Finitistic dimension of properly stratified algebras

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Dedicated to Yuri Drozd on the occasion of his 60th birthday.

Abstract

We prove that the finitistic dimension of a properly stratified algebra having a simple preserving duality and for which every tilting module is cotilting, equals twice the projective dimension of the characteristic tilting module. As a corollary we get that the global dimension of a quasi-hereditary algebra with duality equals twice the projective dimension of the characteristic tilting module. As another corollary we obtain an affirmative answer to the conjecture of Erdmann and Parker. Finally, we calculate the finitistic dimension of the blocks of certain parabolic generalizations of the category $\mathcal{O}$.

1 Introduction and the main result

The notion of properly stratified algebras, which appeared in [Dl], seems to be the most elaborated notion, generalizing the classical notion of quasi-hereditary algebras, [CPS1], (see [CPS2] for a more general class of standardly stratified algebras). On the other hand, certain algebras, which arise in parabolic generalizations of the BGG category $\mathcal{O}$, [FKM], or algebras, describing certain blocks of the category of Harish-Chandra bimodules, [KKM], turned out to be not quasi-hereditary but properly stratified.

An important homological invariant of algebras (and their module categories) is the homological (or global) dimension. For quasi-hereditary algebras it is finite and was studied by many authors, see for example [BGG, CPS2, Ko2, Ko3, Pa1, Pa2, To, Xi].

In the case of properly stratified algebras the global dimension is infinite, unless the algebra is quasi-hereditary. However, it was shown in [AHLU1] that the finitistic dimension of a properly stratified algebra is finite, which makes this notion an appropriate substitution of the global dimension in the case of quasi-hereditary algebras. Some upper bounds for the finitistic dimension of certain properly stratified algebras were obtained in [AHLU1, AHLU2, MP].

Almost all quasi-hereditary and properly stratified algebras, which naturally appear in applications (for example algebras of the category $\mathcal{O}$, Schur algebras), possess a simple preserving duality. In all known examples it was noted that the finitistic (global) dimension of such algebras equals $2k$ for some integer $k \geq 0$ and this k has various interpretations in terms of the combinatorial data.

This motivated the authors of [MP] to formulate the conjecture that this number k coincides with the projective dimension of the characteristic tilting module (see [Ri, AHLU2]). In the quasi-hereditary case even a stronger conjecture was formulated in [EP].

In the present paper we prove both conjectures for properly stratified algebras with duality, for which tilting and cotilting modules coincide (the last property holds for all quasi-hereditary algebras, and for properly stratified algebras it can be considered as an analogue of self-injectivity for local algebras). Our main result is the following theorem:

Theorem 1. *Assume that A is a properly stratified algebra having a simple preserving duality, and such that every tilting A-module is cotilting. Then* $\mathrm{fin.dim}(A) = 2\,\mathrm{p.d.}(T)$*, where T is the characteristic tilting module.*

Corollary 1. *Assume that A is a quasi-hereditary algebra having a simple preserving duality. Then* $\mathrm{gl.dim}(A) = 2\,\mathrm{p.d.}(T)$*, where T is the characteristic tilting module.*

We remark that in [KKM] the conjecture [MP, Conjecture 1] is proved for certain properly stratified algebras with duality, in which tilting and cotilting modules do not coincide.

The paper is organized as follows: in Section 2 we introduce necessary definitions and notation. Theorem 1 is proved in Section 3. Further, in Section 4 we give some applications of this result, in particular, we describe its connection with the notion of good filtration dimension in Subsection 4.2 and with the notion of exact Borel subalgebra in Subsection 4.6. We prove the conjecture of Erdmann and Parker in Subsection 4.4, and calculate the finitistic dimension of the properly stratified algebras, which appear as parabolic generalizations of the BGG category $\mathcal{O}$, in Subsection 4.5. In Subsection 4.3 we discuss some homological invariants for modules, inspired by the conjecture of Erdmann and Parker. These invariants are then used in the proof of this conjecture.

2 Some definitions, notation and conventions

Let A be a finite-dimensional associative algebra over an algebraically closed field, $\Bbbk$. By a module we will always mean left module, and by A-mod we denote the category of finite-dimensional left A-modules. For a primitive idempotent, $e = e_i \in A$, we will denote by $L(i)$ the corresponding simple module, by $P(i)$ the projective cover of $L(i)$ and by $I(i)$ the injective envelope of $L(i)$. We say that A is *properly stratified* (see [Dl]) if the following properties hold:

1. There is a linear order $\leq$ on a complete set $\{e_1, e_2, \dots, e_n\}$ of primitive orthogonal idempotents of A, which we assume to be given by the natural ordering of the indexes.

2. There is a family, $\{\Delta(i)\}$, $i = 1, 2, \dots, n$, of A-modules, such that the module $P(j)$ surjects on $\Delta(j)$ for every j, and the kernel of this map is filtered by $\Delta(i)$, $i > j$.

3. There is a family, $\{\overline{\Delta}(i)\}$, $i = 1, 2, \dots, n$, of A-modules, such that the module $\overline{\Delta}(j)$ surjects on $L(j)$ for every j, with the kernel of this map being filtered by $L(i)$, $i < j$, and such that the module $\Delta(j)$ is filtered by $\overline{\Delta}(j)$ for every j.

The modules $\Delta(i)$ are called *standard modules* and the modules $\overline{\Delta}(i)$ are called *proper standard modules.* In the dual way we also define *costandard modules* $\nabla(i)$ and *proper costandard modules* $\overline{\nabla}(i)$. We denote by $\mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\overline{\Delta})$, resp. $\mathcal{F}(\nabla)$, resp. $\mathcal{F}(\overline{\nabla})$) the full subcategory in the category of all A-modules, consisting of all modules, filtered by $\Delta(i)$ (resp. $\overline{\Delta}(i)$, resp. $\nabla(i)$, resp. $\overline{\nabla}(i)$), $i = 1, \dots, n$. We denote by $T(i)$ and $C(i)$ the *indecomposable tilting* and *cotilting* modules respectively, which correspond to i (see [AHLU2]). We set $I = \oplus_{i=1}^{n} I(i)$, $P = \oplus_{i=1}^{n} P(i)$ and $\nabla = \oplus_{i=1}^{n} \nabla(i)$.

For an A-module, N, we denote by $\mathrm{add}(N)$ the full subcategory in A-mod, which consists of all modules M, isomorphic to a direct summand of some N^l, $l \geq 0$. We have $\mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\nabla}) = \mathrm{add}(T)$ and $\mathcal{F}(\overline{\Delta}) \cap \mathcal{F}(\nabla) = \mathrm{add}(C)$, where $T = \oplus_{i=1}^{n} T(i)$ and $C = \oplus_{i=1}^{n} C(i)$ are the *characteristic tilting* and *cotilting* modules respectively.

Throughout the paper we assume that A has a *simple preserving duality*, that is there exists an exact, involutive, and contravariant equivalence $^\circ$ on A-mod, which preserves isomorphism classes of simple modules. In this case $^\circ$ swaps $\Delta(i)$ with $\nabla(i)$, $\overline{\Delta}(i)$ with $\overline{\nabla}(i)$, $P(i)$ with $I(i)$, and $T(i)$ with $C(i)$. Throughout the paper we also assume that $T(i) \cong C(i)$ for all i.

For an A-module M we denote by $\mathrm{p.\,d.}(M)$ (resp. $\mathrm{i.\,d.}(M)$) the projective (resp. injective) dimension of M. By $\mathrm{fin.\,dim}(A)$ (resp. $\mathrm{gl.\,dim}(A)$) we denote the projectively defined finitistic (resp. global) dimension of A.

A properly stratified algebra A of finite global dimension is quasi-hereditary and vice versa. Moreover, $\mathrm{gl.\,dim}(A) < \infty$ if and only if $\Delta(i) \simeq \overline{\Delta}(i)$ for all i. In this case we automatically obtain $T(i) \simeq C(i)$ for all i.

Denote by $B = \mathrm{End}_A(T)$ the *Ringel dual* of A. The algebra B is always stratified. For B-modules we will use the same notation as for A-modules, adding the subscript B. The functor $F(\_) = \mathrm{Hom}_A(T, \_) : A\text{--mod} \to B\text{-mod}$ maps $T(i)$ to the indecomposable projective module $P_B(i)$ over B (see [Ri, AHLU2]).

As usually we will use Com, K and D to denote the category of complexes, the homotopic category and the derived category respectively. For an A-module M we denote by $M^\bullet$ the complex in $\mathrm{Com}^b(A)$ such that $M^0 = M$ and $M^i = 0$ for all $i \neq 0$. We call a complex, $\mathcal{C}^\bullet \in \mathrm{Com}(A)$, *minimal* provided that it does not contain direct summands of the form

$$\dots \to 0 \to N \xrightarrow{\mathrm{Id}} N \to 0 \to \dots, \tag{1}$$

where $N \in A$-mod. Since $F(\mathrm{Com}(\mathrm{add}(T))) \cong \mathrm{Com}(\mathrm{add}({}_B B))$, we have that a complex $\mathcal{C}^\bullet \in \mathrm{Com}(\mathrm{add}(T))$ is minimal if and only if all differentials in $F(\mathcal{C}^\bullet)$ are zero modulo the radical. A complex $\mathcal{C}^\bullet = \{\mathcal{C}_i : i \in \mathbb{Z}\}$ such that $\mathcal{C}_i = 0$ for all $i < 0$ will be called *positive.*

3 Proof of the main result

Under assumptions of Theorem 1 we have the inequality $\mathrm{fin.dim}(A) \leq 2\,\mathrm{p.d.}(T)$ by [MP, Corollary 2] (see also Appendix for a different proof). Set $k = \mathrm{p.d.}(T)$. To prove Theorem 1 it is enough to show that there exist $M, N \in A$-mod such that $\mathrm{Ext}_A^{2k}(M,N) \neq 0$. In fact, after some preparation, we will prove that there exists i such that $\mathrm{Ext}_A^{2k}(\nabla(i), \Delta(i)) \neq 0$.

Let $\mathcal{M} = \mathrm{add}(T)$ or $\mathcal{M} = \mathrm{add}(C)$ or $\mathcal{M} = \mathrm{add}(A)$. Following [Ha, Chapter III(2), Lemma 2.1] the canonical functor $K^b(\mathcal{M}) \to D^b(A)$ is full and faithful, hence for every $\mathcal{C}_1^\bullet, \mathcal{C}_2^\bullet \in \mathrm{Com}^b(\mathcal{M})$ we have

$$\mathrm{Hom}_{D^b(A)}(\mathcal{C}_1^\bullet, \mathcal{C}_2^\bullet) \cong \mathrm{Hom}_{K^b(A)}(\mathcal{C}_1^\bullet, \mathcal{C}_2^\bullet). \tag{2}$$

Proposition 1. *Let* $M \in A-\mathrm{mod}$ *and assume that there exists a minimal complex*

$$\mathcal{C}_\nabla^\bullet : \qquad \dots \longrightarrow 0 \longrightarrow T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} \dots \xrightarrow{f_{s-1}} T_s \longrightarrow 0 \longrightarrow \dots, \tag{3}$$

in $\mathrm{Com}(\mathrm{add}(T))$, *such that* $H^t(\mathcal{C}_\nabla^\bullet) \cong M$ *for some* $t \in \{0, \dots, s\}$, *and* $H^p(\mathcal{C}_\nabla^\bullet) = 0$ *for all* $p \neq t$. *Then* $\mathrm{Ext}_A^{2t}(M, M^\circ) \neq 0$.

Proof. First we apply $^\circ$ to (3) and obtain the following complex:

$$\mathcal{C}_\Delta^\bullet : \qquad \dots \longrightarrow 0 \longrightarrow T_s^\circ \xrightarrow{f_{s-1}^\circ} T_{s-1}^\circ \xrightarrow{f_{s-2}^\circ} \dots \xrightarrow{f_0^\circ} T_0^\circ \longrightarrow 0 \longrightarrow \dots, \tag{4}$$

where T_0° stays in degree 0. Since $T_i^\circ \cong T_i$ for all i we get that $\mathcal{C}_\Delta^\bullet \in \mathrm{Com}(\mathrm{add}(T))$. Moreover, we have that the unique non-zero homology of $\mathcal{C}_\Delta^\bullet$ is $H^{-t}(\mathcal{C}_\Delta^\bullet) \cong M^\circ$. Hence in the derived category $D^b(A)$ the complexes $(M^\circ)^\bullet$ and $M^\bullet$ are isomorphic to the complexes $\mathcal{C}_\Delta^\bullet[-t]$ and $\mathcal{C}_\nabla^\bullet[t]$ respectively. Then we have

$$\mathrm{Ext}_A^{2t}(M, M^\circ) \cong \mathrm{Hom}_{D^b(A)}(\mathcal{C}_\nabla^\bullet, \mathcal{C}_\Delta^\bullet).$$

Using (2) and applying the functor F thereafter, we get

$$\mathrm{Hom}_{D^b(A)}(\mathcal{C}_\nabla^\bullet, \mathcal{C}_\Delta^\bullet) = \mathrm{Hom}_{K^b(A)}(\mathcal{C}_\nabla^\bullet, \mathcal{C}_\Delta^\bullet) = \mathrm{Hom}_{K^b(B)}(F(\mathcal{C}_\nabla^\bullet), F(\mathcal{C}_\Delta^\bullet)).$$

Hence, to complete the proof it is enough to construct a non-zero morphism from $F(\mathcal{C}_\nabla^\bullet)$ to $F(\mathcal{C}_\Delta^\bullet)$ in $K^b(B)$.

Lemma 1. *Let* $P_B^{(j)} = F(T_j)$, $Q_B^{(j)} = F(T_j^\circ)$, $j = 0, \dots, s$, *and* $g_j = F(f_j)$, $h_j = F(f_j^\circ)$, $j = 0, \dots, s-1$. *Then the diagram*

$$\begin{array}{ccccccccccccc}
\cdots \to & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & P_B^{(0)} & \xrightarrow{g_0} & P_B^{(1)} & \xrightarrow{g_1} & \cdots & \xrightarrow{g_{s-1}} & P_B^{(s)} & \to \cdots \\
 & \downarrow & & \downarrow & \overset{\alpha}{\swarrow} & \downarrow \Phi & \overset{\beta}{\swarrow} & \downarrow & & & & \downarrow & \\
\cdots \to & Q_B^{(s)} & \xrightarrow{h_{s-1}} \cdots \xrightarrow{h_1} & Q_B^{(1)} & \xrightarrow{h_0} & Q_B^{(0)} & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \to \cdots
\end{array}$$

where Φ *is any isomorphism, represents a non-zero morphism in* $K^b(B)$.

Proof. Let us first note that all morphisms in both rows of the diagram are radical morphisms because of the minimality of (3) and (4). Hence $\mathrm{Im}\, g_0 \subset \mathrm{rad}\, P_B^{(1)}$ and $\mathrm{Im}\, h_0 \subset \mathrm{rad}\, Q_B^{(0)}$. This implies that for any α and β as depicted on the diagram we have $\mathrm{Im}(h_0 \circ \alpha + \beta \circ g_0) \subset \mathrm{rad}\, Q_B^{(0)}$. Therefore $h_0 \circ \alpha + \beta \circ g_0 \neq \Phi$, which completes the proof. □

Applying the duality to [AHLU2, Proposition 2.2] we get that every $\nabla(i)$ has a finite resolution by cotilting (=tilting) modules. Let i_0 be such that the following minimal tilting resolution of $\nabla(i_0)$ has the maximal possible length m:

$$0 \longrightarrow T_m \xrightarrow{f_m} T_{m-1} \xrightarrow{f_{m-1}} \dots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \longrightarrow \nabla(i_0) \longrightarrow 0. \tag{5}$$

Lemma 2. *$m = k$, that is the length of the resolution* (5) *equals* $\mathrm{p.\,d.}(T)$.

Proof. For $X \in \mathcal{F}(\nabla)$ we denote by $\mathtt{m}_X$ the minimal l for which there exits an exact sequence $0 \to Y_l \to \dots \to Y_0 \to X \to 0$, where all Y_i are tilting modules. Applying Ringel duality, we see that $\mathtt{m}_X$ equals the maximal i such that $\mathrm{Ext}^i_A(X, T) \neq 0$. In particular, for any $X \in \mathcal{F}(\nabla)$ we have $\mathtt{m}_X \leq \mathtt{m}_\nabla$. Using [Ha, Section III, 3.2] we have $\mathtt{m}_I = \mathrm{p.\,d.}(T) = k$. Applying $\mathrm{Hom}_A({}_-, T)$ to the short exact sequence $\nabla \hookrightarrow I \twoheadrightarrow K$, where $K \in \mathcal{F}(\nabla)$, we get $\mathtt{m}_I = \mathtt{m}_\nabla$. The last implies $m = \mathtt{m}_\nabla = \mathtt{m}_I = k$. □

Theorem 1 now follows by applying Proposition 1 to $M = \nabla(i_0)$, $t = k$, with $\mathcal{C}^\bullet_\nabla$ being the resolution, obtained from (5). □

4 Applications

4.1 Quasi-hereditary algebras

As we have already noted in the introduction, Theorem 1 immediately implies the following result, which covers many results from [EP, KKM, MP, Pa1, Pa2]:

Corollary 2. *Assume that A is a quasi-hereditary algebra having a simple preserving duality. Then* $\mathrm{gl.\,dim}(A) = 2\,\mathrm{p.\,d.}(T)$.

4.2 Connection with good filtration dimension

Let $\mathcal{M}$ be any family of A-modules. Let N be an A-module and assume that there exists a (possibly infinite) exact sequence

$$0 \longrightarrow M_j \longrightarrow \dots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow N \longrightarrow 0, \tag{6}$$

where $M_i \in \mathcal{M}$ for all i. The minimal j for which a sequence of the form (6) exists is called *$\mathcal{M}$-filtration dimension* of N and will be denoted by $\dim_{\mathcal{M}}(N)$. Dually, the minimal j for which there exists (if any) a (possibly infinite) exact sequence of the form

$$0 \longrightarrow N \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \dots \longrightarrow M_j \longrightarrow 0,$$

where $M_i \in \mathcal{M}$ for all i, is called $\mathcal{M}$-*filtration codimension* of N and will be denoted by $\mathrm{codim}_{\mathcal{M}}(N)$. These concepts were introduced in [FP].

If A is properly stratified then both $\dim_{\mathcal{F}(\overline{\Delta})}(N)$ and $\mathrm{codim}_{\mathcal{F}(\overline{\nabla})}(N)$ are finite for every $N \in A$-mod, see [MP, Corollary 1]. Hence we can define $\dim_{\mathcal{F}(\overline{\Delta})}(A)$ and $\mathrm{codim}_{\mathcal{F}(\overline{\nabla})}(A)$ as the supremum of the values of $\dim_{\mathcal{F}(\overline{\Delta})}(N)$ and $\mathrm{codim}_{\mathcal{F}(\overline{\nabla})}(N)$ on $N \in A$-mod respectively. We refer the reader to [CZ, MP, Pa1, Pa2] for details and properties of these notions.

From Theorem 1 and [MP, Theorem 1] we immediately get:

Corollary 3. *Let A be as in Theorem 1. Then*

$$\mathrm{fin.}\dim(A) = 2\dim_{\mathcal{F}(\overline{\Delta})}(A) = 2\,\mathrm{codim}_{\mathcal{F}(\overline{\nabla})}(A).$$

4.3 More on good filtration dimension

In contrast with the other parts of the paper, in this subsection we work under the assumption that A is any properly stratified algebra.

Lemma 3. *1. Let*

$$\mathcal{X}^\bullet: \qquad \dots \longrightarrow 0 \longrightarrow X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \tag{7}$$

be a (possibly infinite) complex in $\mathrm{Com}(\mathrm{add}(T))$ *or in* $\mathrm{Com}(\mathrm{add}(C))$. *Assume that $H^i(\mathcal{X}^\bullet) = 0$ for all $i > 0$. Then $H^0(\mathcal{X}^\bullet) \in \mathcal{F}(\overline{\Delta})$.*

2. Let

$$\mathcal{X}^\bullet: \qquad \dots \longrightarrow X_{-1} \xrightarrow{f_{-1}} X_0 \longrightarrow 0 \longrightarrow \dots$$

be a (possibly infinite) complex in $\mathrm{Com}(\mathrm{add}(T))$ *or in* $\mathrm{Com}(\mathrm{add}(C))$. *Assume that $H^i(\mathcal{X}^\bullet) = 0$ for all $i < 0$. Then $H^0(\mathcal{X}^\bullet) \in \mathcal{F}(\overline{\nabla})$.*

Proof. We prove the first statement and the second one is proved by dual arguments.

Recall that $\mathcal{F}(\overline{\Delta}) = \{M \in A\text{-mod} : \mathrm{Ext}^1_A(M, \nabla) = 0\}$ (see [Dl, Theorem 5(v)]). Applying $\mathrm{Hom}_A(\_, \nabla)$ to the short exact sequence

$$0 \to \ker(f_i) \to X_i \to \ker(f_{i+1}) \to 0, \qquad i \geq 0,$$

and using $T, C \in \mathcal{F}(\overline{\Delta})$, we get $\mathrm{Ext}^k_A(\ker(f_i), \nabla) \cong \mathrm{Ext}^{k+1}_A(\ker(f_{i+1}), \nabla)$ for all $k \geq 1$. Since $\mathrm{i.d.}(\nabla) < \infty$, the proof follows by induction. □

Unfortunately to ensure that every $M \in \mathcal{F}(\overline{\Delta})$ appears as $H^0(\mathcal{X}^\bullet)$ for some $\mathcal{X}^\bullet$ as in (7) requires an additional assumption. To proceed we will need the following standard statement:

Lemma 4. *1. Let $\mathcal{Y}^\bullet$ be a positive complex in* $\mathrm{Com}(\mathcal{F}(\Delta))$. *Then there exists a positive complex $\mathcal{J}^\bullet \in$* $\mathrm{Com}(\mathrm{add}(T))$, *which is quasi-isomorphic to $\mathcal{Y}^\bullet$. Moreover, for $\mathcal{Y}^\bullet \in$* $\mathrm{Com}^b(\mathcal{F}(\Delta))$ *we get $\mathcal{J}^\bullet \in$* $\mathrm{Com}^b(\mathrm{add}(T))$.

2. *Let $\mathcal{Y}^\bullet$ be a positive complex in $\mathrm{Com}(\mathcal{F}(\nabla))$. Then there exists a positive complex $\mathcal{J}^\bullet \in \mathrm{Com}(\mathrm{add}(C))$, whose shift is quasi-isomorphic to $\mathcal{Y}^\bullet$. Moreover, for $\mathcal{Y}^\bullet \in \mathrm{Com}^b(\mathcal{F}(\nabla))$ we get $\mathcal{J}^\bullet \in \mathrm{Com}^b(\mathrm{add}(C))$.*

Proof. We again will prove only the first statement. The proof of the second one is similar. Let

$$\mathcal{Y}^\bullet : \qquad \dots \longrightarrow 0 \longrightarrow Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \dots \xrightarrow{g_{j-1}} Y_j \xrightarrow{g_j} \dots .$$

For a complex $\mathcal{X}^\bullet$ and $j \in \mathbb{Z}$ we denote by $\mathfrak{t}_j\mathcal{X}^\bullet$ the j-th truncation of $\mathcal{X}^\bullet$, which is the complex defined via: $\mathfrak{t}_j\mathcal{X}^i = \mathcal{X}^i$, $i \leq j$; $\mathfrak{t}_j\mathcal{X}^i = 0$, $i > j$, and the differential on all corresponding places in $\mathcal{X}^\bullet$ and $\mathfrak{t}_j\mathcal{X}^\bullet$ coincide.

Set $\mathcal{Y}^\bullet_j = \mathfrak{t}_j\mathcal{Y}^\bullet$. First we show that there exists a positive complex $\mathcal{J}^\bullet_j \in \mathrm{Com}^b(\mathrm{add}(T))$, which is quasi-isomorphic to $\mathcal{Y}^\bullet_j$, and such that $\mathfrak{t}_j\mathcal{J}^\bullet_N = \mathcal{J}^\bullet_j$ for all $j < N$.

We prove this statement by induction in j. As $Y_i \in \mathcal{F}(\Delta)$ for all i, we can choose for every i a finite tilting coresolution, $\varphi_i : Y^\bullet_i \to \mathcal{T}^\bullet_{Y_i} \in \mathrm{Com}(\mathrm{add}(T))$. Existence of this coresolution proves the statement for $j = 0$.

Let $\mathcal{J}^\bullet_{j-1} \in \mathrm{Com}^b(\mathrm{add}(T))$ be a positive complex, which is quasi-isomorphic to $\mathcal{Y}^\bullet_{j-1}$. The morphism g_{j-1} induces the following triangle in $K^b(A)$:

$$\mathcal{Y}^\bullet_j \longrightarrow \mathcal{Y}^\bullet_{j-1} \xrightarrow{g^\bullet_{j-1}} Y^\bullet_j[-j+1] \longrightarrow \mathcal{Y}^\bullet_j[1].$$

By inductive assumptions we have quasi-isomorphisms $\Phi_{j-1} : \mathcal{Y}^\bullet_{j-1} \to \mathcal{J}^\bullet_{j-1}$ and $\varphi_j[-j+1] : Y^\bullet_j[-j+1] \to \mathcal{T}^\bullet_{Y_j}[-j+1]$. Using the formula (2) from Section 3, there exists a morphism of complexes $\psi_j : \mathcal{J}^\bullet_{j-1} \to \mathcal{T}^\bullet_{Y_i}[-j+1]$, which represents in $D^b(A)$ the morphism $\varphi_j[-j+1] \circ g^\bullet_{j-1} \circ \Phi^{-1}_{j-1}$. The morphism ψ_j makes the sub-diagram, represented by solid arrows in the diagram (8) below, commutative in $D^b(A)$:

$$\begin{array}{ccccccc}
\mathcal{Y}^\bullet_j & \dashrightarrow & \mathcal{Y}^\bullet_{j-1} & \xrightarrow{g^\bullet_{j-1}} & Y^\bullet_j[-j+1] & \dashrightarrow & \mathcal{Y}^\bullet_j[1] \\
\Big\downarrow{\scriptstyle \Phi_j} & & \Big\downarrow{\scriptstyle \Phi_{j-1}} & & \Big\downarrow{\scriptstyle \varphi_j[-j+1]} & & \Big\downarrow{\scriptstyle \Phi_j[1]} \\
\mathrm{Cone}(\psi_j)[-1] & \dashrightarrow & \mathcal{J}^\bullet_{j-1} & \xrightarrow{\psi_j} & \mathcal{T}^\bullet_{Y_j}[-j+1] & \dashrightarrow & \mathrm{Cone}(\psi_j)
\end{array} \qquad (8)$$

Taking cone of ψ_j we extend the diagram (8) to a morphism of distinguished triangles in $D^b(A)$, which happens to be an isomorphism since both Φ_{j-1} and φ_j are. Obviously, $\mathrm{Cone}(\psi_j)[-1]$ is a positive complex and Φ_j is a quasi-isomorphism, and we set $\mathcal{J}^\bullet_{j+1} = \mathrm{Cone}(\psi_j)[-1]$. Moreover, by construction we get $\mathfrak{t}_j\mathcal{J}^\bullet_{j+1} = \mathcal{J}^\bullet_j$.

The property $\mathfrak{t}_j\mathcal{J}^\bullet_N = \mathcal{J}^\bullet_j$ for $N > j$ allows us to define the limit complex $\mathcal{J}^\bullet$, such that $\mathfrak{t}_j\mathcal{J}^\bullet = \mathcal{J}^\bullet_j$, $j \geq 1$, together with a quasi-isomorphism $\Phi : \mathcal{Y}^\bullet \to \mathcal{J}^\bullet$. The statement that $\mathcal{Y}^\bullet \in \mathrm{Com}^b(\mathcal{F}(\Delta))$ implies $\mathcal{J}^\bullet \in \mathrm{Com}^b(\mathrm{add}(T))$ is obvious. □

Lemma 5. *Assume that all tilting A-modules are cotilting.*

1. *For any $M \in \mathcal{F}(\overline{\Delta})$ there exists a (possibly infinite) complex*

$$\mathcal{X}^\bullet : \qquad \dots \longrightarrow 0 \longrightarrow X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \qquad (9)$$

in $\mathrm{Com}(\mathrm{add}(T))$ such that $H^i(\mathcal{X}^\bullet) = 0$ for all $i > 0$ and $H^0(\mathcal{X}^\bullet) \cong M$.

2. For any $M \in \mathcal{F}(\overline{\nabla})$ there exists a (possibly infinite) complex

$$\mathcal{X}^\bullet : \qquad \dots \longrightarrow X_{-1} \xrightarrow{f_{-1}} X_0 \longrightarrow 0 \longrightarrow \dots$$

in $\mathrm{Com}(\mathrm{add}(T))$ *such that $H^i(\mathcal{X}^\bullet) = 0$ for all $i < 0$ and $H^0(\mathcal{X}^\bullet) \cong M$.*

Proof. We prove again only the first statement and the proof of the second one is dual. Let $\mathcal{I}_M^\bullet$ be an injective coresolution of M. All injective modules admit a finite resolution by cotilting modules, and hence by tilting modules. Moreover, the length of such resolution is always bounded by $\mathrm{i.d.}(C) = \mathrm{i.d.}(T)$ (see the proof of [AHLU2, Proposition 2.2]). Apllying the second statement of Lemma 4 we get for some $j \geq 0$ the complex

$$\mathcal{Y}^\bullet : \qquad \dots \longrightarrow 0 \longrightarrow Y_{-j} \xrightarrow{f_{-j}} Y_{-j+1} \xrightarrow{f_{-j+1}} \dots, \tag{10}$$

in $\mathrm{Com}(\mathrm{add}(T))$ which is quasi-isomorphic to $M^\bullet$.

From Lemma 3 it follows that $\ker(f_1) \cong Y_0/\ker(f_0) \in \mathcal{F}(\overline{\Delta})$. From this and $M \in \mathcal{F}(\overline{\Delta})$ we get $Y_0/\mathrm{Im}(f_{-1}) \in \mathcal{F}(\overline{\Delta})$. Since $\mathcal{F}(\nabla)$ is closed under taking cokernels of monomorphisms, we get $\mathrm{Im}(f_{-1}) \in \mathcal{F}(\nabla)$ by induction. Now $\mathrm{Im}(f_{-1}) \in \mathcal{F}(\nabla)$ and $Y_0/\mathrm{Im}(f_{-1}) \in \mathcal{F}(\overline{\Delta})$ implies that the the short exact sequence

$$0 \longrightarrow \mathrm{Im}(f_{-1}) \longrightarrow Y_0 \longrightarrow Y_0/\mathrm{Im}(f_{-1}) \longrightarrow 0.$$

splits. That is $Y_0/\mathrm{Im}(f_{-1})$ is a tilting module. This implies that the complex

$$\dots \longrightarrow 0 \longrightarrow Y_0/\mathrm{Im}(f_{-1}) \xrightarrow{\hat{f}_0} Y_1 \xrightarrow{f_1} \dots,$$

where $\hat{f}_0$ is the induced map, consists of tilting modules, and has a unique non-zero homology, which is concentrated in degree 0 and isomorphic to M. □

The lemmas above are necessary to study the following invariants of A-modules. Let $\mathcal{M}$ be a family of A-modules and $M \in A$-mod. Assume that for some finite t there exists a (possibly infinite) s and a complex

$$\mathcal{X}_M^\bullet : \qquad \dots \longrightarrow 0 \longrightarrow X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{s-1}} X_s \longrightarrow 0 \longrightarrow \dots, \tag{11}$$

such that $X_i \in \mathcal{M}$ for all i, $H^t(\mathcal{X}_M^\bullet) \simeq M$ and $H^p(\mathcal{X}_M^\bullet) = 0$ for all $p \neq t$. Then the minimal possible t for which a complex of the form (11) exists will be called $\mathcal{M}$*-invariant* of M and denoted by $\mathrm{inv}_{\mathcal{M}}(M)$.

Dually, if for some finite t there exists a (possibly infinite) s and a complex

$$\mathcal{Y}_M^\bullet : \qquad \dots \longrightarrow 0 \longrightarrow Y_{-s} \xrightarrow{f_{-s}} \dots \xrightarrow{f_{-2}} Y_{-1} \xrightarrow{f_{-1}} Y_0 \longrightarrow 0 \longrightarrow \dots, \tag{12}$$

such that $Y_i \in \mathcal{M}$ for all i, $H^{-t}(\mathcal{Y}_M^\bullet) \simeq M$ and $H^p(\mathcal{Y}_M^\bullet) = 0$ for all $p \neq t$, then the minimal possible t for which a complex of the form (12) exists will be called $\mathcal{M}$*-coinvariant* of M and denoted by $\mathrm{coinv}_{\mathcal{M}}(M)$. In both cases we will call the minimal possible s the *degree* of the (co)invariant.

Lemma 6. *Let $M \in A$-mod.*

1. $\mathrm{inv}_{\mathrm{add}(T)}(M)$ *is defined and has a finite degree if and only if* $\mathrm{p.d.}(M) < \infty$ *if and only if* $\dim_{\mathcal{F}(\Delta)}(M) < \infty$. *Moreover, in this case* $\mathrm{inv}_{\mathrm{add}(T)}(M) = \dim_{\mathcal{F}(\Delta)}(M)$.

2. $\mathrm{coinv}_{\mathrm{add}(C)}(M)$ *is defined and has a finite degree if and only if* $\mathrm{i.d.}(M) < \infty$ *if and only if* $\dim_{\mathcal{F}(\nabla)}(M) < \infty$. *Moreover, in this case* $\mathrm{coinv}_{\mathrm{add}(C)}(M) = \dim_{\mathcal{F}(\nabla)}(M)$.

Proof. We prove the first statement and one proves the second statement by dual arguments. It is obvious that $\mathrm{p.d.}(M) < \infty$ if and only if $\dim_{\mathcal{F}(\Delta)}(M) < \infty$.

Let $\mathrm{inv}_{\mathrm{add}(T)}(M) = t < \infty$ and assume that it has finite degree s. Then from $\mathcal{X}_M^\bullet$ we get an exact sequence

$$0 \longrightarrow X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{t-2}} X_{t-1} \xrightarrow{f_{t-1}} \ker(f_t) \longrightarrow M \to 0. \tag{13}$$

Recall that $\mathcal{F}(\Delta)$ is closed under taking the kernels of epimorphisms. Since $\mathcal{X}_M^\bullet$ is bounded and consists of tilting modules, which, in fact, belong to $\mathcal{F}(\Delta)$, we obatin $\ker(f_t) \in \mathcal{F}(\Delta)$ and thus $\dim_{\mathcal{F}(\Delta)}(M) \leq t$.

On the other hand, assume that $\dim_{\mathcal{F}(\Delta)}(M) = t < \infty$ and let

$$0 \longrightarrow Z_0 \xrightarrow{g_0} Z_1 \xrightarrow{g_1} \dots \xrightarrow{g_{t-2}} Z_{t-1} \xrightarrow{g_{t-1}} Z_t \longrightarrow M \to 0 \tag{14}$$

be an exact sequence such that all $Z_i \in \mathcal{F}(\Delta)$. Applying the first statement of Lemma 4 to the resolution of M, obtained from (14), we get a complex $\mathcal{Y}^\bullet \in \mathrm{Com}^b(\mathrm{add}(T))$, which is positive and quasi-isomorphic to the complex $M^\bullet[-t]$. This implies $\mathrm{inv}_{\mathrm{add}(T)}(M) \leq t$, moreover, the degree of this invariant is finite. □

Lemma 7. *Assume that all tilting modules for A are cotilting.*

1. $\mathrm{inv}_{\mathrm{add}(T)}(M)$ *is defined for all $M \in A$-mod and* $\mathrm{inv}_{\mathrm{add}(T)}(M) = \dim_{\mathcal{F}(\overline{\Delta})}(M)$.

2. $\mathrm{coinv}_{\mathrm{add}(T)}(M)$ *is defined for all $M \in A$-mod and* $\mathrm{coinv}_{\mathrm{add}(T)}(M) = \dim_{\mathcal{F}(\overline{\nabla})}(M)$.

Proof. We prove the first statement and one proves the second statement by dual arguments. As in the proof of Lemma 5, considering an injective coresolution of M, we obtain a complex in $\mathrm{Com}(\mathrm{add}(T))$ of the form (10) such that $H^i(\mathcal{Y}^\bullet) = 0$ for all $i \neq 0$ and $H^0(\mathcal{Y}^\bullet) \cong M$. Hence $\mathrm{inv}_{\mathrm{add}(T)}(M)$ is defined.

Assume now that $\dim_{\mathcal{F}(\overline{\Delta})}(M) = t < \infty$. If $t = 0$, then $M \in \mathcal{F}(\overline{\Delta})$ and we have $\mathrm{inv}_{\mathrm{add}(T)}(M) = 0$ by Lemma 5. Let now $t > 0$. Using the dimension shift, from the short exact sequence

$$0 \longrightarrow \mathrm{Im}(f_{-1}) \longrightarrow \ker(f_0) \longrightarrow M \longrightarrow 0$$

we have $\dim_{\mathcal{F}(\overline{\Delta})}(\mathrm{Im}(f_{-1})) = t - 1$ by [MP, Lemma 1]. Now for all $i = 1, \dots, t-1$ from the short exact sequences

$$0 \longrightarrow \mathrm{Im}(f_{-i-1}) = \ker(f_{-i}) \longrightarrow Y_{-i} \longrightarrow \mathrm{Im}(f_{-i}) \longrightarrow 0$$

we analogously obtain $\dim_{\mathcal{F}(\overline{\Delta})}(\ker(f_{-t+1})) = 0$ that is $\ker(f_{-t+1}) \in \mathcal{F}(\overline{\Delta})$. Since $\mathcal{F}(\nabla)$ is closed under taking cokernels of monomorphsims, we get $\mathrm{Im}(f_{-t}) = \ker(f_{-t+1}) \in \mathcal{F}(\nabla)$ by induction and thus $\ker(f_{-t+1})$ is a (co)tilting module. This implies that the complex

$$\mathcal{Y}^\bullet_M : \qquad \dots \longrightarrow 0 \to \ker(f_{-t+1}) \hookrightarrow Y_{-t+1} \stackrel{f_{-t+1}}{\longrightarrow} Y_{-t+2} \stackrel{f_{-t+2}}{\longrightarrow} \dots$$

belongs to $\mathrm{Com}(\mathrm{add}(T))$ and satisfies $H^i(\mathcal{Y}^\bullet_M) = 0$ for all $i \neq 0$ and $H^0(\mathcal{Y}^\bullet_M) \cong M$, implying $\mathrm{inv}_{\mathrm{add}(T)}(M) \leq t$.

Finally, let $\mathrm{inv}_{\mathrm{add}(T)}(M) = t < \infty$. Then from $\mathcal{X}^\bullet_M$ (see (11)) we get an exact sequence

$$0 \longrightarrow X_0 \stackrel{f_0}{\longrightarrow} X_1 \stackrel{f_1}{\longrightarrow} \dots \stackrel{f_{t-2}}{\longrightarrow} X_{t-1} \stackrel{f_{t-1}}{\longrightarrow} \ker(f_t) \longrightarrow M \to 0 \qquad (15)$$

and an exact sequence

$$0 \longrightarrow \ker(f_t) \longrightarrow X_t \longrightarrow X_{t+1} \longrightarrow \dots$$

By Lemma 3 we have $\ker(f_t) \in \mathcal{F}(\overline{\Delta})$ and thus $\dim_{\mathcal{F}(\overline{\Delta})}(M) \leq t$ from (15). This completes the proof. □

Corollary 4. *Assume that all tilting modules for A are cotilting. Then for all $M \in A$*-mod *with* p. d.$(M) < \infty$ *holds* $\dim_{\mathcal{F}(\Delta)}(M) = \dim_{\mathcal{F}(\overline{\Delta})}(M)$.

Corollary 5. *Let A be as in Theorem 1 and $M \in A$*-mod. *Then*

$$\mathrm{p.\,d.}(M) \leq \dim_{\mathcal{F}(\Delta)}(M) + \mathrm{p.\,d.}(T).$$

Proof. It is obvious that p. d.(M) is finite if and only if $\dim_{\mathcal{F}(\Delta)}(M)$ is. Having M, for which $\dim_{\mathcal{F}(\Delta)}(M) = t < \infty$, we consider the bounded tilting complex (11) for M. Analogously to Lemma 4 we construct a new complex, in which we substitute all tilting modules X_i by their projective resolutions and the statement follows. □

4.4 Conjecture of Erdmann and Parker

In [EP, 5.3] the following conjecture is formulated: let A be a quasi-hereditary algebra having a simple preserving duality $^\circ$, and M be an A-module such that $\mathrm{codim}_{\mathcal{F}(\nabla)}(M) = t$. Then $\mathrm{Ext}^{2t}_A(M^\circ, M) \neq 0$.

Let A be as in Theorem 1 and $M \in A$-mod. Combining Lemma 6 with Proposition 1 we immediately obtain an affirmative answer to the conjecture of Erdmann and Parker:

Corollary 6. *Let $M \in A$-mod and* $\dim_{\mathcal{F}(\Delta)}(M) = t < \infty$. *Then* $\mathrm{Ext}^{2t}_A(M, M^\circ) \neq 0$.

We remark that, following the proof of Proposition 1, one can get the following:

Proposition 2. *Let A be as in Theorem 1. Assume that there exists an indecomposable tilting module $T(i)$ such that $k =$* p. d.$(T(i)) >$ p. d.$(T(j))$ *for all $j \neq i$. Let $M, N \in A$-mod be such that* $\dim_{\mathcal{F}(\Delta)}(M) = \mathrm{codim}_{\mathcal{F}(\nabla)}(N) = k$. *Then* $\mathrm{Ext}^{2k}_A(M, N) \neq 0$.

The conditions of Proposition 2 are satisfied for example for the algebras of the regular blocks of the BGG category $\mathcal{O}$.

4.5 $\mathcal{S}$-subcategories in $\mathcal{O}$

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a fixed triangular decomposition of $\mathfrak{g}$ and $\mathcal{O}$ be the BGG-category $\mathcal{O}$ for $\mathfrak{g}$. We refer the reader to [MP] or [FKM] for detailed notation. Fix dominant integral weights λ and μ and denote by W_μ the stabilizer of μ in the Weyl group W. Denote by $\mathcal{O}^\lambda$ the block of $\mathcal{O}$ corresponding to λ. Let $I_1, \dots, I_s$ be a complete list of pairwise non-isomorphic indecomposable W_μ-antidominant injective modules from $\mathcal{O}^\lambda$ and $I = \oplus_{i=1}^s I_i$. Denote by $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$ the full subcategory of $\mathcal{O}^\lambda$, consisting of all modules M, which have a copresentation $0 \to M \to I^a \to I^b$. This category is a block of an $\mathcal{S}$-subcategory in $\mathcal{O}$, associated with Enright's completion functor, see [FKM]. In particular, in [FKM, Section 5] it was shown that $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$ is equivalent to the module category of a properly stratified algebra. This equivalence endows $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$ with an abelian structure, which is not inherited from that on $\mathcal{O}$, but the usual duality on $\mathcal{O}$ naturally induces a duality on $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$.

Tilting modules in $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$ are described in [FKM, Section 6] and are exactly the W_μ-dominant tilting modules in $\mathcal{O}^\lambda$. In particular, they are self-dual and hence cotilting. As an immediate corollary of Theorem 1 we get the following strengthening of [MP, Theorem 3]:

Corollary 7. *The following numbers are equal.*

(i) The finitistic dimension of $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$.

(ii) Twice the projective dimension of the characteristic tilting module in $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$.

(iii) Twice the $\overline{\Delta}$*-filtration dimension of* $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$.

(iv) Twice the projective dimension of the self-dual standard module in $\mathcal{O}^{\mathcal{S}}(\lambda, \mu)$.

4.6 Connection with exact Borel subalgebras

Let A be a quasi-hereditary algebra having a simple preserving duality, and B be an exact Borel subalgebra of A in the sense of [Ko1]. From [Ko1, Proposition 2.6] and [CZ, Theorem 3.3] it follows that projective dimensions of the characteristic tilting modules for A and B coincide, and this common value equals the global dimension of B. As an immediate corollary of Theorem 1 we obtain the following strengthening of [CZ, Theorem 3.3].

Corollary 8. $\mathrm{gl.dim}(A) = 2\,\mathrm{gl.dim}(B)$.

We remark that it is shown in [Ov] that for every quasi-hereditary algebra A there exists a Morita equivalent algebra A', having an exact Borel subalgebra. So Corollary 8 can always be used to compute the global dimension of A.

5 Appendix: $\mathrm{fin.\,dim}(A) \le 2\,\mathrm{p.\,d.}(T)$

In this Appendix we would like to present a short proof of the inequality $\mathrm{fin.\,dim}(A) \le 2\,\mathrm{p.\,d.}(T)$. This inequality was originally proved in [MP, Corollary 2], following the ideas of [CZ]. We assume that A is an in Theorem 1.

Since all tilting modules are cotilting, we have that all cotilting modules, in particular $\Delta(1) = \nabla(1)$, have finite projective dimension. From the short exact sequence

$$0 \to K \to C(i) \to \nabla(i) \to 0,$$

where K is filtered by $\nabla(j)$, $j < i$, we get, by induction in i, that all $\nabla(i)$ have finite projective dimension. This implies that all $I(i) \in \mathcal{F}(\nabla)$ have finite projective dimension. Using $^\circ$ we get that all $P(i)$ have finite injective dimension. Since every module embeds into an injective module, it follows that $\mathrm{fin.\,dim}(A)$ equals the maximal j such that $\mathrm{Ext}^j_A(I,P) \neq 0$. So, to complete the proof we have only to show that $\mathrm{Ext}^j_A(I,P) = 0$ for all $j > 2\,\mathrm{p.\,d.}(T)$. From the tilting coresolution for P (see proof of [AHLU2, Proposition 2.2]) we get $\mathrm{codim}_{\mathcal{F}(\nabla)}(P) \le \mathrm{p.\,d.}(T)$ and, dually, $\dim_{\mathcal{F}(\Delta)}(I) \le \mathrm{p.\,d.}(T)$. Let $\mathcal{X}^\bullet_I$ (resp. $\mathcal{Y}^\bullet_P$) be the complex (11) (resp. (12)) for I (resp. P), for $\mathcal{M} = \mathrm{add}(T)$. Using (2) we get $\mathrm{Hom}_{D^b(A)}(\mathcal{X}^\bullet_I, \mathcal{Y}^\bullet_P[l]) = 0$ for all $l > 0$ and thus for $s = 2\,\mathrm{p.\,d.}(T) - \mathrm{codim}_{\mathcal{F}(\nabla)}(P) - \dim_{\mathcal{F}(\Delta)}(I) \ge 0$ we have

$$\mathrm{Ext}^j_A(I,P) = \mathrm{Hom}_{D^b(A)}(\mathcal{X}^\bullet_I, \mathcal{Y}^\bullet_P[j - 2\,\mathrm{p.\,d.}(T) + s]) = 0$$

for all $j > 2\,\mathrm{p.\,d.}(T)$, completing the proof.

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