Categories of induced modules for Lie algebras with triangular decomposition

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Abstract

We study categories of modules for Lie algebras with triangular decomposition, which contain certain generalized Verma modules and are analogous to classical category $\mathcal{O}$. We relate blocks of these categories to module categories over finite-dimensional algebras, which turn out to be projectively stratified. Moreover, we study tilting modules. Finally we show how to relate some of these situations to similar ones over certain proper subalgebras of the given Lie algebra.

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1 Introduction

In [FKM1, FKM2, FKM3] we defined and studied certain categories of induced modules over simple complex finite-dimensional Lie algebras generalizing the celebrated BGG-category $\mathcal{O}$ ([BGG]). In particular, it was shown that under certain natural assumptions the blocks of such categories are equivalent to module categories over projectively stratified algebras. Moreover, there exists an analogue of Soergel's combinatorial description (see [S1, S2, S3] for original results) and a satisfactory theory of tilting modules for these blocks.

The main aim of this paper is to extend the results of [FKM1, FKM2, FKM3] to the case of Lie algebras with triangular decomposition in the sense of [RW, MP]. Our goal is to go far enough to prove an analogue of the BGG-reciprocity, to relate the local situation to projectively stratified algebras and to develop a theory of tilting modules. Additionally we describe indecomposable blocks and injective objects in our categories.

In the case of infinite-dimensional affine Lie algebras there exist three essentially different types of parabolic subalgebras ([F1]). It happens only for two types of parabolic subalgebras that the corresponding Levi factor is a reductive finite-dimensional algebra. We restrict our attention to these two types of parabolics. In the first part of the paper we study (in a more general situation than that of affine Lie algebras) the so-called standard

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parabolic subalgebras, which can be viewed as a direct analogue of parabolic subalgebras for simple finite-dimensional Lie algebras. In the second part of the paper we show that the study of the parabolic subalgebras of the second possible type in most of cases reduces to the standard parabolics over some well-defined subalgebra.

The paper is organized as follows. In Section 2 we present our basic setup. In Section 3 we define the main object of the paper – category $\mathcal{O}(\mathcal{P}, \Lambda, H)$. Moreover, we prove an analogue of the BGG-reciprocity. In Section 4 we study a block decomposition of $\mathcal{O}(\mathcal{P}, \Lambda, H)$. In Section 5 we relate the local situation in $\mathcal{O}(\mathcal{P}, \Lambda, H)$ to projectively stratified finite-dimensional algebras (as in [FKM1, Section 5]). In Section 6 we establish two equivalences of categories in the case of induction from a well-embedded $sl(2, \mathbb{C})$-subalgebra. In Section 7 we develop a theory of tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda, H)$. Section 8 is devoted to the study of non-standard generalized Verma modules for affine Lie algebras. These modules are induced from a non-standard parabolic subalgebra having a finite-dimensional Levi factor. The structure of such modules in the case when the central element is non-degenerate is studied in Section 9. Finally, in Section 10 we define a category $\mathcal{O}(\mathfrak{P}, \Lambda, H)$ of modules for affine Lie algebra associated with a non-standard parabolic subalgebra and establish the equivalence between $\mathcal{O}(\mathfrak{P}, \Lambda, H)$ and a category $\mathcal{O}(\mathcal{P}', \Lambda', H')$ for a certain subalgebra in the case when the central element acts injectively on every object. Hence, in this case all the results about $\mathcal{O}(\mathcal{P}, \Lambda, H)$ can be easily extended to the category $\mathcal{O}(\mathfrak{P}, \Lambda, H)$.

2 Setting

Let $\mathfrak{G}$ be a complex Lie algebra with a triangular decomposition $(\mathfrak{H}, \mathfrak{G}_+, \mathcal{Q}_+, \sigma)$ ([MP]), where $\mathfrak{H}$ is a Cartan subalgebra, $\mathcal{Q}_+$ is the set of roots of the positive part $\mathfrak{G}_+$ of $\mathfrak{G}$, and σ is an antiinvolution on $\mathfrak{G}$. Let $\mathcal{Q} = \mathcal{Q}_+ \cup -\mathcal{Q}_+$ denote the set of all roots of $\mathfrak{G}$. For $\alpha \in \mathcal{Q}$ denote by $\mathfrak{G}_\alpha$ the corresponding root space in $\mathfrak{G}$. Fix a basis π of $\mathcal{Q}_+$ and a subset S of π such that the subalgebra $\mathfrak{A}'$, generated by $\mathfrak{H}$ and all $\mathfrak{G}_\alpha$, where α is an integral linear combination of elements from S, is reductive finite-dimensional. Denote by $\mathcal{Q}_+^S$ the set of positive roots of $\mathfrak{A}'$. Let $\mathfrak{A}' = \mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}$, where $\mathfrak{A}$ is semi-simple and $\mathfrak{H}_{\mathfrak{A}}$ is central in $\mathfrak{A}'$. Set $\mathcal{P} = \mathfrak{A}' + \mathfrak{G}_+$ and let $\mathfrak{N}$ be a subalgebra of $\mathcal{P}$ generated by all $\mathfrak{G}_\alpha$, $\alpha \in \mathcal{Q}_+ \setminus \mathcal{Q}_+^S$. One has $\mathcal{P} = \mathfrak{A}' \oplus \mathfrak{N}$.

As in [FKM1, Section 3], we start with a category, Λ, of $\mathfrak{A}'$-modules, satisfying the following conditions:

- Λ is a full subcategory of the category of finitely generated and $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable $\mathfrak{A}'$-modules;
- Λ carries an abelian structure, such that the endomorphism rings of simple objects are $\mathbb{C}$ (this abelian structure may differ from the one on the category of $\mathfrak{A}'$-modules);
- for any simple finite-dimensional $\mathfrak{A}'$-module F, the functor $F \otimes \_$ is an exact endofunctor on Λ.

A category, satisfying all the conditions above will be called *admissible*.

For any $V \in \Lambda$ we may consider an induced module $M(V) = M_{\mathcal{P}}(V) = U(\mathfrak{G}) \otimes_{U(\mathcal{P})} V$, where $\mathfrak{N}V = 0$. Usually the module $M(V)$ is called a *generalized Verma module* (GVM), provided the module V is simple (see, for example [FKM1, Section 2] and the references therein). We will say that $M(V)$ is a GVM (resp. a *standard* module) if V is a simple (resp. projective) object in Λ. We remark that it is possible for simple objects in Λ to be non-simple $\mathfrak{A}'$-modules (see [FKM2, Section 5] for an example).

It follows directly from the construction, that any $M(V)$ is $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable. Denote by p_S the projection of $\mathfrak{H}^*$ on $\mathfrak{H}_{\mathfrak{A}}^*$ with respect to the dual to the Cartan subalgebra of $\mathfrak{A}$. If V is a simple object in Λ, then $\mathfrak{H}_{\mathfrak{A}}$ acts on V via some $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$ and the $\mathfrak{H}_{\mathfrak{A}}$-support of $M(V)$ coinsides with the set $P(\lambda)$ consisting of all $\mathfrak{H}_{\mathfrak{A}}$-weights $\lambda - \mu$, where $\mu = 0$ or $\mu = p_S(\alpha)$ for a non-negative integral linear combination α of roots from $\mathfrak{N}$. Moreover, any GVM $M(V)$ has a unique maximal submodule, which decomposes into a direct sum of modules from Λ, when viewed as an $\mathfrak{A}'$-module. Let $L(V) = L_{\mathcal{P}}(V)$ denote the corresponding quotient.

Now we are in position to make our basic assumptions:

(A) The algebra $\mathfrak{G}$ is an integrable $\mathfrak{A}$-module under the adjoint action.

(B) Any object in Λ has finite length (in particular, for any $M \in \Lambda$ and for any finite-dimensional $\mathfrak{A}'$-module F the module $F \otimes M$ belonging to Λ has finite length in Λ).

We note that, from the assumption (A), it follows automatically that all $\mathfrak{H}_{\mathfrak{A}}$-weight spaces of $\mathfrak{G}$ are finite-dimensional.

3 Category $\mathcal{O}(\mathcal{P}, \Lambda, H)$

In this Section we define and study a category of $\mathfrak{G}$-modules induced from $\mathfrak{A}$ via the process of parabolic induction. Our main result is Theorem 1, which establishes an analogue of the BGG-reciprocity principle for such category.

Fix a finite subset H in $\mathfrak{H}_{\mathfrak{A}}^*$ and denote by $\mathcal{O}(\mathcal{P}, \Lambda, H)$ the full subcategory of the category of $\mathfrak{G}$-modules, which consists of all modules M satisfying the following conditions:

1. M is $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable;
2. the $\mathfrak{H}_{\mathfrak{A}}$ support of M is a subset of $P = \cup_{\lambda \in H} P(\lambda)$;
3. any $\mathfrak{H}_{\mathfrak{A}}$-weight space of M is an $\mathfrak{A}'$-module of finite length;
4. viewed as an $\mathfrak{A}'$-module M decomposes into a direct sum of objects from Λ.

Assume that $\mathcal{O}(\mathcal{P}, \Lambda, H)$ carries an abelian structure induced from that on Λ.

Lemma 1. *$\mathcal{O}(\mathcal{P}, \Lambda, H)$ is closed under operations of taking finite direct sums.*

Proof. Obvious. □

Lemma 2. *Let V be a simple object in Λ having the $\mathfrak{H}_{\mathfrak{A}}$-weight $\lambda \in P$. Then both $M(V)$ and $L(V)$ are objects in $\mathcal{O}(\mathcal{P}, \Lambda, H)$ and $L(V)$ is a simple object. Moreover, the set of all such $L(V)$ exhausts the set of simple objects in $\mathcal{O}(\mathcal{P}, \Lambda, H)$ (up to isomorphism).*

Proof. Follows from the assumption (A) by standard arguments (see [FKM1, Proposition 2]). $\square$

Lemma 3. *Let $M \in \mathcal{O}(\mathcal{P}, \Lambda, H)$ and F be a $\mathfrak{H}_{\mathfrak{A}}$-weight $\mathfrak{G}$-module with finite-dimensional $\mathfrak{H}_{\mathfrak{A}}$-weight spaces such that the $\mathfrak{H}_{\mathfrak{A}}$-support of $F \otimes M$ is a subset of P. Then $F \otimes M$ belongs to $\mathcal{O}(\mathcal{P}, \Lambda, H)$.*

Proof. Follows from the assumptions (A), (B) and admissibility of Λ. $\square$

Denote by $\mathcal{O}^f(\mathcal{P}, \Lambda, H)$ the full subcategory of $\mathcal{O}(\mathcal{P}, \Lambda, H)$ consisting of all finitely generated modules. Clearly, all $M(V)$ and $L(V)$ as in Lemma 2 are objects in $\mathcal{O}^f(\mathcal{P}, \Lambda, H)$.

Proposition 1. *Assume that Λ has enough projective modules. Then every object of $\mathcal{O}^f(\mathcal{P}, \Lambda, H)$ is a quotient of a projective in $\mathcal{O}(\mathcal{P}, \Lambda, H)$.*

Proof. Fix an indecomposable projective $V \in \Lambda$ with an $\mathfrak{H}_{\mathfrak{A}}$-weight $\lambda \in P$. Denote by A the subalgebra of $U(\mathfrak{G}_+)$, generated by all graded components $U(\mathfrak{G}_+)_\mu$ such that $\lambda + p_S(\mu) \notin P$. Clearly, $U(\mathfrak{G}_+)/A$ is a completely reducible finite-dimensional $\mathfrak{A}'$-module under the adjoint action. Moreover, it is a $\mathcal{P}$-module. Now set

$$I(V) = U(\mathfrak{G}) \otimes_{U(\mathcal{P})} \left((U(\mathfrak{G}_+)/A) \otimes_{U(\mathcal{P})} V \right).$$

Then $\mathrm{Hom}_{\mathcal{O}(\mathcal{P},\Lambda,H)}(I(V), M) \simeq \mathrm{Hom}_{\Lambda}(V, M_\lambda)$ and hence $I(V)$ is projective in $\mathcal{O}(\mathcal{P}, \Lambda, H)$, since V is projective in Λ. Now the rest is standard. $\square$

It is clear that each $I(V)$, constructed in Proposition 1, in fact, belongs to $\mathcal{O}^f(\mathcal{P}, \Lambda, H)$ and has a finite standard filtration, i.e. a finite filtration, whose subquotients are standard modules.

Corollary 1. *There is a bijection between simple objects in $\mathcal{O}(\mathcal{P}, \Lambda, H)$ and indecomposable projectives in $\mathcal{O}(\mathcal{P}, \Lambda, H)$ which are objects of $\mathcal{O}^f(\mathcal{P}, \Lambda, H)$. Moreover, each indecomposable projective as above has a standard filtration.*

Proof. Analogous to that of [RW, Corollary 13]. $\square$

Let V be a simple object in Λ such that $L(V)$ is a simple object in $\mathcal{O}(\mathcal{P}, \Lambda, H)$. We will denote by $P(V)$ the corresponding projective cover, given by Corollary 1.

The notion of composition series in $\mathcal{O}(\mathcal{P}, \Lambda, H)$ is not well-defined, however there is a natural notion of the *multiplicity* of a simple object L in $M \in \mathcal{O}(\mathcal{P}, \Lambda, H)$ (see [S3, Section 4]). We define $(M : L)$ as the supremum of all numbers of occurrences of L as a subquotient in a finite filtration (in $\mathcal{O}(\mathcal{P}, \Lambda, H)$) of M. It is easy to see that this number is always finite, moreover, one also has $(M : L(V)) = \dim \mathrm{Hom}_{\mathcal{O}(\mathcal{P},\Lambda,H)}(P(V), M)$.

Lemma 4. *Let $M \in \mathcal{O}(\mathcal{P}, \Lambda, H)$ have a standard filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

such that $M_i/M_{i-1} \simeq M(V_i)$, $i = 1, 2, \dots, n$. Then, as an $\mathfrak{A}'$-module, $M/(\sigma(\mathfrak{N})M) \simeq \oplus_{i=1}^n V_i$.

Proof. Analogous to that of [RW, Lemma 1]. □

Assume that $M \in \mathcal{O}(\mathcal{P}, \Lambda, H)$ has a standard filtration and $M(V)$ is a standard module in $\mathcal{O}(\mathcal{P}, \Lambda, H)$. We can define $[M : M(V)]$ to be the number of occurrences of $M(V)$ in a standard filtration of M. This number is independent of the choice of filtration by Lemma 4. As any module in Λ has finite length, each standard module $M(V)$ can be filtered by generalized Verma modules and for simple W we set $[M : M(W)]$ to be the number of occurrences of $M(W)$ in a *generalized Verma flag*, i.e. a filtration, whose quotients are GVMs, of M. Now we can formulate the main result of this Section, which generalizes the celebrated BGG-reciprocity principle ([BGG, RW]). We recall ([FKM1, Theorem 5]) that Λ is said to satisfy the *duality condition* if for any two simple objects X and Y in Λ there exists a constant $\mathfrak{i}(X, Y)$ such that for any simple finite-dimensional $\mathfrak{A}'$-module F holds $((F \otimes X) : Y) = \mathfrak{i}(X, Y)((F \otimes Y) : X)$.

Theorem 1. *Assume that Λ satisfies the duality condition and decomposes into a direct sum of full subcategories, each of them being the module category of some quasi-directed (i.e projectively stratified with projective standard modules, [CPS2]) algebra. Then for any two simple objects $L(V)$ and $L(W)$ from $\mathcal{O}(\mathcal{P}, \Lambda, H)$ holds*

$$[P(V) : M(W)] = \mathfrak{i}(V, W)(\hat{V} : V)(M(W) : L(V)),$$

where $\hat{V}$ is the projective cover of V in Λ.

Proof. Analogous to that of [FKM1, Theorem 5]. □

Example 1. *Let Λ be an admissible category of generic Gelfand-Zetlin modules from [FKM1, Section 11]. It was shown there that Λ satisfies all conditions, necessary for Theorem 1 with $\mathfrak{i}(V, W) = (\hat{W} : W)/(\hat{V} : V)$. Hence, in this case the BGG-reciprocity can be rewritten, as*

$$[P(V) : M(W)] = (\hat{W} : W)(M(W) : L(V)).$$

We also note, that $(\hat{W} : W)$ coincides with the cardinality of the orbit of $\lambda \in \mathfrak{H}^*$ under the Weyl group action, where λ is such that the central character of W coincides with the central character of the Verma module (over $\mathfrak{A}$), parametrized by λ.

Remark 1. *Example 1 contains, in particular, the example of admissible Λ associated with $sl(2, \mathbb{C})$ induction described in [FKM1, Section 10].*

4 Decomposition of $\mathcal{O}(\mathcal{P}, \Lambda, H)$ into blocks

In the previous Section we obtained an analogue of the BGG-reciprocity for category $\mathcal{O}(\mathcal{P}, \Lambda, H)$. In [FKM1] it has been shown that for finite-dimensional $\mathfrak{G}$ under some natural conditions the indecomposable blocks of $\mathcal{O}(\mathcal{P}, \Lambda, H)$ are equivalent to the module categories of projectively stratified finite-dimensional algebras (see [FKM1, Section 5] for definition). In the general case this is not true, since an indecomposable block of $\mathcal{O}(\mathcal{P}, \Lambda, H)$ does not necessarily contain only finitely many simple objects. In Section 6 we will "locally" relate the general situation to projectively stratified algebras. As a first step, which we will make here, decompose $\mathcal{O}(\mathcal{P}, \Lambda, H)$ into blocks in a natural way, as it was done in [DGK] in the classical situation. From now on we assume that Λ satisfies all the conditions of Theorem 1.

We begin with the following Lemma generalizing the Extension Lemma [H, Lemma 2.3].

Lemma 5. *Let $M(V)$ (resp. $M(W)$) be GVM's corresponding to a simple V (resp. W) from Λ. If $\mathrm{Ext}_{\mathcal{O}(\mathcal{P},\Lambda,H)}(M(V), M(W)) \neq 0$ then $L(V)$ is a subquotient of $M(W)$.*

Proof. By virtue of Theorem 1 the proof is analogous to that of [H, Lemma 2.3]. $\square$

Lemma 5 motivates the following definitions. Let $\mathcal{A} = \mathcal{A}(H)$ denote the set of quivalence classes of simple objects V from Λ such that $M(V) \in \mathcal{O}(\mathcal{P}, \Lambda, H)$. This means that $L(V)$, $V \in \mathcal{A}$ is a complete list of simple modules in $\mathcal{O}(\mathcal{P}, \Lambda, H)$. We introduce an equivalence relation $\sim$ on $\mathcal{A}$ as the transitive closure of the relation $\hat{\sim}$ defined as follows: $V \hat{\sim} W$ if and only if there exists $Z \in \mathcal{A}$ such that both $L(V)$ and $L(W)$ are subquotients of $M(Z)$. Let $\mathcal{A}^{\sim}$ be the set of equivalence classes of $\sim$ on $\mathcal{A}$. Now we are in position to state the basic decomposition theorem for $\mathcal{O}(\mathcal{P}, \Lambda, H)$ (see [DGK, Theorem 4.2] and [S3, Theorem 4.2]).

Theorem 2. *For $\chi \in \mathcal{A}^{\sim}$ let $\mathcal{O}(\mathcal{P}, \Lambda, H)_\chi$ denote the full subcategory of $\mathcal{O}(\mathcal{P}, \Lambda, H)$ which consists of all modules M, having simple subquotients $L(V)$, $V \in \chi$ only. Then*

$$\mathcal{O}(\mathcal{P}, \Lambda, H) = \bigoplus_{\chi \in \mathcal{A}^{\sim}} \mathcal{O}(\mathcal{P}, \Lambda, H)_\chi.$$

Proof. Analogous to that of [S3, Theorem 4.2]. $\square$

5 Relation to projectively stratified algebras

To relate a local situation in $\mathcal{O}(\mathcal{P}, \Lambda, H)$ to projectively stratified algebras we will use the classical arguments from the corresponding problem for quasi-hereditary (resp. stratified) algebras, [CPS1, Theorem 3.5] (resp. [CPS2, Theorem 2.2.6]).

Fix a finite set $\hat{H}$ in P and let $\hat{P} = \cup_{\lambda \in \hat{H}} \hat{P}(\lambda)$, where $\hat{P}(\lambda)$ consists of all $\mathfrak{H}_{\mathfrak{A}}$-weights $\lambda + \mu$ with $\mu = 0$ or $\mu = p_S(\alpha)$ for a non-negative integral linear combination α of roots from $\mathfrak{N}$. Fix an indecomposable block χ of $\mathcal{O}(\mathcal{P}, \Lambda, H)$ and let $T = T(\chi, \hat{H})$ be the poset of all simple objects $V \in \chi$ whose $\mathfrak{H}_{\mathfrak{A}}$-weight belongs to $\hat{P}$. Our main result in this Section is the following Theorem.

Theorem 3. *Assume that Λ is a direct sum of module categories of projectively stratified algebras and T is finite. Then the algebra*

$$A = \mathrm{End}_{\mathfrak{G}}\left(\bigoplus_{V \text{ simple in } T} P(V)\right)$$

is projectively stratified (see [FKM1, Section 5]).

Proof. Consider the functor $\mathbb{A} = \mathrm{Hom}_{\mathfrak{G}}(P, \_)$ from $\mathcal{O}^f(\mathcal{P}, \Lambda, H)$ to the category of A-modules. It is exact and sends a projective object $P(V)$, (for V simple in T) to a projective module, and a simple object V to a simple module. Define standard objects $\hat{M}(W)$, W projective in Λ with $L(W) \in T$, as $\mathbb{A}(M(W))$. Then $\mathbb{A}$ transfers a standardly filtered object from $\mathcal{O}(\mathcal{P}, \Lambda, H)$ to a standardly filtered object in the category of A-modules. Induction on the poset T as in [CPS1, Theorem 3.5] shows that all axioms of a projectively stratified algebra are satisfied by this choice of standard modules and hence A is projectively stratified. □

Corollary 2. *Assume that Λ is a sum of module categories of projectively stratified algebras, T is finite and the algebra A from Theorem 3 has a duality. Then for any simples $L(V)$, $L(W)$ in T holds*

$$[P(V) : M(W)] = (\hat{W} : W)(M(W) : L(V)),$$

where $\hat{W}$ is the projective cover of W in Λ.

Proof. It follows from the exactness of $\mathbb{A}$ that $[P(V) : M(W)] = [\mathbb{A}(P(V)) : \mathbb{A}(M(W))]$ and $(M(W) : L(V)) = (\mathbb{A}(M(W)) : \mathbb{A}(L(V)))$. Now the statment follows from Theorem 3 and [FKM1, Theorem 4]. □

Using Corollary 2 one can extend the arguments from the previous Section to a more general situation, where Λ is a sum of module categories of projectively stratified algebras.

6 Induction from $sl(2, \mathbb{C})$: two equivalences

Assume that $|S| = 1$, i.e. $S = \{\alpha\}$ and $\mathfrak{A} \simeq sl(2, \mathbb{C})$. For our convenience we fix a standard basis $X_{\pm\alpha}, H_\alpha$ in $\mathfrak{A}$. In this case there is a natural example of an admissible category, $\Lambda = \Lambda(V(a, b))$, $a, b \in \mathbb{C}$, associated with an indecomposable dence $\mathfrak{A}$-module $V(a, b)$ with one dimensional weight spaces, on which $0 \neq X_{-\alpha} \in \mathfrak{G}_{-\alpha}$ acts bijectively ([FKM1, Section 10]). Such module is uniquely determined by its weight a and the eigenvalue b of the Casimir operator, $\mathfrak{c} = (H_\alpha + 1)^2 + 4X_{-\alpha}X_\alpha$, acting on it. For a finite-dimensional $\mathfrak{G}$ it has been shown in [FKM2, Sections 3-5] that the corresponding categories of induced modules have quite similar structure, moreover, they are equivalent to certain full subcategories of $\mathcal{O}$. The aim of this Section is to generalize these results to $\mathcal{O}(\mathcal{P}, \Lambda, H)$. Note, that it was shown in [FKM1, Section 10] that Λ satisfies all conditions of Theorem 1.

According to the assumption (A), $X_{-\alpha}$ acts locally ad-nilpotently on $\mathfrak{G}$. Let U_α denote the localization of $U(\mathfrak{G})$ with respect to the powers of $X_{-\alpha}$ ([M, Lemma 4.2]). Let θ_x, $x \in \mathbb{C}$ be the unique polynomial family of automorphisms of U_α, such that $\theta_x(u) = X_{-\alpha}^x u X_{-\alpha}^{-x}$ for all $u \in U_\alpha$ and all $x \in \mathbb{Z}$ ([M, Lemma 4.3]).

Theorem 4. *The categories $\mathcal{O}(\mathcal{P}, \Lambda(V(a_1, b)), H)$ and $\mathcal{O}(\mathcal{P}, \Lambda(V(a_2, b)), H)$ are (block-wise) equivalent.*

Proof. The proof repeats one from [FKM2, Theorem 1]. Clearly, we can assume $a_1 \neq a_2$. Choose $x \in \mathbb{C}$ such that $a_1 = a_2 + x\alpha$. From our assumptions we also have $x \notin \mathbb{Z}$. By the definition, $X_{-\alpha}$ acts bijectively on all modules from both $\mathcal{O}(\mathcal{P}, \Lambda(V(a_i, b)), H)$, $i = 1, 2$. Thus, any such module can be trivially extended to a U_α-module.

Let M be a U_α-module and $v \in M$ such that $H_\alpha v = av$ and $\mathfrak{c}v = a'v$. Then, for any integer y, we have $\theta_x(H_\alpha)v = X_{-\alpha}^y H_\alpha X_{-\alpha}^{-y} v = (a + 2y)v$ and $\theta_x(\mathfrak{c})v = X_{-\alpha}^y \mathfrak{c} X_{-\alpha}^{-y} v = a'v$. Since the family θ_y is polynomial in y, $\theta_x(H_\alpha)v = (a+2y)v$ and $\theta_x(\mathfrak{c})v = a'v$ for any $y \in \mathbb{C}$. From this we get that θ_x (resp. θ_{-x}) is a well-defined functor from $\mathcal{O}(\mathcal{P}, \Lambda(V(a_1, b)), H)$ to $\mathcal{O}(\mathcal{P}, \Lambda(V(a_2, b)), H)$ (resp. from $\mathcal{O}(\mathcal{P}, \Lambda(V(a_2, b)), H)$ to $\mathcal{O}(\mathcal{P}, \Lambda(V(a_1, b)), H)$). As the composition of θ_x and θ_{-x} is an identity, we easily conclude that these functors are mutually inverse. The block version follows immediately. $\square$

Fix a and b such that $b = (a + 2l + 1)^2$ for some $l \in \mathbb{Z}$. Under this choice the simple objects in $\Lambda = \Lambda(V(a, b))$ are not simple $\mathfrak{A}'$-modules and hence, they have a non-trivial X_α-finite part. Denote by $\mathcal{O}$ the category of all $\mathfrak{H}$-diagonalizable $\mathfrak{G}$-modules with finite-dimensional weight spaces, whose support is contained into a finite union of supports of Verma modules. For $M \in \mathcal{O}(\mathcal{P}, \Lambda, H)$ let $E(M)$ be the subset of locally X_α-finite elements of M. Clearly, $E(M)$ is a $\mathfrak{G}$-submodule of M.

Lemma 6. *For $M \in \mathcal{O}^f(\mathcal{P}, \Lambda, H)$ holds $E(M) \in \mathcal{O}$.*

Proof. From the definition of Λ and $\mathcal{O}(\mathcal{P}, \Lambda, H)$ it follows that each $M \in \mathcal{O}(\mathcal{P}, \Lambda, H)$ is $\mathfrak{H}$-diagonalizable with finite-dimensional weight spaces. Hence, so is $E(M)$.

Since M is finitely generated, it has a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

where each M_i/M_{i-1} is a subquotient of some GVM $M(V(a_i, b_i))$. Then the support of $E(M)$ is contained in the union of supports of Verma modules $E(M(V(a_i, b_i)))$. This completes the proof. $\square$

By Lemma 6, we can consider E as a functor from $\mathcal{O}^f(\mathcal{P}, \Lambda, H)$ to $\mathcal{O}$ which acts on homomorphisms by restriction. The main property of E is the following theorem. Denote by $\mathcal{O}^f$ the full sucategory of $\mathcal{O}$ consisting of finitely generated modules.

Theorem 5. *The functor E produces an equivalence between $\mathcal{O}^f(\mathcal{P}, \Lambda, H)$ and a full subcategory of $\mathcal{O}^f$.*

The proof essentially repeats the arguments from [FKM2, Sections 4,5] and we divide it into a sequence of lemmas.

Lemma 7. *$E(M) = 0$ if and only if $M = 0$, moreover, for any $\lambda \in \mathfrak{H}^*$ and for any $k \in \mathbb{N}$ big enough holds* $\dim M_{\lambda-k\alpha} = \dim E(M)_{\lambda-k\alpha}$.

Proof. Since M decomposes into a direct sum of objects from Λ it is enough to prove the statment for Λ, which is done in [FKM2, Lemma 1]. $\square$

From the standard $sl(2,\mathbb{C})$ theory one easily derives that $E(L(V(a,b)))$ is a $\mathfrak{G}$-module having exactly one simple subquotient on which $X_{-\alpha}$ acts injectively ([FKM2, Lemma 2]). Denote by $\hat{E}(L(V(a,b)))$ the parameter of this simple highest weight subquotient of the module $E(L(V(a,b)))$. Now we can reduce the problem of calculating the multiplicities of simple subquotients in $M \in \mathcal{O}^f(\mathcal{P},\Lambda,H)$ to the same problem in $\mathcal{O}^f$. In particular, this reduces the problem about multiplicities of simples in a GVM to the corresponding problem for $\mathcal{O}$.

Corollary 3. *For any $M \in \mathcal{O}^f(\mathcal{P},\Lambda,H)$ and any simple $L \in \mathcal{O}^f(\mathcal{P},\Lambda,H)$ holds $(M : L) = (E(M) : L(\hat{E}(L)))$.*

Proof. Standard arguments using induction with respect to the poset of simple modules in $\mathcal{O}^f(\mathcal{P},\Lambda,H)$ ordered with respect to their appearance as subquotients in GVMs. $\square$

Lemma 8. *Let $\mathcal{O}^f(H)$ be the full subcategory of $\mathcal{O}^f$, consisting of modules, whose $\mathfrak{H}_{\mathfrak{A}}$-support belongs to P. Then E sends projectives from $\mathcal{O}^f(\mathcal{P},\Lambda,H)$ to projectives in $\mathcal{O}^f(H)$.*

Proof. Each projective occures as a direct summand of some $I(V)$ (V projective in Λ), so it is enough to prove that $E(I(V))$ is projective in $\mathcal{O}^f(H)$. This follows directly from the construction of $I(V)$, projectivity of V and the fact, that on $\mathfrak{A}'$-level E commutes with tensoring with finite-dimensional modules ([FKM2, Proposition 1]). $\square$

Lemma 9. *E is full and faithful on morphisms.*

Proof. The second statement is easy. It is sufficient to prove

$$\dim \mathrm{Hom}_{\mathcal{O}^f(\mathcal{P},\Lambda,H)}(M_1, M_2) = \dim \mathrm{Hom}_{\mathcal{O}^f}(E(M_1), E(M_2))$$

only. We know that the image of E belongs to $\mathcal{O}^f(H)$. First we prove our result for projective M_1. Certainly, we can assume $M_1 = P(V)$, where V is simple in Λ. Then $\dim \mathrm{Hom}_{\mathcal{O}^f(\mathcal{P},\Lambda,H)}(P(V), M_2) = (M_2 : L(V))$. Further $E(P(V))$ is projective in $\mathcal{O}^f(H)$. Using the same arguments as in [FKM2, Lemma 7, Proposition 2] one can show that $E(P(V)) \simeq P(\hat{E}(L(V)))$ and hence

$$\dim \mathrm{Hom}_{\mathcal{O}^f}(E(P(V)), E(M_2)) = (E(M_2) : L(\hat{E}(L(V)))).$$

Now the statement follows from Corollary 3. By virtue of Proposition 1, the general case now follows by the same arguments as in [FKM2, Theorem 2]. $\square$

Proof of Theorem 5. Follows from Lemmas above. $\square$

7 Tilting modules

In this section we follow [S3, Section 5] and [AHLU] to study tilting modules in $\mathcal{O}(\mathcal{P},\Lambda,H)$. Our main result is the following Theorem, which is analogous to assertions in [R, D].

Theorem 6. *Assume that Λ has a block decomposition with respect to the action of the center, which is locally finite, with each block being the module category over a projectively stratified algebra. Also assume that $\mathcal{O}(\mathcal{P},\Lambda,H)$ has a duality coming from the standard Hopf algebra structure on $\mathfrak{G}$. Let V be an indecomposable projective object in Λ such that $M(V)\in\mathcal{O}(\mathcal{P},\Lambda,H)$. Then there exists a unique object $T(V)\in\mathcal{O}(\mathcal{P},\Lambda,H)$ such that*

1. $\mathrm{Ext}^1_{\mathcal{O}(\mathcal{P},\Lambda,H)}(X,T(V))=0$ *for any $X\in\mathcal{O}(\mathcal{P},\Lambda,H)$ having a standard filtration;*
2. *$T(V)$ has a (possibly infinite) filtration starting with $M(V)$, whose subquotients are of the form $M(W)$, where W is projective in Λ.*

We will call the filtration from (2) also *standard*. The module $T(V)$ will be called indecomposable *tilting module* corresponding to V. By a *tilting module* one usually understands a direct sum of several $T(W)$.

To prove this Theorem we need some preparation, combined in the following Lemmas. Till the end of the Section we work under the assumptions of Theorem 6. In order to construct tilting modules we follow the procedure given by Ringel ([R]), which has been modified by Soergel ([S3]) to cover filtrations of infinite length as well.

Lemma 10. *Let $\lambda,\mu\in\mathfrak{H}_{\mathfrak{A}}^*$ and V be a simple (resp. indecomposable projective) module in Λ such that λ is an $\mathfrak{H}_{\mathfrak{A}}$-weight of V in $M(V)$. Then there exists only finitely many non-isomorphic simples (resp. indecomp. projectives) $W\in\Lambda$ such that μ is an $\mathfrak{H}_{\mathfrak{A}}$-weight of W in $M(W)$ and* $\mathrm{Hom}_{\mathcal{O}(\mathcal{P},\Lambda,H)}(M(V),M(W))$ *or* $\mathrm{Ext}^1_{\mathcal{O}(\mathcal{P},\Lambda,H)}(M(V),M(W))$ *is non-zero.*

Proof. Clearly, from $\mathrm{Hom}_{\mathcal{O}(\mathcal{P},\Lambda,H)}(M(V),M(W))\neq 0$ it follows $(M(W):L(V))\neq 0$. From Lemma 5 we also know that $\mathrm{Ext}^1_{\mathcal{O}(\mathcal{P},\Lambda,H)}(M(V),M(W))\neq 0$ implies $(M(W):L(V))\neq 0$ and hence it is sufficient to show that there exist only finitely many non-isomorphic simples (resp. indecomposable projectives) $W\in\Lambda$ such that μ is an $\mathfrak{H}_{\mathfrak{A}}$-weight of W and $(M(W):L(V))\neq 0$. As an $\mathfrak{A}$-module we have that $M(W)_\lambda\simeq W\otimes F$ for a fixed finite-dimensional module F. Recall that Λ has a block decomposition with respect to the action of the center, hence V belongs to some block Λ_i. Since F is fixed, there exist only finitely many blocks Λ_j, $j\in J$, such that $\hat{W}\otimes F$ has V as a composition factor for some $\hat{W}\in\Lambda_j$ ([K, BG]). As each Λ_j has only finitely many simples (resp. indecomposable projectives) we obtain the statement of the Lemma. $\square$

Lemma 11. *For all objects $V,W\in\Lambda$ the vector spaces* $\mathrm{Hom}_{\mathcal{O}(\mathcal{P},\Lambda,H)}(M(V),M(W))$ *and* $\mathrm{Ext}^1_{\mathcal{O}(\mathcal{P},\Lambda,H)}(M(V),M(W))$ *are finite-dimensional.*

Proof. Any $\mathfrak{H}_{\mathfrak{A}}$-weight space of any module in $\mathcal{O}(\mathcal{P},\Lambda,H)$ is a module of finite length in Λ. Let λ be the $\mathfrak{H}_{\mathfrak{A}}$-weight of V. Then we have a natural inclusion

$$\mathrm{Hom}_{\mathcal{O}(\mathcal{P},\Lambda,H)}(M(V),M(W))\hookrightarrow\mathrm{Hom}_{\Lambda}(V,M(W)_\lambda)$$

where the last space is finite-dimensional, since each block of Λ is a module category over a finite-dimensional algebra. Let $\hat{V}$ be a projective cover of V in Λ. Consider the corresponding projective module $P(\hat{V})$ from $\mathcal{O}(\mathcal{P},\Lambda,H)$, i.e. a projective in $\mathcal{O}(\mathcal{P},\Lambda,H)$ such that there is an exact sequence

$$0 \to K \to P(\hat{V}) \to M(\hat{V}) \to 0$$

with K having a standard filtration. From this we get a surjection

$$\mathrm{Hom}_{\mathcal{O}(\mathcal{P},\Lambda,H)}(K, M(W)) \to \mathrm{Ext}^1_{\mathcal{O}(\mathcal{P},\Lambda,H)}(M(V), M(W)).$$

Now everything follows from the first half of the Lemma. □

Let $\Omega = \{\lambda_i \,|\, i \in \mathbb{N}\}$ be a sequence of elements from P. For $n \in \mathbb{N}$ set $\Omega_n = \{\lambda_i \,|\, 1 \leq i \leq n\}$ and $\hat{P}_n = \cup_{\lambda\in\Omega_n} \hat{P}(\lambda)$.

Lemma 12. *Let Ω be as above, and let V be a projective module in Λ such that $M(V) \in \mathcal{O}(\mathcal{P},\Lambda,H)$. Then for any $n \in \mathbb{N}$ there exists exactly one (up to isomorphism) indecomposable object $T = T(\Omega, n, V)$ such that*

1. $\mathrm{Ext}^1_{\mathcal{O}(\mathcal{P},\Lambda,H)}(M(W), T) = 0$ *for all indecomposable projective $W \in \Lambda$ with $\mathfrak{H}_{\mathfrak{A}}$-weight from $\hat{P}_n \cap P$;*
2. *There is an inclusion $M(V) \hookrightarrow T$, whose cokernel has a finite standard filtration with all subquotients of the form $M(W)$, W projective in Λ with $\mathfrak{H}_{\mathfrak{A}}$-weight from $\hat{P}_n \cap P$.*

Proof. Using Lemmas 11 and 10 the proof repeats the one from [S3, Proposition 5.6]. □

Lemma 13. *Under the conditions of Lemma 12 for any $n > m \in \mathbb{N}$ there exists an inclusion $T(\Omega, m, V) \hookrightarrow T(\Omega, n, V)$ and the cokernel of each such inclusion has a standard filtration with all subquotients of the form $M(W)$, W projective in Λ with $\mathfrak{H}_{\mathfrak{A}}$-weight from $(\hat{P}_n \setminus \hat{P}_m) \cap P$.*

Proof. Analogous to that of [S3, Proposition 5.7]. □

Lemma 14. *Let $V \in \Lambda$ and F be a finite-dimensional $\mathfrak{A}$-module. Then* $\mathrm{Hom}_{\mathbb{C}}(F, V) \in \Lambda$.

Proof. Denote by * the duality on Λ induced from that on $\mathcal{O}(\mathcal{P},\Lambda,H)$. We know that this duality comes from the standard Hopf algebra structure on $\mathfrak{A}$. We have $\mathrm{Hom}_{\mathbb{C}}(F, V) \simeq (F \otimes V^*)^*$ as an $\mathfrak{A}$-module and everything follows from admissibility of Λ. □

For $\mathfrak{H}_{\mathfrak{A}}$-weight $\mathfrak{G}$-modules M_1 and M_2 let $\mathcal{H}\mathrm{om}(M_1, M_2)$ denote the $\mathfrak{H}_{\mathfrak{A}}$-weight set of $\mathfrak{H}_{\mathfrak{A}}$-graded morphisms from M_1 to M_2. From Lemma 14 it follows, in particular, that $M(V)^*$ is isomorphic to $\mathcal{H}\mathrm{om}_{\mathcal{P}}(U(\mathfrak{G}), V)$. We also know that for simple V, $L(V)$ is the socle of $M(V)^*$. Unfortunately, in general the module $M(V)^*$ does not belong to $\mathcal{O}^f(\mathcal{P},\Lambda,H)$.

Lemma 15. *In $\mathcal{O}(\mathcal{P},\Lambda,H)$ there are enough injective modules.*

Proof. For $M \in \mathcal{O}(\mathcal{P}, \Lambda, H)$ the biggest $\mathfrak{H}_{\mathfrak{A}}$-weight $\mathfrak{G}$-submodule of $\mathcal{H}om_{\mathfrak{A}'}(U(\mathfrak{G}), M)$, which lies in $\mathcal{O}(\mathcal{P}, \Lambda, H)$ (Lemma 14), is an injective module, containing M. $\square$

Lemma 16. *Let $M \in \mathcal{O}(\mathcal{P}, \Lambda, H)$ such that $\mathrm{Ext}^1_{\mathcal{O}(\mathcal{P},\Lambda,H)}(M(W), M) = 0$ for all standard $M(W) \in \mathcal{O}(\mathcal{P}, \Lambda, H)$. Then $\mathrm{Ext}^1_{\mathcal{O}(\mathcal{P},\Lambda,H)}(N, M) = 0$ for all $N \in \mathcal{O}(\mathcal{P}, \Lambda, H)$ having a standard filtration.*

Proof. Standard by virtue of Lemma 15 (see [S3, Lemma 5.10]). $\square$

Proof of Theorem 6. Let Ω be as above. Assume that $P = \cup_{n\geq 1}(\hat{P}_n \cap P)$. Put $T(V) = \lim_{n\to\infty} T(\Omega, n, V)$. It is straightforward, that $T(V)$ satisfies (1) and (2).

Using Lemma 16, the proof of the uniqueness of $T(V)$ is standard (see [S3, Theorem 5.2]). In particular, $T(V)$ is independent of the choice of Ω. $\square$

Remark 2. *It is easy to see that for $\Lambda = \Lambda(V(a, b))$ (see Section 6) the functor E sends tilting modules from $\mathcal{O}(\mathcal{P}, \Lambda, H)$ to tilting modules in $\mathcal{O}$. So, in the same way as in [FKM2, Section 10] the charcter of a tilting module from $\mathcal{O}(\mathcal{P}, \Lambda, H)$ can be computed using the character of the corresponding tilting module in $\mathcal{O}$, which is known in many cases ([S3]).*

During the proof of Theorem 6 we have obtained additional information about injective modules in $\mathcal{O}(\mathcal{P}, \Lambda, H)$. In the following Corollary we combine those parts of it concerning simples.

Corollary 4. *Each simple L in $\mathcal{O}(\mathcal{P}, \Lambda, H)$ has an injective hull $I(L) \in \mathcal{O}(\mathcal{P}, \Lambda, H)$. Each $I(L)$ has an increasing filtration with quotients isomorphic to duals of standard modules and under conditions of Corollary 2 there is a reciprocity formula*

$$[I(L) : M(W)^*] = (\hat{W} : W)(M(W) : L)$$

for any simple $L(W) \in \mathcal{O}(\mathcal{P}, \Lambda, H)$.

Proof. Follows from the properties of the duality. $\square$

8 Non-standard generalized Verma modules for affine Lie algebras

In this section we assume that $\mathfrak{G}$ is an affine Lie algebra with a 1-dimensional center Z spanned by an element c. Let δ be an indivisible imaginary root of $\mathcal{Q}$. Then the set of all imaginary roots is $\mathcal{Q}^{im} = \{k\delta | k \in \mathbb{Z} \setminus \{0\}\}$. Denote by G a Heisenberg subalgebra of $\mathfrak{G}$ generated by the root spaces $\mathfrak{G}_{k\delta}$.

A subalgebra $\mathfrak{P} \subset \mathfrak{G}$ is called parabolic if $\mathfrak{P} \supset \mathfrak{H}$ and $\mathfrak{P} + \sigma(\mathfrak{P}) = \mathfrak{G}$. The classification and the structure of all parabolic subalgebras in $\mathfrak{G}$ was described in [F1]. Every parabolic subalgebra has a Levi decomposition $\mathfrak{P} = \mathfrak{A}' \oplus \mathfrak{N}$ where $\mathfrak{A}'$ is either finite-dimensional reductive Lie algebra (type I) or an extension of a sum of some affine Lie subalgebras by a central subalgebra and by a certain subalgebra of the Heisenberg algebra G, generated by the imaginary root spaces of $\mathfrak{G}$, (type II). The subalgebra $\mathfrak{N}$ is called the radical of $\mathfrak{P}$. Note that $\mathfrak{N}$ is solvable only in the case II.

We will work with a fixed parabolic subalgebra $\mathfrak{P} = \mathfrak{A}' \oplus \mathfrak{N}$ of type I. Thus $\mathfrak{A}'$ is a finite-dimensional reductive Lie algebra. Let $\mathfrak{A}' = \mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}$ where $\mathfrak{A}$ is semisimple and $\mathfrak{H}_{\mathfrak{A}}$ is central in $\mathfrak{A}'$. For a subset $T \subset \mathcal{Q}$ we will denote by $\mathfrak{G}(T)$ the subalgebra of $\mathfrak{G}$ generated by the root subspaces $\mathfrak{G}_\alpha$ where $\alpha \in T$. Finally, let $\mathfrak{H}(T) = \mathfrak{H} \cap \mathfrak{G}(T)$.

Let π be a basis of the root system $\mathcal{Q}$, $\mathcal{Q}_+(\pi)$ be the set of positive roots with respect to π and $\delta \in \mathcal{Q}_+(\pi)$ be the indivisible imaginary root. For a subset $T \subset \pi$ we will denote by $< T >_\pi$ a root subsystem generated by T and δ. Set $< T >_\pi^+ = < T >_\pi \cap \mathcal{Q}_+(\pi)$. Let also $\mathcal{Q}_+(T) \subset \mathcal{Q}_+(\pi)$ denote the set of positive roots generated by T.

It is easy to see that there are parabolic subalgebras $\mathfrak{P}$ which do not correspond to any triangular decomposition of $\mathfrak{G}$ ([F2]). In fact, if $\mathfrak{P}$ contains $\mathfrak{G}_{\alpha+k\delta}$ for some α and infinitely many both positive and negative integers k then this parabolic subalgebra does not correspond to any triangular decomposition of $\mathfrak{G}$. On the other hand we have the following

Proposition 2. *Let $T \subset \pi$.*

(i) If T is connected, then $(\mathfrak{H}(< T >_\pi), \mathfrak{G}(< T >_\pi^+), < T >_\pi^+, \sigma)$ is a triangular decomposition of $\mathfrak{G}(< T >_\pi)$;

(ii) If T is not connected, then $(\mathfrak{H}(< T >_\pi), \mathfrak{G}(< T >_\pi^+), Q_+, \sigma)$ is a triangular decomposition of $\mathfrak{G}(< T >_\pi)$;

(iii) Let $T = \cup_i T_i$ with all sets T_i connected. Then $\mathfrak{G}(< T >_\pi) = \mathfrak{G}^T + G(T)$, where $\mathfrak{G}^T = \sum_i \mathfrak{G}^i$, $[\mathfrak{G}^i, \mathfrak{G}^j] = 0$, $i \neq j$, $\mathfrak{G}^i$ is the derived algebra of an affine Lie algebra of rank $|T_i| + 1$ for each i, $[\mathfrak{G}^T, G(T)] = 0$, $G(T) \subset G$, $G(T) + (G \cap \mathfrak{G}^T) = G$, $\mathfrak{G}^T \cap G(T) = \cap_i \mathfrak{G}^i = Z$.

Proof. Statements (i) and (ii) are obvious. Note that in the case when T is not connected, the root subsystem $< T >_\pi$ has no basis consisting of real roots (see [C, Remark 1.4]). Statement (iii) follows from [F2, Proposition 3.2.]. □

Set $\tilde{\mathfrak{G}}^T = \mathfrak{G}^T + \mathfrak{H}$ and $\mathfrak{m}_T = \tilde{\mathfrak{G}}^T + G(T)$.

Proposition 3. *Let $\mathfrak{P} = \mathfrak{A}' \oplus \mathfrak{N}$ be a parabolic subalgebra of $\mathfrak{G}$ of type I, i.e. $\mathfrak{A}'$ is finite-dimensional reductive Lie algebra. Assume that $\mathfrak{G}_\delta \subset \mathfrak{N}$ and $\mathfrak{G}_{\alpha+k\delta} \subset \mathfrak{N}$ for a real root α and all integer k such that $\alpha + k\delta \in \mathcal{Q}$. Then there exist a basis π of $\mathcal{Q}$, a root $\alpha_0 \in \pi$ and a subset $T \subset \pi_0 = \pi \setminus \{\alpha_0\}$ such that*

(i) The set T contains a basis of the root system of $\mathfrak{A}'$ and $\delta \in \mathcal{Q}_+(\pi)$;

(ii) $\mathfrak{P} = \mathcal{P}(T) \oplus \mathfrak{N}_T$ where $\mathcal{P}(T) = \mathfrak{A}' + \mathfrak{G}(< T >_\pi^+)$ is a parabolic subalgebra of $\mathfrak{m}_T$, $\mathfrak{N}_T$ is generated by the spaces $\mathfrak{G}_{\beta+k\delta}$ for all $\beta \in \mathcal{Q}_+(\pi_0) \setminus \mathcal{Q}_+(T)$ and all integers k for which $\beta + k\delta \in \mathcal{Q}$.

Proof. The statements follow from [F2], Theorem 2.5. $\square$

From now on we will assume that $\mathfrak{P} = \mathfrak{A}' \oplus \mathfrak{N}$ is fixed, $\mathfrak{G}_\delta \subset \mathfrak{N}$ and that there exists a real root α such that $\mathfrak{G}_{\alpha+k\delta} \subset \mathfrak{N}$ if $\alpha + k\delta \in \mathcal{Q}$. In this case $\mathfrak{P}$ is called non-standard parabolic subalgebra. Then by Proposition 3, (ii), $\mathfrak{P} = \mathcal{P}(T) \oplus \mathfrak{N}_T$ for a certain T. It is clear that $\mathfrak{G} = \sigma(\mathfrak{N}_T) \oplus \mathfrak{m}_T \oplus \mathfrak{N}_T$. Denote $\mathcal{P}_T = \mathfrak{m}_T \oplus \mathfrak{N}_T$. This is a parabolic subalgebra of $\mathfrak{G}$ of type II. It follows from Proposition 2 that algebra $\mathfrak{m}_T$ has a triangular decomposition $\mathfrak{m}_T^- \oplus \mathfrak{H} \oplus \mathfrak{m}_T^+$.

Let Λ be an admissible category of $\mathfrak{A}'$-modules and $V \in \Lambda$. We will make V into a $\mathfrak{P}$-module with a trivial action of the radical $\mathfrak{N}$. The module

$$M_{\mathfrak{P}}(V) = U(\mathfrak{G}) \otimes_{U(\mathfrak{P})} V$$

is called *non-standard generalized Verma module* ([F2]). As modules $M_{\mathcal{P}}(V)$, $M_{\mathfrak{P}}(V)$ is $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable. If V is a simple object, then $M_{\mathfrak{P}}(V)$ has a unique maximal submodule, which decomposes into a direct sum of modules in Λ, when viewed as $\mathfrak{A}'$-module. We denote by $L_{\mathfrak{P}}(V)$ the simple quotient of $M_{\mathfrak{P}}(V)$.

Consider the subspace $M^T(V) = U(\mathfrak{m}_T) \otimes_{U(\mathfrak{P}) \cap U(\mathfrak{m}_T)} V$ which is an $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable $\mathfrak{m}_T$-module. Module $M^T(V)$ is a generalized Verma module over $\mathfrak{m}_T$ if V is simple. In this case we will denote by $L^T(V)$ the simple quotient of $M^T(V)$. We can also view it as a $\mathcal{P}(T)$-module with a trivial action of $\mathfrak{N}_T$.

Since $M_{\mathfrak{P}}(V)$ is $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable module and c is central then c is diagonalizable on $M_{\mathfrak{P}}(V)$. We say that $M_{\mathfrak{P}}(V)$ is non-degenerate if c is non-degenerate on V.

Let $T = \cup_i T_i$ be a decomposition of T into connected components and let $\mathfrak{G}^T = \sum_i \mathfrak{G}^i$ be the corresponding decomposition of $\mathfrak{G}^T$. Consider the tensor algebra $A_T = \otimes_i U(\mathfrak{G}^i) \otimes U(G(T))$. Clearly, $M^T(V)$ is an A_T-module. Algebra $G(T)$ has a natural triangular decomposition $G(T) = G(T)_- \oplus Z \oplus G(T)_+$ induced by the choice of positive roots $\mathcal{Q}_+(\pi)$. Let $M(c_T)$ be a $G(T)_-$-free rank one $G(T)$-module on which c acts via the scalar c_T. Then we have the following standard fact.

Proposition 4. *Let $\mathfrak{P} = \mathfrak{A}' \oplus \mathfrak{N} = \mathcal{P}(T) \oplus \mathfrak{N}_T$, $\mathfrak{A}' = \mathfrak{A} + \mathfrak{H}$, $\mathfrak{A} = \oplus_{j \in J} \mathfrak{A}_j$ where all $\mathfrak{A}_j$ are simple Lie algebras. Suppose that $T = \cup_{j \in J} T_j \cup_{i \in I} T_i'$ with all T_j and T_i' being connected and each T_j containing a basis of the root system of $\mathfrak{A}_j$. Denote $\mathfrak{H}_i = \mathfrak{H} \cap \mathfrak{G}^i$, $i \in I$. If $V \in \Lambda$ is simple then V is a $\mathcal{P}(T)$-module with a trivial action of $\mathfrak{G}_\beta$ for all $\beta \in < T >_\pi^+$ which are not the roots of $\mathfrak{A}$, $V \simeq \otimes_{j \in J} V_j$, where V_j is simple (in a proper category of $\mathfrak{A}_j$-modules) and $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable, $\mathfrak{H}_i$ acts on V via a weight $\lambda_i \in \mathfrak{H}_i^*$ for each $i \in I$ and*

$$M^T(V) \simeq \otimes_{j \in J} M_{\mathfrak{P}_j}(V_j) \otimes_{i \in I} M(\lambda_i) \otimes M(c_T)$$

as A_T-modules, where $\mathfrak{P}_j = \mathfrak{A}_j + \mathfrak{H} + \mathfrak{G}(< T >_\pi^+)$ and $M(\lambda_i)$ is the Verma module with highest weight λ_i over $\mathfrak{G}^i$ with respect to the induced triangular decomposition of $\mathfrak{G}^i$.

Proof. Follows from the construction of $M_{\mathfrak{P}}(V)$. $\square$

Let Q be the abelian group generated by π and let $\hat{Q}$ be the abelian group generated by π_0. Every element of $\beta \in Q$ can be written in the form $\beta = \alpha + k\delta$ for some $\alpha \in \hat{Q}$ and we set $\mathrm{ht}^{im}(\beta) = |k|$. If $\mu = \varphi + k\delta$ with $\varphi \in \hat{Q}$ and $k \in \mathbb{Z}$, then we denote by $\mathrm{ht}\,\mu$ (respectively $\mathrm{ht}_1\,\mu$) the number of elements of π_0 (respectively $\pi_0 \setminus T$) in the decomposition of φ. The universal enveloping algebra $U(\mathfrak{G})$ has a natural grading by the elements of Q. If $u \in U(\mathfrak{G})_\mu$ and $\mu = \varphi + k\delta$, $\varphi \in \hat{Q}$, $k \in \mathbb{Z}$ then we set $\mathrm{ht}_1\,u = \mathrm{ht}_1\,\mu$. The following is the principle result of this Section showing the role of $M^T(V)$.

Lemma 17. *If $N \neq 0$ is a submodule of $M_{\mathfrak{P}}(V)$ then $N \cap M^T(V) \neq 0$.*

Proof. The proof is analogous to the proof of Lemma 5.4 in [F2]. Without loss of generality we can assume that V is simple in Λ. Let $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$ be the $\mathfrak{H}_{\mathfrak{A}}$-weight of V and $0 \neq v \in N_{\lambda-\mu}$ where $\mu = \varphi + k\delta$, $\varphi \in \hat{Q}$, $k \in \mathbb{Z}$. Then we can write $v = \sum_{i=1}^r u_i v_i$, where $u_i \in U(\sigma(\mathfrak{N}_T))$ are linearly independent, $v_i \in M^T(V)_{\lambda-\mu_i}$, $\mu_i = \nu_i + l_i\delta$, $\nu_i \in \hat{Q}$, $l_i \in \mathbb{Z}$. Clearly, $\mathrm{ht}_1\,u_i = \mathrm{ht}_1\,u_j$ for all i and j. We will denote this number by $\mathrm{ht}_1\,v$ and carry out the proof by induction in $\mathrm{ht}_1\,v$. Assume that $\mathrm{ht}_1\,v = 1$. Then $u_i \in \mathfrak{G}_{\varphi_i + k_i\delta}$ with $\varphi_i \in \mathcal{Q}_+(\pi_0)$, $\mathrm{ht}_1\,\varphi_i = 1$. We can also assume that $ht\varphi_i \geq ht\varphi_j$ if $i < j$. Choose sufficiently large $m \in \mathbb{Z}_+$ such that $\varphi_r - m\delta \in \mathcal{Q}$, $m > |k_i|$ for all i and $m - k_t > \sum_{i=1}^r |l_i|$ for all those t for which $ht\varphi_t = \mathrm{ht}\,\varphi_r$. Now let $0 \neq x \in \mathfrak{G}_{\varphi_r - m\delta}$. We have that $xv_i = 0$ for all i and $xv \neq 0$. Clearly, $xv \in M^T(V)$ which completes the proof in this case. If $ht_1 v > 1$ then similar arguments (see the proof of [F2, Lemma 5.4]) and induction in $\mathrm{ht}_1\,v$ complete the proof of the Lemma. ☐

As an immediate consequence of Lemma 17 we have

Theorem 7. *Let V be a simple $\mathfrak{A}'$-module. Module $M_{\mathfrak{P}}(V)$ is simple if and only if $M^T(V)$ is simple as a $\mathfrak{m}_T$-module.*

Hence, Theorem 7 reduces the problem of simplicity of a non-standard generalized Verma module $M_{\mathfrak{P}}(V)$ to the problem of simplicity of a generalized Verma module $M(V_j)$ (see [KM] for several examples) and Verma modules $M(\lambda_i)$ and $M(c_T)$ ([MP]).

9 Non-standard non-degenerate generalized Verma modules

As in the previous Section we work here with a fixed non-standard parabolic subalgebra $\mathfrak{P}$ of $\mathfrak{G}$ of type I. We study further properties of module $M_{\mathfrak{P}}(V)$ in the case when c is non-degenerate on V. As in the case of Verma type modules worked out in [F2, Section 5.3], under this condition the structure of $M_{\mathfrak{P}}(V)$ is completely determined by the structure of m_T-module $M^T(V)$.

Lemma 18. *(i) Let $c_T \neq 0$. Then the Verma module $M(c_T)$ is irreducible.*

(ii) *Let $V \in \Lambda$ such that $0 \notin \operatorname{Spec}_V c$, N be a submodule of $M^T(V)$, $0 \neq v \in N$ and $v = \sum_{j\in J} u_j v_j$ where $v_j \in V$, $u_j = \sum_i u_{ij}u'_{ij} \in U(\mathfrak{m}_T^-)$, $u_{ij} \in \mathfrak{G}_{\beta_{ij}}$, $\beta_{ij} \in -\mathcal{Q}_+(\pi)$, $\beta_{ij} \neq \beta_{ik}$ for $j \neq k$, and $\operatorname{ht}^{im}(\beta_{ij})$ is sufficiently large for all i and j. Then $u'_{ij}v_j \in N$ for all i, j.*

Proof. Statement (i) is standard. The proof of (ii) is analogous to the proof of [F2, Lemma 5.13]. Without loss of generality we may assume that $|J| = 1$. The general case is treated similarly. Let $v = \sum_{i\in I} u_i u'_i v_0$, where $u_i \in \mathfrak{G}_{\beta_i}$, $\beta_i \neq \beta_j$ if $i \neq j$, $\beta_i = \varphi_i - k_i\delta$, $\varphi_i \in \hat{Q}$, $k_i \in \mathbb{Z}$. Again without loss of generality we may assume that all u'_i are homogeneous, $u'_i \in U(\mathfrak{m}_T^-)_{\nu_i - l_i\delta}$, $k_i >> l_j$ for all i and j. Suppose first that $|I| = 1$. Then $v = uu'v_0$ with $u \in \mathfrak{G}_{\varphi - m\delta}$, $\varphi \in \hat{Q}$, $u' \in U(\mathfrak{m}_T^-)_{\beta - l\delta}$, $m >> l \geq 0$. If $\varphi = 0$ then choose $x \in \mathfrak{G}_{m\delta}$ such that $[x, u] \neq 0$. Since $m > l$, $xu'v_0 = 0$. Taking into account that 0 is not an eigenvalue of c on V, we obtain that $xv \in N$ is proportional to $u'v_0$. Now consider the case $\varphi \neq 0$. Choose a sufficiently large positive k such that $k < m$, $\varphi - k\delta \in \mathcal{Q}$, $k > l$ and $m - k > l$. If m is sufficiently large, such k always exists. For $0 \neq x \in \mathfrak{G}_{-\varphi + k\delta}$ we have $xv = xuu'v_0 = \tilde{u}u'v_0 \neq 0$, where $\tilde{u} \in \mathfrak{G}_{(k-m)\delta}$. Thus we came to the first case. Now suppose that $|I| > 1$. Choose $\varphi_{i_0} \in \{\varphi_i, i \in I\}$ with minimal $\operatorname{ht}\varphi_{i_0}$. If $\operatorname{ht}\varphi_{i_0} = 0$ then choose k such that $l_i < k < k_i$ for all i. Then for $0 \neq x \in \mathfrak{G}_{k\delta}$, $0 \neq xv = \sum_{i\in I\setminus\{i_0\}} \tilde{u}_i u'_i v_0$ where $\tilde{u} = [x, u_i] \in \mathfrak{G}_{\beta_i + k\delta}$. Induction on $|I|$ completes the proof in this case. If $\operatorname{ht}\varphi_{i_0} > 0$ then choose k such that $\varphi - k\delta \in \mathcal{Q}$, $l_i < k < k_i$ and $l_i < k_{i_0} - k$ for all $i \in I$. Now for $0 \neq x \in \mathfrak{G}_{-\varphi_{i_0} + k\delta}$ we have $xv = \tilde{u}_{i_0}u'_{i_0}v_0 + \sum_{i\in I\setminus\{i_0\}}[x, u_i]u'_i v_0$, where $\tilde{u}_{i_0} \in \mathfrak{G}_{(k - k_{i_0})\delta}$. If we reduced the number of summands then we can apply the induction in $|I|$. If not we use the fact that the minimal height is 0 now and proceed as in the case above. This completes the proof. □

Theorem 8. *Let $V \in \Lambda$, N be a submodule of $M_{\mathfrak{P}}(V)$ and $N^T = N \cap M^T(V)$. If $\operatorname{Spec}_V c$ does not contain 0 then*

(i) *$N \simeq U(\mathfrak{G}) \otimes_{U(\mathcal{P}_T)} N^T$ where $\mathfrak{N}_T$ acts trivially on N^T, in particular, $N \simeq U(\sigma(\mathfrak{N}_T)) \otimes_{\mathbb{C}} N^T$ as a vector space;*

(ii) *If V is a simple object in Λ, then $L_{\mathfrak{P}}(V) \simeq U(\mathfrak{G}) \otimes_{U(\mathcal{P}_T)} L^T(V)$, where $\mathfrak{N}_T$ acts trivially on L^T, in particular, $L_{\mathfrak{P}}(V) \simeq U(\sigma(\mathfrak{N}_T)) \otimes_{\mathbb{C}} L^T(V)$ as a vector space.*

Proof. Lemma 17 guarantees that $N^T \neq 0$ if $n \neq 0$. Also it is clear that the $\mathfrak{m}_T$-module N^T can be viewed as a $\mathcal{P}_T$-module with a trivial action of $\mathfrak{N}_T$. Hence all the tensor products are well-defined. The proof of statement (i) follows the general lines of the proof of [F2, Theorem 5.14]. Let $0 \neq v \in N$. Then $v \in \sum_{j\in J} u_j v_j$ where $u_j \in U(\sigma(\mathfrak{N}))$ and $v_j \in V$. Without loss of generality we may assume that $|J| = 1$. Hence, $v = \sum_{i\in I} u_i u'_i v_0$ with $u_i \in U(\sigma(\mathfrak{N}_T))_{\mu_i}$, $u'_i \in U(\mathfrak{m}_T^-)$ and all u_i's are not multiples of each other. We can also assume that $\operatorname{ht}_1 \mu_i = \operatorname{ht}_1 \mu_j$ for all i and j and denote this number by $\operatorname{ht}_1 v$. We will show by induction on $\operatorname{ht}_1 v$ that $u'_i v_0 \in N$ for all i which would imply statement (i). Suppose first that $\operatorname{ht}_1 v = 1$. Then $u_i \in \mathfrak{G}_{-\varphi_i + m_i\delta}$, $\varphi_i \in \hat{Q}$, $\operatorname{ht}_1 \varphi_i = 1$ for all i. Let $\varphi_{i_0} \in \{\varphi_i\}$ and m is sufficiently large positive integer. Consider $0 \neq x \in \mathfrak{G}_{\varphi_{i_0} - m\delta}$. Then $xu'_i v_0 = 0$ for all i and $0 \neq xv = \tilde{u}u'_{i_0}v_0 + \sum_{i\in I\setminus\{i_0\}} \tilde{u}_i u'_i v_0$ where $\tilde{u} \in \mathfrak{G}_{(m_{i_0}-m)\delta}$, $\tilde{u}_i = [x, u_i]$, $\mathrm{ht}_1\, \tilde{u}_i = 0$. Applying Lemma 18 we conclude that $\tilde{u}u'_{i_0}v_0 \in N$. Taking into acount that 0 is not an eigenvalue of c on V, we obtain that $u_{i_0}v_0 \in N$ just by applying an element $y \in \mathfrak{G}_{-\varphi_{i_0}+m\delta}$ such that $[y, \tilde{u}] \neq 0$. It remains to apply the induction in $|I|$. Suppose now that $\mathrm{ht}_1\, v > 1$. In this case the standard arguments (see the proof of [F2, Theorem 5.14]) show that there exist $\varphi \in \hat{Q}$ with $\mathrm{ht}_1\, \varphi \neq 0$, sufficiently large positive integer m and $x \in \mathfrak{G}_{\varphi-m\delta}$ such that $xv \neq 0$ and $\mathrm{ht}_1\, xv < \mathrm{ht}_1\, v$. Then we proceed by the induction in $\mathrm{ht}_1\, v$ and in $|I|$ using also the fact that c is non-degenerate on V. Statement (ii) is an immediate consequence of (i). □

Corollary 5. *Let $\mathfrak{P} = \mathfrak{A}' \oplus \mathfrak{N} = \mathcal{P}(T) \oplus \mathfrak{N}_T$, V and V' be simple objects from Λ, $c_T(V) \neq 0$ and $c_T(V') \neq 0$. Then*

$$\mathrm{Hom}\,_{\mathfrak{G}}(M_{\mathfrak{P}}(V), M_{\mathfrak{P}}(V')) \simeq \mathrm{Hom}\,_{\mathfrak{m}_T}(M^T(V), M^T(V')).$$

10 An equivalence of categories

In this Section we study a category of $\mathfrak{G}$-modules which contains non-standard generalized Verma modules in the case when c is non-degenerate. Let $\mathfrak{P} = \mathfrak{A}' \oplus \mathfrak{N} = \mathcal{P}(T) \oplus \mathfrak{N}_T$, Λ be an admissible category of $\mathfrak{A}'$-modules that satisfy condition (B). Fix a finite set $H \in \mathfrak{H}_{\mathfrak{A}}^*$ and consider the category $\mathcal{O}(\mathcal{P}(T), \Lambda, H)$ of $\mathfrak{m}_T$-modules as in Section 3. We assume that $\mathcal{O}(\mathcal{P}(T), \Lambda, H)$ carries a natural abelian structure induced from Λ. If V is a simple module in Λ with $\mathfrak{H}_{\mathfrak{A}}$-weights in P then, by Lemma 2, both $M^T(V)$ and $L^T(V)$ are objects in $\mathcal{O}(\mathcal{P}(T), \Lambda, H)$. Moreover, modules of type $L^T(V)$ exhaust all simple objects. Further, $\mathcal{O}^f(\mathcal{P}(T), \Lambda, H)$ will denote the full subcategory of $\mathcal{O}(\mathcal{P}(T), \Lambda, H)$ consisting of all finitely generated modules.

Denote by K_T the set of all integral linear combinations with positive coefficients of elements from the set $(< T >_{\pi}^{+} \cup \{\beta + k\delta | \beta \in \mathcal{Q}_+(\pi_0) \setminus \mathcal{Q}_+(T)\, k \in \mathbb{Z}\}) \cap (\mathcal{Q} \setminus \mathcal{Q}_+^S)$ (see Section 2).

Let $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$ and $\tilde{P}(\lambda) = \lambda - K_T$. We will assume H to be chosen in such a way that $\lambda(c) \neq 0$ for all $\lambda \in H$ and $\tilde{P}(\lambda) \not\subset \tilde{P}(\mu)$ for all $\lambda, \mu \in H$.

Now we define a full subcategory, $\mathcal{O}(\mathfrak{P}, \Lambda, H)$, of the category of $\mathfrak{G}$-modules, which is consisting of all modules M satisfying the following conditions:

1. The $\mathfrak{H}_{\mathfrak{A}}$ support of M is a subset of $\tilde{P} = \cup_{\lambda \in H} \tilde{P}(\lambda)$;
2. $M^T = \sum_{\mu \in P} M_\mu$ is an object in $\mathcal{O}(\mathcal{P}(T), \Lambda, H)$;
3. The module M is generated by M^T.

If V is a simple object in Λ having an $\mathfrak{H}_{\mathfrak{A}}$-weight in $\tilde{P}$, then both $M_{\mathfrak{P}}(V)$ and $L_{\mathfrak{P}}(V)$ are objects of $\mathcal{O}(\mathfrak{P}, \Lambda, H)$. Also, any simple object of $\mathcal{O}(\mathfrak{P}, \Lambda, H)$ is isomorphic to $L_{\mathfrak{P}}(V)$ for some simple $V \in \Lambda$ that has an $\mathfrak{H}_{\mathfrak{A}}$ weight in $\tilde{P}$.

Let $M \in \mathcal{O}(\mathfrak{P}, \Lambda, H)$ and L be a simple object in the same category. Then, as in the case of $\mathcal{O}(\mathcal{P}, \Lambda, H)$, we have a well–defined notion of multiplicity $(M : L)$ of L in M. It follows from Theorem 8 that $(M_{\mathfrak{P}}(V) : L_{\mathfrak{P}}(V)) = (M^T(V) : L^T(V))$.

It is obvious that any $\mathfrak{m}_T$-module $W \in \mathcal{O}(\mathcal{P}(T), \Lambda, H)$ has a *generalized highest weight series*, i.e. a filtration

$$0 = W_0 \subset W_1 \subset W_2 \subset \dots$$

of submodules of W such that $\cup_i W_i = W$ and W_i/W_{i-1} is a homomorphic image of $M^T(V_i)$ for some $V_i \in \Lambda$. A standard argument ([F2, Proposition 14.3]) shows that the same holds for any object in $\mathcal{O}(\mathfrak{P}, \Lambda, H)$, i.e. any object M has a filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots$$

of submodules such that $\cup M_i = M$ and M_i/M_{i-1} is a homomorphic image of $M_{\mathfrak{P}}(V_i)$ for some $V_i \in \Lambda$.

Proposition 5. *The category $\mathcal{O}(\mathfrak{P}, \Lambda, H)$ is closed under operations of taking finite direct sums, submodules, which decompose into a direct sum of modules from Λ, when viewed as $\mathfrak{A}'$-module, and the corresponding quotients.*

Proof. The statement about direct sums is obvious. Let $M \in \mathcal{O}(\mathfrak{P}, \Lambda, H)$ and $N \subset M$ be a submodule, which decomposes into a direct sum of modules from Λ, when viewed as $\mathfrak{A}'$-module. It is enough to show that $N \in \mathcal{O}(\mathfrak{P}, \Lambda, H)$. Let $N^T = \sum_{\mu \in P} N_\mu$. Since N^T is an $\mathfrak{m}_T$-submodule of M^T, $N^T \in \mathcal{O}(\mathcal{P}(T), \Lambda, H)$. Consider a generalized highest weight series, $0 = M_0 \subset M_1 \subset \dots$, of M, where M_i/M_{i-1} is a homomorphic image of $M_{\mathfrak{P}}(V_i)$, $V_i \in \Lambda$. Set $N_i = N \cap M_i$, $\bar{N}_i = N_i/N_{i-1}$ and let $\bar{N}_i^T$ be the T-part of $\bar{N}_i$. Then $\cup_i N_i = N$ and if $\bar{N}_i \neq 0$ then $\bar{N}_i^T \neq 0$ by Lemma 17. It follows from Theorem 8 that $\bar{N}_i$ is generated by $\bar{N}_i^T$ implying immediately that N_i is generated by N_i^T as a $\mathfrak{G}$-module. Thus N is generated by N^T and the statement follows. $\square$

Theorem 9. *If $\lambda(c) \neq 0$ for all $\lambda \in H$, then the categories $\mathcal{O}(\mathfrak{P}, \Lambda, H)$ and $\mathcal{O}(\mathcal{P}(T), \Lambda, H)$ are equivalent.*

Proof. Define an exact functor $F : \mathcal{O}(\mathfrak{P}, \Lambda, H) \to \mathcal{O}(\mathcal{P}(T), \Lambda, H)$ where $F(M) = M^T$ and $F(f) = f|_{M^T}$ for any $M \in \mathcal{O}(\mathfrak{P}, \Lambda, H)$ and any $f \in Hom_{\mathfrak{G}}(M, M')$. If $W \in \mathcal{O}(\mathcal{P}(T), \Lambda, H)$ is an $\mathfrak{m}_T$-module then we can view it as a $\mathcal{P}_T$-module with a trivial action of $\mathfrak{N}_T$ and define a $\mathfrak{G}$-module $Y(W) = U(\mathfrak{G}) \otimes_{U(\mathcal{P}_T)} W \in \mathcal{O}(\mathfrak{P}, \Lambda, H)$. Hence, Y is an exact functor from $\mathcal{O}(\mathcal{P}(T), \Lambda, H)$ to $\mathcal{O}(\mathfrak{P}, \Lambda, H)$. If $X \in \mathcal{O}(\mathfrak{P}, \Lambda, H)$ is an arbitrary homomorphic image of a module $M_{\mathfrak{P}}(V)$ then $Y \circ F(X) \simeq X$ by Theorem 8. For an arbitrary object $M \in \mathcal{O}(\mathfrak{P}, \Lambda, H)$ consider a generalized highest weight series $0 = M_0 \subset M_1 \subset \dots$. Induction on i shows that $Y \circ F(M_i) \simeq M_i$ for all i and therefore $Y \circ F(M) \simeq M$. Now Corollary 5 completes the proof. $\square$

Using Theorem 9 all the results from Section 3 to Section 7 can be easily transferred to the category $\mathcal{O}(\mathfrak{P}, \Lambda, H)$. In particular, there is an analogue of the BGG-reciprocity, a decomposition into blocks, a relation to projectively stratified algebras and a theory of tilting modules.

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Note, that referring to [S3] we mean the last version of the paper, available from the homepage of W.Soergel: `http://sunpool.mathematik.uni-freiburg.de/home/soergel/`.

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