A combinatorial description of blocks in $\mathcal{O}(\mathcal{P}, \Lambda)$ associated with $sl(2)$-induction

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Abstract

We study the category $\mathcal{O}(\mathcal{P}, \Lambda)$, where Λ is an admissible category of dense weight $sl(2)$-modules. We give a combinatorial description of projectively stratified algebras, arising from $\mathcal{O}(\mathcal{P}, \Lambda)$ and prove a double centralizer property. Moreover, we determine the characters of tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$ and prove that the finite-dimensional algebra associated with the principal block of our $\mathcal{O}(\mathcal{P}, \Lambda)$ is its own Ringel dual.

1 Introduction

Together with its definition in [BGG], two basic facts of the category $\mathcal{O}$, associated with a simple complex finite-dimensional Lie algebra $\mathfrak{G}$, were established. The first states that $\mathcal{O}$ decomposes into blocks, each of which is a module category over a finite-dimensional algebra (these algebras belong to the class of quasi-hereditary algebras, defined later in [CPS1]). The second one is the celebrated BGG reciprocity between simple, projective and Verma modules in $\mathcal{O}$. Another crucial result about $\mathcal{O}$ was obtained much later by Soergel in [S2]. There a combinatorial description of the finite-dimensional algebras which correspond to blocks of $\mathcal{O}$ was given. In the case of the principal block of $\mathcal{O}$, the combinatorial datum of this description is the so-called coinvariant algebra together with the big projective module (Soergel has obtained in [S2] a combinatorial description of this module structure over the coinvariant algebra).

Verma modules are produced by inflating a one–dimensional module over a Cartan subalgebra to a Borel subalgebra and then inducing up to $\mathfrak{G}$. A well–known generalization involves replacing the Cartan subalgebra by a (larger) reductive subalgebra and the corresponding Borel subalgebra by a parabolic subalgebra. Such generalized Verma modules have been studied both for the special case where the input for inflation and induction still is a finite dimensional module and for more general cases where one starts already with an infinite dimensional module. In this way, many new simple $\mathfrak{G}$-modules can be produced as quotients of generalized Verma modules and, together with a new class of modules, constructed by Mathieu ([M]), this completes the classification of simple weight modules with finite-dimensional weight spaces ([F, M]).

In [FKM] we dealt with the question of generalizing the definition of $\mathcal{O}$ in such a way that generalized Verma modules are included. We proposed a natural generalization $\mathcal{O}(\mathcal{P},\Lambda)$ of $\mathcal{O}$, which corresponds to an admissible category Λ of (infinite-dimensional in general) modules over a parabolic subalgebra $\mathcal{P}$ of $\mathfrak{G}$. In fact, we have shown that under some natural conditions, the obtained categories decompose into blocks, each of which is a module category over a finite dimensional algebra. In contrast to the classical case, this finite dimensional algebra usually is not quasi–hereditary. However, it is projectively stratified and thus the theory of stratified algebras (developped by Cline, Parshall and Scott [CPS2] for quite different sorts of examples) can be applied. We also found an analogue of BGG reciprocity.

The aim of this paper is to obtain an analogue of Soergel's combinatorial description for $\mathcal{O}(\mathcal{P},\Lambda)$ in the case, when the semisimple part of the Levi factor of $\mathcal{P}$ is isomorphic to $sl(2,\mathbb{C})$ (see [FKM, Section 10]). In particular, we show (Theorem 3 and Corollary 7 in Section 6), that the combinatorial datum is again the coinvariant algebra and the big projective module in the principal block of $\mathcal{O}(\mathcal{P},\Lambda)$.

Theorem A. *Let $P(L)$ be the big projective module in the principal block of the category $\mathcal{O}(\mathcal{P},\Lambda(V(\lambda,\gamma)))$. Then* End $_{\mathcal{O}(\mathcal{P},\Lambda)}(P(L))$ *is the coinvariant algebra.*

Moreover, Theorem 4 in Section 7 provides us with a double centralizer property.

Theorem B. *Let B denote the (projectively stratified finite-dimensional) algebra associated with the principal block $\mathcal{O}(\mathcal{P},\Lambda)_{triv}$. Then B is isomorphic to the endomorphism algebra of the big projective module, viewed as a module over its endomorphism ring.*

We also construct tilting modules in $\mathcal{O}(\mathcal{P},\Lambda)$ and determine their characters. In Section 10 (Theorem 7) we establish Ringel self–duality.

Theorem C. *The projectively stratified algebra of the principal block $\mathcal{O}(\mathcal{P},\Lambda)_{triv}$ is its own Ringel dual.*

The paper is organized as follows. In Section 2 we introduce our main objects. In Section 3 we use Mathieu's localization technique to reduce the study of an arbitrary category $\mathcal{O}(\mathcal{P},\Lambda)$ to a special case $\Lambda = \Lambda(V(\mathfrak{l},\gamma))$. In Section 4 we define a functor $E : \mathcal{O}(\mathcal{P},\Lambda(V(\mathfrak{l},\gamma))) \to \mathcal{O}$ and establish its basic properties. In fact, we show that it transfers generalized Verma modules to Verma modules and projective objects to projective objects. In Section 5 we prove that $\mathcal{O}(\mathcal{P},\Lambda)$ is equivalent to a full subcategory of $\mathcal{O}$. In Section 6 we determine the endomorphism algebra of the big projective module in the principal block of $\mathcal{O}(\mathcal{P},\Lambda)$. In fact, we prove that this is the coinvariant algebra. In Section 7 we establish the double centralizer property for $\mathcal{O}(\mathcal{P},\Lambda)$. In Section 8 we define and investigate a subclass of tilting modules in $\mathcal{O}$ which we call strong tilting modules. Further, in Section 9 we define a notion of tilting module in $\mathcal{O}(\mathcal{P},\Lambda)$ and prove their existence and uniqueness. In fact, we show, that E transfers a tilting module in $\mathcal{O}(\mathcal{P},\Lambda)$ into a strong tilting module in $\mathcal{O}$ and this map is bijective. Finally, in Section 10 we determine the multiplicities of generalized Verma modules occuring in a standard filtration of a tilting module in $\mathcal{O}(\mathcal{P},\Lambda)$ and prove that the projectively stratified algebra associated with the principal block of $\mathcal{O}(\mathcal{P},\Lambda)$ is isomorphic to its Ringel dual.

2 Main objects

Let $\mathfrak{A}$ denote the Lie algebra $sl(2, C)$ with a fixed root basis $e = X_\alpha$, $f = X_{-\alpha}$, $h = H_\alpha$, where α is a root of $\mathfrak{A}$. For $\gamma \in \mathbb{C}$ and $\lambda \in \mathbb{C}/2\mathbb{Z}$ let $V(\lambda, \gamma)$ denote the unique weight $\mathfrak{A}$-module (see [FM1]), satisfying the following conditions:

1. λ is the support of $V(\lambda, \gamma)$ and all weight spaces of $V(\lambda, \gamma)$ are one-dimensional,
2. γ is the unique eigenvalue of the Casimir operator $C = (h+1)^2 + 4fe$ on $V(\lambda, \gamma)$,
3. f acts bijectively on $V(\lambda, \gamma)$.

Clearly, $V(\lambda, \gamma)$ is an indecomposable $\mathfrak{A}$-module, generated by any $V(\lambda, \gamma)_{\mu+k\alpha}$, $\mu \in \lambda$ for $k \in \mathbb{N}$ big enough.

Call a weight $\mathfrak{A}$-module V with finite-dimensional weight spaces admissible, provided f acts bijectively on V. By definition, any $V(\lambda, \gamma)$ is admissible.

Let $\tilde{\Lambda} = \tilde{\Lambda}(V(\lambda, \gamma))$ denote the category of $\mathfrak{A}$-modules, defined as follows: the objects of $\tilde{\Lambda}$ are all admissible submodules and all admissible quotients of all modules having the form $V(\lambda, \gamma) \otimes F$, where F is a finite-dimensional $\mathfrak{A}$-module; the homomorphisms of $\tilde{\Lambda}$ are those homomorphisms of $\mathfrak{A}$-modules, whose kernel is an admissible module. Clearly, $\tilde{\Lambda}$ is an abelian category (i.e. that it is closed under operations of taking admissible submodules and quotients), moreover, $\tilde{\Lambda}$ is closed under taking finite direct sums.

Remark 1. *In the case, when $V(\lambda, \gamma)$ is simple (this means $\gamma \neq (l+1)^2$ for all $l \in \lambda$), any submodule in $V(\lambda, \gamma) \otimes F$, where F is finite-dimensional, is admissible (see, for example, [CF]). The objects of $\tilde{\Lambda}$ are all the quotients and submodules of modules $V(\lambda, \gamma) \otimes F$.*

In the case, when $V(\lambda, \gamma)$ is not simple, $\tilde{\Lambda}$ still is a full subcategory of the category of $\mathfrak{A}$-modules. It is easy to see, that $\tilde{\Lambda}$ inherits an abelian structure from the category of all $\mathfrak{A}$-modules. In fact, let M_1 and M_2 be two weight modules with finite dimensional weight spaces and let $\varphi : M_1 \to M_2$ be a morphism. Then f acts injectively on $\ker(\varphi) \subset M_1$. *Using the bijective action of f on M_2 (which has finite dimensional weight spaces) we also get that f acts surjectively on* $\ker(\varphi)$. *In a similar way, one can check that f acts bijectively on* $\mathrm{coker}(\varphi)$.

Now let $\mathfrak{G}$ be a complex simple finite-dimensional Lie algebra and P be a parabolic subalgebra of $\mathfrak{G}$ such that $P = (\mathfrak{A} \oplus \mathfrak{H}_\mathfrak{A}) \oplus \mathfrak{N}$, $\mathfrak{A} \oplus \mathfrak{H}_\mathfrak{A}$ reductive, $\mathfrak{N}$ nilpotent, $\mathfrak{H}_\mathfrak{A}$ abelian and $\mathfrak{A}$ as above. The category $\tilde{\Lambda}$ can be extended in a unique way to a category $\Lambda = \Lambda(V(\lambda, \gamma))$ of $\tilde{\mathfrak{A}} = \mathfrak{A} \oplus \mathfrak{H}_\mathfrak{A}$-modules, which satisfies the following conditions:

1. any $M \in \Lambda$ belongs to $\tilde{\Lambda}$, when viewed as an $\mathfrak{A}$-module,
2. any $M \in \Lambda$ is $\mathfrak{H}_\mathfrak{A}$-diagonalizable,
3. for any $M \in \Lambda$ and any $\mathfrak{H}_\mathfrak{A}$-diagonalizable finite-dimensional $\tilde{\mathfrak{A}}$-module F the module $M \otimes F$ decomposes into a direct sum of indecomposable modules from Λ,

4. the homomorphisms in Λ are those homomorphisms of $\tilde{\mathfrak{A}}$-modules, whose restriction to $\mathfrak{A}$ belongs to $\tilde{\Lambda}$.

Following [FKM, Section 3], we define $\mathcal{O}(\mathcal{P},\Lambda)$ to be the category of $\mathfrak{G}$-modules, whose objects are finitely generated and $\mathfrak{N}$-finite $\mathfrak{G}$-modules, which decompose into a direct sum of modules from Λ, when wieved as $\tilde{\mathfrak{A}}$-modules and whose homomorphisms are those homomorphisms of $\mathfrak{G}$-modules, whose kernel decomposes into a direct sum of modules from Λ, when viewed as an $\tilde{\mathfrak{A}}$-module. By [FKM, Section 4], $\mathcal{O}(\mathcal{P},\Lambda)$ has a block decomposition (with finitely many simples in each block) and by [FKM, Section 10], this decomposition can be choosen such that each block is equivalent to the module category over a projectively stratified finite-dimensional algebra. Moreover, if γ is not a square of an integer, this algebra is quasi-hereditary and in all cases there is an analogue of the BGG-reciprocity. From Remark 1 it follows, that the category $\mathcal{O}(\mathcal{P},\Lambda)$ is a full subcategory of the category of all $\mathfrak{G}$-modules and it inherits the abelian structure from the last category. Further, $\mathcal{O}(\mathcal{P},\Lambda)$ is closed under taking finite direct sums and under tensoring with finite-dimensional $\mathfrak{G}$-modules. Moreover, tensoring with a finite–dimensional $\mathfrak{G}$–module is an exact functor also with respect to the new abelian structure.

3 Mathieu's localization and the first equivalence

The aim of this section is to prove the following result:

Theorem 1. *The categories $\mathcal{O}(\mathcal{P},\Lambda(V(\lambda_1,\gamma)))$ and $\mathcal{O}(\mathcal{P},\Lambda(V(\lambda_2,\gamma)))$ are (blockwise) equivalent (i.e. they are independent on λ).*

To prove this we will use Mathieu's localization, proposed in [M] as a tool for classifying simple dense modules with finite-dimensional weight spaces. We refer the reader to [M] for all technical details.

Denote by U_f the localization of $U(\mathfrak{G})$ with respect to the powers of $f = X_{-\alpha}$ and let θ_x, $x \in \mathbb{C}$ be the unique polynomial family of automorphisms of U_f, such that $\theta_x(u) = f^x u f^{-x}$ for all $u \in U_f$ and all $x \in \mathbb{Z}$.

Proof. We can assume that $\lambda_1 \neq \lambda_2$. Since $\mathbb{C}$ is one-dimensional over itself, there exists $x \in \mathbb{C}$ such that $\lambda_1 = \lambda_2 + x\alpha$. Moreover, $x \notin \mathbb{Z}$ according to our assumption.

By definition of $\mathcal{O}(\mathcal{P},\Lambda)$, f acts bijectively on any module $V \in \mathcal{O}(\mathcal{P},\Lambda)$. Thus any V can be trivially extended to an U_f-module.

Now suppose that M is an U_f-module and $0 \neq v \in M$ such that $H_\alpha v = av$ for some $a \in \mathbb{C}$. Then for any integer y we have $\theta_y(H_\alpha)v = f^y H_\alpha f^{-y} v = (a+2y)v$. Since the family θ_y is polynomial (in y) by definition, we have, that $\theta_y(H_\alpha)v = (a+2y)v$ for any $y \in \mathbb{C}$. From this it follows immediately, that the twist by θ_{-x} (resp. θ_x) is a well-defined functor from $\mathcal{O}(\mathcal{P},\Lambda(V(\lambda_1,\gamma)))$ to $\mathcal{O}(\mathcal{P},\Lambda(V(\lambda_2,\gamma)))$ (resp. from $\mathcal{O}(P,\Lambda(V(\lambda_2,\gamma)))$ to $\mathcal{O}(\mathcal{P},\Lambda(V(\lambda_1,\gamma))))$. Since the composition of θ_x and θ_{-x} is an identity, we easily conclude that these fuctors are mutually inverse. The block version follows immediately. This completes the proof. □

4 From $\mathcal{O}(\mathcal{P},\Lambda)$ to $\mathcal{O}$

According to Theorem 1, the properties of the category $\mathcal{O}(\mathcal{P},\Lambda(V(\lambda,\gamma)))$ do not depend on λ. Now we recall that for a fixed γ there exists at least one $V(\lambda,\gamma)$, which is not simple. More precisely, if γ is a square of an integer, such $V(\lambda,\gamma)$ is unique and if γ is not a square of an integer, there are precisely two non-isomorphic non-simple modules $V(\lambda',\gamma)$ and $V(\lambda'',\gamma)$. Let $V(\mathfrak{l},\gamma)$ be a non-simple module. The aim of this section is to define and investigate a functor from $\mathcal{O}(\mathcal{P},\Lambda(V(\mathfrak{l},\gamma)))$ to $\mathcal{O}$.

For $M\in\mathcal{O}(\mathcal{P},\Lambda(V(\mathfrak{l},\gamma)))$ denote by $E(M)$ the space of locally e-finite elements of M. Since e is locally ad-nilpotent, $E(M)$ is a $\mathfrak{G}$-submodule of M. On morphisms, E is restriction of a homomorphism $\varphi: M\to N$ to $E(\varphi): E(M)\to E(N)$. We note, that from [FKM, Section 10, Section 4] it follows that any object M in $\mathcal{O}(\mathcal{P},\Lambda(V(\mathfrak{l},\gamma)))$ has finite length (as a $\mathfrak{G}$-module). Hence $E(M)$ also has a finite length. Thus we obtain, that E is a well-defined functor from $\mathcal{O}(\mathcal{P},\Lambda(V(\mathfrak{l},\gamma)))$ to $\mathcal{O}$. Our main goal in this and some of the next sections is to study the properties of this functor E. We note that an analogous functor was used in [M, Lemma A1].

Lemma 1. *$E(M)=0$ if and only if $M=0$.*

Proof. We have to prove the "only if" part. Since $E(M)$ is defined as the locally e-finite part of M and e is locally ad-nilpotent, $E(M)$ is an $\mathfrak{A}$-module (moreover, it is a $\mathfrak{G}$-module). By definition of $\mathcal{O}(\mathcal{P},\Lambda(V(\mathfrak{l},\gamma)))$, M decomposes into a direct sum of modules from $\tilde{\Lambda}$, thus it is sufficient to prove our statement for indecomposable modules in Λ.

Suppose, that γ is not a square of an integer. Then any indecomposable in Λ has the form $V(\hat{\lambda},\hat{\gamma})$ for some $\hat{\lambda}\in\mathbb{C}/2\mathbb{Z}$ and $\hat{\gamma}\in\mathbb{C}$ and is not simple (see [FKM, Section 10] or [FM2, Example 2] or directly apply [K, Theorem 5.1]) by our assumption on Λ. Since $V(\hat{\lambda},\hat{\gamma})$ is not simple and f acts bijectively on it, it should have a non-zero highest weight submodule. Hence its subspace of locally e-finite elements is non-zero.

Now suppose that γ is a square of an integer. Then, by [FKM, Section 10], any indecomposable module in Λ is either some non-simple $V(\hat{\lambda},\hat{\gamma})$ or is a self-extension of some $V(\hat{\lambda},\hat{\gamma})$, moreover, this self-extension in Λ exists if and only if $V(\hat{\lambda},\hat{\gamma})$ itself has length 3. For $V(\hat{\lambda},\hat{\gamma})$ everything is clear (analogous to the previous case). To complete the proof, consider a self-extension $V\in\Lambda$ of some $V(\hat{\lambda},\hat{\gamma})$ of length 3. By definition of Λ, there is a finite-dimensional $\tilde{\mathfrak{A}}$-module F such that V is a direct summand of $F\otimes V(\hat{\lambda},\hat{\gamma})$. Since $V(\hat{\lambda},\hat{\gamma})$ has a non-zero highest weight submodule and tensoring with a finite-dimensional module is an exact functor, we conclude, that the space of locally e-finite vectors in V is non-zero. □

Corollary 1. *Under the notations of Lemma 1 the following holds: For any $\mu\in\operatorname{supp}M$ and any $k\in\mathbb{N}$ big enough $M_{\mu-k\alpha}$ belongs to the locally e-finite part of V.*

Proof. Follows directly from the proof of Lemma 1. □

We recall, that any module in $\mathcal{O}(\mathcal{P},\Lambda)$ is a weight $\mathfrak{G}$-module with finite-dimensional weight spaces ([CF]). Recall, that any $M\in\mathcal{O}(\mathcal{P},\Lambda)$ has finite length. From this and Corollary 1 we deduce the following:

Corollary 2. *Let* $M \in \mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$*. Then for any* $\mu \in \operatorname{supp} M$ *and for any* $k \in \mathbb{N}$ *big enough we have* $\dim M_{\mu-k\alpha} = \dim E(M)_{\mu-k\alpha}$.

Lemma 2. *Let* $L \in \mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$ *be a simple object. Then* $E(L)$ *contains a unique simple subquotient on which* $X_{-\alpha}$ *acts injectively.*

Proof. First we remark that, according to our choice of $V(\mathfrak{l}, \gamma)$, $E(L)$ is not zero. Let $\hat{L}$ be a simple subquotient of $E(L)$ on which $X_{-\alpha}$ acts injectively. Let λ be the highest weight of $E(L)$ and λ' be the highest weight of $\hat{L}$. If $\lambda - \lambda' \notin \mathbb{Z}\alpha$, then, inducing our modules back to U_f, we get a non-trivial submodule of L on which X_α acts bijectively. This is impossible because L is assumed to be simple. Hence $\lambda - \lambda' \in \mathbb{N}\alpha$, and, as $E(L)$ is a quotient of a Verma module, $X_{-\alpha}$ acts on $E(L)/\hat{L}$ locally nilpotent. This implies the uniqueness of $\hat{L}$. □

For a simple object $W \in \Lambda$ we will denote by $L_P(W)$ the unique simple (as an object in $\mathcal{O}(\mathcal{P}, \Lambda)$) quotient of $M_P(W)$. We note, that $L_P(W)$ is a simple $\mathfrak{G}$-module if and only if W is a simple $\mathfrak{A}$-module. The next statement one more times generalizes the ideas used in the proof of Lemma 1:

Lemma 3. *Let* $M \in \mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$ *and* F *be a finite-dimensional* $\mathfrak{G}$*-module. Then* $E(M \otimes F) \simeq E(M) \otimes F$ *(as* $\mathfrak{G}$*-modules).*

Proof. The inclusion $E(M) \otimes F \to M \otimes F$ factors through $E(M \otimes F)$. Exactness of $\_ \otimes F$ for a finite-dimensional F implies the assertion. □

Lemma 4. *Let* W *be a simple object in* $\Lambda(V(\mathfrak{l}, \gamma))$ *and* $M_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$*. Then* $E(M_{\mathcal{P}}(W))$ *is a Verma module in* $\mathcal{O}$*.*

Proof. Recall that W is not simple and isomorphic to some $V(\hat{\lambda}, \hat{\gamma})$. We have that $E(M_{\mathcal{P}}(W)) \simeq M_{\mathcal{P}}(\hat{W})$, where $\hat{W}$ is the locally e-finite part of W. Since $\hat{W}$ is a Verma module over $\mathfrak{A}$ we obtain that $M_{\mathcal{P}}(\hat{W})$ is a Verma module over $\mathfrak{G}$. □

The following statement describes the key property of E.

Proposition 1. *The functor* E *sends projective objects from* $\mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$ *to projective modules in* $\mathcal{O}$*.*

Proof. First we consider the case $\mathfrak{G} = \mathfrak{A}$. According to [FKM, Section 10], an indecomposable projective object in Λ is either some $V(\hat{\lambda}, \hat{\gamma})$ or its selfextension of length two, which appears as a direct summand in $F \otimes V(\mathfrak{l}, \gamma)$ for some finite-dimensional F. If $V(\hat{\lambda}, \hat{\gamma})$ itself is projective, then $\hat{\gamma}$ is not the square of an integer and hence the corresponding e-finite part is the unique simple (=projective =Verma) module in the indecomposable block of category $\mathcal{O}$. This means that in this case the statement is true. If γ is the square of an integer, we can assume that $\gamma = 0$. Thus $V(\mathfrak{l}, \gamma)$ is projective in Λ (see [FKM, Section 10] again) and its e-finite part is projective in $\mathcal{O}$. In this case to obtain the statement we only need to recall that the functor $F \otimes \_$ is exact.

Now consider the general case. Recall the construction of projective modules in $\mathcal{O}$ and $\mathcal{O}(\mathcal{P},\Lambda)$ ([BGG, Section 4], [FKM, Section 4]). In $\mathcal{O}(\mathcal{P},\Lambda)$ any projective occures as a direct summund in the pojection on the corresponding block of a module having the form

$$P(V,k) = U(\mathfrak{G}) \bigotimes_{U(\mathcal{P})} \left((U(\mathfrak{N})/(U(\mathfrak{N})\mathfrak{N}^k)) \otimes V \right),$$

where k is a big enough positive integer and V is a projective in Λ. Clearly, E commutes with the induction from $\mathcal{P}$ to $\mathfrak{G}$. Applying now Lemma 3 we have

$$E(P(V,k)) = U(\mathfrak{G}) \bigotimes_{U(\mathcal{P})} \left((U(\mathfrak{N})/(U(\mathfrak{N})\mathfrak{N}^k)) \otimes E(V) \right).$$

Now $E(V)$ is projective in the corresponding $sl(2)$ category $\mathcal{O}$. ¿From this and the construction of $P(V,k)$ it follows that if the projection of $P(V,k)$ on the block of $\mathcal{O}(\mathcal{P},\Lambda)$ is projective, then the corresponding projection of $E(P(V,k))$ on the block of $\mathcal{O}$ is also projective. To complete the proof, we just have to mention that E commutes with projections on blocks. □

5 Further properties of E: the second equivalence

The aim of this section is to study E in more detail. In fact we will prove that E is a full functor and that it induces an equivalence between $\mathcal{O}(\mathcal{P},\Lambda)$ (Λ as in Section 4) and a full subcategory of $\mathcal{O}$. Till the end of the section we assume that $\Lambda = \Lambda(V(\mathfrak{l},\gamma))$, where $V(\mathfrak{l},\gamma)$ is a non-simple module as in Section 4.

Lemma 5. *Let M and N be in $\mathcal{O}(\mathcal{P},\Lambda)$ and $\varphi : M \to N$ be a non-zero homomorphism (of $\mathfrak{G}$-modules). Then the $\varphi(E(M)) \neq 0$.*

Proof. Let $m \in M$ be such that $\varphi(m) \neq 0$. Recall that f acts injectively (even bijectively) on M and N, and by Corollary 2 there exists a positive integer k such that $f^k \cdot m \in E(M)$. Now $\varphi(f^k \cdot m) = f^k \cdot \varphi(m) \neq 0$. □

Corollary 3. *E produces an equivalence of $\mathcal{O}(\mathcal{P},\Lambda)$ with a (not necessary full) subcategory of $\mathcal{O}$.*

So far we have no evidence, why E should be a full functor. The rest of this section will be devoted to establishing this fact. According to Lemma 2, for any simple object $L \in \mathcal{O}(\mathcal{P},\Lambda)$ there exists a unique simple subquotient $\hat{L}$ of $E(L)$ on which f acts injectively. Since $E(L)$ belongs to $\mathcal{O}$, it is characterized by its highest weight, which we will denote by $\hat{E}(L)$. So, using the usual notation for the simple quotient of a Verma module ([D]), we can write that $\hat{L} \simeq L(\hat{E}(L))$. As usual, for a $\mathfrak{G}$-weight λ by $P(\lambda)$ we will denote the projective cover of $L(\lambda)$ (in $\mathcal{O}$). For a simple $L \in \mathcal{O}(\mathcal{P},\Lambda)$ denote by $P(L)$ its projective cover in $\mathcal{O}(\mathcal{P},\Lambda)$.

Lemma 6. *For any $M \in \mathcal{O}(\mathcal{P}, \Lambda)$ and any simple L in $\mathcal{O}(\mathcal{P}, \Lambda)$ holds $(M : L) = (E(M) : L(\hat{E}(L)))$.*

Proof. We can restrict our attention to a block of $\mathcal{O}(\mathcal{P}, \Lambda)$, which is a module category over a projectively stratified algebra ([FKM, Theorem 3]). Now the statement follows from Corollary 2 by induction with respect to the poset indexing simple modules in our block. □

From Lemma 6 one can easily deduce an analogue of the Kazhdan-Lusztig (Conjecture=) Theorem for generalized Verma modules $M_{\mathcal{P}}(W)$, where W is a simple object in Λ (this was proved first time in [KM]). In fact, we just reduce the problem to the known result for category $\mathcal{O}$ (see [BB, BK, S2]).

Corollary 4. *Let W be a simple object in Λ and L be a simple object in $\mathcal{O}(\mathcal{P}, \Lambda)$. Then $(M_{\mathcal{P}}(W) : L) = (M(\hat{E}(L_{\mathcal{P}}(W))) : L(\hat{E}(L)))$.*

Proof. First we note that $E(M_{\mathcal{P}}(W))$ is a Verma module and $M(\hat{E}(L_{\mathcal{P}}(W))) \subset E(M_{\mathcal{P}}(W))$. Moreover, $E(M_{\mathcal{P}}(W))/M(\hat{E}(L_{\mathcal{P}}(W)))$ is a direct sum of finite-dimensional $\mathfrak{A}$-modules, hence $(E(M_{\mathcal{P}}(W)) : L(\hat{E}(L))) = (M(\hat{E}(L_{\mathcal{P}}(W))) : L(\hat{E}(L)))$. The rest follows from Lemma 6. □

Lemma 7. *$E(M)$ is indecomposable if and only if M is indecomposable.*

Proof. Clearly, it is enough to prove that $E(M)$ is indecomposable as soon as M is indecomposable. Suppose that $E(M) = N_1 \oplus N_2$. Since f acts injectively on $E(M)$ it acts injectively on both N_i, $i = 1, 2$. Recall, that N_i, $i = 1, 2$ are $\mathfrak{G}$-submodules in M. Let M_i, $i = 1, 2$ denote the set of all $v \in M$ such that $f^k(v) \in N_i$ for some k big enough. From Corollary 1 it follows that $M = M_1 \oplus M_2$ as a vector space. Since f is a locally adnilpotent element in $\mathfrak{G}$ it follows that both M_i, $i = 1, 2$ are $\mathfrak{G}$-submodules. This contradicts the assumtion, that M is indecomposable. □

Proposition 2. *$E(P(L)) \simeq P(\hat{E}(L))$.*

Proof. By Lemma 7, $E(P(L))$ is indecomposable, since so is $P(L)$. Hence we only need to compute the unique simple quotient N of $E(P(L))$. Let $L \simeq L_P(W)$ for some simple $W \in \Lambda$ and $P(W)$ be the projective cover of W in Λ. Clearly, N is isomorphic to the unique simple quotient of $E(M_{\mathcal{P}}(P(W)))$ and hence to the unique simple quotient of $M_{\mathcal{P}}(E(P(W)))$. Let W' be the top of $E(P(W))$. From the proof of Proposition 1 it follows that $M_P(W') \simeq M(\hat{E}(L))$ and thus $N \simeq L(\hat{E}(L))$. □

Corollary 5. *Let $M \in \mathcal{O}(\mathcal{P}, \Lambda)$ and Q be a projective object in $\mathcal{O}(\mathcal{P}, \Lambda)$. Then*

$$\dim \mathrm{Hom}_{\mathcal{O}(\mathcal{P},\Lambda)}(Q, M) = \dim \mathrm{Hom}_{\mathcal{O}}(E(Q), E(M)).$$

Proof. Clearly, we can assume, that $Q \simeq P(L)$ for some simple $L \in \mathcal{O}(\mathcal{P}, \Lambda)$. Then we have $\dim \mathrm{Hom}_{\mathcal{O}(\mathcal{P},\Lambda)}(P(L), M) = (M : L)$ and $\dim \mathrm{Hom}_{\mathcal{O}}(E(P(L)), E(M)) = (E(M) : L(\hat{E}(L)))$ by Proposition 2. Now the statement follows from Lemma 6. □

Corollary 5 and Lemma 5 suggest that E should be a full functor. Now we are able to prove this.

Theorem 2. *For any $M, N \in \mathcal{O}(\mathcal{P}, \Lambda)$ holds*

$$\dim \mathrm{Hom}_{\mathcal{O}(\mathcal{P},\Lambda)}(M, N) = \dim \mathrm{Hom}_{\mathcal{O}}(E(M), E(N)).$$

Proof. First we note that by Corollary 5 the statement is true, when M is a projective module in $\mathcal{O}(\mathcal{P}, \Lambda)$. Clearly, we can assume that M is indecomposable. Let $P(M)$ be a projective cover of M. By virtue of Lemma 5 it is enough to prove that for any $\varphi : E(M) \to E(N)$ there is ψ in $\mathcal{O}(\mathcal{P}, \Lambda)$ such that $\varphi = E(\psi)$. We have that $E(P(M))$ is a projective cover of $E(M)$. Let $a : E(P(M)) \to E(M)$ be a canonical epimorphism. By Lemma 5 and Corollary 5, there exists an epimorphism $x : P(M) \to M$ and a homomorphism $y : P(M) \to N$ such that $a = E(x)$ and $\varphi \circ a = E(y)$. For $m \in M$ set $\psi(m) = y \circ x^{-1}(m)$. We have to show that this is a well-defined map. But $\ker a \subset \ker \varphi \circ a$, hence $\ker x \subset \ker y$ since f acts bijectively on $P(M)$ and E acts on homomorphisms by restriction. This means that ψ is well-defined. Since both x and y are $\mathfrak{G}$-morphisms we deduce that ψ is also a $\mathfrak{G}$-morphism. Clearly, $E(\psi) = \varphi$, since E is just a restriction. This completes the proof of our theorem. ☐

Corollary 6. *E is a full functor. In particular, $\mathcal{O}(\mathcal{P}, \Lambda)$ is equivalent to a full subcategory of $\mathcal{O}$. Moreover, the image of a block of $\mathcal{O}(\mathcal{P}, \Lambda)$ is contained in a block of $\mathcal{O}$.*

6 Analogue for $\mathcal{O}(\mathcal{P}, \Lambda)$ of Soergel's Endomorphism Theorem

Recall ([S2]), that the principal block of $\mathcal{O}$ is the block $\mathcal{O}_{triv}$, containing the trivial (one-dimensional) $\mathfrak{G}$-module. Let $L(\mu_i)$, $i = 1, 2, \ldots, n$ be a complete list of simple modules in $\mathcal{O}_{triv}$. Note, that $A = \{\mu_i\}$ coinsides with the orbit of 0 under the dot action of the Weyl group of $\mathfrak{G}$ (see, for example [D] or [S2]). Following [S2] we will call the projective module $P(w_0 \cdot 0)$, where w_0 is the longest element in the Weyl group, the big projective module. Let γ be the square of an integer and denote by $\mathcal{O}(\mathcal{P}, \Lambda)_{triv}$ the direct summand of $\mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$ which has a non-trivial image in $\mathcal{O}_{triv}$ under E. Since E acts blockwise and $\mathcal{O}_{triv}$ is indecomposable, such indecomposable $\mathcal{O}(\mathcal{P}, \Lambda)_{triv}$ is unique. Let L be a simple object in $\mathcal{O}(\mathcal{P}, \Lambda)$ such that $\hat{E}(L) = w_0 \cdot 0$. It exists, since f acts injectively on $L(w_0 \cdot 0) = M(w_0 \cdot 0)$. Now the results of the previous section (in fact, Corollary 5 and Lemma 5) make it possible to describe the endomorphism algebra of the "big projective" module $P(L)$ in $\mathcal{O}(\mathcal{P}, \Lambda)_{triv}$. This is a direct analogue of the Soergel's result for $\mathcal{O}_{triv}$ (which we also use in the proof).

Theorem 3. *Let $\Lambda = \Lambda(V(\mathfrak{l}, \gamma))$. Then* $\mathrm{End}_{\mathcal{O}(\mathcal{P},\Lambda)}(P(L)) \simeq \mathrm{End}_{\mathcal{O}}(P(w_0 \cdot 0))$. *In fact,* $\mathrm{End}_{\mathcal{O}(\mathcal{P},\Lambda)}(P(L))$ *is the coinvariant algebra (see [S2]).*

Proof. Follows from Corollary 5, Corollary 6, Lemma 5 and [S2, Endomorphismensatz 3]. □

Consider $\mathcal{O}(\mathcal{P}, \Lambda(V(\lambda,\gamma)))$ for γ being the square of an integer and $\lambda \in \mathbb{C}/2\mathbb{Z}$. Let $x \in \mathbb{C}$ be such that θ_x moves $\mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l},\gamma)))$ to $\mathcal{O}(P, \Lambda(V(\lambda,\gamma)))$. Call $\theta_x(P(L))$ the big projective module in the principal block $\theta_x(\mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l},\gamma)))_{triv})$ of $\mathcal{O}(\mathcal{P}, \Lambda(V(\lambda,\gamma)))$.

Corollary 7. End $_{\mathcal{O}(\mathcal{P},\Lambda(V(\lambda,\gamma)))}(\theta_x(P(L)))$ *is the coinvariant algebra.*

Proof. Follows from Theorem 3 and Theorem 1. □

Remark 2. *It is easy to see, that, as in the category $\mathcal{O}$, the big projective module in $\mathcal{O}(\mathcal{P}, \Lambda)$ can be characterized as the unique indecomposable projective in the principal block of $\mathcal{O}(\mathcal{P}, \Lambda)$ such that any Verma (or, in notation of [FKM], generalized Verma) module from the principal block occurs as a subquotient in the standard filtration of this projective module.*

7 Analogue for $\mathcal{O}(\mathcal{P}, \Lambda)$ of Soergel's Double Centralizer Theorem

The next result is analogous to Soergel's description of the algebra corresponding to the principal block of $\mathcal{O}$ (see [S2]). It is usually called the *double centralizer property.*

Theorem 4. *Let B denote the (projectively stratified finite-dimensional) algebra associated with $\mathcal{O}(\mathcal{P}, \Lambda)_{triv}$. Then B is isomorphic to the endomorphism algebra of the big projective module, viewed as a module over its endomorphism ring.*

It is more convenient to prove this theorem in "abstract" notations which we are going to introduce now. Let A (resp. B) denote the algebra associated with the principal block of $\mathcal{O}$ (resp. $\mathcal{O}(\mathcal{P}, \Lambda)$). We recall, that according to Section 5, B is a (matrix) subalgebra of A. Let e be the primitive idempotent of A such that Ae is the big projective module in $\mathcal{O}_{triv}$. Then Be is the big projective module in $\mathcal{O}(\mathcal{P}, \Lambda)_{triv}$ and $C = eAe = eBe$ is the coinvarian algebra, which is the endomorphism algebra of Ae and Be. Let $T = \text{Hom}_{\,A}(Ae, \_)$ denote the Soergel's functor ([S2]). Recall, that by Soergel's Theorem ([S2, Struktursatz 2]), for any $M \in \mathcal{O}_{triv}$ and any projective $Q \in \mathcal{O}_{triv}$ holds

$$\text{Hom}_{\,A}(M, Q) \simeq \text{Hom}_{\,C=eAe}(T(M), T(Q)).$$

Proof of Theorem 4. We start from $B = \text{Hom}_{\,B}(B, B)$. Applying the results from Section 5, we have $\text{Hom}_{\,B}(B, B) \simeq \text{Hom}_{\,A}(E(B), E(B))$. Now applying the mentioned result by Soergel we obtain $\text{Hom}_{\,A}(E(B), E(B)) \simeq \text{Hom}_{\,eAe}(T(E(B)), T(E(B)))$. We know from Theorem 3, that $eAe = eBe$. Recall, that $E(Be) = Ae$, hence $T(E(B)) = \text{Hom}_{\,A}(Ae, E(B)) = \text{Hom}_{\,A}(E(Be), E(B)) \simeq \text{Hom}_{\,B}(Be, B) = eB$. Finally,

$$\text{Hom}_{\,eAe}(T(E(B)), T(E(B))) \simeq \text{Hom}_{\,eBe}(eB, eB).$$

Now we note, that B is a matrix subalgebra of A and we can apply the duality on A to the last endomorphism ring, obtaining $\mathrm{Hom}\ _{eBe}(eB, eB) \simeq \mathrm{Hom}\ _{eBe}(Be, Be)$, which completes the proof. $\square$

Remark 3. *According to [FP] or [FKM, Section 12] there is a canonical duality on B which can also be applied directly in the proof of Theorem 4.*

8 Strong tilting modules in $\mathcal{O}$

The rest of the paper will be devoted to the construction and study of tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$. Since by [FKM, Theorem 3] any finite block of $\mathcal{O}(\mathcal{P}, \Lambda)$ corresponds to a projectively stratified algebras, one can just use an abstract result from [AHLU] to state the existence and uniqueness of (characteristic) tilting module for each block of $\mathcal{O}(\mathcal{P}, \Lambda)$. Their result generalizes Ringel's fundamental theorem [R] from quasi–hereditary to standardly stratified algebras.

However we are going to use a slightly more symmetric definition of tilting module, which is more natural in our case. This means that we will not be able to apply results from [AHLU] directly, in particular, we will have to prove the existence of tilting modules. Finally, we will determine the multiplicities of generalized Verma modules occuring in a standard filtration of an indecomposable tilting module, thus determining the character of this tilting module. This generalizes the recent result of Soergel ([S3]).

First we recall the notion of tilting modules for $\mathcal{O}$ and study them from another point of view. The Chevalley anti–involution σ on $\mathfrak{G}$ give rises to a duality (i.e. an exact contravariant equivalence, preserving simple objects) on $\mathcal{O}$ (see, for example [J, Section 4.10]). For $M \in \mathcal{O}$ we will denote by M^* the corresponding dual module in $\mathcal{O}$. We note, that $L(\lambda)^* \simeq L(\lambda)$ for any $\lambda \in \mathfrak{H}^*$. Let $\mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\bigtriangledown)$) denote the full subcategory of $\mathcal{O}$, which consists of all modules in $\mathcal{O}$ having a Verma flag, i.e. a filtration whose subquotients are Verma modules (resp. a dual Verma flag, i.e. a filtration, whose subquotients are $M(\lambda)^*$, $\lambda \in \mathfrak{H}^*$). A module $M \in \mathcal{O}$ is called tilting module, if it belongs to $\mathcal{F}(\Delta) \cap \mathcal{F}(\bigtriangledown)$. It is known, that indecomposable tilting modules are naturaly parametrized by simple modules, hence by $\lambda \in \mathfrak{H}^*$ ([R]). We will denote by $T(\lambda)$ the indecomposable tilting module which corresponds to $\lambda \in \mathfrak{H}^*$ (i.e. whose Verma flag starts with $M(\lambda)$).

Let α denote the simple root of $\mathfrak{G}$ which corresponds to the subalgebra $\mathfrak{A}$ and s_α denote the corresponding simple reflection on $\mathfrak{H}^*$. Suppose that $\lambda \in \mathfrak{H}^*$ is such that $s_\alpha(\lambda) = \lambda + k\alpha$ for some $k \in \mathbb{N}$. Consider the indecomposable projective module $P(\lambda) \in \mathcal{O}$, Clearly, there exists a Verma flag $P(\lambda) = P_0 \supset P_1 \supset P_2 \supset \dots$ of $P(\lambda)$ such that $P_0/P_1 \simeq M(\lambda)$ and $P_1/P_2 \simeq M(s_\alpha(\lambda))$. Set $\hat{P}(\lambda) = P(\lambda)/P_2$. Then $\hat{P}(\lambda)$ has a Verma flag with $M(\lambda)$ on the top and $M(s_\alpha(\lambda))$ on the bottom. Define a class $K(\alpha)$ of modules in $\mathcal{O}$ as follows: if $\lambda \in \mathfrak{H}^*$ is such that $s_\alpha(\lambda) - \lambda \notin (\mathbb{Z}\alpha \setminus \{0\})$ then $K(\alpha)$ contains $M(\lambda)$; in the other case $K(\alpha)$ contains $\hat{P}(\lambda)$ if $s_\alpha(\lambda) - \lambda \in \mathbb{N}\alpha$ and $K(\alpha)$ contains $\hat{P}(s_\alpha(\lambda))$ if $\lambda - s_\alpha(\lambda) \in \mathbb{N}\alpha$.

Denote by $\mathcal{F}(\Delta)_\alpha$ (resp. $\mathcal{F}(\bigtriangledown)_\alpha$) the full subcategory of $\mathcal{O}$, containing all modules which admit a filtration with subquotients from $K(\alpha)$ (resp. with subquotients of the form M^*, $M \in K(\alpha)$). Since any module in $K(\alpha)$ has a Verma flag, we have $\mathcal{F}(\Delta)_\alpha \subset \mathcal{F}(\Delta)$,

$\mathcal{F}(\nabla)_\alpha \subset \mathcal{F}(\nabla)$ and $\mathcal{O}(K(\alpha)) = \mathcal{F}(\Delta)_\alpha \cap \mathcal{F}(\nabla)_\alpha \subset \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. Hence any module in $\mathcal{O}(K(\alpha))$ (if there is any) is a tilting module. So to determine $\mathcal{O}(K(\alpha))$ we have to find out which indecomposable tilting modules belong to it.

Lemma 8. *For any $M \in K(\alpha)$ (resp. M such that $M^* \in K(\alpha)$) and any finite-dimensional $\mathfrak{G}$-module F the module $F \otimes M$ belongs to $\mathcal{F}(\Delta)_\alpha$ (resp. $\mathcal{F}(\nabla)_\alpha$).*

Proof. Follows from the exactness of $F \otimes \_$ by standard arguments combined with the observation that we are considering objects which are induced from projective objects. □

Proposition 3. *$T(\lambda) \in \mathcal{O}(K(\alpha))$ if and only if either $s_\alpha(\lambda) - \lambda \notin \mathbb{Z}\alpha \setminus \{0\}$ or $\lambda - s_\alpha(\lambda) \in \mathbb{N}\alpha$.*

Proof. If $s_\alpha(\lambda) - \lambda \notin (\mathbb{Z}\alpha\setminus\{0\})$ then, according to the definition of $K(\alpha)$, any Verma module (resp. dual Verma module), occuring in the Verma flag (resp. dual Verma flag) of $T(\lambda)$ belongs to $K(\alpha)$ (resp. is of the form M^* for some $M \in K(\alpha)$). Hence $T(\lambda) \in \mathcal{O}(K(\alpha))$.

Recall that any increasing Verma flag of $T(\lambda)$ starts with $M(\lambda)$. ¿From the definition of $K(\alpha)$ it follows that for any $\lambda \in \mathfrak{H}^*$ such that $s_\alpha(\lambda) - \lambda \in \mathbb{N}\alpha$ there are no modules in $K(\alpha)$ such that their increasing Verma flag starts with $M(\lambda)$. Since any filtration with quotients from $K(\alpha)$ can be extended to a Verma flag, we obtain that in the case $s_\alpha(\lambda) - \lambda \in \mathbb{N}\alpha$ the module $T(\lambda)$ cannot belong to $K(\alpha)$.

So we only have to prove that $T(\lambda) \in K(\alpha)$ in case $\lambda - s_\alpha(\lambda) \in \mathbb{N}\alpha$. This will follow easily if we recall the inductive construction of tilting modules via tensoring with finite-dimensional modules. Suppose that λ is such that $\lambda - s_\alpha(\lambda) \in \mathbb{N}\alpha$ and $M(s_\alpha(\lambda))$ is simple. Then $T(\lambda) \simeq \hat{P}(s_\alpha(\lambda))$ by the construction of $\hat{P}(s_\alpha(\lambda))$ and hence $T(\lambda) \in \mathcal{F}(\Delta)_\alpha$. But $T(\lambda)$ is also self-dual as a tilting module in the category $\mathcal{O}$, hence $T(\lambda) \in \mathcal{F}(\nabla)_\alpha$. Finally, $T(\lambda) \in \mathcal{O}(K(\alpha))$.

Now we note, that from Lemma 8 it follows that $\mathcal{O}(K(\alpha))$ is stable under tensoring with finite-dimensional modules. In particular, it means that if we fix λ as in the previous paragraph, then $T(\lambda) \otimes F$ belongs to $\mathcal{O}(K(\alpha))$ for any finite-dimensional $\mathfrak{G}$-module F. To complete the proof we only need to recall that any $T(\mu)$ with $\mu - s_\alpha(\mu) \in \mathbb{N}\alpha$ occurs as a direct summand in $T(\lambda) \otimes F$ for some λ as in the previous paragraph and some finite-dimensional F ([CI]). □

The modules in $\mathcal{O}(K(\alpha))$ will be called *strong tilting modules*. Later we will see that they are closely related to tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$.

Corollary 8. *The big projective module is a strong tilting module.*

Proof. Obvious. □

9 Tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$

Let $\Lambda = \Lambda(V(\mathfrak{l}, \gamma))$ as in the previous sections. In order to introduce the notion of a tilting module in $\mathcal{O}(\mathcal{P}, \Lambda)$ we need a natural duality on $\mathcal{O}(\mathcal{P}, \Lambda)$. This can be easily done using σ for $\mathcal{O}(\mathcal{P}, \Lambda(V(\lambda, \gamma)))$ in the case, when $V(\lambda, \gamma)$ is a simple $\mathfrak{A}$-module (see [FP] or [FKM, Section 12]). The same direct construction for the case $\mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$ does not work, because dualization does not preserve the bijectivity of the action of f. In fact, e acts bijectively on the dual module. There are two ways to solve this problem. The first way is to fix a non-integer x and to define a duality * on $\mathcal{O}(\mathcal{P}, \Lambda)$ as the composition of θ_x, the natural duality on $\theta_x(\mathcal{O}(\mathcal{P}, \Lambda))$, which can be constructed via σ (here everything works since both e and f act bijectively on $\theta_x(\mathcal{O}(\mathcal{P}, \Lambda))$), and θ_{-x}. The second way is to compose σ with the natural automorphism of $\mathfrak{G}$ corresponding to the simple reflection s_α. We choose the second way and from now on for $M \in \mathcal{O}(\mathcal{P}, \Lambda)$ we will denote by M^* the corresponding dual module.

Let $\mathcal{G}(\Delta)$ (resp. $\mathcal{G}(\nabla)$) denote the full subcategory of $\mathcal{O}(\mathcal{P}, \Lambda)$ which consists of all modules having standard filtration, i.e. a filtration, whose subquotients are isomorphic to $M_{\mathcal{P}}(W)$, where W is projective in Λ (resp. a dual standard filtration, i.e. a filtration, whose subquotients are isomorphic to $M_{\mathcal{P}}(W)^*$, where W is projective in Λ). A module $M \in \mathcal{O}(\mathcal{P}, \Lambda)$ will be called a tilting module if $M \in \mathcal{G}(\Delta) \cap \mathcal{G}(\nabla)$.

So far we do not know if there is any tilting module in $\mathcal{O}(\mathcal{P}, \Lambda)$. The aim of this section is to describe all tilting module in $\mathcal{O}(\mathcal{P}, \Lambda)$. We recall that our definition of tilting module does not coincide with the general definition, used in [AHLU]. The difference is in the definition of $\mathcal{G}(\nabla)$. In [AHLU], the existence of a filtration is required, whose subquotients are isomorphic to $M_{\mathcal{P}}(W)^*$, where W is simple in Λ. Our condition is more restrictive. Taking into account the uniqueness of characteristic tilting module for standardly stratified algebras (this class includes, in particular, projectively stratified algebras) in [AHLU], we only have to show that for any simple $L = L_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P}, \Lambda)$ there exists an indecomposable tilting module $T(L) \in \mathcal{O}(\mathcal{P}, \Lambda)$ such that the standard filtration of $T(L)$ starts with $M_{\mathcal{P}}(W)$.

Lemma 9. *For any $M \in \mathcal{G}(\Delta)$ (resp. $M \in \mathcal{G}(\nabla)$) and any submodule N occuring in a standard filtration (resp. dual standard filtration) of M holds $E(N) \subset E(M)$ and $E(M/N) \simeq E(M)/E(N)$.*

Proof. Follows from the definition of E and the fact that $M \simeq N \oplus (M/N)$ as an $\mathfrak{A}$-module. □

Lemma 10. *Let T be a tilting module in $\mathcal{O}(\mathcal{P}, \Lambda)$. Then $E(T)$ is a strong tilting module in $\mathcal{O}$.*

Proof. ¿From the definition of $K(\alpha)$ it follows immediately, that for any projective $W \in \Lambda$ holds $E(M_{\mathcal{P}}(W)) \in K(\alpha)$. Now, by Lemma 9, the standard (resp. dual standard) filtration of T is sent by E to a filtration with subquotients from $K(\alpha)$ (resp. with subquotients, dual to modules in $K(\alpha)$). This completes the proof. □

Lemma 11. *For any strong tilting module $T' \in \mathcal{O}$ there exists a tilting module $T \in \mathcal{O}(\mathcal{P}, \Lambda)$ such that $E(T) \simeq T'$.*

Proof. Clearly, it is enough to prove this statement for indecomposable T', so we can suppose that $T' = T(\lambda)$. First, assume that $M(s_\alpha(\lambda))$ is a simple $\mathfrak{G}$-module. Clearly, $M(\lambda)$ belongs to the image of E, hence $M(\lambda) = E(M_{\mathcal{P}}(W))$ for some simple object $W \in \Lambda$. Let W' be the projective cover of W. From the definition of $K(\alpha)$ one immediately obtains $T(\lambda) = E(M_{\mathcal{P}}(W'))$. Now the statement follows from Lemma 3, the inductive construction of strong tilting modules as in the proof of Proposition 3 and the remark that tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$ are self-dual. □

Theorem 5. *For any simple object $L = L_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P}, \Lambda)$ there exists a unique indecomposable tilting module $T(L) \in \mathcal{O}(\mathcal{P}, \Lambda)$ such that the standard filtration of $T(L)$ starts with $M_{\mathcal{P}}(W')$, where W' is a projective cover of W in Λ. The set $T(L)$, where L runs through simple modules in $\mathcal{O}(\mathcal{P}, \Lambda)$ is a complete set of indecomposable tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$. Any tilting module is a finite direct sum of indecomposable tilting modules.*

Proof. Existence follows from Lemma 11. The rest follows from [AHLU, 2.1 and 2.2]. □

10 Characters of tilting modules: Analogue for $\mathcal{O}(\mathcal{P}, \Lambda)$ of Soergel's Character Formula

In Section 9 we proved the existence of tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$. The aim of this section in to determine the formal character of a tilting module. Clearly, it is sufficient to do this for an indecomposable module $T(L)$, where L is a simple module in $\mathcal{O}(\mathcal{P}, \Lambda)$. Further, by the definition, $T(\lambda)$ has a standard filtration, hence it has a filtration by $M_{\mathcal{P}}(W)$, where $W \in \Lambda$ is a simple object. Since $M_{\mathcal{P}}(W)$ is an extension of two Verma modules (with respect to a different basis in $\mathfrak{G}$), its character is known. So the problem is to determine the multiplicities $[T(L) : M_{\mathcal{P}}(W)]$. We solve this problem by reducing it to the recently solved analogous problem for $\mathcal{O}$ (see [S3]).

Theorem 6. *Let W_1 and W_2 be simple objects in Λ. Denote by $l(W_2')$ the length of the projective cover W_2' of W_2 in Λ. Then*

$$[T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2)] = l(W_2')[E(T(L_{\mathcal{P}}(W_1))) : E(M_{\mathcal{P}}(W_2))].$$

Proof. Set $m = [T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2)]$. Then $m = l(W_2')[T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2')]$ and by Lemma 9, $[T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2')] = [E(T(L_{\mathcal{P}}(W_1))) : E(M_{\mathcal{P}}(W_2'))]$. If $l(W_2') = 1$ then $W_2 = W_2'$ and we are done. Otherwise, it follows from the definition of $K(\alpha)$, that the number of Verma modules in a Verma flag of $E(M_{\mathcal{P}}(W_2'))$ equals 2, moreover, $[E(M_{\mathcal{P}}(W_2')) : E(M_{\mathcal{P}}(W_2))] = 1$. This completes the proof. □

Remark 4. *According to Lemma 10, $E(T(L_{\mathcal{P}}(W_1)))$ is a strong tilting module. In particular, it is a tilting module in $\mathcal{O}$. Furthermore, $E(M_{\mathcal{P}}(W_2))$ is a Verma module in $\mathcal{O}$, hence, the multiplicity $[E(T(L_{\mathcal{P}}(W_1))) : E(M_{\mathcal{P}}(W_2))]$ can be computed by Soergel's Theorem, [S3, Theorem 5.12 and Theorem 6.7].*

Remark 5. *Applying the functors θ_x one extends the above results to an arbitrary category $\mathcal{O}(\mathcal{P}, \Lambda(V(l,\gamma)))$.*

Finally, if one looks at the proof of [S3, Theorem 2.1], one sees that it implies another interesting result for the principal block $\mathcal{O}(\mathcal{P}, \Lambda)_{triv}$ of $\mathcal{O}(\mathcal{P}, \Lambda)$. We keep the notation from [S3]. Let $S = S_\delta$ denote the semi-regular bimodule, associated with a semi-infinite character δ. As it was shown in [S3], the composition of the functor $S \otimes_{U(\mathfrak{G})} \_$ with the graded duality D maps the indecomposable projective $P(w(\lambda))$, λ dominant, into the indecomposable tilting module $T(ww_0(\lambda)) \in \mathcal{O}$, where w_0 is the longest element of the Weyl group. Comparing Proposition 2 with the definition of strong tilting module we see that for any indecomposable projective module $P(L) \in \mathcal{O}(\mathcal{P}, \Lambda)$ the module $D(S \otimes_{U(\mathfrak{G})} E(P(L)))$ is an indecomposable strong tilting module. If we recall Lemma 10, Lemma 11 and the fact that $S \otimes_{U(\mathfrak{G})} \_$ is an equivalence of certain categories ([S3, Section 2]) which preserves short exact sequences, we obtain the following result:

Theorem 7. *The projectively stratified algebra of $\mathcal{O}(\mathcal{P}, \Lambda)_{triv}$ is its own Ringel dual (see [R, KK] for detail).*

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