important for understanding the structure of its module category. In this paper, we discuss generalizations of the concepts of good filtration dimension and Weyl filtration dimension, introduced by Friedlander and Parshall, [FP], to properly stratified algebras. The structure of a properly stratified algebra requires the existence of four different families of modules, called standard, proper standard, costandard and proper costandard modules, and we study the notion of filtration dimensions with respect to all these families. The notions of filtration dimensions with respect to proper standard and proper costandard modules were recently introduced and studied by Zhu and Caenepeel in [CZ]. In [CZ] these notions are related to the projective dimension of the characteristic tilting modules, however, no relation to, for example, the finitistic dimension was given.

In the present paper, we relate the filtration dimensions with respect to proper standard and proper costandard modules with the finitistic dimension of the properly stratified algebra, obtaining some upper bound for the last one. These bounds are usually much better than the classical bounds, obtained in [AHLU1]. Moreover, we conjecture that the bound we obtain is exact when the algebra has a simple preserving duality. It also happens that the finitistic dimension of a properly stratified algebra is related to the finitistic filtration dimensions with respect to the standard and costandard modules.

We give several examples of well-known quasi-hereditary algebras where our bound is optimal, including many of the Schur algebras, and all blocks of the BGG category $\mathcal{O}$. The last example is considered in detail and in many cases we give an explicit combinatorial formula for the global dimension. Unfortunately, we did not manage to find an elegant argument for the category $\mathcal{O}$ and were forced to use the (very non-trivial) Koszul duality theorem of Beilinson-Ginzburg-Soergel, [BGS], and Backelin, [Ba], and the recent description of Koszul quasi-hereditary algebras due to Ágoston-Dlab-Lukács, [ADL].

As one of the corollaries of our results we also get upper bounds for the finitistic dimension of the parabolic generalizations of $\mathcal{O}$, considered in [FKM]. It is known that these categories appear as categories of Harish-Chandra bimodules, [KM].

2 Properly stratified algebras

Throughout the paper A will denote a basic finite-dimensional associative algebra over an algebraically closed field, k. By a module we will always mean left module. For a primitive idempotent, $e = e_i$, we will denote by $L(i)$ the corresponding simple module, by $P(i)$ the projective cover of $L(i)$ and by $I(i)$ the injective envelope of $L(i)$. We say that A is *properly stratified* if the following properties hold:

1. There is a given linear order, $\leq$, on a complete set, $\{e_1, e_2, \dots, e_n\}$ of primitive orthogonal idempotents. We will always assume that this order is given by the natural ordering of the indices.

2. There is a family, $\{\Delta(i)\}$, $i = 1, 2, \dots, n$, of A-modules, such that the module $P(j)$ surjects on $\Delta(j)$ for every j, and the kernel of this map is filtered by $\Delta(i)$, $i > j$.

3. There is a family, $\{\overline{\Delta}(i)\}$, $i = 1, 2, \dots, n$, of A-modules, such that the module $\overline{\Delta}(j)$ surjects on $L(j)$ for every j, with the kernel of this map being filtered by $L(i)$, $i < j$; and such that the module $\Delta(j)$ is filtered by $\overline{\Delta}(j)$ for all j.

The modules $\Delta(i)$ are called *standard modules* and the modules $\overline{\Delta}(j)$ are called *proper standard modules*. In a dual way we also define *costandard modules* $\nabla(i)$ and *proper costandard modules* $\overline{\nabla}(i)$. We denote by $\mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\overline{\Delta})$, resp. $\mathcal{F}(\nabla)$, resp. $\mathcal{F}(\overline{\nabla})$) the full subcategory in the category of all A-modules, consisting of all modules, filtered by $\Delta(i)$ (resp. $\overline{\Delta}(j)$, resp. $\nabla(i)$, resp. $\overline{\nabla}(i)$).

From now on, throughout the paper all algebras will be assumed to be properly stratified.

An A-module $M \in \mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\nabla})$ is called *tilting* and an A-module $M \in \mathcal{F}(\overline{\Delta}) \cap \mathcal{F}(\nabla)$ is called *cotilting*. By [AHLU2] (see also [Ri] for the setting of quasi-hereditary algebras) every tilting (resp. cotilting) module is a direct sum of indecomposable tilting (resp. cotilting) modules, the later being in a natural bijection with simple modules. For $i = 1, \dots, n$ we denote by $T(i)$ (resp. $S(i)$) the indecomposable tilting (resp. cotilting) modules, whose every Δ (resp. ∇) filtration starts (resp. ends) with $\Delta(i)$ (resp. $\nabla(i)$). The modules $T = \oplus_{i=1}^{n} T(i)$ and $S = \oplus_{i=1}^{n} S(i)$ are called the *characteristic tilting* and the *characteristic cotilting* modules respectively.

A properly stratified algebra, A, as above is quasi-hereditary if and only if it has a finite global dimension, and it has finite global dimension if and only if $\Delta(i) \simeq \overline{\Delta}(i)$ for all j.

We say M has a Δ-resolution (resp. $\overline{\Delta}$-resolution) of (possibly infinite) length l if we have a resolution

$$0 \to X_l \to X_{l-1} \to \cdots \to X_1 \to X_0 \to M \to 0$$

with all $X_i \in \mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\overline{\Delta})$). Dually we say M has a ∇-resolution (resp. $\overline{\nabla}$-resolution) of (possibly infinite) length l if we have a resolution

$$0 \to M \to X_0 \to X_1 \to \cdots \to X_{l-1} \to X_l \to 0$$

with all $X_i \in \mathcal{F}(\nabla)$ (resp. $\mathcal{F}(\overline{\nabla})$).

Since the projectives $P(i) \in \mathcal{F}(\Delta) \subseteq \mathcal{F}(\overline{\Delta})$ and the injectives are in $\mathcal{F}(\nabla) \subseteq \mathcal{F}(\overline{\nabla})$, these resolutions exist for all A-modules.

We define the Δ-filtration dimension of M, abbreviated $\Delta.\mathrm{f.d.}(M)$, to be the minimal length of a finite Δ-resolution of M, should one exist and $\Delta.\mathrm{f.d.}(M) = \infty$ otherwise. We similarly define $\overline{\Delta}.\mathrm{f.d.}(M)$, $\nabla.\mathrm{f.d.}(M)$ and $\overline{\nabla}.\mathrm{f.d.}(M)$.

For the properly stratified algebra A we also define

$$\overline{\nabla}.\mathrm{f.d.}(A) = \sup\{\overline{\nabla}.\mathrm{f.d.}(M) \mid M \in A-\mathrm{Mod}\},$$
$$\overline{\Delta}.\mathrm{f.d.}(A) = \sup\{\overline{\Delta}.\mathrm{f.d.}(M) \mid M \in A-\mathrm{Mod}\}.$$

As we will see in Corollary 1 both $\overline{\nabla}.\mathrm{f.d.}(A)$ and $\overline{\Delta}.\mathrm{f.d.}(A)$ are finite for any properly stratified algebra A.

If a properly stratified algebra A is not quasi-hereditary then there are A-modules, for which the Δ-filtration dimension is infinite and there are A-modules, for which the ∇-filtration dimension is also infinite. Indeed, it is fairly obvious that the projective dimension of any A-module of finite Δ-filtration dimension is finite since the projective dimension of every $\Delta(i)$ is finite. Dually, the injective dimension of any A-module of finite ∇-filtration dimension is finite since the injective dimension of every $\nabla(i)$ is finite. Now the desired conclusion follows from the remark above that A is not quasi-hereditary if and only if the global dimension of A is infinite. Taking this into account it makes sense to define for the properly stratified algebra A

$$\mathrm{fin}.\nabla.\mathrm{f.d.}(A) = \sup\{\nabla.\mathrm{f.d.}(M) \mid M \in A-\mathrm{Mod} \text{ with } \nabla.\mathrm{f.d.}(M) < \infty\},$$
$$\mathrm{fin}.\Delta.\mathrm{f.d.}(A) = \sup\{\Delta.\mathrm{f.d.}(M) \mid M \in A-\mathrm{Mod} \text{ with } \Delta.\mathrm{f.d.}(M) < \infty\}.$$

Since $\Delta(i) \in \mathcal{F}(\overline{\Delta})$ and $\nabla(i) \in \mathcal{F}(\overline{\nabla})$ for all i it follows immediately that $\overline{\Delta}.\mathrm{f.d.}(M) \leq \Delta.\mathrm{f.d.}(M)$ and $\overline{\nabla}.\mathrm{f.d.}(M) \leq \nabla.\mathrm{f.d.}(M)$ for all A-modules M. Since all projective modules belong to $\mathcal{F}(\Delta)$ and all injective modules belong to $\mathcal{F}(\nabla)$, we also have $\Delta.\mathrm{f.d.}(M) \leq \mathrm{p.d.}(M)$ and $\nabla.\mathrm{f.d.}(M) \leq \mathrm{i.d.}(M)$ for all A-modules M.

Remark 1. *Note that* $\overline{\nabla}.\,\mathrm{f.\,d.}(A)$ *is defined for* A *as an algebra not for* $A = {}_AA$ *considered as a left* A*-module. In fact, since all projective* A *modules belong, by definition, to* $\mathcal{F}(\Delta) \subset \mathcal{F}(\overline{\Delta})$*, we always have* $\Delta.\,\mathrm{f.\,d.}({}_AA) = 0 = \overline{\Delta}.\,\mathrm{f.\,d.}({}_AA)$*. However, later on we will see that* $\overline{\nabla}.\,\mathrm{f.\,d.}(A) = \overline{\nabla}.\,\mathrm{f.\,d.}({}_AA)$*.*

3 Main results

We start with the following lemma.

Lemma 1.

$$\Delta.\,\mathrm{f.\,d.}(M) = \sup\{d \mid \mathrm{Ext}^d(M, \overline{\nabla}(i)) \neq 0 \textit{ for some } i\}$$
$$\overline{\Delta}.\,\mathrm{f.\,d.}(M) = \sup\{d \mid \mathrm{Ext}^d(M, \nabla(i)) \neq 0 \textit{ for some } i\}$$
$$\nabla.\,\mathrm{f.\,d.}(M) = \sup\{d \mid \mathrm{Ext}^d(\overline{\Delta}(i), M) \neq 0 \textit{ for some } i\}$$
$$\overline{\nabla}.\,\mathrm{f.\,d.}(M) = \sup\{d \mid \mathrm{Ext}^d(\Delta(i), M) \neq 0 \textit{ for some } i\}$$

Proof. The proof of these statements follows exactly as in the quasi-hereditary algebra case [FP, Proposition 3.4] using dimension shifting once we have that these statements are true in degree zero. That is we have

$$M \in \mathcal{F}(\Delta) \text{ if and only if } \mathrm{Ext}^i(M, \overline{\nabla}(j)) = 0 \text{ for all } i > 0 \text{ and all } j \in \Gamma$$

and similarly for the other cases. But these properties are true for properly stratified algebras using, for example, [Dl, Theorem 5] (see also [DR1] for the setting of quasi-hereditary algebras). □

Corollary 1. *1. We have,*

$$\overline{\Delta}.\,\mathrm{f.\,d.}(A) = \sup\{\mathrm{i.\,d.}(\nabla(j)) \mid j \in \Gamma\} = \mathrm{i.\,d.}\,S < \infty$$
$$\overline{\nabla}.\,\mathrm{f.\,d.}(A) = \sup\{\mathrm{p.\,d.}(\Delta(j)) \mid j \in \Gamma\} = \mathrm{p.\,d.}\,T < \infty$$

where S *and* T *are the respective characteristic cotilting and tilting modules for* A *(see [AHLU2]).*

2. We have that $\overline{\nabla}.\,\mathrm{f.\,d.}(A) = \overline{\nabla}.\,\mathrm{f.\,d.}({}_AA)$*.*

Proof. It is clear using the previous lemma that $\overline{\Delta}.\,\mathrm{f.\,d.}(A) = \sup\{\mathrm{i.\,d.}(\nabla(j)) \mid j \in \Gamma\}$. This is also equal to $\mathrm{i.\,d.}(S)$ where S is the characteristic cotilting module for A using the dual version of [CZ, Proposition 2.2]. Similar remarks hold for $\overline{\nabla}.\,\mathrm{f.\,d.}(A)$. Now by [AHLU1, Proposition 1.8] and its dual version, all $\Delta(j)$ and $\nabla(j)$ have finite projective and injective dimensions respectively, hence the first statement.

The second statement now follows from the first one and the remark that, because of the fact that $\mathrm{Hom}(\Delta(i), \_)$ is covariant, the long exact sequence associated with the sequence $0 \to K(i) \to P(i) \to L(i) \to 0$ implies that the (finite) supremum

$$\sup\{d \mid \mathrm{Ext}^d(\Delta(i), M) \neq 0 \text{ for some } i\}$$

is achieved on some projective A-module. □

Remark 2. *In the case when* $\mathrm{i.d.}({}_AA) < \infty$ *one also gets that* $\mathrm{fin.}\nabla.\mathrm{f.d.}(A) = \nabla.\mathrm{f.d.}({}_AA)$ *by the same arguments as used in the proof of the second statement of Corollary 1.*

Lemma 2. *Let A be a properly stratified algebra, then we have the following inequalities*

$$\mathrm{fin.}\Delta.\mathrm{f.d.}(A) \le \overline{\Delta}.\mathrm{f.d.}(A)$$

$$\mathrm{fin.}\nabla.\mathrm{f.d.}(A) \le \overline{\nabla}.\mathrm{f.d.}(A).$$

Proof. We prove the first statement, the second is similar.

Let M be an A-module with finite Δ-filtration dimension. Then we have a finite Δ-resolution for M. Since each $\Delta(\lambda)$ is filtered by $\overline{\Delta}(\mu)$'s, such a resolution is also a $\overline{\Delta}$-resolution for M. Hence $\overline{\Delta}.\mathrm{f.d.}(M) \le \Delta.\mathrm{f.d.}(M) < \infty$.

Now suppose M has finite Δ-filtration dimension d. So there exists i such that $\mathrm{Ext}^d(M, \overline{\nabla}(i))$ is non-zero. We claim that $\mathrm{Ext}^d(M, \nabla(i))$ is also non-zero. Since $d = \Delta.\mathrm{f.d.}(M) \ge \overline{\Delta}.\mathrm{f.d.}(M)$ we have that $\mathrm{Ext}^j(M, \overline{\nabla}(i))$ and $\mathrm{Ext}^j(M, \nabla(i))$ are zero for $j > d$. Since A is properly stratified we have the following short exact sequence for $\nabla(i)$

$$0 \to K \to \nabla(i) \to \overline{\nabla}(i) \to 0 \tag{1}$$

where K (possibly zero) is filtered by $\overline{\nabla}(i)$'s. We now apply $\mathrm{Ext}^*(M, -)$ to the short exact sequence (1) to get

$$\cdots \to \mathrm{Ext}^d(M, K) \to \mathrm{Ext}^d(M, \nabla(i)) \to \mathrm{Ext}^d(M, \overline{\nabla}(i)) \to 0$$

where the last zero follows by the argument above. But by assumption on i, $\mathrm{Ext}^d(M, \overline{\nabla}(i))$ is non-zero. Hence $\mathrm{Ext}^d(M, \nabla(i))$ is also non-zero. Thus $\overline{\Delta}.\mathrm{f.d.}(M) = d = \Delta.\mathrm{f.d.}(M)$ if $\Delta.\mathrm{f.d.}(M)$ is finite. Hence we have $\mathrm{fin.}\Delta.\mathrm{f.d.}(A) \le \overline{\Delta}.\mathrm{f.d.}(A)$. □

We denote the projectively defined finitistic dimension of A by $\mathrm{fin.dim}(A)$. We here note that the left-right symmetry of the concepts in the following theorem means that it applies equally well to the injectively defined finitistic dimension.

Theorem 1. *Let A be a properly stratified algebra, then we have the following*

$$\mathrm{fin.dim}(A) \le \mathrm{fin.}\Delta.\mathrm{f.d.}(A) + \overline{\nabla}.\mathrm{f.d.}(A) \le$$
$$\le \overline{\nabla}.\mathrm{f.d.}(A) + \overline{\Delta}.\mathrm{f.d.}(A) = \mathrm{p.d.}(T) + \mathrm{i.d.}(S).$$

Proof. Let $M \in A-\mathrm{Mod}$ with finite projective dimension. Then M has finite Δ-filtration dimension and so has a finite Δ-resolution of length at most $\mathrm{fin.}\Delta.\mathrm{f.d.}(A)$. The end term of this resolution has a Δ-filtration and hence has finite projective dimension bounded by $\overline{\nabla}.\mathrm{f.d.}(A)$. Hence the projective dimension of M is bounded by $\mathrm{fin.}\Delta.\mathrm{f.d.}(A) + \overline{\nabla}.\mathrm{f.d.}(A)$. The other inequalities follow using Corollary 1 and Lemma 2. □

Remark 3. *There are "classical" bounds for the global dimension of a quasi-hereditary algebras and the finitistic dimension of a stratified algebras, obtained in [DR2] and [AHLU1] respectively. These bounds are given as $2m-2$, where m denotes the number of pairwise-non-isomorphic simple modules. The number m can be even substituted by one plus the height of the minimal possible partial order on simple modules, which allows the prescribed stratified structure. However, many examples are known (see [ADL, AHLU1, AHLU2, DR3]), when one can show that the classical bound is not exact. One good example is the following algebra $\hat{A}_n$, which is the path algebra of the quiver*

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, bend left, "a_1"] & \overset{2}{\bullet} \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & \overset{3}{\bullet} \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] & \cdots \arrow[l, bend left, "b_3"] & \cdots \arrow[r, bend left, "a_{n-2}"] & \overset{n-1}{\bullet} \arrow[l, bend left, "b_{n-2}"] \arrow[r, bend left, "a_{n-1}"] & \overset{n}{\bullet} \arrow[l, bend left, "b_{n-1}"] \end{tikzcd}$$

modulo the relations $a_ib_i = 0$ for all possible i.

This algebra is quasi-hereditary and the number of simple modules, n, for this algebra coincides with one plus the height of the minimal possible partial order on simple modules, which allows the prescribed quasi-hereditary structure. However, it is easy to calculate that the global dimension of this algebra is in fact 2. At the same time it is also quite easy to calculate that for this algebra we have $T = S$ and $\mathrm{p.d.}(T) = \mathrm{i.d.}(T) = 1$ *and therefore* $\mathrm{fin.dim}(\hat{A}_n) = 2 = 2 \cdot \mathrm{p.d.}(T) = \mathrm{p.d.}(T) + \mathrm{i.d.}(T)$. *This shows that the bound obtained in Theorem 1 can be more effective than the "classical" one.*

Actually, it is easy to see that $\mathrm{p.d.}(T) = \sup\{\mathrm{p.d.}(\Delta(i)) \mid 1 \leq i \leq n\} \leq n-1$ *for any properly stratified algebra with n simples, arguing exactly in the same way as the quasi-hereditary case. (See [DR1].) We dually have* $\mathrm{i.d.}(S) \leq n-1$ *and thus* $\mathrm{p.d.}(T) + \mathrm{p.d.}(S) \leq 2n-2$.

Suppose A has a simple preserving duality, that is an exact involutive and contravariant equivalence preserving simple modules. Thus it swaps $\Delta(i)$ with $\nabla(i)$ and $\overline{\Delta}(i)$ with $\overline{\nabla}(i)$. Then $\mathrm{fin.}\Delta\mathrm{.f.d.}(A) = \mathrm{fin.}\nabla\mathrm{.f.d.}(A) \leq \overline{\nabla}\mathrm{.f.d.}(A) = \overline{\Delta}\mathrm{.f.d.}(A) = \mathrm{p.d.}(T) = \mathrm{i.d.}(S)$. Theorem 1 then gives us that $\mathrm{fin.dim}(A) \leq 2\,\mathrm{p.d.}(T)$. As we have a large class of examples for which equality holds we formulate the following conjecture.

Conjecture 1. *Suppose A is properly stratified and has a simple preserving duality. Then*

$$\mathrm{fin.dim}(A) = 2\,\mathrm{p.d.}(T).$$

In the case where A is quasi-hereditary this reduces to showing that the global dimension of A is twice its ∇-filtration dimension. We can at least show this when $\nabla\mathrm{.f.d.}(A) = 1$.

Lemma 3. *Let A be a quasi-hereditary algebra with simple preserving duality denoted $^\circ$ and with $\nabla\mathrm{.f.d.}(A) = 1$. Then the global dimension of A is two.*

Proof. Since $\nabla\mathrm{.f.d.}(A) = 1$ there is a simple $L(i)$ for A with $\nabla\mathrm{.f.d.}(L(i)) = 1$. The result now follows using [Pa1, Lemma 2.6] which says that $\mathrm{Ext}^2(L(i), L(i)) \cong \mathrm{Hom}(Q^\circ, Q) \neq 0$ where Q is the quotient $\nabla(i)/L(i)$ and Q° is its dual. $\square$

Conjecture 1 is also true for quasi-hereditary algebras which satisfy a particular property. Namely that there is a partial order $\leq$ on the set of equivalence classes of primitive idempotents such that the algebra is quasi-hereditary with respect to (arbitrary linear extensions of) this order and for which $\mu < \lambda$ implies $\nabla.\mathrm{f.d.}(L(\mu)) < \nabla.\mathrm{f.d.}(L(\lambda))$. This property holds for generalized Schur algebras whose weights are regular ([Pa2]) and the regular block of category $\mathcal{O}$ (follows from [Ca] or the last remark in [BGG]). Although this property is far from being true in general. A proof of the conjecture for algebras satisfying this property will appear in [EP].

In [CZ, Section 3] it is shown that for the quasi-hereditary algebras having an exact Borel subalgebra (in the sense of König, [Ko]) the characteristic tilting module over the Borel subalgebra induces up to a characteristic tilting module over the original algebra, moreover, the projective dimensions of these two modules coincide. The arguments from [CZ, Section 3] easily extend to properly stratified algebras with an exact Borel subalgebra in the sense of [KlM], where we refer the reader for all definitions and examples. Until the end of this section we assume that A is a properly stratified algebra and B is an exact Borel subalgebra of A. Both A and B are properly stratified. Hence, to distinguish the corresponding structural modules, we will add the subscript A or B to them, respectively.

Lemma 4. $\mathrm{fin.dim}(B) = \mathrm{p.d.}(T_B)$.

Proof. Since B is pyramidal (see [KlM]), the characteristic cotilting B-module is injective. This, together with Theorem 1, implies $\mathrm{p.d.}(T_B) \leq \mathrm{fin.dim}(B) \leq \mathrm{p.d.}(T_B) + \mathrm{i.d.}(S_B) = \mathrm{p.d.}(T_B)$ and hence the statement. □

Lemma 5. *Let B be an exact Borel subalgebra of a properly stratified algebra, A. Then for any B-module M having a standard filtration (as a B-module) the A-module $A \otimes_B M$ has a standard filtration (as an A-module), moreover,* $\mathrm{p.d.}(A \otimes_B M) \leq \mathrm{p.d.}(M)$.

Proof. The idea is the same as in [CZ, Lemma 3.2]. Since B is an exact Borel subalgebra of A, the functor $A \otimes_B \_$ is exact and sends projective B-modules to projective A-modules (see [KlM, Section 4]). Hence it sends the minimal projective resolution of M to some projective resolution of $A \otimes_B M$ implying $\mathrm{p.d.}(A \otimes_B M) \leq \mathrm{p.d.}(M)$.

That $A \otimes_B M$ has a standard filtration follows from the exactness of $A \otimes_B \_$ and the fact that this functor sends standard B-modules to standard A-modules. □

Corollary 2. *Let A be a properly stratified algebra with a duality $^\circ$ and B be an exact Borel subalgebra of A. Denote by T_A and T_B characteristic tilting modules for A and B respectively. Then*

$$\mathrm{fin.dim}(A) \leq 2\,\mathrm{p.d.}(T_A) \leq 2\,\mathrm{p.d.}(T_B) = 2\,\mathrm{fin.dim}(B).$$

Proof. As A has a duality, we have $S_A \simeq T_A^\circ$ and hence $\mathrm{p.d.}(T_A) = \mathrm{i.d.}(T_A^\circ) = \mathrm{i.d.}(S_A)$. Theorem 1 now implies $\mathrm{fin.dim}(A) \leq 2\,\mathrm{p.d.}(T_A)$.

Further, we have $\mathrm{p.d.}(T_A) = \sup\{\mathrm{p.d.}(\Delta_A(i)) : i = 1, \dots, n\}$. That B is an exact Borel subalgebra of A guarantees by definition that $A \otimes_B \Delta_B(i) \simeq \Delta_A(i)$ for all i. Now Lemma 5 implies $\sup\{\mathrm{p.d.}(\Delta_A(i)) : i = 1, \ldots, n\} \leq \sup\{\mathrm{p.d.}(\Delta_B(i)) : i = 1, \ldots, n\}$. Again we use $\sup\{\mathrm{p.d.}(\Delta_B(i)) : i = 1, \ldots, n\} = \mathrm{p.d.}(T_B)$ and finally get $\mathrm{p.d.}(T_B) = \mathrm{fin.dim}(B)$ from Lemma 4. This completes the proof. $\square$

4 Examples

4.1 Schur algebras

Our first example is the Schur Algebra. Let k be an algebraically closed field of characteristic p and $G = GL_n(k)$, the general linear group over k. Let E be the natural n-dimensional representation of G and $E^{\otimes r}$ its r-fold tensor product. Let G act diagonally by $g(e_1 \otimes \cdots \otimes e_r) = g \cdot e_1 \otimes \cdots \otimes g \cdot e_r$. This representation defines a ring homomorphism $\psi : kG \to \mathrm{End}_k(E)$ and the span of the image of ψ is denoted by $S(n, r) = S_k(n, r)$ and is called the *Schur algebra*, [Gr].

We know that the global dimension of $S(n, r)$ is twice its Δ-filtration dimension in various cases.

(i) $r \leq n$ [Do, Section 4.8]

(ii) $n = 2$ or 3 [Pa1, Theorems 3.7 and 5.12]

(iii) $p > n$ or if $p = n$ and r is a multiple of p [Pa2, Theorems 5.8 and 5.9]

4.2 Category $\mathcal{O}$

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, be a fixed triangular decomposition of $\mathfrak{g}$, and W be the Weyl group of $\mathfrak{g}$. The BGG category $\mathcal{O}$, [BGG], is defined as the full subcategory in the category of all $\mathfrak{g}$-modules, which consists of all finitely generated, $\mathfrak{h}$-diagonalizable and $U(\mathfrak{n}_+)$-locally finite modules. With respect to the action of the center $Z(\mathfrak{g})$ of the universal enveloping algebra, the category $\mathcal{O}$ decomposes into a direct sum of blocks $\mathcal{O}^\theta$, $\theta \in Z(\mathfrak{g})^*$, defined as follows:

$$\mathcal{O}^\theta = \left\{ M \in \mathcal{O} | \text{ there exists } k \in \mathbb{N} \text{ such that } (z - \theta(z))^k M = 0 \text{ for all } z \in Z(\mathfrak{g}) \right\}.$$

The characters $\theta \in Z(\mathfrak{g})$ are in a natural bijection with dominant weights $\lambda \in \mathfrak{h}^*$, [Di, Section 7.4]. For a dominant $\lambda \in \mathfrak{h}^*$ the simple modules in the corresponding block $\mathcal{O}^{\theta_\lambda}$ are in a natural bijection with the right cosets of W modulo the stabilizer W^λ of λ in W under the *dot-action* $w \cdot \lambda = w(\lambda + \rho) - \rho$, where ρ is half the sum of all positive roots.

The principal objects of $\mathcal{O}$ are the *Verma modules* $M(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\mu$, where the $\mathfrak{h} \oplus \mathfrak{n}_+$-module structure on $\mathbb{C}_\mu = \mathbb{C}$ is given by $(h + n)c = \mu(h)c$, $h \in \mathfrak{h}$, $n \in \mathfrak{n}$ and $c \in \mathbb{C}$. In the language of quasi-hereditary algebras Verma modules are called *standard* and usually denoted by $\Delta(\mu)$ ([Ri]). The module $M(\mu)$ has a simple top, denoted by $L(\mu)$ and such modules constitute an exhaustive list of simples in $\mathcal{O}$.

The Chevalley anti-involution on $\mathfrak{g}$ naturally extends to a duality, $^\circ$, on $\mathcal{O}$, which is a contravariant, involutive equivalence, preserving simple modules, see e.g. [Ir3]. We set $\nabla(\mu) = M(\mu)^\circ$ and denote by $T(\mu)$ the indecomposable tilting modules, whose any Verma flag starts with $M(\mu)$. We also denote by $T(\theta)$ the characteristic tilting module for $\mathcal{O}^\theta$.

Studying blocks $\mathcal{O}^\theta$ one can always assume that θ is integral, according to the combinatorial description of $\mathcal{O}^\theta$ provided by [So]. For a fixed integral θ let λ_θ denote the dominant highest weight corresponding to θ, which is given by [Di, Section 7.4]. For a dominant integral weight μ we denote by W_μ the stabilizer of μ in W with respect to the dot-action. Let $\mathfrak{a}(\mu)$ be the regular subalgebra of $\mathfrak{g}$, which corresponds to W_μ, i.e. which is generated by root elements representing roots occurring in reflections in W_μ. Using the notation $\lambda = \lambda_\theta$ we denote by $\mathcal{O}(\lambda, \mu)$ the full subcategory of $\mathcal{O}^\theta$, consisting of all modules M, which are locally finite as $\mathfrak{a}(\mu)$-modules. If S denotes the set of simple roots for W_μ, then the category $\mathcal{O}(\lambda, \mu)$ is the block $\mathcal{O}_S^\theta$ of the parabolic category $\mathcal{O}_S$ of Rocha-Caridi, [RC].

Let $\mathfrak{P} = \mathfrak{a}(\mu) + \mathfrak{h} + \mathfrak{n}_+$ be the parabolic subalgebra of $\mathfrak{g}$, $\mathfrak{n}(\mu)$ be its nilpotent radical, and $\mathfrak{h}^\mu \subset \mathfrak{h}$ be the center of the Levi factor of $\mathfrak{P}$. For a simple finite-dimensional module V and $\nu \in (\mathfrak{h}^\mu)^*$ we define a $\mathfrak{P}$-module structure on V via $(a + h + n)v = av + \nu(h)v$, $a \in \mathfrak{a}(\mu)$, $h \in \mathfrak{h}^\mu$, $n \in \mathfrak{n}(\mu)$ and $v \in V$. The module $M(V, \nu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{P})} V$ is called the *generalized Verma module*, associated with V, $\mathfrak{P}$ and ν. Clearly $M(V, \nu) \in \mathcal{O}(\lambda, \mu)$. In the quasi-hereditary structure the modules $M(V, \nu)$ are the standard modules.

Theorem 2. *The following numbers are equal.*

- *(i) The global dimension of the block $\mathcal{O}(\lambda, \mu)$.*
- *(ii) The Loewy length of the direct sum of all projective-injective modules in the category $\mathcal{O}(-w_0(\mu), \lambda)$ minus one.*
- *(iii) Twice the projective dimension of the characteristic tilting module for $\mathcal{O}(\lambda, \mu)$.*
- *(iv) Twice the Δ-filtration dimension of $\mathcal{O}(\lambda, \mu)$.*
- *(v) Twice the projective dimension of the simple (generalized) Verma module in $\mathcal{O}(\lambda, \mu)$.*

Proof. It follows from [RC], that $\mathcal{O}(\lambda, \mu)$ is a module category over a quasi-hereditary algebra, and it is clear that $^\circ$ restricts to a duality on $\mathcal{O}(\lambda, \mu)$. Hence the equality of (iii) and (iv) follows from Corollary 1.

The equality of (iii) and (v) can be deduced by the following arguments. Any simple Verma module in $\mathcal{O}(\lambda, \mu)$ is tilting and hence (v) does not exceed (iii). At the same time, it was shown in [CI], that every tilting module can be obtained as a direct summand of the tensor product of the simple Verma module from $\mathcal{O}(\lambda, \mu)$ with a finite-dimensional module. Since the functor of tensoring with a finite-dimensional module is exact and sends projectives to projectives (see e.g. [Ja]), any projective resolution of the simple Verma module is sent to a projective resolution of the resulting tensor product. Hence the projective dimension of the characteristic tilting module is less than or equal to the projective dimension of the simple Verma module.

Further, the categories $\mathcal{O}(\lambda,\mu)$ and $\mathcal{O}(-w_0(\mu),\lambda)$ are Koszul dual to each other, [BGS, Ba]. Hence the global dimension of $\mathcal{O}(\lambda,\mu)$ coincides with the maximal non-zero degree in the Koszul grading of the quasi-hereditary algebra of $\mathcal{O}(-w_0(\mu),\lambda)$. The Koszul grading on the algebra induces a grading on the projective modules. Since indecomposable projectives have simple tops, this grading gives the radical filtration of the projective modules by [BGS, Proposition 2.4.1]. And the radical filtration is a Loewy filtration. Hence the maximal non-zero degree in the Koszul grading of the quasi-hereditary algebra of $\mathcal{O}(-w_0(\mu),\lambda)$ equals the maximal Loewy length of the indecomposable projective modules minus 1. Any projective module is filtered by generalized Verma modules ([RC]) and hence the simples, occurring in the socle of any projective module are precisely those, which can occur in the socles of generalized Verma modules. According to Irving's self-duality theorem [Ir1], the projective covers of these simple modules are self-dual and hence injective. In particular, every projective module from $\mathcal{O}(-w_0(\mu),\lambda)$ is a submodule of a projective-injective module from $\mathcal{O}(-w_0(\mu),\lambda)$. Therefore the maximal Loewy length is achieved on the direct sum of all projective-injective modules in $\mathcal{O}(-w_0(\mu),\lambda)$. This proves the equality of (i) and (ii).

The inequality $(i)\leq(iv)$ follows from Theorem 1 and to complete the proof it is now enough to show that $(ii)\geq(v)$.

Let $M_1=L_1$ be a simple Verma module in $\mathcal{O}(\lambda,\mu)$ and $\{L_i : i=1,\dots,t\}$ be a complete list of pairwise non-isomorphic simples in $\mathcal{O}(\lambda,\mu)$. Then the module $\mathrm{Ext}^*(L_1,\oplus_{i=1}^t L_i)$ is projective in the Koszul dual of $\mathcal{O}(\lambda,\mu)$, which is $\mathcal{O}(-w_0(\mu),\lambda)$ by [Ba]. Since the simple module L_1 is also standard, the corresponding idempotent can be considered as the minimal element with respect to the partial order equipping $\mathcal{O}(\lambda,\mu)$ with the structure of the module category over a quasi-hereditary algebra. Since both $\mathcal{O}(\lambda,\mu)$ and $\mathcal{O}(-w_0(\mu),\lambda)$ are Koszul and quasi-hereditary, we can apply [ADL, Theorem 2.6] and get that the module $\mathrm{Ext}^*(L_1,\oplus_{i=1}^t L_i)$ corresponds to the maximal index for $\mathcal{O}(-w_0(\mu),\lambda)$ and hence is a projective standard module, N say. Since any standard module has a simple top, the graded Koszul filtration of this module is a Loewy filtration (in fact it is the radical filtration by [BGS, Proposition 2.4.1]) and hence the Loewy length l of N equals the projective dimension p of L_1 plus 1. From [Ir2, Proposition 3.1] it now follows that the Loewy length of some projective module in $\mathcal{O}(-w_0(\mu),\lambda)$ is at least $2l-1=2p+1$ and thus $(ii)\geq(v)$. This completes the proof. □

A combination of (ii) with the results of Irving, [Ir2, Theorem 4.3], on the Loewy length of projective modules in $\mathcal{O}_S$, allows one to give in many cases a combinatorial formula for the global dimension of $\mathcal{O}(\lambda,\mu)$. This can be done in all cases, when [Ir2, Theorem 4.3] is applicable. In particular, according to [IS], this is always the case for $\mathfrak{g}=\mathfrak{sl}(n,\mathbb{C})$.

Let W^μ denote the set of the minimal coset representatives in W/W_μ. For $w\in W^\mu$ we denote by $\overline{w}$ the longest element in the Weyl group, generated by all simple reflections s, satisfying $l(ws)<l(w)$. We set $t(\mu)$ to be the maximal length of $\overline{w}$ for all $w\in W^\mu$.

Corollary 3. *Suppose that the assumptions of [Ir2, Theorem 4.3] are satisfied, then the global dimension of $\mathcal{O}(\lambda,0)=\mathcal{O}^\theta$ equals $2t(\lambda)$.*

Proof. By Theorem 2, the global dimension of $\mathcal{O}(\lambda, 0)$ equals the maximal Loewy length of the indecomposable projective injective module in $\mathcal{O}^\theta$. So if the assumptions of [Ir2, Theorem 4.3] are satisfied, we can apply this theorem and get that the maximal Loewy length equals $2t(\lambda) + 1$. $\square$

Let w_0^μ and w_0 denote the longest element in W^μ or W respectively.

Corollary 4. *The global dimension of $\mathcal{O}(0, \lambda)$ equals $2(l(w_0) - l(w_0^\mu))$.*

Proof. By Theorem 2, the global dimension of $\mathcal{O}(0, \lambda)$ equals the Loewy length of the unique projective injective module in $\mathcal{O}^\theta$. Now the result follows from [Ir2, Section 3]. $\square$

4.3 $\mathcal{S}$-subcategories in $\mathcal{O}$

We retain the notation from the previous subsection. Let λ and μ be dominant integral, $\lambda = \lambda_\theta$ for $\theta \in Z(\mathfrak{g})^*$. Let further $I_1, \dots, I_s$ be a complete list of pairwise non-isomorphic indecomposable W_μ-antidominant injective modules from $\mathcal{O}^\theta$ and $I = \oplus_{i=1}^s I_i$. Denote by $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$ the full subcategory of $\mathcal{O}^\theta$, consisting of all modules M, which has a copresentation $0 \to M \to I^a \to I^b$. This category is a block of an $\mathcal{S}$-subcategory in $\mathcal{O}$, associated with Enright's completion functor, which was defined and studied in [FKM]. In particular, in [FKM, Section 5] it was shown that $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$ is equivalent to the module category of a properly stratified algebra. This equivalence gives $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$ an abelian structure, which is not inherited from $\mathcal{O}$. In fact, simple objects in $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$ are not simple $\mathfrak{g}$-modules in general. It is also known, [KM], that $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$ is equivalent to certain blocks of the category of Harish-Chandra $\mathfrak{g}$-bimodules, [BG], with the diagonal right action of the center of $\mathfrak{g}$. It the latter category simple objects are precisely simple $\mathfrak{g}$-bimodules.

Tilting modules in $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$ are described in [FKM, Section 6] and are exactly the W_μ-dominant tilting modules in $\mathcal{O}^\theta$. As an immediate corollary of the previous section we get.

Theorem 3. *The following numbers are equal and give an upper bound for the finitistic dimension of a block of $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$.*

(*i*) *Twice the projective dimension of the characteristic tilting module in the block of $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$.*

(*ii*) *Twice the $\overline{\Delta}$-filtration dimension of the block of $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$.*

(*iii*) *Twice the projective dimension of the self-dual standard module in the block of the category $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$.*

Proof. The equality of (*i*) and (*ii*) follows from Corollary 1. Using [FKM, Section 6], the equality of (*ii*) and (*iii*) is proved by the same arguments as the analogous statement in Theorem 2. $\square$

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Volodymyr Mazorchuk, Department of Mathematics, Uppsala University, Box 480, 751 06, Uppsala, SWEDEN, e-mail: `mazor@math.uu.se`, web: "http://www.math.uu.se/~mazor/".

Alison E. Parker, School of Mathematics and Statistics F07, University of Sydney NSW 2006, AUSTRALIA, e-mail: `alisonp@maths.usyd.edu.au`,
web: "http://www.maths.usyd.edu.au:8000/u/alisonp/".