Categories of induced modules and projectively stratified algebras

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Abstract

We construct a generalization of the BGG–category $\mathcal{O}$, whose blocks correspond to projectively stratified algebras. We prove reciprocity formulae in these categories and present two classes of examples.

1 Introduction

Since its definition by Bernstein, Gelfand and Gelfand ([BGG]) the category $\mathcal{O}$ has become a basic object of study. It motivated the introduction of concepts such as quasi-hereditary algebras ([CPS1]), BGG-algebras ([I]) and Harish-Chandra bimodules ([S]). Generalizations of $\mathcal{O}$ have been studied by many authors (see, for example, [CF, FP, FM, R, RW]). Typically, the definition of Verma module is generalized by inducing from a parabolic subalgebra or by using infinite dimensional modules as input for the induction process. In particular, the so-called category $\mathcal{O}^{\alpha}$ defined in [CF] and studied in [FP] discusses such a setup, involving objects which are quite different from those in $\mathcal{O}$. However, in [FP] it is claimed that the situation still is described by quasi-hereditary algebras. Unfortunately, there is a gap in the proof of Proposition 3.4 in [FP], and the BGG-duality claimed in [FP] is in fact not true. To repair the situation one has to leave the class of quasi-hereditary algebras and pass to what we call projectively stratified algebras. In this way, one can keep the main features of the classical construction. The categories obtained in this way contain the chosen class of generalized Verma modules. They have enough projective objects. These projective objects are filtered by generalized Verma modules. And there is an analogue of the BGG-reciprocity formula.

The aim of this paper is to present a general construction that allows one to construct analogues of the category $\mathcal{O}$ associated with a parabolic subalgebra of a simple finite-dimensional complex Lie algebra. As an input for the induction process we choose simple modules (not necessarily finite-dimensional). We study the obtained categories and show that under some natural conditions they lead to projectively stratified algebras, which generalize quasi-hereditary algebras. Moreover, some analogue of BGG-duality holds. In particular, our construction can be applied to obtain the classical category $\mathcal{O}$, its generalization by Rocha-Caridi ([R]) and the categories from [FM]. Moreover, it can also be used to enlarge the mentioned $\mathcal{O}^\alpha$ category in such a way that we obtain projectively stratified algebras and an analogue of the BGG-reciprocity (this example is discussed in detail in Section 10). We present two different analogues of the BGG-reciprocity, and in particular, we obtain some structural results on simple modules over Lie algebras by comparing these analogues.

The paper is organized as follows: in Section 2 we collect basic information and describe the set-up. In Section 3 we define two main objects of the paper: an admissible category Λ (from which parabolic induction starts) and the category $\mathcal{O}(\mathcal{P},\Lambda)$ "derived" from Λ. In Section 4 we describe conditions that lead to the existence of projective modules in $\mathcal{O}(\mathcal{P},\Lambda)$. In Section 5 we show that under some natural assumptions $\mathcal{O}(\mathcal{P},\Lambda)$ decomposes into blocks such that each block is the module category of a finite-dimensional algebra. Moreover, we show that assuming such a decomposition for Λ with blocks being projectively stratified algebras we obtain that the blocks of $\mathcal{O}(\mathcal{P},\Lambda)$ also correspond to projectively stratified algebras. In Section 6 we prove an analogue of BGG-reciprocity for stratified algebras having a duality. In Section 7 we show that under some arithmetical conditions on the behavior of simple modules in Λ there is another natural analogue of BGG-reciprocity. In Section 8 we compare these reciprocities and show that together they give some arithmetical information about simple modules in Λ. In Section 9 we prove some auxiliary lemmas necessary to construct examples of admissible Λ and $\mathcal{O}(\mathcal{P},\Lambda)$ having a block decomposition. In Section 10 we describe in detail an example in which Λ is a category of dense $sl(2,\mathbb{C})$ modules, which covers, in particular, the main example in [FM] and improves the $\mathcal{O}^\alpha$ category from [FP]. In Section 11 we discuss an example of a category of Gelfand-Zetlin modules. Finally, in Section 12 we apply the main result of Section 8 to obtain some results about Gelfand-Zetlin modules.

2 Preliminaries

For a Lie algebra $\mathfrak{A}$ we will denote by $U(\mathfrak{A})$ the universal enveloping algebra of $\mathfrak{A}$ and by $Z(\mathfrak{A})$ the center of $U(\mathfrak{A})$.

Let B be an abelian subalgebra of $\mathfrak{G}$. A $\mathfrak{G}$-module V is called a *weight* module (with respect to B) if $V=\oplus_{\lambda\in B^*}V_\lambda$ where $V_\lambda=\{v\in V\,|\,hv=\lambda(h)v \text{ for all } h\in B\}$.

Let $\mathfrak{H}$ be a Cartan subalgebra of the semisimple complex Lie algebra $\mathfrak{G}$ and let $\mathcal{P}$ be a parabolic subalgebra of $\mathfrak{G}$ with Levi decomposition $\mathcal{P}=(\mathfrak{A}\oplus\mathfrak{H}_\mathfrak{A})\oplus\mathfrak{N}$ where $\mathfrak{A}$ is a semisimple Lie algebra, $\mathfrak{H}_\mathfrak{A}\subset\mathfrak{H}$, $[\mathfrak{A},\mathfrak{H}_\mathfrak{A}]=0$ and $\mathfrak{N}$ is nilpotent. Let Δ be the root system of $\mathfrak{G}$, $\mathfrak{G}=\mathfrak{H}\oplus\sum_{\alpha\in\Delta}\mathfrak{G}_\alpha$ be the root decomposition of $\mathfrak{G}$ and $\mathfrak{N}=\sum_{\alpha\in\Delta(\mathfrak{N})}\mathfrak{G}_\alpha$ be the root decomposition of $\mathfrak{N}$. Set $\mathfrak{N}_-=\sum_{\alpha\in-\Delta(\mathfrak{N})}\mathfrak{G}_\alpha$.

Let $\Omega_\mathcal{P}$ denote the set of representatives of the isomorphism classes of simple $\mathfrak{A}\oplus\mathfrak{H}_\mathfrak{A}$-modules. Since $\mathfrak{H}_\mathfrak{A}$ is abelian and central it acts on any simple $V\in\Omega_\mathcal{P}$ via some $\lambda\in\mathfrak{H}_\mathfrak{A}^*$, i.e. $hv=\lambda(h)v$ for all $v\in V$ and $h\in\mathfrak{H}_\mathfrak{A}$. Let V be an $\mathfrak{A}\oplus\mathfrak{H}_\mathfrak{A}$-module (not necessarily simple or indecomposable). We can consider V as a $\mathcal{P}$-module with the trivial action of $\mathfrak{N}$ and construct a $\mathfrak{G}$-module

$$M_\mathcal{P}(V)=U(\mathfrak{G})\otimes_{U(\mathcal{P})}V.$$

If V is simple the module $M_{\mathcal{P}}(V)$ is usually called a *generalized Verma module.* The main properties of the modules $M_{\mathcal{P}}(V)$ are collected in the following proposition ([CF]).

Proposition 1. *Let V be an $\mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}$-module. Then:*

1. *$M_{\mathcal{P}}(V)$ is a free $U(\mathfrak{N}_-)$-module isomorphic to $U(\mathfrak{N}_-) \otimes V$ as a vector space.*
2. *Assume that $\mathfrak{H}_{\mathfrak{A}}$ acts on V via some $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$. Then $M_{\mathcal{P}}(V)$ is a weight module with respect to $\mathfrak{H}_{\mathfrak{A}}$ and $M_{\mathcal{P}}(V)_\lambda \simeq V$.*
3. *If V is simple then $M_{\mathcal{P}}(V)$ has a unique maximal submodule.*
4. *Let W be a $\mathfrak{G}$-module generated by a simple $\mathcal{P}$-submodule V on which $\mathfrak{N}$ acts trivially. Then W is a homomorphic image of $M_{\mathcal{P}}(V)$.*

Proof. Follows from the PBW-Theorem, the construction of $M_{\mathcal{P}}(V)$ and universal properties of the tensor product. $\square$

For $V \in \Omega_{\mathcal{P}}$ we will denote by $L_{\mathcal{P}}(V)$ the unique irreducible quotient of $M_{\mathcal{P}}(V)$ which occurs with multiplicity one in a composition series of $M_{\mathcal{P}}(V)$.

For a fixed basis S of the root system of $\mathfrak{A}$ one can consider the S-Harish-Chandra homomorphism φ_S (or *generalized Harish-Chandra homomorphism*) defined in [DFO]. Let $S(\mathfrak{H}_{\mathfrak{A}})$ denote the symmetric algebra of $\mathfrak{H}_{\mathfrak{A}}$ and $K = Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}})$. Let $i : Z(\mathfrak{G}) \to K$ be the restriction of φ_S to $Z(\mathfrak{G})$. It induces a natural map $i^* : K^* \to Z(\mathfrak{G})^*$ and the cardinal $|(i^*)^{-1}(\theta)|$ is finite for any $\theta \in Z(\mathfrak{G})^*$.

A category Λ of Lie algebra modules is said to have a *block decomposition* if $\Lambda = \oplus_i \Lambda_i$ is a direct sum of full subcategories Λ_i, each of which has only finitely many non-isomorphic simple modules.

3 Admissible categories and category $\mathcal{O}(\mathcal{P}, \Lambda)$

Let $\tilde{\mathfrak{A}}$ be a reductive complex finite-dimensional Lie algebra with semisimple part $\mathfrak{A}$ and let Λ be a full subcategory of the category of all finitely generated $\tilde{\mathfrak{A}}$-modules.

Definition 1. *The category Λ will be called* admissible *if the following conditions are satisfied:*

1. *Λ has an abelian structure (which is not necessary inherited from the category of all modules) and the endomorphism ring of any simple object in Λ is $\mathbb{C}$.*
2. *Any $M \in \Lambda$ is weight with respect to the center of $\tilde{\mathfrak{A}}$.*
3. *For any finite-dimensional simple $\tilde{\mathfrak{A}}$-module F, $F \otimes \_$ is an exact endofunctor on Λ.*

We note that, according to the last assumption, $F \otimes M$ has finite length for any $M \in \Lambda$. So far it is not known whether this assumption is superfluous (see [K]).

In what follows we will always assume that Λ is an admissible category and will often consider the objects in Λ only as $\mathfrak{A}$-modules. Let $\mathcal{P} = (\mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}) \oplus \mathfrak{N}$ be a parabolic subalgebra of $\mathfrak{G}$ with the Levi factor $\mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}$, where $\mathfrak{A}$ is semisimple, $\mathfrak{H}_{\mathfrak{A}}$ is abelian, $[\mathfrak{A}, \mathfrak{H}_{\mathfrak{A}}] = 0$ and $\tilde{\mathfrak{A}} \simeq \mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}$.

Let Λ be an admissible category of $\tilde{\mathfrak{A}}$-modules. Denote by $\mathcal{O}(\mathcal{P}, \Lambda)$ the full subcategory of the category of $\mathfrak{G}$-modules consisting of modules which are

1. finitely generated;

2. $\mathfrak{N}$-finite;

3. direct sums of modules from Λ, when viewed as $\tilde{\mathfrak{A}}$-modules.

Assume that the abelian structure on Λ naturally extends to an abelian structure on $\mathcal{O}(\mathcal{P}, \Lambda)$ and for any finite-dimensional $\mathfrak{G}$-module F, $F \otimes \_$ is an exact endofunctor on $\mathcal{O}(\mathcal{P}, \Lambda)$ with respect to this abelian structure.

Proposition 2. *1. The modules $M_{\mathcal{P}}(W)$ and $L_{\mathcal{P}}(W)$ are objects of $\mathcal{O}(\mathcal{P}, \Lambda)$ for any simple $W \in \Lambda$.*

2. If V is a simple module in $\mathcal{O}(\mathcal{P}, \Lambda)$ then $V \simeq L_{\mathcal{P}}(W)$ for some simple $W \in \Lambda$.

Proof. To prove the first statement it is enough to show that as an $\mathfrak{A}$-module $M_{\mathcal{P}}(W)$ decomposes into a direct sum of modules from Λ. This follows from the fact that $M_{\mathcal{P}}(W) \simeq U(\mathfrak{N}_-) \otimes W$ as a vector space by Proposition 1 and this isomorphism carries over the decomposition of $U(\mathfrak{N}_-)$ as a direct sum of finite-dimensional $\mathfrak{A}$-modules with respect to the adjoint action. We conclude that $M_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P}, \Lambda)$ and also $L_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P}, \Lambda)$.

Let V be a simple module in $\mathcal{O}(\mathcal{P}, \Lambda)$. Since V is $\mathfrak{N}$-finite and $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable there exists a non-zero element $v \in V$ such that $\mathfrak{N}v = 0$ and $hv = \lambda(h)v$ for all $h \in \mathfrak{H}_{\mathfrak{A}}$ and some $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$. Put $V_\lambda = U(\mathfrak{A})v$. Then $\mathfrak{N}w = 0$ for any $w \in V_\lambda$ implying that V_λ is a simple $\mathfrak{A}$-module and $V \simeq L_{\mathcal{P}}(V_\lambda)$ by Proposition 1. This completes the proof. $\square$

The module $M_{\mathcal{P}}(V)$ will be called *standard module* if V is an indecomposable projective in Λ. A module $M \in \mathcal{O}(\mathcal{P}, \Lambda)$ is said to have a *standard filtration* if there is a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$$

such that each M_i/M_{i-1} is a standard module.

Proposition 3. *Let Λ be admissible and let V be a projective module in Λ. Fix a non-negative integer k and consider $(U(\mathfrak{N})/(U(\mathfrak{N})\mathfrak{N}^k)$ as a $\mathcal{P}$-module under adjoint action. Then the module*

$$P(V, k) = U(\mathfrak{G}) \bigotimes_{U(\mathcal{P})} \left((U(\mathfrak{N})/(U(\mathfrak{N})\mathfrak{N}^k)) \otimes V \right)$$

has a standard filtration.

Proof. Since V is an $\mathfrak{H}_{\mathfrak{A}}$-weight module, so is $P(V,k)$. Moreover, since k is finite, among all the weights of $P(V,k)$ there exists a maximal, say λ, with respect to a natural order. Consider the $\mathfrak{A}$-module $P(V,k)_\lambda$. The PBW theorem guarantees that the $U(\mathfrak{G})$–submodule generated by $P(V,k)_\lambda$ in $P(V,k)$ is $U(\mathfrak{N}_-)$-free. Since $U(\mathfrak{G})$ is a direct sum of finite-dimensional $\mathfrak{A}$-modules under the adjoint action, it follows that $P(V,k)_\lambda$ is isomorphic to $V \otimes F$ as an $\mathfrak{A}$-module for some finite-dimensional module F. Since Λ is admissible, $V \otimes F \in \Lambda$. Further, as tensoring with a finite-dimensional module is an exact functor, we conclude that $V \otimes F = \oplus_t X(t)$ and each $X(t)$ is projective in Λ. Since λ is a maximal weight it follows that all $M_{\mathcal{P}}(X(t))$ are submodules in $P(V,k)$ so we can construct the first steps of our filtration. Now one has just to proceed by induction. This completes the proof. $\square$

For a given standard filtration of a module $M \in \mathcal{O}(\mathcal{P},\Lambda)$ we will denote by $[M : M_{\mathcal{P}}(V)]$ the number of occurrences of $M_{\mathcal{P}}(V)$ as a subquotient of this filtration. Note that this may depend on the choice of a standard filtration.

4 Projective objects in $\mathcal{O}(\mathcal{P},\Lambda)$

The module $P(V,k)$ constructed in Proposition 3 need not be projective in $\mathcal{O}(\mathcal{P},\Lambda)$ in general. In order to construct projective modules in $\mathcal{O}(\mathcal{P},\Lambda)$ we have to assume that $\mathcal{O}(\mathcal{P},\Lambda)$ has certain properties.

A direct summand (or block) $\mathcal{O}_i$ of $\mathcal{O}(\mathcal{P},\Lambda)$ is said to be *quasi-finite* (respectively *finite*) if the set of highest weights (with respect to $\mathfrak{H}_{\mathfrak{A}}$) of all simple modules in $\mathcal{O}_i$ is finite (respectively $\mathcal{O}_i$ contains only finitely many simple objects up to isomorphism). We will say that $\mathcal{O}(\mathcal{P},\Lambda)$ has a *quasi block decomposition* if $\mathcal{O}(\mathcal{P},\Lambda)$ decomposes into a direct sum of quasi-finite full subcategories. For example, by virtue of the finiteness of $|(i^*)^{-1}(\theta)|$ (see the note about the generalized Harish-Chandra homomorphism in Section 2) it is easy to see that $\mathcal{O}(\mathcal{P},\Lambda)$ has a quasi block decomposition if any module in Λ is locally finite over $Z(\mathfrak{A})$. In this case $\Lambda = \oplus_{\theta \in Z(\mathfrak{A})^*} \Lambda_\theta$, where Λ_θ consists of those $M \in \Lambda$ on which the ideal of $Z(\mathfrak{A})$ corresponding to θ acts locally nilpotent. By the same argument, if each Λ_θ has only finitely many non-isomorphic simple modules, $\mathcal{O}(\mathcal{P},\Lambda)$ has a block decomposition.

Theorem 1. *Suppose that $\mathcal{O}_i$ is a quasi-finite block of $\mathcal{O}(\mathcal{P},\Lambda)$ and V is an indecomposable projective in Λ such that $M_{\mathcal{P}}(V) \in \mathcal{O}_i$. Then for any k big enough, the $\mathcal{O}_i$–projection P_i of $P(V,k)$ is projective in $\mathcal{O}(\mathcal{P},\Lambda)$.*

Proof. Let λ be an $\mathfrak{H}_{\mathfrak{A}}$-weight of V. Since $\mathcal{O}_i$ is quasi-finite, there exist a positive integer N such that $\mathfrak{N}^{(N)} M_\lambda = 0$ holds for all $M \in \mathcal{O}_i$. Let $k > N$. From the construction of $P(V,k)$ it follows that there is a canonical isomorphism between $\mathrm{Hom}_{\mathfrak{G}}(P_i, M)$ and $\mathrm{Hom}_{\mathfrak{A}\oplus\mathfrak{H}_{\mathfrak{A}}}(V, M_i)$ for any $M \in \mathcal{O}(\mathcal{P},\Lambda)$ (here M_i denotes the direct summand of M lying in $\mathcal{O}_i$). Since V is projective in Λ we conclude that P_i is projective in $\mathcal{O}(\mathcal{P},\Lambda)$, as stated. $\square$

Corollary 1. *Suppose that Λ has enough projective modules (i.e. any simple module is a quotient of a projective module) and $\mathcal{O}_i$ is a quasi-finite block of $\mathcal{O}(\mathcal{P},\Lambda)$, then*

1. *$\mathcal{O}_i$ has enough projective modules;*

2. *Every projective in $\mathcal{O}_i$ has a standard filtration;*

3. *There is a one-to-one correspondence between the simple objects in $\mathcal{O}_i$ and the indecomposable projective objects in $\mathcal{O}_i$.*

Proof. The second statement follows from Proposition 3. The first and the third ones follow from Theorem 1 using the same arguments as in [BGG, Corollary 1]. □

Corollary 2. *Suppose that Λ has enough projective modules and $\mathcal{O}(\mathcal{P},\Lambda)$ has a quasi block decomposition, then*

1. *$\mathcal{O}(\mathcal{P},\Lambda)$ has enough projective modules;*

2. *Every projective in $\mathcal{O}(\mathcal{P},\Lambda)$ has a standard filtration;*

3. *There is a one-to-one correspondence between the simple objects in $\mathcal{O}(\mathcal{P},\Lambda)$ and the indecomposable projective modules in $\mathcal{O}(\mathcal{P},\Lambda)$.*

5 Finite-dimensional algebras arising from $\mathcal{O}(\mathcal{P},\Lambda)$

Theorem 2. *Suppose that Λ has enough projective modules and $\mathcal{O}_i$ is a finite block of $\mathcal{O}(\mathcal{P},\Lambda)$. Then $\mathcal{O}_i$ is equivalent to the module category over a finite-dimensional algebra.*

Proof. Consider the endomorphism algebra of the sum of projective covers of all simple modules in $\mathcal{O}_i$. □

Corollary 3. *Suppose that Λ has enough projective modules and $\mathcal{O}(\mathcal{P},\Lambda)$ has a block decomposition. Then each block of $\mathcal{O}(\mathcal{P},\Lambda)$ is equivalent to the module category over a finite-dimensional algebra.*

Now we discuss which finite-dimensional algebras can appear in this way.

Definition 2. *Let A be a finite dimensional algebra. A two-sided ideal J in A is called* projectively stratifying *if J is generated (as a two-sided ideal) by a primitive idempotent and J is projective as a left A-module.*

The algebra A is called projectively stratified *if there exists an ordering $e_1,\dots,e_n$ of the equivalence classes of primitive idempotents of A such that for each l the idempotent e_l generates a projectively stratifying ideal in the quotient algebra $A/<e_1,\dots,e_{l-1}>$.*

This is equivalent to requiring that each projective module P has a filtration of the following form: $0=M_0\subset M_1\subset M_2\subset\cdots\subset M_n=P$ where M_{l+1}/M_l is a direct sum of copies of the module $(A/<e_1,\dots,e_l>)\cdot e_{l+1}$ which is projective over the quotient algebra $A/<e_1,\dots,e_l>$.

A projectively stratified algebra is a stratifying endomorphism algebra in the sense of Cline, Parshall and Scott [CPS2, ADL]. Hence its derived category admits a stratification, that is a sequence of recollements, where the local algebras are the endomorphism rings E_l of the modules $(A/ < e_1, \ldots, e_{l-1} >) \cdot e_l$.

Any quasi-hereditary algebra is projectively stratified. A projectively stratified algebra A is quasi-hereditary if and only if all the rings E_l are semisimple if and only if A has finite global dimension ([CPS2, ADL]).

Theorem 3. *Assume that Λ is a sum of module categories of projectively stratified algebras. Then any finite block of $\mathcal{O}(\mathcal{P}, \Lambda)$ also is the module category of a projectively stratified algebra.*

Proof. By Proposition 2 there is a natural bijection between simple objects in Λ and $\mathcal{O}(\mathcal{P}, \Lambda)$. The induction process can glue several blocks of Λ together into one block of $\mathcal{O}(\mathcal{P}, \Lambda)$. Assume that a finite block of $\mathcal{O}(\mathcal{P}, \Lambda)$ is given and call it $\mathcal{O}_i$. Fix the direct summand Λ_i of Λ (in general, this is a product of several blocks) such that the above bijection restricts to a bijection between Λ_i-simples and $\mathcal{O}_i$-simples.

The functors occurring in the construction of projective objects in $\mathcal{O}(\mathcal{P}, \Lambda)$ are exact and hence transport filtrations from Λ to $\mathcal{O}(\mathcal{P}, \Lambda)$. Start with a module V which contains (up to an isomorphism) at least one copy of each isomorphism class of each non-isomorphic indecomposable projectives in Λ_i. Then the tensor product $(U(\mathfrak{N})/(U(\mathfrak{N})\mathfrak{N}^k)) \otimes V$ again is projective (in Λ) and maps onto all projectives in Λ_i. Hence it contains at least one copy of each isomorphism class of indecomposable projectives in Λ_j. A filtration of this module as in the definition of projectively stratified algebra yields a similar filtration of the induced $U(\mathfrak{G})$-module. Since the number of isomorphism classes of indecomposable projectives in the block $\mathcal{O}_i$ equals the number of indecomposable projectives in Λ_j, the resulting filtration has the correct length. □

Note that we do not know in general how filtration multiplicities change during the tensoring process.

Corollary 4. *Under the conditions of Theorem 3 the following are equivalent for a finite block $\mathcal{O}_i$ of $\mathcal{O}(\mathcal{P}, \Lambda)$:*

1. *The block $\mathcal{O}_i$ is equivalent to the module category of a quasi-hereditary algebra.*
2. *The block $\mathcal{O}_i$ has finite global dimension.*
3. *For any simple $L(V) \in \mathcal{O}_i$ the module V is projective in Λ.*

Proof. Obvious. □

In Section 10 we will construct an example of a projectively stratified non quasi-hereditary algebra, arising in the way described above.

6 First analogue of BGG-reciprocity

For projectively stratified algebras there is an 'abstract' version of BGG-reciprocity (see also [ADL, GM] for some analogous results).

Theorem 4. *Let A be a projectively stratified algebra over an algebraically closed field k. Assume that A has a duality (i.e. a contravariant exact equivalence, which preserves isomorphism classes of simple objects). Assume also that each projective A–module has a filtration by "Verma modules" $M(i)$ (indexed by i in I, the set of isomorphism classes of indecomposable projective A–modules) satisfying $(M(i) : L(i)) = 1$ and $(M(i) : L(j)) \neq 0$ implies $j \leq i$. Denote by $l(i)$ the number $[P(i) : M(i)]$ of occurrences of $M(i)$ in a mentioned filtration of $P(i)$, which coincides with the dimension of the endomorphism ring of the i-th standard module $\Delta(i)$ (see [CPS2]). Then for all $i, j \in I$ there is a BGG-reciprocity (or Brauer-Humphreys-BGG reciprocity):*

$$[P(i) : M(j)] = l(j)(M(j) : L(i)).$$

We note that the only properties of Verma modules needed here are the ones mentioned in the assumptions. No universality is needed.

Proof. We proceed by induction along the filtration of A which makes it a projectively stratified algebra. Let j be a maximal index. Write $P(i) = Ae$ and $P(j) = Af$ for some primitive idempotents e and f. By the choice of j the trace ideal AfA is projective as a left module. We have $Ae \cap AfA = (Af)^l$ for some l which can be computed as $l = \dim_k \mathrm{Hom}_A(Af, Ae)/l(j)$. By the condition on Verma modules, all occurrences of $M(j)$ in a filtration of A are inside the ideal AfA. Hence $[P(i) : M(j)] = l \cdot l(j) = \dim_k \mathrm{Hom}_A(Af, Ae) = \dim_k(fAe)$. Applying the duality on A we get $\dim_k(fAe) = \dim_k(eAf) = (P(j) : L(i))$. Again by the defining condition on Verma modules we have $(P(j) : L(i)) = l(j) \cdot (M(j) : L(i))$. $\square$

When all $l(i) = 1$ we will obtain a quasi-hereditary algebra and the classical BGG-reciprocity.

7 Second analogue of BGG-reciprocity

Here is another analogue of BGG-reciprocity, which covers not only the classical case but also the examples to be discussed in Sections 10 and 11.

Theorem 5. *Assume that Λ a block decomposition with each block being the module category of a local algebra and $\mathcal{O}(\mathcal{P}, \Lambda)$ has a block decomposition. Assume also that for any simple modules X and Y in Λ there exists a constant $\mathfrak{i}(X, Y)$ such that $((F \otimes X) : Y) = \mathfrak{i}(X, Y)((F \otimes Y) : X)$ holds for all finite-dimensional $\mathfrak{A}$-module F. (We will call this condition the* duality condition.*) Then*

$$[P(L(V)) : M_{\mathcal{P}}(W)] = \mathfrak{i}(V, W)\mathfrak{l}(\tilde{V})(M_{\mathcal{P}}(W) : L(V))$$

holds for any two simple modules V and W in Λ, where $P(L(X))$ is the projective cover of $L(X)$ in $\mathcal{O}(\mathcal{P},\Lambda)$ and $\mathfrak{l}(\tilde{X})$ is the multiplicity of a simple module X in its projective cover $\tilde{X}$ in Λ (which coincides with the length of $\tilde{X}$).

Proof. By Theorem 3 each block of $\mathcal{O}(\mathcal{P},\Lambda)$ corresponds to a projectively stratified finite-dimensional algebra. First we note that $\dim \mathrm{Hom}(P(L(V)),M) = (M : L(V))$ for any module $M \in \mathcal{O}(\mathcal{P},\Lambda)$. Thus we have only to show that

$$[P(L(V)) : M_{\mathcal{P}}(W)] = \mathfrak{i}(V,W)\mathfrak{l}(\tilde{V}) \dim \mathrm{Hom}(P(L(V)), M_{\mathcal{P}}(W))$$

for any two simple modules V and W in Λ. Fix a block $\mathcal{O}_j$. Clearly, we need to check our equality inside $\mathcal{O}_j$ only, so we can assume that $L(V)$ and $L(W)$ belong to $\mathcal{O}_j$. Let k be big enough. Let $P(V,k)_j$ be the direct summand of $P(V,k)$ in $\mathcal{O}_j$. Then

$$P(V,k)_j = \sum_{L(K)\in \mathrm{Irr}(\mathcal{O}_j)} n_K(V) P(L(K))$$

and

$$n_K(V) = \dim \mathrm{Hom}_{\mathfrak{G}}(P(V,k)_j, L(K)) = \dim \mathrm{Hom}_{\mathfrak{A}\oplus\mathfrak{H}_{\mathfrak{A}}}(\tilde{V}, L(K)).$$

In particular $n_K(V) = 0$ if $V \not< K$ with respect to the order induced from $\mathfrak{H}_{\mathfrak{A}}$ and $n_V(V) = 1$. This allows us to proceed by induction. From the linearity of our formula (in the induction step) we obtain that it is enough to prove that $[P(V,k)_j : M_{\mathcal{P}}(W)] = \mathfrak{i}(V,W)\mathfrak{l}(\tilde{V}) \dim \mathrm{Hom}(P(V,k)_j, M_{\mathcal{P}}(W))$. Further it is clear that we only have to check that $[P(V,k) : M_{\mathcal{P}}(W)] = \mathfrak{i}(V,W)\mathfrak{l}(\tilde{V}) \dim \mathrm{Hom}(P(V,k), M_{\mathcal{P}}(W))$. Clearly, from the construction of $P(V,k)$ it follows that there exists a finite-dimensional $\mathfrak{A}$-module F such that $[P(V,k) : M_{\mathcal{P}}(W)] = ((F \otimes \tilde{V}) : W) = \mathfrak{l}(\tilde{V})((F \otimes V) : W)$. On the other hand $\dim \mathrm{Hom}(P(V,k), M_{\mathcal{P}}(W)) = \dim \mathrm{Hom}_{\mathfrak{A}\oplus\mathfrak{H}_{\mathfrak{A}}}(\tilde{V}, M_{\mathcal{P}}(W)) = \dim \mathrm{Hom}_{\mathfrak{A}}(\tilde{V}, F \otimes W) = ((F \otimes W) : V)$ by the projectivity of $\tilde{V}$. Application of the duality condition for Λ completes the proof. □

8 Comparing these reciprocities

Comparing Theorem 4 with Theorem 5 one obtains the following result characterizing the behavior of simple modules.

Theorem 6. *Assume that Λ has a block decomposition with each block being the module category of a local algebra and the duality condition is satisfied. Assume also that there are only finitely many simples in Λ having the same central character and $\mathcal{O}(\mathcal{P},\Lambda)$ has a duality (for some $\mathfrak{G}$ and $\mathcal{P}$). Then $\mathfrak{i}(X,Y) = \mathfrak{l}(\tilde{Y})/\mathfrak{l}(\tilde{X})$.*

Proof. It is easy to see that all the conditions of both Theorem 4 and Theorem 5 are satisfied, hence the statement follows by comparing the two reciprocity formulae. □

9 How to construct examples

In this section we give some technical results needed later for constructing examples of categories Λ and $\mathcal{O}(\mathcal{P},\Lambda)$. Let $\mathfrak{A}$ be a semisimple complex finite-dimensional Lie algebra and let V be a simple $\mathfrak{A}$-module. Suppose that for any finite-dimensional $\mathfrak{A}$-module F the module $F\otimes V$ has finite length. Denote by $\tilde{\Lambda}=\tilde{\Lambda}(V)$ the set of isomorphism classes of indecomposable direct summands of modules $F\otimes V$, where F runs through the set of all simple finite-dimensional $\mathfrak{A}$-modules and let $\Lambda=\Lambda(V)$ be the closure of $\tilde{\Lambda}$ under operations of taking direct sums, submodules and quotients.

Lemma 1. *Any module in Λ has finite length. Moreover, Λ is closed under tensor products with finite-dimensional modules.*

Proof. Assme first that there exists a finite-dimensional $\mathfrak{A}$-module F' such that $V\otimes F'\simeq N\oplus N'$ for some $N'\in\Lambda$. Then $N\otimes F$ is a direct summand in $(N\oplus N')\otimes F$ and the last is isomorphic to $V\otimes F'\otimes F$. Since $F\otimes F'$ is finite-dimensional the statement follows from Weyl's theorem on complete reducibility of finite-dimensional $\mathfrak{A}$-modules. For submodules and quotients everything now follows from the exactness of tensor product with a finite-dimensional module. $\square$

Using the above lemma one can easily produce admissible categories over reductive algebras having $\mathfrak{A}$ as the semisimple part. By abuse of notation we will also denote the obtained category by $\Lambda=\Lambda(V)$. By virtue of the Kostant Theorem ([K, Theorem 5.2]), each module of $\Lambda(V)$ is locally finite over $Z(\mathfrak{A})$.

Lemma 2. *Any module in $\mathcal{O}(\mathcal{P},\Lambda)$ is locally finite over $Z(\mathfrak{G})$. Moreover, if, for any given central character, Λ contains only finitely many simple modules with that central character then any module in $\mathcal{O}(\mathcal{P},\Lambda)$ has finite length.*

Proof. As it was mentioned in Section 4, $\mathcal{O}(\mathcal{P},\Lambda)$ has a quasi-block decomposition. Let O_i be a quasi-finite block. By induction in $\mathfrak{H}_{\mathfrak{A}}$-weights one see that each $M\in O_i$ has a filtration

$$0\subset M_1\subset\cdots\subset M_k=M$$

such that each M_i/M_{i-1} is a quotient of some $M_{\mathcal{P}}(V_i)$. So, it is enough to prove the statment for all $M_{\mathcal{P}}(V_i)$. Using the generalized Harish-Chandra homomorphism, the action of $Z(\mathfrak{G})$ on $M_{\mathcal{P}}(V_i)$ can be computed from the actions of $\mathfrak{H}_{\mathfrak{A}}$ and $Z(\mathfrak{A})$ on V_i, which are locally finite, and the first statment follows.

If Λ contains only finitely many simple modules with that central character, then, as observed in Section 4, $\mathcal{O}(\mathcal{P},\Lambda)$ has a block decomposition. Since any module M in $\mathcal{O}(\mathcal{P},\Lambda)$ is finitely generated we have that each M_λ, $\lambda\in\mathfrak{H}_{\mathfrak{A}}^*$ is an $\mathfrak{A}$-module of finite length. Now the statement follows from the properties of the generalized Harish-Chandra homomorphism by the same arguments as in [D, Proposition 7.6.1]. $\square$

In the case of a reasonable choice for V one can prove the assumptions required in Corollary 2, Theorem 3, Theorem 3 or Theorem 5 by direct calculation, which will be illustrated in the subsequent examples.

10 The example $\mathfrak{A} = sl(2, \mathbb{C})$

Let $\mathfrak{A} = sl(2, \mathbb{C})$ with the standard basis $\{e, f, h\}$ and let $c = (h+1)^2 + 4fe$ be a Casimir element. Let Λ' be the category of all weight (with respect to $\mathbb{C}h$), torsion-free (i.e. e and f act injectively) $\mathfrak{A}$-modules. As the centralizer of the Cartan subalgebra in $U(\mathfrak{A})$ is commutative and generated by h and c, from [DFO, Lemma 3] it follows that simple weight torsion modules are parametrized by pairs $(\tilde{\lambda}, \gamma)$ where $\tilde{\lambda} \in \mathbb{C}/2\mathbb{Z}$ is the set of all eigenvalues of h, γ is the unique eigenvalue of c and $\gamma \neq (\lambda+1)^2$ for all $\lambda \in \tilde{\lambda}$.

Let X_1 and X_2 be simple modules in Λ' parametrized by $(\tilde{\lambda}_1, \gamma_1)$ and $(\tilde{\lambda}_2, \gamma_2)$ respectively. Suppose that $\mathrm{Ext}^1(X_1, X_2) \neq 0$. Since c belongs to the centre of $U(\mathfrak{A})$ we immediately obtain that $\gamma_1 = \gamma_2$. Also note that if V is an indecomposable weight $\mathfrak{A}$-module then $\mathrm{supp}\, V \subset \tilde{\mu}$ for some $\tilde{\mu} \in \mathbb{C}/2\mathbb{Z}$ implying that $\tilde{\lambda}_1 = \tilde{\lambda}_2$ and $X_1 \simeq X_2$. Also, there are no non-trivial self-extensions of a simple module in Λ' having an infinitesimal character.

Let V be a simple module in Λ'. Then applying [K, Theorem 5.1] one obtains that for any finite-dimensional $\mathfrak{A}$-module F the module $F \otimes V$ decomposes into a direct sum of indecomposable modules of length not greater than 2. It is easy to see that if none of the two simple modules X and Y is isomorphic to $V(\tilde{\lambda}, 0)$ then the duality condition (with $\mathfrak{i}(X, Y) = 1$) is equivalent to the following: for a fixed non-zero complex a and b and for any non-negative integer n holds $|\{a, -a\} \cap \{b - n + 2i | 0 \leq i \leq n\}| = |\{b, -b\} \cap \{a - n + 2i | 0 \leq i \leq n\}|$, which can be easily verified. It follows by direct calculation that $\mathfrak{i}(V(\tilde{\lambda}, 0), V(\tilde{\lambda}, k^2)) = 2$, $k \neq 0$, thus the duality condition is satisfied. To claim that $\Lambda(V)$ (see Section 9) satisfies all conditions necessary for Corollary 1 we have only to check the existence of projectives. If $V = V(\tilde{\lambda}, \gamma)$ and γ not a square of an integer, then by [K, Theorem 5.1] the module $F \otimes V$ is completely reducible for any finite-dimensional $\mathfrak{A}$-module F, hence there are enough projectives. Assume that $\gamma = k^2$, $k \in \mathbb{Z}_+$. Then it is easy to see that the simple modules in $\Lambda(V)$ are those of the form $V_{2l} = V(\tilde{\lambda}, (k+2l)^2)$, $l \in \mathbb{Z}$ and $V_{2l+1} = V(\widetilde{\lambda+1}, (k+2l+1)^2)$, $l \in \mathbb{Z}$. Hence, we can assume $k = 0$. Denote by F_l the unique $l+1$-dimensional simple $\mathfrak{A}$-module and define an indecomposable extension, $\tilde{V}_l$, of V_l by

$$F_l \otimes V(\tilde{\lambda}, 0) \simeq \tilde{V}_l \oplus V',$$

where $\mathrm{Hom}(V', V_l) = 0$. Clearly, $\tilde{V}_l$ has all subquotients isomorphic to V_l and a straightforward calculation shows that $\tilde{V}_l$ is indecomposable. It follows immediately from the following proposition that the modules $\tilde{V}_l$ are those needed.

Proposition 4. *For any $l, t \in \mathbb{Z}_+$ all indecomposable summands of $V_l \otimes F_t$ are of the form V_s or $\tilde{V}_s$ for $s \in \mathbb{Z}$.*

Proof. If $t \leq l$ then by [K, Theorem 5.1] the module $V_l \otimes F_t$ is completely reducible, hence our statement is true. Now we use an induction in $t - l = k$. We have already seen that for $k < 0$ our statement is true. Clearly, for $t = 1$ the statement is also true since the only $V_l \otimes F_1$ which is not completely reducible is $V_0 \otimes F_1$ for which the statement is trivial. Now $V_l \otimes F_t$ is a direct summand in $V_l \otimes (F_{t-1} \otimes F_1)$ since $F_{t-1} \otimes F_1 \simeq F_t \oplus F_{t-2}$ for $t > 1$, so it is enough to prove the statement for $V_l \otimes (F_{t-1} \otimes F_1) \simeq (V_l \otimes F_1) \otimes F_{t-1}$. The module

$V_l \otimes F_1$ either decomposes into a direct sum of some V_s or is isomorphic to $\tilde{V}_1$. For $V_s \otimes F_{t-1}$ everything follows from the inductive assumptions. Further $\tilde{V}_1 \otimes F_{t-1} \simeq V_0 \otimes F_1 \otimes F_{t-1}$ and the statement follows. $\square$

Thus $\Lambda(V)$ is admissible. It follows from [FM, Section 3, Example 2] (or from the arguments above) that for $V = V(\tilde{\lambda}, \gamma)$ with $\gamma \neq k^2$ for any $k \in \mathbb{Z}$, any simple module in $\Lambda(V)$ is projective. Suppose that $V = V(\tilde{\lambda}, k^2)$ for some $k \in \mathbb{Z}$. It follows from direct calculation, that for any $l \in \mathbb{Z}$ the modules $V(\tilde{\lambda}, (k+2l)^2)$ and $V(\widetilde{\lambda+1}, (k+2l+1)^2)$ are in $\Lambda(V)$, from which we deduce that $\Lambda(V)$ does not depend on the choice of $k \mod 2$. We can assume $k = 0$. Set $\Lambda(\tilde{\lambda}) = \Lambda(V)$, and from our restriction on $\tilde{\lambda}$ we have $\tilde{\lambda} \not\subset \mathbb{Z}$. Now it is easy to see that the modules $V(\tilde{\lambda}, (k+2l)^2)$, $V(\widetilde{\lambda+1}, (k+2l+1)^2)$, $l \in \mathbb{Z}$ exhaust the set of simple modules in $\Lambda(\tilde{\lambda})$. Moreover, any indecomposable module in $\Lambda(\tilde{\lambda})$ is isomorphic to either V_s or $\tilde{V}_s$ for $s \in \mathbb{Z}$.

Let $\alpha \in \Delta$, $\mathfrak{A} \simeq sl(2)$ be a subalgebra of $\mathfrak{G}$ generated by $\mathfrak{G}_{\pm\alpha}$ and let $\mathcal{P} = (\mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}) \oplus \mathfrak{N}$ be a parabolic subalgebra of $\mathfrak{G}$. Let Λ be an admissible category $\Lambda(\tilde{\lambda})$ (the case $\Lambda = \Lambda(V(\tilde{\lambda}, c))$ for c not a square integer has been considered in [FM, Section 5.3] and there it has been shown that in this case all blocks are highest weight categories). It follows by the arguments of Section 4 that $\mathcal{O}(\mathcal{P}, \Lambda)$ has a block decomposition. Applying Theorem 3 we obtain that any block of $\mathcal{O}(\mathcal{P}, \Lambda)$ is the module category over a finite-dimensional projectively stratified algebra. Applying Theorem 5 we obtain that for a simple $W \in \Lambda$ which is not isomorphic to $V(\tilde{\lambda}, 0)$ as an $\mathfrak{A}$-module the following two equalities are satisfied

$$[P(L(V)) : M_{\mathcal{P}}(W)] = 2(M_{\mathcal{P}}(W) : L(V)),$$

and

$$[P(L(V)) : M_{\mathcal{P}}(V(\tilde{\lambda}, 0))] = (M_{\mathcal{P}}(V(\tilde{\lambda}, 0)) : L(V)).$$

11 Gelfand-Zetlin example

Let $\mathfrak{A} \simeq sl(n, \mathbb{C})$ and X_i, Y_i, $i = 1, 2, \dots, n-1$ be the set of canonical generators of $\mathfrak{A}$. Fix a doubly indexed complex vector $[l] = (l_{i,j})_{i=1,2,\dots,n}^{j=1,2,\dots,i}$ satisfying the following conditions:

- $l_{i,j} - l_{i,k} \notin \mathbb{Z}$ for all $i = 1, 2, \dots, n-1$, $1 \leq j < k \leq i$;
- $l_{i,j} - l_{i+1,k} \notin \mathbb{Z}$ for all $i = 1, 2, \dots, n-1$, $j = 1, 2, \dots i$, $k = 1, 2, \dots, i+1$.

Set $S([l])$ to be the set of all tableaux $[t]$ satisfying the following conditions:

- $t_{n,j} = l_{n,j}$ for all $j = 1, 2, \dots, n$;
- $t_{i,j} - l_{i,j} \in \mathbb{Z}$ for all $i = 1, 2, \dots, n-1$, $j = 1, 2, \dots i$.

Let $V([l])$ be the complex space with basis $S([l])$. It is known ([DFO2]) that the formulae

$$X_i[t] = \sum_{j=1}^{i} -\frac{\prod_k (t_{i+1,k} - t_{i,j})}{\prod_{k\neq j}(t_{i,k} - t_{i,j})}([t] + [\delta^{ij}]), \quad Y_i[t] = \sum_{j=1}^{i} \frac{\prod_k (t_{i-1,k} - t_{i,j})}{\prod_{k\neq j}(t_{i,k} - t_{i,j})}([t] - [\delta^{ij}])$$

define on $V([l])$ the structure of a simple $\mathfrak{A}$-module. Moreover, it follows from a direct calculation with Kostant's theorem ([K, Theorem 5.1]), that the only simple subquotients which can appear in $F \otimes V([l])$ for finite-dimensional F are of the form $V([s])$, where $s_{i,j} = l_{i,j}$, $i < n$ and $s_{n,j} - l_{n,j} \in \mathbb{Z}$. Let $\Lambda = \Lambda(V([l]))$ be constructed as in Section 9. As non-isomorphic simples in Λ have different central character ([DFO2, Section 2]), Λ has a block decomposition with respect to central characters with a unique simple in each block. Further, it is also known ([DFO2, Proposition 31]) that self-extensions of $V([l])$ depend only on the action of the so-called Gelfand-Zetlin subalgebra, generated by $\mathfrak{H}$ and the union of centers of $sl(m,\mathbb{C})$, $m = 1, 2, \dots, n$ embedded into the left upper corner (i.e. $sl(m,\mathbb{C})$ is generated by X_i, Y_i, $i = 1, 2, \dots, m-1$). Applying [K, Theorem 5.1] to each center separately we obtain (from the choice of $[l]$) that all Ext are finite-dimensional and controlled by $Z(\mathfrak{A})$. Moreover, it also follows from [K, Theorem 5.1] that the lengths of indecomposable modules in $F \otimes V([l])$ are bounded for all finite-dimensional F. From this we easily deduce that Λ has enough projective modules, hence, taking into account the arguments in Section 4 and Section 9 we conclude that all conditions of Theorem 3 are satisfied. Thus any block of $\mathcal{O}(\mathcal{P}, \Lambda)$ corresponds to a projectively stratified finite-dimensional algebra.

Now we are going to show that the technical condition of Theorem 5 is also satisfied, hence an analogue of the BGG-reciprocity holds. For this we need some auxiliary results.

Lemma 3. *Let $\mathfrak{A}$ be a simple finite-dimensional complex Lie algebra, $\mathfrak{H}$ be its Cartan subalgebra and W be the Weyl group. For $\lambda, \mu \in \mathfrak{H}^*$ set $\lambda \sim \mu$ if and only if $\lambda \in W \cdot \mu$ (here $w \cdot \mu = w(\mu + \rho) - \rho$ is the standard dot-action of W). Then for any $\lambda, \mu \in \mathfrak{H}^*$ and any simple finite-dimensional $\mathfrak{A}$-module F holds*

$$|W \cdot \lambda| \sum_{\nu : \nu + \lambda \sim \mu} \dim F_\nu = |W \cdot \mu| \sum_{\nu : \nu + \mu \sim \lambda} \dim F_\nu.$$

Proof. Let W_λ (resp. W_μ) be the subgroup of W stabilizing λ (resp. μ). Then we can rewrite our equality in the form

$$|W_\mu| \sum_{\nu : \nu + \lambda \sim \mu} \dim F_\nu = |W_\lambda| \sum_{\nu : \nu + \mu \sim \lambda} \dim F_\nu.$$

Let $w_1, w_2, \dots, w_k$ be all the elements of W and define ν_i by $\nu_i + \lambda = w_i \cdot \mu$. We can rewrite the last equality as $w_i^{-1} \cdot \lambda = \mu - w_i^{-1}(\nu_i)$. Since $\dim F_\xi = \dim F_{w(\xi)}$ for all $\xi \in \mathfrak{H}^*$ and $w \in W$ we conclude that both the left hand side of the desired equality coincide with $\sum_{i=1}^{n} \dim F_{\nu_i}$ and the right hand side coincide with $\sum_{i=1}^{n} \dim F_{-\nu_i}$. The statment now follows from $\dim F_\nu = \dim F_{w_0(\nu)} = \dim F_{-\nu}$, where w_0 is the longest element in the Weyl group. □

For $\theta \in Z(\mathfrak{A})^*$ and $\lambda \in \mathfrak{H}^*$ we will write $\theta = \theta(\lambda)$ if θ is the central character of the Verma module $M(\lambda)$ (with highest weight $\lambda - \rho$). We recall that any simple module in $\Lambda = \Lambda(V([l]))$ is uniquely determined by its central character.

Lemma 4. *Let $V([s])$ and $V([t])$ be two simple modules in Λ and F a simple finite-dimensional $\mathfrak{A}$-module. Assume that $\theta(\mu_1)$ and $\theta(\mu_2)$ are the central characters of $V([s])$ and $V([t])$ respectively. Then the number of simple subquotients of the module $V([s]) \otimes F$ isomorphic to $V([t])$ equals $\sum_\nu \dim F_\nu$, where the sum is taken over all ν such that $\mu_1 + \nu \sim \mu_2$.*

Proof. Let Γ be the Gelfand-Zetlin subalgebra of $\mathfrak{G}$ ([MO]). First we note, that from [DFO, Proposition 21] it follows that each simple finite-dimensional $\mathfrak{G}$-module decomposes into a direct sum of simple (and thus one-dimensional, since Γ is commutative) non-isomorphic Γ-modules. Moreover, the same is true for any $V([l])$ as defined in this section (follows from [DFO, Proposition 21, Section 2.3]).

As the second step we remark, that from the construction of $V([l])$ and [DFO, Proposition 21] it follows that two modules $V([l^1])$ and $V([l^2])$ such that $l^1_{i,j} = l^2_{i,j}$ for all $1 \leq i \leq n-1$ and all j are isomorphic if and only if their central characters coincide.

Now, as it was noted above, by a direct application of [K, Theorem 5.1] one can easily determine the possible simple subquotients of $V([s]) \otimes F$, for example in the following way: a module $V([s'])$ with $s'_{i,j} = s_{i,j}$ for all $1 \leq i \leq n-1$ and all j, having a central chracter $\theta(\mu_{s'})$ can occur as a simple subquotient in $V([s]) \otimes F$ if and only if $W\mu_{s'}$ intersects the set $\mu_1 + P(F)$, where $P(F)$ states for the support of the module F. We will also denote by $\tilde{P}(F)$ the corresponding multi-support, i.e. the support, in which all weights are counted with their multiplicities.

Denote by Γ' a natural subalgebra of Γ, which is the Gelfand-Zetlin subalgebra of $U(sl(n-1))$. By construction, the module $V([l])$ is dense with respect to Γ' (i.e. its Γ'-support coincides with a weight lattice) and Γ'-weight subspaces are one-dimensional. Since tensoring with a finite-dimensional module preserves the weight lattice, we conclude that $V([s]) \otimes F$ is a dense module and all non-trivial Γ'-weight subspaces of it are of dimension $\dim F$. Hence, applying [DFO, Corollary 33], we obtain that the length of $V([s]) \otimes F$ equals $\dim F$.

Now we want to substitute $V([s])$ by a finite-dimensional module E. Suppose that E lies far enough from the walls, i.e. the length of $E \otimes F$ equals $\dim F$. We will call a Γ'-weight subspace of E generic provided the dimension of this weight subspace in $E \otimes F$ equals $\dim F$. Clearly, any E lying far from the walls has a generic Γ'-weight subspace. Fix E lying far from the walls and a generic Γ'-weight χ (i.e. E_χ is a generic Γ'-weight subspace). Fix $z \in Z(\mathfrak{G})$. Choose a basis in $(E \otimes F)_\chi$ and write the characteristic polynomial $f_z(X)$ of z in this basis. Let λ be a highest weight of E. ¿From the Littlewood-Richardson rule we obtain

$$f_z(X) = \pm \left(\prod_{\lambda' \in \tilde{P}(F)} (X - \theta(\lambda + \lambda')(z)). \right) \tag{1}$$

Note that from the Gelfand-Zetlin formulae, which define the action of the generators of $\mathfrak{G}$ on finite-dimensional modules, it follows that the coefficients of $f_z(X)$ are just the rational functions in the entries of the tableau corresponding to χ. Since we can find sufficiently many modules E lying far from the walls and sufficiently many generic Γ'-weight subspaces in E we conclude for any generic Γ'-weight χ in any simple module M, defined using Gelfand-Zetlin formulae, the polynomial $f_z(X)$ has also the form (1) (where $\theta(\lambda)$ is the central character of M).

To complete the proof we only have to recall that the modules $V([l])$ were constructed using Gelfand-Zetlin formulae and, as it was mentioned above, any Γ'-weight subspace of $V([l])$ is generic. □

Combining Lemma 3 and Lemma 4 we obtain that for two simple modules $V([s])$ and $V([t])$ from Λ having central characters $\theta(\mu_1)$ and $\theta(\mu_2)$ respectively holds

$$\mathfrak{i}(V([s]), V([t])) = |W \cdot (\mu_2)|/|W \cdot (\mu_1)|.$$

Thus all conditions of Theorem 5 are also satisfied and the corresponding analogue of BGG-duality holds.

12 Application of Theorem 6 to the last example

Fix $[l]$ as in Section 11 such that $l_{n,i} \in \mathbb{Z}$ for all $1 \leq i \leq n$. Now we are going to apply Theorem 6 to obtain some structural results about $\Lambda = \Lambda(V([l]))$. To be able to apply Theorem 6 we only have to check that the category $\mathcal{O}(\mathcal{P}, \Lambda)$ constructed in Section 11 has a duality. For this we note (see [DFO]) that each module in $\mathcal{O}(\mathcal{P}, \Lambda)$ decomposes into a direct sum of finite-dimensional modules with respect to a commutative subalgebra K in $U(\mathfrak{G})$ generated by the flag of centers mentioned in Section 11. Further the canonical Chevalley involution on $\mathfrak{G}$ stabilizes K pointwise. Thus we can define a duality on $\mathcal{O}(\mathcal{P}, \Lambda)$ by standard arguments (dualizing each finite-dimensional space separately, see [FM, Section 5.5]). Applying now Theorem 6 we get

$$\mathfrak{i}(V([s]), V([t])) = \mathfrak{l}(P(V([t])))/\mathfrak{l}(P(V([s]))) = |W \cdot (\mu_2)|/|W \cdot (\mu_1)|.$$

Moreover, this enables us to compute $\mathfrak{l}(P(V([t])))$ precisely. In fact, take $[s]$ such that $s_{n,i} = 0$ for all i. In this case $|W \cdot (\mu_1)| = 1$ and, clearly, $\mathfrak{l}(P(V([s]))) = 1$. Hence $\mathfrak{l}(P(V([t]))) = |W \cdot (\mu_2)|$ and we can combine this into the following corollary.

Corollary 5. *Let $\Lambda = \Lambda(V([l]))$, for $[l]$ as in Section 11 such that $l_{n,i}$ are integers for all i, and let $V([s])$ be a simple module in Λ having the central character χ_λ (i.e. the central character of the simple highest weight module with the highest weight λ). Then the length of the projective cover of $V([s])$ in Λ equals $|W \cdot (\lambda)|$.*

This allows us to reformulate the BGG-reciprocity for the Gelfand-Zetlin example:

Corollary 6. *Keep the notation of Section 11. Then there holds the reciprocity formula*

$$[P(V([t])) : M_{\mathcal{P}}(V([s]))] = |W \cdot (\lambda)|(M_{\mathcal{P}}(V([s])) : L(V([t]))),$$

where χ_λ is the central charachter of $V([s])$.

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References

[ADL] *I.Agoston, V.Dlab, E.Lukacs*, Stratified algebras, C. R. Math. Acad. Sci. Soc. R. Can. 20 (1998), no. 1, 22-28.

[BGG] *I.N.Bernstein, I.M.Gelfand, S.I.Gelfand*, A certain category of $\mathfrak{G}$-modules. (Russian) Funkcional. Anal. i Priložen. 10 (1976), no. 2, 1–8.

[CPS1] *E.Cline, B.Parshall, L.Scott*, Finite-dimensional algebras and highest weight categories. J. Reine Angew. Math. 391 (1988), 85–99.

[CPS2] *E.Cline, B.Parshall, L.Scott*, Stratifying endomorphism algebras, Mem. Amer. Math. Soc. 124 (1996), n. 591.

[CF] *A.J.Coleman, V.Futorny*, Stratified L-modules. J. Algebra 163 (1994), no. 1, 219–234.

[D] *J.Dixmier*, Algèbres Enveloppantes, Paris, 1974.

[DFO] *Yu. A. Drozd, V.M. Futorny, S.A. Ovsienko*, The Harish-Chandra S-homomorphism and $\mathfrak{G}$-modules generated by a semiprimitive element. (Russian) Ukrain. Mat. Zh. 42 (1990), no. 8, 1031–1037; translation in Ukrainian Math. J. 42 (1990), no. 8, 919–924 (1991)

[DFO2] *Yu. A. Drozd, V.M. Futorny, S.A. Ovsienko*, Harish-Chandra subalgebras and Gelfand-Zetlin modules, in: Finite-Dimensional algebras and related topics (Ottawa, On, 1992), 79-93, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 424, Kluwer Acad. Publ., Dordrecht, 1994.

[FP] *V.Futorny, D.Pollack*, A new category of Lie algebra modules satisfying the BGG reciprocity principle. Comm. Algebra 22 (1994), no. 1, 213–227.

[FM] *V.Futorny, V.Mazorchuk*, Highest weight categories of Lie algebra modules. J. Pure Appl. Algebra 138 (1999), no. 2, 107–118.

[GM] *X.Gomez, V.Mazorchuk*, On an analogue of BGG-reciprocity, Preprint CPT-99/P.3882, Marseille University.

[I] *R.Irving*, BGG algebras and the BGG reciprocity principle. J. Algebra 135 (1990), no. 2, 363–380.

[K] *B.Kostant*, On the tensor product of a finite and an infinite dimensional representation. J. Functional Analysis 20 (1975), no. 4, 257–285.

[MO] *V.Mazorchuk, S.Ovsienko*, Submodule structure of generalized Verma modules induced from generic Gelfand-Zetlin modules. Algebr. Represent. Theory 1 (1998), no. 1, 3–26.

[R] *A.Rocha-Caridi*, Splitting criteria for $\mathfrak{G}$-modules induced from a parabolic and the Bernstein - Gelfand - Gelfand resolution of a finite-dimensional, irreducible $\mathfrak{G}$-module. Trans. Amer. Math. Soc. 262 (1980), no. 2, 335–366.

[RW] *A. Rocha-Caridi, N. Wallach*, Projective modules over graded Lie algebras. I. Math. Z. 180 (1982), no. 2, 151–177.

[S] *W.Soergel*, The combinatorics of Harish-Chandra bimodules. J. Reine Angew. Math. 429 (1992), 49–74.

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