

## Stratified algebras arising in Lie theory

Volodymyr Mazorchuk

Department of Mathematics, Uppsala University, Box 480, S-751 06, Uppsala, SWEDEN,  
mazor@math.uu.se

**Abstract.** In this survey we show what kind of stratified associative algebras arise in parabolic generalizations of the category  $\mathcal{O}$ , associated with a triangular decomposition of a semi-simple finite-dimensional complex Lie algebra.

*Dedicated to Vlastimil Dlab on the occasion of his 70th birthday*

### 1 Quasi-hereditary algebras and stratified algebras

Let  $A$  be a finite-dimensional associative algebra over a field  $k$ , and  $\leq$  be a partial order on the set  $\Gamma$  of indexes for isomorphism classes  $\{L(i) \mid i \in \Gamma\}$  of simple  $A$ -modules. The pair  $(A, \leq)$  is said to be a *quasi-hereditary algebra*, [8], provided that there exists a family,  $\{\Delta(i) \mid i \in \Gamma\}$ , of  $A$ -modules, called *standard* modules, such that the following two conditions are satisfied:

- (QH1)  $\Delta(i)$  surjects onto  $L(i)$ , and any composition subfactor,  $L(j)$ , of the kernel of this surjection satisfies  $j < i$ ;
- (QH2) the indecomposable projective cover  $P(i)$  of  $L(i)$  surjects onto  $\Delta(i)$ , and the kernel of this surjection has a filtration, whose subquotients are modules  $\Delta(j)$  with  $i < j$ .

The principal example, where quasi-hereditary algebras appear, is the celebrated category  $\mathcal{O}$  of Bernstein, Gelfand and Gelfand, associated with a triangular decomposition of a semi-simple finite-dimensional complex Lie algebra, [5], which will be discussed in the next section. For other example, where quasi-hereditary algebras naturally appear, we refer the reader to [25].

In 1996 the same authors introduced a more general concept of standardly stratified algebras, [9], which looks as follows. Recall first that a *pre-order* (or *quasi-order*) is a reflexive and transitive relation. One has to note that for a pre-order,  $\preceq$  say, the inequalities  $i \preceq j$  and  $j \preceq i$  do not imply  $i = j$  in general. Now, instead of the partial order  $\leq$  we fix a partial pre-order,  $\preceq$ , on  $\Gamma$ , and call the pair  $(A, \preceq)$  a *standardly stratified algebra* provided that there exists a family,

---

2000 *Mathematics Subject Classification.* Primary 16G10, 17B10, Secondary 16D90, 22E47.  
Partially supported by The Swedish Research Council.

$\{\Delta(i) \mid i \in \Gamma\}$ , of  $A$ -modules, called again *standard* modules, such that the following two conditions are satisfied:

- (SS1)  $\Delta(i)$  surjects onto  $L(i)$ , and any composition subfactor,  $L(j)$ , of the kernel of this surjection satisfies  $j \preceq i$ ;
- (SS2) the indecomposable projective cover  $P(i)$  of  $L(i)$  surjects onto  $\Delta(i)$ , and the kernel of this surjection has a filtration, whose subquotients are modules  $\Delta(j)$  with  $i \prec j$ .

Independently, V.Dlab introduced in [12] the notion of  $\Delta$ -*filtered algebras*, which differs from the standardly stratified algebras above only by requiring that  $\preceq$  is still a partial order. In Section 6 we will see that the absence of this requirement is very convenient in some applications. Remark that the notion of stratifying ideal, used in [9] for the (alternative) definition of standardly stratified algebras, appears already in [2, Section 1] under the name *strong idempotent ideal*<sup>1</sup>.

It is obvious that every quasi-hereditary algebra is standardly stratified and the reverse is certainly not true. In fact, any finite-dimensional algebra can be given a standard stratification by choosing  $\preceq$  to be the trivial pre-order (full relation). In this case  $\Delta(i) = P(i)$  for all  $i \in \Gamma$ .

Recently, a special subclass of stratified algebra, called properly stratified algebras, was introduced in [13]. The difference with the usual stratified algebras is that, first of all,  $\leq$  is required to be a partial order (as for quasi-hereditary algebras), which is more restrictive if compared to the requirement for stratified algebras of having a pre-order. Secondly, one also additionally requires the existence of two different families of modules, the so-called standard and costandard modules. One of the motivations for this definition is an attempt to restore the lost left-right symmetry: an algebra,  $A$ , is quasi-hereditary if and only if the opposite algebra  $A^{op}$  is quasi-hereditary, whereas an analogous statement for standardly stratified algebras does not hold.

So, following [13], a pair,  $(A, \leq)$ , is called a *properly stratified algebra* provided that there exists a family,  $\{\Delta(i) \mid i \in \Gamma\}$ , of  $A$ -modules, called *standard* modules, and a family,  $\{\overline{\Delta}(i) \mid i \in \Gamma\}$ , of  $A$ -modules, called *proper standard* modules, such that the following three conditions are satisfied:

- (PS1)  $\overline{\Delta}(i)$  surjects onto  $L(i)$  and any composition subfactor,  $L(j)$ , of the kernel of this surjection satisfies  $j < i$ ;
- (PS2) the indecomposable projective cover  $P(i)$  of  $L(i)$  surjects onto  $\Delta(i)$ , and the kernel of this surjection has a filtration, whose subquotients are modules  $\Delta(j)$  with  $i < j$ ;
- (PS3) for every  $i \in \Gamma$  the module  $\Delta(i)$  is filtered by  $\overline{\Delta}(i)$ .

It is straightforward that every properly stratified algebra is standardly stratified, and it is easy to see that an algebra,  $A$ , is properly stratified if and only if the opposite algebra  $A^{op}$  is.

On the level of stratification with respect to the induced partial order, one can say that a quasi-hereditary algebra is stratified by semi-simple algebras (the endomorphism algebras of the standard modules), a properly stratified algebra is stratified by local algebras (again the endomorphism algebras of the standard modules), and a standardly stratified algebra is stratified by more or less arbitrary

---

<sup>1</sup>I am in debt to V.Dlab for this reference.

finite-dimensional algebras (the endomorphism algebras of the direct sums of standard modules, which correspond to “equal” parameters, i.e. those  $i$  and  $j$  for which  $i \preceq j$  and  $j \preceq i$ ).

## 2 Principal example: Category $\mathcal{O}$

Let  $\mathfrak{g}$  be a semi-simple complex finite-dimensional Lie algebra and  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$  be a fixed triangular decomposition of  $\mathfrak{g}$ , where  $\mathfrak{h}$  is a Cartan subalgebra. The BGG category  $\mathcal{O}$ , [5], associated with this triangular decomposition, is defined as the full subcategory in the category of all  $\mathfrak{g}$ -modules, which consists of all finitely generated,  $\mathfrak{h}$ -diagonalizable and  $U(\mathfrak{n}_+)$ -locally finite modules (here and later a module,  $M$ , over an associative algebra,  $A$ , is said to be *A-locally finite* provided that the dimension of the vector space  $Av$  is finite for every element  $v \in M$ ).

Very important objects in  $\mathcal{O}$  are *Verma modules*  $M(\lambda)$ , defined as follows: for  $\lambda \in \mathfrak{h}^*$  we consider  $\mathbb{C}_\lambda = \mathbb{C}$  as an  $\mathfrak{h} \oplus \mathfrak{n}_+$ -module under the action  $(h+n)(c) = \lambda(h)c$ ,  $n \in \mathfrak{n}$ ,  $h \in \mathfrak{h}$ ,  $c \in \mathbb{C}$ , and set  $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda$  (see [11, Chapter 7] for details). The module  $M(\lambda)$  is indecomposable and has the unique simple quotient  $L(\lambda)$ . Moreover, the set  $\{L(\lambda) | \lambda \in \mathfrak{h}^*\}$  is a complete set of simple objects in  $\mathcal{O}$ . The principal result of [5] is the following.

- Theorem 2.1**
1. *Category  $\mathcal{O}$  has enough projective modules, i.e. every module in  $\mathcal{O}$  is a quotient of a projective module.*
  2. *Every projective object in  $\mathcal{O}$  is filtered by Verma modules (or has a so-called Verma flag).*
  3. *The number  $[P(\lambda) : M(\mu)]$  of occurrences of the Verma module  $M(\mu)$  as a subquotient in any Verma flag of the projective cover  $P(\lambda)$  of the module  $L(\lambda)$  equals the composition multiplicity of  $L(\lambda)$  in  $M(\mu)$ , i.e.  $[P(\lambda) : M(\mu)] = (M(\mu) : L(\lambda))$ .*

We remark that here and further the expression *module  $M$  is filtered by modules  $\{N_i | i \in I\}$*  means that there exists a filtration of  $M$ , such that every subquotient of this filtration is isomorphic to some  $N_i$ .

The last statement of Theorem 2.1 is known as the *BGG-reciprocity principle*. Theorem 2.1 already suggests that category  $\mathcal{O}$  should lead to quasi-hereditary algebras with Verma modules being standard. Indeed, because of the  $\mathfrak{h}$ -weight decomposition we always have that  $(M(\lambda) : L(\mu)) \neq 0$  implies  $\mu \leq \lambda$  with respect to the natural order on  $\mathfrak{h}^*$ . At the same time from this and from the BGG-reciprocity one immediately gets that  $[P(\lambda) : M(\mu)] \neq 0$  implies  $\lambda \leq \mu$ . Moreover,  $(M(\lambda) : L(\lambda)) = 1$  also implies  $[P(\lambda) : M(\lambda)] = 1$  by the BGG-reciprocity. Hence to get a finite-dimensional algebra we just need to take a direct summand of  $\mathcal{O}$  having finitely many simples. This is quite standard using the following arguments with the central character. It is easy to see that the action of the center  $Z(\mathfrak{g})$  of the universal enveloping algebra  $U(\mathfrak{g})$  on every module from  $\mathcal{O}$  is locally finite and hence  $\mathcal{O}$  decomposes into a direct sum of full subcategories  $\mathcal{O}^\theta$ ,  $\theta \in Z(\mathfrak{g})^*$ , where

$$\mathcal{O}^\theta = \{M \in \mathcal{O} \mid \text{there exists } k \in \mathbb{N} \text{ such that } (z - \theta(z))^k M = 0, z \in Z(\mathfrak{g})\}.$$

From the Harish-Chandra isomorphism theorem ([11, Proposition 7.4.7]) it follows that simples in  $\mathcal{O}^\theta$  are indexed by elements in some orbit of the action of the Weyl group  $W$  on  $\mathfrak{h}^*$ . Hence  $\mathcal{O}^\theta$  has only finitely many simple modules (up to isomorphism) and we get.

**Corollary 2.1** *Every  $\mathcal{O}^\theta$  is equivalent to the module category of a quasi-hereditary algebra, or, equivalently, if  $\lambda$  is a dominant weight, then the algebra  $\text{End}_{\mathfrak{g}}(\oplus_{w \in W} P(w \cdot \lambda))$  is quasi hereditary.*

### 3 Parabolic generalizations of $\mathcal{O}$

The interest to study parabolic generalizations of the category  $\mathcal{O}$  was motivated by different questions coming from algebra, analysis and combinatorics. Possibly the first generalization of  $\mathcal{O}$  was introduced in 1980 by Rocha-Caridi in [31]. This category, which by definition is a subcategory of  $\mathcal{O}$ , will be discussed later on in Subsection 4.1. The first example of a parabolic category, lying outside  $\mathcal{O}$ , was proposed in [10]. Later on in [16] and recently in [17] some attempts were made to present a general approach to constructing parabolic generalizations of  $\mathcal{O}$ , and these approaches can be put in the following framework.

Let  $\mathfrak{p} \supset \mathfrak{h} \oplus \mathfrak{n}_+$  be a parabolic subalgebra of  $\mathfrak{g}$ . Then we can write  $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{h}_{\mathfrak{a}} \oplus \mathfrak{n}$ , where  $\mathfrak{n}$  is the nilpotent radical of  $\mathfrak{p}$ ,  $\mathfrak{a} = \mathfrak{a} \oplus \mathfrak{h}_{\mathfrak{a}}$  is the reductive Levi factor, the algebra  $\mathfrak{a}$  is semi-simple and the algebra  $\mathfrak{h}_{\mathfrak{a}}$  is the commutative center of  $\mathfrak{a}$ . For any full subcategory  $\Lambda$  of the category of all  $\mathfrak{a}$ -modules we denote by  $\mathcal{O}(\mathfrak{p}, \Lambda)$  the full subcategory of the category of all  $\mathfrak{g}$ -modules, which consists of all finitely generated,  $\mathfrak{h}_{\mathfrak{a}}$ -diagonalizable and  $U(\mathfrak{n})$ -locally finite modules, which decompose into a direct sum of modules from  $\Lambda$ , when viewed as  $\mathfrak{a}$ -modules.

Certainly, without any restrictions on  $\Lambda$  one can not even guarantee that the category  $\mathcal{O}(\mathfrak{p}, \Lambda)$  is not trivial. However, if  $\Lambda$  consists of finitely generated modules and is stable under tensoring with all finite-dimensional  $\mathfrak{a}$ -modules, the category  $\mathcal{O}(\mathfrak{p}, \Lambda)$  always contains induced modules  $M_{\mathfrak{p}}(V, \lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$ , where  $\lambda \in \mathfrak{h}_{\mathfrak{a}}^*$  and the  $\mathfrak{p}$ -module structure on  $V \in \Lambda$  is defined by  $\mathfrak{n}V = 0$  and  $hv = \lambda(h)v$  for  $v \in V$  and  $h \in \mathfrak{h}_{\mathfrak{a}}$ . If the module  $V$  is simple (as an  $\mathfrak{a}$ -module), the module  $M_{\mathfrak{p}}(V, \lambda)$  is usually called the *generalized Verma module*, associated with  $\mathfrak{p}$ ,  $V$  and  $\lambda$ , see [29] for details. Every generalized Verma module has a unique simple quotient, and these simple modules are usually denoted by  $L_{\mathfrak{p}}(V, \lambda)$ .

Putting some more restrictions on  $\Lambda$  one can get categories  $\mathcal{O}(\mathfrak{p}, \Lambda)$  with further nice properties, in particular, one can arrive at the categories which are described by quasi-hereditary algebras (this situation will be discussed in Section 4), or by properly stratified algebras (respectively Section 5), or, finally, by general standardly stratified algebras (this will be considered in Section 6). In all the situations we will try to list the basic properties of the corresponding algebras connected with the stratifications, describe the example and list several applications, in particular to the study of generalized Verma modules.

### 4 Quasi-hereditary algebras and $\mathcal{O}(\mathfrak{p}, \Lambda)$

Certainly, in the case  $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}_+$  the category  $\mathcal{O}(\mathfrak{p}, \Lambda)$  coincides with  $\mathcal{O}$  and hence one gets that the first series of quasi-hereditary algebras associated with  $\mathcal{O}(\mathfrak{p}, \Lambda)$  is the series of quasi-hereditary algebras, associated with  $\mathcal{O}$ . The second classical example was constructed and investigated by Rocha-Caridi in [31] (and extended to infinite-dimensional algebras in [32]) and looks as follows.

**4.1 The category  $\mathcal{O}_S$  of Rocha-Caridi.** In [31] the category  $\mathcal{O}_S$  (where  $S$  indexes the algebra  $\mathfrak{p}$  by denoting the set of simple roots of the algebra  $\mathfrak{a}$ ) is defined as the category  $\mathcal{O}(\mathfrak{p}, \Lambda)$ , where  $\Lambda = \mathcal{F}_{\mathfrak{a}}$  is the category of all finite-dimensional  $\mathfrak{a}$ -modules. It follows immediately that  $\mathcal{O}(\mathfrak{p}, \mathcal{F}_{\mathfrak{a}})$  is a full subcategory of  $\mathcal{O}$ , in

particular, the category  $\mathcal{O}(\mathfrak{p}, \mathcal{F}_\alpha)$  inherits the decomposition into a direct sum of full subcategories  $\mathcal{O}(\mathfrak{p}, \mathcal{F}_\alpha)^\theta$ ,  $\theta \in Z(\mathfrak{g})^*$ , where  $\mathcal{O}(\mathfrak{p}, \mathcal{F}_\alpha)^\theta = \mathcal{O}(\mathfrak{p}, \mathcal{F}_\alpha) \cap \mathcal{O}^\theta$ . If  $V$  is a simple object in  $\mathcal{F}_\alpha$ , then the induced generalized Verma module  $M_{\mathfrak{p}}(V, \lambda)$  belongs to  $\mathcal{O}(\mathfrak{p}, \mathcal{F}_\alpha)$  and the modules  $\{L_{\mathfrak{p}}(V, \lambda)\}$  constitute an exhaustive list of simple modules (objects) in  $\mathcal{O}(\mathfrak{p}, \mathcal{F}_\alpha)$ .

- Theorem 4.1**
1. *Category  $\mathcal{O}(\mathfrak{p}, \mathcal{F}_\alpha)$  has enough projective modules.*
  2. *Every projective object in  $\mathcal{O}(\mathfrak{p}, \mathcal{F}_\alpha)$  has a filtration, whose subquotients are generalized Verma modules  $M_{\mathfrak{p}}(V, \lambda)$ , where  $V$  is a simple object in  $\Lambda$  (or has a so-called generalized Verma flag).*
  3. *The number  $[P_{\mathfrak{p}}(V, \lambda) : M_{\mathfrak{p}}(N, \mu)]$  of occurrences of the generalized Verma module  $M_{\mathfrak{p}}(N, \mu)$  as a subquotient in any generalized Verma flag of the projective cover  $P_{\mathfrak{p}}(V, \lambda)$  of the module  $L_{\mathfrak{p}}(V, \lambda)$  equals the composition multiplicity of  $L_{\mathfrak{p}}(V, \lambda)$  in  $M_{\mathfrak{p}}(N, \mu)$ , i.e.  $[P_{\mathfrak{p}}(V, \lambda) : M_{\mathfrak{p}}(N, \mu)] = (M_{\mathfrak{p}}(N, \mu) : L_{\mathfrak{p}}(V, \lambda))$ .*

Calling generalized Verma modules standard and repeating the arguments from Section 2 we get.

**Corollary 4.1** *Every  $\mathcal{O}(\mathfrak{p}, \mathcal{F}_\alpha)^\theta$  is equivalent to the module category of a quasi-hereditary algebra.*

**4.2 The general case.** The proof of Theorem 4.1 and that of Theorem 2.1 go along the same scheme, analyzing which one can formulate the following rather general result, see [16].

**Theorem 4.2** *Assume that every object of  $\Lambda$  is semi-simple, has a finite length (but not necessarily finite-dimensional as an  $\mathfrak{a}$ -module!), and that  $\Lambda$  is stable under tensoring with finite dimensional  $\mathfrak{a}$ -modules. Then*

1. *Category  $\mathcal{O}(\mathfrak{p}, \Lambda)$  has enough projective modules.*
2. *The modules  $\{L_{\mathfrak{p}}(V, \lambda) | V \text{ is simple in } \Lambda\}$  constitute an exhaustive list of simple modules in the category  $\mathcal{O}(\mathfrak{p}, \Lambda)$ .*
3. *Every projective object in  $\mathcal{O}(\mathfrak{p}, \Lambda)$  is filtered by generalized Verma modules  $M_{\mathfrak{p}}(V, \lambda)$ , where  $V$  is a simple object in  $\Lambda$ .*
4. *Assume additionally that  $\mathcal{O}(\mathfrak{p}, \Lambda)$  has a duality, i.e. that there exists an exact contravariant involutive self-equivalence on  $\mathcal{O}(\mathfrak{p}, \Lambda)$ , which preserves simple modules. Then the number  $[P_{\mathfrak{p}}(V, \lambda) : M_{\mathfrak{p}}(N, \mu)]$  of occurrences of the generalized Verma module  $M_{\mathfrak{p}}(N, \mu)$  as a subquotient of any generalized Verma flag of the projective cover  $P_{\mathfrak{p}}(V, \lambda)$  of the module  $L_{\mathfrak{p}}(V, \lambda)$  equals the composition multiplicity of  $L_{\mathfrak{p}}(V, \lambda)$  in  $M_{\mathfrak{p}}(N, \mu)$ , i.e.  $[P_{\mathfrak{p}}(V, \lambda) : M_{\mathfrak{p}}(N, \mu)] = (M_{\mathfrak{p}}(N, \mu) : L_{\mathfrak{p}}(V, \lambda))$ .*

Using the notion of the  $S$ -homomorphism of Harish-Chandra, [14], and standard properties of the translation functors, [22], one can extend the arguments about the central character to  $\mathcal{O}(\mathfrak{p}, \Lambda)$  to get the following:

**Corollary 4.2** *The category  $\mathcal{O}(\mathfrak{p}, \Lambda)$  decomposes into a direct sum of full subcategories (usually called blocks), each of which is equivalent to the module category of a quasi-hereditary algebra.*

## 5 Properly stratified algebras and $\mathcal{O}(\mathfrak{p}, \Lambda)$

Several examples of properly stratified algebras, which appear for certain  $\mathcal{O}(\mathfrak{p}, \Lambda)$  were constructed in [17, 18, 19, 23, 26, 27] and in this section we describe some of them. We start from the general result from [17].

**5.1 The general case.** Let us now assume that the category  $\Lambda$  consists of finitely generated and locally  $Z(\mathfrak{a})$ -finite modules, has enough projectives, and at most one simple object for every central character, which, in addition, has trivial endomorphism ring. In particular, every  $\Lambda^\chi$ ,  $\chi \in Z(\mathfrak{a})^*$ , is the module category over a local algebra. Furthermore, we assume that these local algebras are self-injective,  $\Lambda$  is stable under tensoring with finite-dimensional  $\mathfrak{a}$ -modules and such tensoring is an exact functor with respect to the natural abelian structure in the module category. We will call such  $\Lambda$  *properly admissible*, and throughout this subsection we will keep the assumption that  $\Lambda$  is properly admissible. Let  $V$  be a simple object in  $\Lambda$  (as we will see it may happen that  $V$  is not simple as an  $\mathfrak{a}$ -module) and  $\tilde{V}$  be its indecomposable projective cover in  $\Lambda$ . It is immediate that the induced modules  $M_{\mathfrak{p}}(V, \lambda)$  and  $M_{\mathfrak{p}}(\tilde{V}, \lambda)$  belong to  $\mathcal{O}(\mathfrak{p}, \Lambda)$ . Now one has to proceed in several steps. Combining the construction of projective modules from [32] with the Harish-Chandra  $S$ -homomorphism arguments, one gets

**Proposition 5.1** *Assume that  $\Lambda$  is properly admissible. Then*

1. *The category  $\mathcal{O}(\mathfrak{p}, \Lambda)$  has enough projective modules.*
2. *Every module in  $\mathcal{O}(\mathfrak{p}, \Lambda)$  is locally  $Z(\mathfrak{g})$ -finite. In particular,  $\mathcal{O}(\mathfrak{p}, \Lambda)$  decomposes into a direct sum of full subcategories  $\mathcal{O}(\mathfrak{p}, \Lambda)^\theta$ ,  $\theta \in Z(\mathfrak{g})^*$ , where*

$$\mathcal{O}(\mathfrak{p}, \Lambda)^\theta = \{M \in \mathcal{O}(\mathfrak{p}, \Lambda) \mid (z - \theta(z))^k M = 0, z \in Z(\mathfrak{g}), \text{ for some } k \in \mathbb{N}\}.$$

3. *Every  $\mathcal{O}(\mathfrak{p}, \Lambda)$  is equivalent to the module category of a finite-dimensional associative algebra.*

In particular, this gives  $\mathcal{O}(\mathfrak{p}, \Lambda)$  an abelian structure. Because of the exactness of the parabolic induction from  $\mathfrak{a}$  to  $\mathfrak{g}$  (which follows from the Poincaré-Birkhoff-Witt Theorem), and the condition that  $\tilde{V}$  is filtered by  $V$  in  $\Lambda$ , we immediately obtain that (as objects in  $\mathcal{O}(\mathfrak{p}, \Lambda)$ , i.e. with respect to the abelian structure, given by Proposition 5.1) the module  $M_{\mathfrak{p}}(\tilde{V}, \lambda)$  is filtered by  $M_{\mathfrak{p}}(V, \lambda)$ . Because of our restrictions on  $\Lambda$ , we always have that tensoring with finite-dimensional modules preserves projective modules in  $\Lambda$ . It is also easy to see that modules  $M_{\mathfrak{p}}(V, \lambda)$ , where  $V \in \Lambda$  is simple, have a unique simple quotient as objects of  $\mathcal{O}(\mathfrak{p}, \Lambda)$  (but not as  $\mathfrak{g}$ -modules in general). Let us denote by  $L_{\mathfrak{p}}(V, \lambda)$  this unique simple quotient. The modules  $\{L_{\mathfrak{p}}(V, \lambda) \mid V \text{ is simple in } \Lambda\}$  constitute an exhaustive list of simple objects in the category  $\mathcal{O}(\mathfrak{p}, \Lambda)$ . Now, inductive arguments, analogous to the original arguments of BGG, give us.

**Proposition 5.2** *Every projective module in  $\mathcal{O}(\mathfrak{p}, \Lambda)$  has a filtration, whose quotients are modules of the form  $M_{\mathfrak{p}}(\tilde{V}, \lambda)$ .*

Extending the natural order on  $\mathfrak{h}_{\mathfrak{a}}^*$  to the set of all parameters, one easily checks the remaining ordering conditions for the properly stratified algebra and thus gets.

**Theorem 5.1** *Every block  $\mathcal{O}(\mathfrak{p}, \Lambda)^\theta$  is equivalent to the module category of a properly stratified algebra. With respect to this properly stratified structure, the modules  $M_{\mathfrak{p}}(V, \lambda)$  or  $M_{\mathfrak{p}}(\tilde{V}, \lambda)$  are proper standard and standard modules respectively.*

Now we see the role, played by  $\Lambda$  in the properly stratified structure. Namely, simple objects in  $\Lambda$  are induced up to the proper standard modules, and projective objects — to the standard modules. If  $\Lambda$  is semi-simple (Section 4) we get that simple modules in  $\Lambda$  are projective and thus proper standard and standard modules in  $\mathcal{O}(\mathfrak{p}, \Lambda)$  coincide resulting the fact that  $\mathcal{O}(\mathfrak{p}, \Lambda)$  is quasi-hereditary. The algebras of the block of  $\Lambda$  become in  $\mathcal{O}(\mathfrak{p}, \Lambda)$  the endomorphism algebras of the standard modules. This visualizes the parabolic induction, which combines several blocks of  $\Lambda$  (module categories over local algebras) into a properly stratified algebra, representing a block of  $\mathcal{O}(\mathfrak{p}, \Lambda)$ .

Not surprisingly that one can even get several analogs of the BGG reciprocity for properly stratified algebra of  $\mathcal{O}(\mathfrak{p}, \Lambda)$  (we refer the reader to [3, 13] for the general case of properly stratified algebras).

**Theorem 5.2** *Assume that  $\mathcal{O}(\mathfrak{p}, \Lambda)$  has a duality and let  $P_{\mathfrak{p}}(V, \lambda)$  denote the projective cover of  $L_{\mathfrak{p}}(V, \lambda)$ . Then the following reciprocity formulae hold:*

$$[P_{\mathfrak{p}}(V, \lambda) : M_{\mathfrak{p}}(\tilde{N}, \mu)] = (M_{\mathfrak{p}}(N, \mu) : L_{\mathfrak{p}}(V, \lambda)).$$

$$[P_{\mathfrak{p}}(V, \lambda) : M_{\mathfrak{p}}(N, \mu)] = (M_{\mathfrak{p}}(\tilde{N}, \mu) : L_{\mathfrak{p}}(V, \lambda)).$$

$$[P_{\mathfrak{p}}(V, \lambda) : M_{\mathfrak{p}}(\tilde{N}, \mu)][M_{\mathfrak{p}}(\tilde{N}, \mu) : M_{\mathfrak{p}}(N, \mu)] = (M_{\mathfrak{p}}(\tilde{N}, \mu) : L_{\mathfrak{p}}(V, \lambda)).$$

$$[P_{\mathfrak{p}}(V, \lambda) : M_{\mathfrak{p}}(N, \mu)] = [M_{\mathfrak{p}}(\tilde{N}, \mu) : M_{\mathfrak{p}}(N, \mu)](M_{\mathfrak{p}}(N, \mu) : L_{\mathfrak{p}}(V, \lambda)).$$

**5.2  $\mathcal{S}$ -subcategories in  $\mathcal{O}$ .** The first example, which perfectly embeds into the scheme, presented in the previous subsection, is what was considered in [19] under the name  $\mathcal{S}$ -subcategories in  $\mathcal{O}$  (the name was motivated by the fact, proved in [19], that such categories possess a combinatorial description, analogous to Soergel's combinatorial description of the blocks of the category  $\mathcal{O}$ , see [34]). In this example we define  $\Lambda = \Lambda^{\mathcal{S}}$  as follows. Let  $M_0$  denote the Verma  $\mathfrak{a}$ -module with the most degenerate central character, i.e. the unique projective simple integral Verma module. Define  $\Lambda^{\mathcal{S}}$  as the category of all  $\mathfrak{a}$ -modules  $V$ , which have copresentation  $0 \rightarrow V \rightarrow M_0 \otimes F_1 \rightarrow M_0 \otimes F_2$ , where  $F_1$  and  $F_2$  are finite dimensional. By definition,  $\Lambda^{\mathcal{S}}$  is a subcategory in the category  $\mathcal{O}$ , in particular, it inherits decomposition with respect to the central character. The modules  $M_0 \otimes F$ , where  $F$  is finite-dimensional, are projective-injective in  $\mathcal{O}$ . Moreover, every integral block (which corresponds to some central character) contains exactly one indecomposable projective-injective module, which we denote by  $P^{\chi}$ ,  $\chi \in Z(\mathfrak{a})^*$ . Now standard abstract arguments (see e.g. [1]) imply that every  $(\Lambda^{\mathcal{S}})^{\chi}$ ,  $\chi \in Z(\mathfrak{a})^*$ , is the module category over  $\text{End}(P^{\chi})$ , the latter being a local algebra, moreover, from the injectivity of  $M_0 \otimes F$  it also follows that this algebra is self-injective. The simple objects in  $\Lambda^{\mathcal{S}}$  are projective integral Verma modules (mostly non-simple as  $\mathfrak{a}$ -modules), and the projective objects in  $\Lambda^{\mathcal{S}}$  are projective-injective integral modules from  $\mathcal{O}$ .

From the definition we immediately get that the behavior of the functors  $F \otimes _$  is as desired and hence  $\Lambda^{\mathcal{S}}$  is properly admissible. The corresponding category  $\mathcal{O}(\mathfrak{p}, \Lambda^{\mathcal{S}})$  is a subcategory of the category  $\mathcal{O}$  (now for  $\mathfrak{g}$ ) and it is easy to see that the usual duality on  $\mathcal{O}$  canonically restricts to  $\mathcal{O}(\mathfrak{p}, \Lambda^{\mathcal{S}})$ . This yields that  $\mathcal{O}(\mathfrak{p}, \Lambda^{\mathcal{S}})$  has all properties, described in the previous subsection.

In [27] it was shown that in the case, when  $\mathfrak{a}$  is a direct sum of some  $\mathfrak{sl}(n_i, \mathbb{C})$ , the category  $\Lambda^{\mathcal{S}}$  can be realized as a certain category of the so-called Gelfand-Zetlin modules. This realization has an advantage that in the obtained category the notions of a simple  $\mathfrak{a}$ -module and a simple object coincide.

**5.3 Relative  $\mathfrak{sl}(3, \mathbb{C})$ - $\mathfrak{sl}(2, \mathbb{C})$  example.** Choosing  $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$  with the standard triangular decomposition, and  $\mathfrak{a} = \mathfrak{sl}(2, \mathbb{C})$ , embedded with respect to the upper left corner, we get that the block of the  $\mathcal{S}$ -subcategory  $\mathcal{O}(\mathfrak{p}, \Lambda^S)$  corresponding to the trivial central character (the central character of the one-dimensional module) contains three simple modules, say indexed by 1, 2 and 3 with the natural order. The graded (with respect to the grading inherited from  $\mathcal{O}$ , see [34]) filtrations of the projective ( $P(i)$ ), standard ( $\Delta(i)$ ) and proper standard ( $\overline{\Delta}(i)$ ) modules in this block then look as follows:

$$\begin{aligned} \overline{\Delta}(3) &= \begin{array}{c} 3 \\ 2 \\ 1 \end{array}, & \overline{\Delta}(2) &= \begin{array}{c} 2 \\ 1 \end{array}, & \overline{\Delta}(1) &= 1. \\ \\ \Delta(3) = P(3) &= \begin{array}{ccccc} & & 3 & & \\ & & 2 & & \\ & 1 & 3 & & 2 \\ & & 2 & & \\ & & 1 & & \end{array}, & \Delta(2) &= \begin{array}{cc} 1 & 2 \\ & 1 \end{array}, & \Delta(1) &= \begin{array}{c} 1 \\ 1 \end{array}. \\ \\ P(1) &= \begin{array}{ccccccc} & & 1 & & & & \\ & & 1 & & 2 & & \\ 1 & & 1 & & 2 & & 3 \\ & & 1 & & 2 & & \\ & & & & 2 & & \\ & & & & 1 & & \\ & & & & & & 2 \\ & & & & & & 3 \end{array}, & P(2) &= \begin{array}{ccccccc} & & & & 1 & & 2 \\ & & & & 1 & & 3 \\ 1 & & & & 1 & & 2 \\ & & & & 1 & & 3 \\ & & & & & & 2 \\ & & & & & & 1 \\ & & & & & & 2 \\ & & & & & & 3 \end{array}, \end{aligned}$$

We have to note that the module  $P(1)$  above is not rigid in the sense that its socle filtration is different from its radical filtration. Indeed, one can show by direct calculation that the Loewy length of  $P(1)$  is 7 and thus the graded filtration presented above is in fact a Loewy filtration. However,  $P(1)$  has a filtration, whose quotients are standard modules, and hence are isomorphic to  $\Delta(i)$ ,  $i = 1, 2, 3$ . From this one gets that  $\Delta(1)$  is a quotient of  $P(1)$ . But the Loewy length of  $\Delta(1)$  is two. This implies that the socle of  $\Delta(1)$ , which is isomorphic to the simple module 1, must contribute to the top of the radical of  $P(1)$ . Hence the graded filtration of  $P(1)$  above does not coincide with the radical filtration and thus  $P(1)$  is not rigid. We refer the reader to [24] for more details. The original projective-injective module in the category  $\mathcal{O}$  is known to be rigid. This shows that the structure of the  $\mathcal{S}$ -subcategories in  $\mathcal{O}$  is usually more complicated than that of  $\mathcal{O}$ .

**5.4 Harish-Chandra bimodules.** A connection between  $\mathcal{S}$ -subcategories in  $\mathcal{O}$  and certain categories of Harish-Chandra bimodules was established in [26]. First, it was shown that the category  $\mathcal{O}(\mathfrak{p}, \Lambda^S)$  has the following alternative description:

**Proposition 5.3**  *$\mathcal{O}(\mathfrak{p}, \Lambda^S)$  is the full subcategory of  $\mathcal{O}$ , which consists of all modules  $M$ , which can be copresented by  $\mathfrak{a}$ -antidominant injective modules.*

Let  $\mathcal{HC}$  denote the category of all finitely generated  $\mathfrak{g}$ -bimodules, the diagonal action of  $\mathfrak{g}$  on which is locally finite. This category is called the category of *Harish-Chandra bimodules*. In the famous paper [4] Bernstein and Gelfand have shown that certain direct summands of the subcategory  $\mathcal{HC}^1$  of  $\mathcal{HC}$ , which consists



of all bimodules, the right action of the  $Z(\mathfrak{g})$  on which is diagonalizable, are equivalent to subcategories of  $\mathcal{O}$ , consisting of modules, which have a presentation by  $\mathfrak{a}$ -antidominant projective modules. Usual duality between projective and injective modules now relates Harish-Chandra bimodules to  $\mathcal{S}$ -subcategories in  $\mathcal{O}$ , giving the following.

**Theorem 5.3** *The category  $\mathcal{HC}^1$  decomposes into a direct sum of full subcategories, each of which is equivalent to the module category of a properly stratified algebra. Moreover, these properly stratified algebras are precisely the same, which appear in  $\mathcal{S}$ -subcategories in  $\mathcal{O}$ .*

In Theorem 5.3 we certainly understand that the algebras in question can sometimes be quasi-hereditary, however, all quasi-hereditary algebras are properly stratified.

**5.5 Modules with minimal annihilator.** In [23] it is shown that categories, which appear as  $\mathcal{S}$ -subcategories in  $\mathcal{O}$ , also appear in the study of generalized Verma modules with minimal annihilators. These categories have the following description. We start with a simple  $\mathfrak{a}$ -module,  $V$ , whose annihilator coincides with the annihilator of some projective and simple Verma module, and consider the category  $\Lambda = \text{Coker}(V)$ , which is a full subcategory in  $\mathfrak{a}\text{-mod}$  and consists of all  $\mathfrak{a}$ -modules  $M$ , which have a presentation of the form  $V \otimes F_1 \rightarrow V \otimes F_2 \rightarrow M \rightarrow 0$ , where  $F_1$  and  $F_2$  are finite-dimensional. The arguments from [4, 30] imply that this category is isomorphic to the subcategory of  $\mathcal{HC}^1$  (for  $\mathfrak{a}$ ), associated with the most degenerate central character (right action of the center). In particular, one gets that  $\text{Coker}(V)$  is properly admissible. The same arguments can also be applied to the category  $\mathcal{O}(\mathfrak{p}, \text{Coker}(V))$  and one gets the following statement.

**Theorem 5.4** *The category  $\mathcal{O}(\mathfrak{p}, \text{Coker}(V))$  decomposes into a direct sum of full subcategories, each of which is equivalent to the module category of a properly stratified algebra. Moreover, these properly stratified algebras are precisely the same, which appear in  $\mathcal{S}$ -subcategories in  $\mathcal{O}$ .*

Because of the categorical definition of the generalized Verma modules (they are proper standard modules), the equivalence, given by Theorem 5.4 can be applied to derive the information about the structure of generalized Verma modules, induced from simple modules with minimal annihilators (which is the ideal, generated by some central character). In particular, one obtains the following.

**Theorem 5.5** *Let  $V$  be a simple  $\mathfrak{a}$ -module with a minimal annihilator and  $\hat{V}$  be a simple Verma module over  $\mathfrak{a}$  with the same annihilator. Then the module  $M_{\mathfrak{p}}(V, \lambda)$  is simple if and only if the module  $M_{\mathfrak{p}}(\hat{V}, \lambda)$  (the latter module being a Verma module over  $\mathfrak{g}$ ) is simple.*

**5.6 Thick category  $\mathcal{O}$ .** Properly stratified algebras can also be obtained from the categories  $\mathcal{HC}^n$  of Harish-Chandra bimodules, defined using the condition that the right action of the center of  $U(\mathfrak{g})$  on these bimodules is given by Jordan cells of degree at most  $n$ , see [33]. This corresponds to the so-called *thick category*  $\mathcal{O}$ , that is the generalization of  $\mathcal{O}$ , in which one allows that the action of the Cartan subalgebra on modules is not necessarily diagonalizable, but is given by Jordan cells of degree at most  $n$ . Although the local algebras, which appear in the category  $\Lambda$ , corresponding to this situation, are not self-injective in general, one easily sees that all arguments can be carried over to this case. Moreover, this further generalizes to

the category  $\mathcal{HC}^I$ , where  $I$  is an ideal of  $Z(\mathfrak{g})$  of finite codimension, which consists of all bimodules  $M \in \mathcal{HC}$  such that  $MI = 0$ . Finally, one can easily generalize the proof of Theorem 5.1 to obtain the following statement:

**Theorem 5.6** *Let  $I$  be an ideal of  $Z(\mathfrak{g})$  of finite codimension. Then the regular (with respect to the left action of the center) part of the category  $\mathcal{HC}^I$  decomposes into a direct sum of full subcategories, each of which is equivalent to the module category of a properly stratified algebra.*

## 6 Stratified algebras and $\mathcal{O}(\mathfrak{p}, \Lambda)$

As we have already seen, many categories  $\mathcal{O}(\mathfrak{p}, \Lambda)$  lead to properly stratified algebras. In particular, classical categories of Harish-Chandra bimodules are described by properly stratified algebras. However, it is not very difficult to find an example of  $\mathcal{O}(\mathfrak{p}, \Lambda)$ , which can not correspond to properly stratified algebras. Moreover, the study of generalized Verma modules  $M_{\mathfrak{p}}(V, \lambda)$  produces such an example in a natural way, and this goes as follows. During the study of  $M_{\mathfrak{p}}(V, \lambda)$  one necessarily comes to the point, where one has to study the category of all subquotients of the modules of the form  $F \otimes V$ ,  $F$  finite dimensional, or the corresponding category  $\text{Coker}(V)$  (see definition in Subsection 5.5). If  $V$  has minimal annihilator, the category  $\text{Coker}(V)$  can always be described by local algebras and thus the corresponding parabolic category  $\mathcal{O}(\mathfrak{p}, \text{Coker}(V))$  is described by properly stratified algebras. However, even starting from a simple highest weight module, in the case of a bigger annihilator one usually obtains that the category  $\text{Coker}(V)$  does not correspond to local algebras. Here is an example.

**6.1  $\mathfrak{sl}(3, \mathbb{C})$ -example.** Let  $\mathfrak{a} = \mathfrak{sl}(3, \mathbb{C})$ ,  $\alpha, \beta$  be simple roots,  $L(\lambda)$  be a simple highest weight module, such that  $\lambda$  is integral and  $(\lambda, \alpha) \in \mathbb{N}$ ,  $(\lambda, \alpha + \beta) = -1$ . Define  $\Lambda = \text{Coker}(L(\lambda))$  as the category of all modules  $V$ , which have presentation  $V_2 \rightarrow V_1 \rightarrow V \rightarrow 0$ , where both  $V_i$  are direct summands in some  $F_i \otimes L(\lambda)$ ,  $F_i$  finite dimensional. Then  $\Lambda$  decomposes into a direct sum of full subcategories each of which is equivalent to the module category over an associative algebra. In some exceptional cases these algebras are semi-simple, but in the general case these algebras are Morita equivalent to the algebra of the following quiver with relations:

$$\bullet \begin{array}{c} \xrightarrow{x} \\ \xleftarrow{y} \end{array} \bullet \quad xyx = yxy = 0.$$

This algebra is not local and there is no way for the blocks of the corresponding category  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\lambda))$ , which is a subcategory in  $\mathcal{O}$ , to correspond to properly stratified algebras. However, as it will be shown in the next section, this can be restored if one uses the general notion of stratified algebras.

We remark that the phenomenon, described above, is not exceptional: the general problem can be formulated as follows: Let  $\mathfrak{a}$  be a semi-simple Lie algebra and  $L(\lambda)$  be a simple highest weight  $\mathfrak{a}$ -module. Studying generalized Verma modules, one usually has to construct a “reasonable” category, containing  $L(\lambda)$  (or some appropriate lift of this module), which is stable under tensoring with finite dimensional  $\mathfrak{a}$ -modules. The best candidate for this category is the Coker-category of Milićić and Soergel, which we described in Subsection 5.5, see [30] for more details. The example above is in fact an example of such category. The main point of this subsection is that the blocks of this Coker-category are not described by

local algebras in general. In fact, for “good” values of  $\lambda$  the blocks of this Coker-category are described by the endomorphism algebras of the direct sum of self-dual projective modules in a block of the parabolic category  $\mathcal{O}_S$  of Rocha-Caridi, for some  $S$  (see Subsection 4.1). One additional difficulty is that not all  $\lambda$  are “good” and for those, which are not “good”, I even do not know any candidate for the corresponding “reasonable” category.

**6.2 A set-up for stratified algebras.** Assume now that the category  $\Lambda$  consists of finitely generated and locally  $Z(\mathfrak{a})$ -finite modules, has enough projectives, and at most finitely many simple object for every central character, all of them having trivial endomorphism rings. As an example of such category one can take the category  $\text{Coker}(L(\lambda))$  of  $\mathfrak{sl}(3, \mathbb{C})$ -modules from Subsection 6.1. More general, for any  $\mathfrak{a}$  and any parabolic subalgebra  $\mathfrak{p}$  in  $\mathfrak{a}$  one can take the full subcategory in the corresponding category  $\mathcal{O}_S$ , which consists of all modules, having a presentation by self-dual projective modules.

If  $\Lambda$  satisfies the above conditions, we immediately have that every  $\Lambda^\chi$ ,  $\chi \in Z(\mathfrak{a})^*$ , is the module category of a finite-dimensional algebra, and we assume that these algebras are self-injective. Furthermore, we assume that  $\Lambda$  is stable under tensoring with finite-dimensional  $\mathfrak{a}$ -modules and that such tensoring defines an exact functor with respect to the natural abelian structure in the module category. We will call such  $\Lambda$  *s-admissible*.

Let  $V$  be a simple object in  $\Lambda$  and  $\tilde{V}$  be its indecomposable projective cover in  $\Lambda$ . It is immediate that the induced modules  $M_{\mathfrak{p}}(V, \lambda)$  and  $M_{\mathfrak{p}}(\tilde{V}, \lambda)$  belong to  $\mathcal{O}(\mathfrak{p}, \Lambda)$ . Now using standard arguments one obtains.

**Theorem 6.1** *Assume that  $\Lambda$  is s-admissible. Then*

1. *The category  $\mathcal{O}(\mathfrak{p}, \Lambda)$  has enough projective modules.*
2. *Every module in  $\mathcal{O}(\mathfrak{p}, \Lambda)$  is locally  $Z(\mathfrak{g})$ -finite. In particular,  $\mathcal{O}(\mathfrak{p}, \Lambda)$  decomposes into a direct sum of full subcategories  $\mathcal{O}(\mathfrak{p}, \Lambda)^\theta$ ,  $\theta \in Z(\mathfrak{g})^*$ , where*  

$$\mathcal{O}(\mathfrak{p}, \Lambda)^\theta = \{M \in \mathcal{O}(\mathfrak{p}, \Lambda) : (z - \theta(z))^k M = 0, z \in Z(\mathfrak{g}), \text{ for some } k \in \mathbb{N}\}.$$
3. *Every  $\mathcal{O}(\mathfrak{p}, \Lambda)$  is equivalent to the module category of a finite-dimensional associative algebra. In particular, this gives  $\mathcal{O}(\mathfrak{p}, \Lambda)$  an abelian structure.*
4. *Every projective module in the category  $\mathcal{O}(\mathfrak{p}, \Lambda)$  is filtered by modules of the form  $M_{\mathfrak{p}}(\tilde{V}, \lambda)$ .*

By the same arguments as in Subsection 5.1 we get that modules  $M_{\mathfrak{p}}(V, \lambda)$ , where  $V \in \Lambda$  is simple, have a unique simple quotient as objects of  $\mathcal{O}(\mathfrak{p}, \Lambda)$ , and we denote this simple quotient by  $L_{\mathfrak{p}}(V, \lambda)$ . The modules  $\{L_{\mathfrak{p}}(V, \lambda) | V \text{ is simple in } \Lambda\}$  constitute an exhaustive list of simple objects in the category  $\mathcal{O}(\mathfrak{p}, \Lambda)$ . However, in contrast with Subsection 5.1, it is not true in general that the module  $M_{\mathfrak{p}}(\tilde{V}, \lambda)$  is filtered by  $M_{\mathfrak{p}}(V, \lambda)$ . In fact,  $M_{\mathfrak{p}}(\tilde{V}, \lambda)$  is filtered by modules  $M_{\mathfrak{p}}(N, \lambda)$ , where  $N$  runs through the set of all simple subquotients of  $\tilde{V}$  in  $\Lambda$ . However, if we now introduce the natural pre-order on the set of parameters of simple modules in  $\mathcal{O}(\mathfrak{p}, \Lambda)$ , with respect to which the parameters of  $L_{\mathfrak{p}}(V, \lambda)$  and  $L_{\mathfrak{p}}(N, \lambda)$  are equal if and only if modules  $V$  and  $N$  are taken from one indecomposable block of  $\Lambda$ , we immediately get

**Theorem 6.2** *Every block  $\mathcal{O}(\mathfrak{p}, \Lambda)^\theta$  is equivalent to the module category of a standardly stratified algebra. With respect to this standard stratification the modules  $M_{\mathfrak{p}}(\tilde{V}, \lambda)$  are standard modules.*

**6.3 General BGG-reciprocity.** As in the classical case, assuming that the category  $\mathcal{O}(\mathfrak{p}, \Lambda)$  has a duality, one can also obtain the following analogue of the BGG reciprocity ([20]) in the case, when  $\mathcal{O}(\mathfrak{p}, \Lambda)$  corresponds to the stratified algebras.

**Theorem 6.3** *Retain all the notations from Subsections 6.2. Assume that  $\mathcal{O}(\mathfrak{p}, \Lambda)^\theta$  is as in Theorem 6.2 and has a duality. Then we have the following reciprocity:*

$$[P_{\mathfrak{p}}(V, \lambda) : M_{\mathfrak{p}}(\tilde{N}, \mu)] = (M_{\mathfrak{p}}(\tilde{N}, \mu) : L_{\mathfrak{p}}(V, \lambda)),$$

where  $P_{\mathfrak{p}}(V, \lambda)$  denotes the indecomposable projective cover of  $L_{\mathfrak{p}}(V, \lambda)$ .

Without additional assumptions on  $\Lambda$ , one even can not reformulate this reciprocity as it was done in Theorem 5.2. However, one more restriction on  $\Lambda$  gives the following.

**Corollary 6.1** *If, in addition to the assumptions of Theorem 6.3, the Cartan matrix of the arbitrary block of  $\Lambda$  is symmetric, i.e.  $(\tilde{V} : N) = (\tilde{N} : V)$  for all simple  $V, N \in \lambda$ , then we have the following reciprocity:*

$$[P_{\mathfrak{p}}(V, \lambda) : M_{\mathfrak{p}}(N, \mu)] = (M_{\mathfrak{p}}(\tilde{N}, \mu) : L_{\mathfrak{p}}(V, \lambda)).$$

**Proof** Denote by  $\Lambda^s$  the set of isomorphism classes of simple objects in  $\Lambda$ . Then, using the exactness of the parabolic induction, the symmetry of the Cartan matrix of  $\Lambda$ , and Theorem 6.3, we have

$$\begin{aligned} [P_{\mathfrak{p}}(V, \lambda) : M_{\mathfrak{p}}(N, \mu)] &= \sum_{S \in \Lambda^s} [P_{\mathfrak{p}}(V, \lambda) : M_{\mathfrak{p}}(\tilde{S}, \mu)](\tilde{S}, N) = \\ &= \sum_{S \in \Lambda^s} (M_{\mathfrak{p}}(S, \mu) : L_{\mathfrak{p}}(V, \lambda))(\tilde{N}, S) = (M_{\mathfrak{p}}(\tilde{N}, \mu) : L_{\mathfrak{p}}(V, \lambda)). \end{aligned}$$

□

We remark that Theorem 6.3 implies that the Cartan matrix of a block of the category  $\mathcal{O}(\mathfrak{p}, \Lambda)$  has form  $C^tDC$ , where  $C$  is triangular and  $D$  is block-diagonal. For properly stratified algebras we get that  $D$  is diagonal and for quasi-hereditary algebras we get that  $D$  is the identity matrix. The diagonal blocks of the matrix  $D$  are Cartan matrices of the blocks of the category  $\Lambda$ , which contribute to the given block of  $\mathcal{O}(\mathfrak{p}, \Lambda)$ . In particular, if the conditions of Corollary 6.1 are satisfied, then the matrix  $D$  and hence the matrix  $C^tDC$  are symmetric.

**6.4 The category of Mathieu and Britten-Futorny-Lemire.** Another example of the categories  $\Lambda$  and  $\mathcal{O}(\mathfrak{p}, \Lambda)$  leading to stratified algebras naturally appears in the context studied in [28] (for  $\Lambda$ ) and [7] (for  $\mathcal{O}(\mathfrak{p}, \Lambda)$ ), where the principal objects of study are the so-called torsion-free  $\mathfrak{a}$ -modules of finite degree. For this set up we have to assume that all simple direct summands of  $\mathfrak{a}$  are of type  $A_n$  or  $C_n$ . A weight  $\mathfrak{a}$ -module,  $V$ , is called *torsion-free* provided that the action of all elements from  $\mathfrak{a} \setminus \mathfrak{h}$  on this module is bijective. If  $V$  is simple and has finite-dimensional weight spaces, then all non-trivial weight spaces have the same dimension, called the *degree* of  $V$ . According to [15], such modules exist only if all simple direct summands of  $\mathfrak{a}$  are of type  $A_n$  or  $C_n$ , and [28] gives a complete classification of such simple modules.

In the paper [7] the authors study the structure of generalized Verma modules  $M_{\mathfrak{p}}(V, \lambda)$ , where  $V$  is a torsion-free weight  $\mathfrak{a}$ -module of finite degree. The main tool

in this description is *Mathieu's twisting functor*, defined and used in [28] to classify all simple torsion-free modules of finite degree. This one is constructed as follows (we follow [28, Section 4]): for a root,  $\alpha$ , one considers the localization  $U_\alpha$  of  $U(\mathfrak{a})$  (or  $U(\mathfrak{g})$ ) with respect to the Ore subset  $\{X_{-\alpha}^n : n \in \mathbb{N}\}$ , where  $X_{-\alpha}$  denotes a non-zero root vector of  $\mathfrak{a}$  (resp.  $\mathfrak{g}$ ), corresponding to the root  $\alpha$ . Further, there exists a unique family of automorphisms  $\Theta_x$ ,  $x \in \mathbb{C}$ , of the algebra  $U_\alpha$  such that  $\Theta_n(r) = X_{-\alpha}^n r X_{-\alpha}^{-n}$  for all  $r \in U_\alpha$  and all  $n \in \mathbb{Z}$ , and such that the map  $x \mapsto \Theta_x(r)$  is polynomial in  $x$  for all  $r \in U_\alpha$ . The Mathieu's twisting functor  $\mathcal{M}_x^\alpha$ ,  $x \in \mathbb{C}$ , is the composition of the tensor induction  $U_\alpha \otimes_-$  with the  $\Theta_x$ -twist. The main result of [7] states that an appropriate product of different  $\mathcal{M}_x^\alpha$  defines a lattice epimorphism from the lattice of submodules of some specific quotient of some Verma module (this one is associated with  $M_{\mathfrak{p}}(V, \lambda)$  in a natural way, prescribed by the results of [28]) to the lattice of submodules of the module  $M_{\mathfrak{p}}(V, \lambda)$ . It is also shown that in some cases (which depend on the choice of  $V$  and on the type of  $\mathfrak{p}$ ) this correspondence is a lattice isomorphisms.

If one now lifts the generalized Verma modules in question up to the categorical level, it is quite easy to see that Mathieu's twisting functor does much more. In fact, it defines an equivalence between some categories  $\mathcal{O}(\mathfrak{p}, \Lambda)$ , under which generalized Verma modules are sent to generalized Verma modules. The idea to use Mathieu's twisting functors to establish such an equivalence goes back to [28, Appendix], where it was used to prove the equivalence of several blocks of the category  $\mathcal{O}$ . This equivalence immediately gives us the results of Britten, Futorny and Lemire and even embeds these results in a much more general picture.

In more details, one proceeds as follows. We start from some simple torsion-free  $\mathfrak{a}$ -module  $N$  of finite degree and consider the category  $\text{Coker}(N)$ , which, because of the bijectivity of the action of all elements from  $\mathfrak{a} \setminus \mathfrak{h}$ , coincides with the category of all subquotients of  $F \otimes N$ , where  $F$  is finite dimensional. Using [21, Sections 3.1] in  $A_n$  case and [6, Section 3] in  $C_n$  case, we even get that the category  $F \otimes N$  contains a simple projective module,  $V$  say. This module is again a simple torsion-free  $\mathfrak{a}$ -module of finite degree and  $N \in \text{Coker}(V)$ . According to [28], every simple torsion free module of finite degree comes with some simple highest weight module, i.e. there exists a simple highest weight module, say  $\hat{V}$ , such that the module  $V$  is obtained from  $\hat{V}$ , using some composition of different  $\mathcal{M}_x^\alpha$  (see [28] for details). Let us denote this composition of Mathieu's twists by  $\mathcal{M}$ .

One can check by direct calculation that the functor  $\mathcal{M}_x^\alpha$  commutes with translation functors (see for example [18, 27]) and thus obtain that  $\hat{V}$  is a simple projective module in the category  $\text{Coker}(\hat{V})$ . We also remark that from the definition of  $\mathcal{M}_x^\alpha$  it follows that the functor  $\mathcal{M}_{x^{-1}}^\alpha$  is inverse to  $\mathcal{M}_x^\alpha$  on the full subcategory of the category of all  $\mathfrak{g}$ -modules, which consists of all modules, on which  $X_{-\alpha}$  acts bijectively. Therefore the functor  $\mathcal{M}$  produces an equivalence of  $\text{Coker}(\hat{V})$  and  $\text{Coker}(V)$ , moreover, this equivalence commutes with translation functors.

**Proposition 6.1** *The categories  $\text{Coker}(\hat{V})$  and  $\text{Coker}(V)$  are  $s$ -admissible.*

**Proof** It is certainly enough to prove the statement for  $\text{Coker}(\hat{V})$ . As usual translation functors send projectives to projectives, we get enough projectives in the category  $\text{Coker}(\hat{V})$  as translations of  $\hat{V}$ . Moreover, since  $\hat{V}$  is injective, all projective modules in  $\text{Coker}(\hat{V})$  are injective. Everything else is standard, see [23].  $\square$

From Theorem 6.1 and Theorem 6.2 we immediately obtain.

**Theorem 6.4** *The categories  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\hat{V}))$  and  $\mathcal{O}(\mathfrak{p}, \text{Coker}(V))$  decompose into a direct sum of full subcategories, each of which is equivalent to the module category of a standardly stratified algebra.*

Moreover, considering Mathieu's functors for  $U(\mathfrak{g})$  instead of  $U(\mathfrak{a})$  we get the following statement.

**Theorem 6.5** *Mathieu's twisting functor defines an equivalence between the parabolic categories  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\hat{V}))$  and  $\mathcal{O}(\mathfrak{p}, \text{Coker}(V))$ . This equivalence commutes with translation functors, and sends generalized Verma modules to generalized Verma modules.*

**Proof** Let  $\mathcal{M}_{\mathfrak{g}}$  denotes the same composition of  $\mathcal{M}_x^\alpha$ , used for the definition of  $\mathcal{M}$ , but now considered as Mathieu's twists for the algebra  $\mathfrak{g}$ . The fact that  $\mathcal{M}_x^\alpha$  (and hence  $\mathcal{M}_{\mathfrak{g}}$ ) sends generalized Verma modules to generalized Verma modules follows immediately from the PBW Theorem for the algebra  $U_\alpha$ . From the definition of  $\mathcal{M}_{\mathfrak{g}}$  it now follows that  $\mathcal{M}_{\mathfrak{g}}$  sends a proper standard generalized Verma module of  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\hat{V}))$  to a proper standard generalized Verma module in the category  $\mathcal{O}(\mathfrak{p}, \text{Coker}(V))$ .

Now recall that every projective module in  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\hat{V}))$  is a direct summand of a translation of a proper standard generalized Verma module. Since  $\mathcal{M}_x^\alpha$  commutes with translation functors, we get that  $\mathcal{M}_{\mathfrak{g}}$  sends projectives from  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\hat{V}))$  to projectives in  $\mathcal{O}(\mathfrak{p}, \text{Coker}(V))$ . As  $\mathcal{M}_{\mathfrak{g}}$  is obviously exact, we get that it maps  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\hat{V}))$  to  $\mathcal{O}(\mathfrak{p}, \text{Coker}(V))$ . By the same arguments, the composition  $\mathcal{M}'_{\mathfrak{g}}$  of inverse Mathieu's twists, which were used to define  $\mathcal{M}_{\mathfrak{g}}$ , sends  $\mathcal{O}(\mathfrak{p}, \text{Coker}(V))$  to  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\hat{V}))$ . It is now obvious that  $\mathcal{M}_{\mathfrak{g}}$  and  $\mathcal{M}'_{\mathfrak{g}}$  are mutually inverse equivalences between  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\hat{V}))$  and  $\mathcal{O}(\mathfrak{p}, \text{Coker}(V))$ , which commute with translation functors.  $\square$

Since the category  $\text{Coker}(V)$  coincides with the category of all subquotients of the modules  $F \otimes V$ ,  $F$  finite-dimensional, the notion of simple object and simple modules in  $\text{Coker}(V)$  and, further, in  $\mathcal{O}(\mathfrak{p}, \text{Coker}(V))$  coincide. In contrast to this, the category  $\text{Coker}(\hat{V})$  is in general different from the category of all subquotients of the modules  $F \otimes \hat{V}$ ,  $F$  finite dimensional, and hence simple objects in  $\text{Coker}(\hat{V})$  and  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\hat{V}))$  are not in general simple  $\mathfrak{a}$ - or  $\mathfrak{g}$ -modules respectively. Standard properties of the translation functors immediately imply that simple objects in  $\text{Coker}(\hat{V})$  are simple  $\mathfrak{a}$ -modules if and only if  $\text{Coker}(\hat{V})$  is semi-simple. When this is the case, can be easily derived from [28]. In particular, this is always the case for symplectic Lie algebras. Further, this is also the case for  $\mathfrak{sl}(n, \mathcal{C})$  if the highest weight of  $\hat{V}$  is nonintegral in the sense of [7] (i.e. its first coordinate with respect to the basis, consisting of fundamental weights, is not an integer). Thus we obtain the following corollary, which is a combination of the two main results of [7], namely [7, Theorem 3] and [7, Theorem 4].

**Corollary 6.2** *The functor  $\mathcal{M}_{\mathfrak{g}}$  defines a lattice epimorphism from the submodule lattice of generalized Verma modules from  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\hat{V}))$  to the submodule lattice of the corresponding generalized Verma modules in  $\mathcal{O}(\mathfrak{p}, \text{Coker}(V))$ . If  $\text{Coker}(\hat{V})$  is semi-simple, then this epimorphism is in fact an isomorphism.*

**Proof** The functor  $\mathcal{M}_{\mathfrak{g}}$  is exact and sends simple objects from  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\hat{V}))$  to simple objects in  $\mathcal{O}(\mathfrak{p}, \text{Coker}(V))$ . Moreover, simple objects in  $\mathcal{O}(\mathfrak{p}, \text{Coker}(V))$

are simple  $\mathfrak{g}$ -modules. Moreover, it is obvious that  $\mathcal{M}_{\mathfrak{g}}$  does not annihilate any simple object from  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\hat{V}))$ . The first statement now follows from the exactness of  $\mathcal{M}_{\mathfrak{g}}$ . If  $\text{Coker}(\hat{V})$  is semi-simple, simple objects in  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\hat{V}))$  are simple  $\mathfrak{g}$ -modules as well, which implies that the map, induced by  $\mathcal{M}_{\mathfrak{g}}$  on the submodule lattice of generalized Verma modules, is bijective, completing the proof.  $\square$

In particular, Corollary 6.2 reduces the problem of composition multiplicities for  $M_{\mathfrak{p}}(V, \lambda)$ , where  $V$  is a simple torsion-free module of finite degree, to the analogous problem for  $M_{\mathfrak{p}}(\hat{V}, \lambda)$ , which is a problem for the category  $\mathcal{O}$ . The last one can be solved by combination of Kazhdan-Lusztig Theorem, Soergel's equivalence ([34]) and BGG-resolution (or Kac-Wakimoto resolution, see [7, Theorem 1]).

It is obvious, that the usual duality on  $\mathcal{O}$  restricts to  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\hat{V}))$ , and hence one can apply Theorem 6.3 to compute the Cartan matrix for all blocks of  $\mathcal{O}(\mathfrak{p}, \text{Coker}(\hat{V}))$ .

### Acknowledgments

I acknowledge the financial support of the Fields Institute for Mathematical Sciences during my visit to the Institute in July-August 2002, and the partial financial support of the Swedish Research Council. I thank Professor V.Dlab for his comments on the paper and Olexandr Khomenko for stimulating discussions. Especially, I would like to thank the referee for many useful comments, which led to the improvements in the paper.

### References

- [1] M.Auslander, *Representation theory of Artin algebras. I.* Comm. Algebra **1** (1974), 177–268.
- [2] M.Auslander, M.Platzeck, G.Todorov, *Homological theory of idempotent ideals.* Trans. Amer. Math. Soc. **332** (1992), no. 2, 667–692.
- [3] I.Ágoston, V.Dlab, E.Lukács, *Stratified algebras.* C. R. Math. Acad. Sci. Soc. R. Can. **20** (1998), no. 1, 22–28.
- [4] I.N.Bernstein, S.I.Gelfand, *Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras.* Compositio Math. **41** (1980), no. 2, 245–285.
- [5] I.N.Bernstein, I.M.Gelfand, S.I.Gelfand, *A certain category of  $\mathfrak{g}$ -modules.* Funkcional. Anal. i Priložen. **10** (1976), no. 2, 1–8.
- [6] D.Britten and F.Lemire, *On modules of bounded multiplicities for the symplectic algebras.* Trans. Amer. Math. Soc. **351** (1999), no. 8, 3413–3431.
- [7] D.Britten, V.Futorny and F.Lemire, *Submodule lattice of generalized Verma modules.* Preprint 2002.
- [8] E.Cline, B.Parshall, L.Scott, *Finite-dimensional algebras and highest weight categories.* J. Reine Angew. Math. **391** (1988), 85–99.
- [9] E.Cline, B.Parshall, L.Scott, *Stratifying endomorphism algebras.* Mem. Amer. Math. Soc. **124** (1996), no. 591,
- [10] A.J.Coleman, V.Futorny, *Stratified  $L$ -modules.* J. Algebra **163** (1994), no. 1, 219–234.
- [11] J.Dixmier, *Algèbres Enveloppantes.* Paris, 1974.
- [12] V.Dlab, *Quasi-hereditary algebras revisited.* in Representation theory of groups, algebras, and orders (Constanța, 1995). An. Științ. Univ. Ovidius Constanța Ser. Mat. **4** (1996), no. 2, 43–54.
- [13] V.Dlab, *Properly stratified algebras.* C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 3, 191–196.
- [14] Yu.Drozd, S.Ovsienko and V.Futorny, *The Harish-Chandra  $S$ -homomorphism and  $\mathfrak{G}$ -modules generated by a semiprimitive element.* Ukrain. Mat. Zh. **42** (1990), no. 8, 1031–1037; translation in Ukrainian Math. J. **42** (1990), no. 8, 919–924 (1991)
- [15] S.L.Fernando, *Lie algebra modules with finite dimensional weight spaces. I.* Trans. Am. Math. Soc. **322**, No.2, 757–781 (1990)

- [16] V.Futorny, V.Mazorchuk, *Highest weight categories of Lie algebra modules*. J. Pure Appl. Algebra **138** (1999), no. 2, 107–118
- [17] V.Futorny, S.König and V.Mazorchuk, *Categories of Induced Modules and Standardly Stratified Algebras*. Algebr. Represent. Theory **5** (2002), no. 3, 259–276.
- [18] V.Futorny, S.König and V.Mazorchuk, *A combinatorial description of blocks in  $\mathcal{O}(\mathcal{P}, \Lambda)$  associated with  $\mathfrak{sl}(2)$ -induction*. J. Algebra **231** (2000), no. 1, 86–103.
- [19] V.Futorny, S.König and V.Mazorchuk,  *$\mathcal{S}$ -subcategories in  $\mathcal{O}$* . Manuscripta Math. **102** (2000), no. 4, 487–503.
- [20] X.Gomez, V.Mazorchuk, *On an analogue of BGG-reciprocity*. Commun. Algebra **29**, No.12, 5329–5334 (2001).
- [21] R.Irving, B.Shelton, *Loewy series and simple projective modules in the category  $\mathcal{O}_S$* . Pacific J. Math. **132** (1988), no. 2, 319–342.
- [22] J.C.Jantzen, *Einhüllende Algebren halbeinfacher Lie-Algebren*. Ergebnisse der Mathematik und ihrer Grenzgebiete **3**, Springer-Verlag, Berlin, 1983.
- [23] O.Khomenko and V.Mazorchuk, *Structure of modules induced from simple modules with minimal annihilator*. Preprint 2002:6, Uppsala University, to appear in Canad. Math. J.
- [24] O.Khomenko and V.Mazorchuk, *Rigidity of generalized Verma modules*. Colloq. Math. **92** (2002), no. 1, 45–57.
- [25] M.Klucznik and S.König, *Characteristic tilting modules over quasi-hereditary algebras*. Ergänzungsreihe 99-004, Bielefeld University, Bielefeld, Germany, 1999.
- [26] S.König and V.Mazorchuk, *Enright's completions and injectively copresented modules*. Trans. Amer. Math. Soc. **354** (2002), no. 7, 2725–2743
- [27] S.König and V.Mazorchuk, *An equivalence of two categories of  $\mathfrak{sl}(n, \mathbb{C})$ -modules*. Algebr. Represent. Theory **5** (2002), no. 3, 319–329.
- [28] O.Mathieu, *Classification of irreducible weight modules*. Ann. Inst. Fourier **50**, No.2, 537–592 (2000).
- [29] V.Mazorchuk, *Generalized Verma modules*. Mathematical Studies Monograph Series, 8. VNTL Publishers, Lviv, 2000.
- [30] D.Miličić, W.Soergel, *The composition series of modules induced from Whittaker modules*. (English) Comment. Math. Helv. **72**, No.4, 503–520 (1997).
- [31] A.Rocha-Caridi, *Splitting criteria for  $\mathfrak{g}$ -modules induced from a parabolic and the Bernstein-Gelfand-Gelfand resolution of a finite-dimensional, irreducible  $\mathfrak{g}$ -module*. Trans. Amer. Math. Soc. **262** (1980), no. 2, 335–366.
- [32] A.Rocha-Caridi and N.Wallach, *Projective modules over graded Lie algebras. I*. Math. Z. **180** (1982), no. 2, 151–177.
- [33] W.Soergel, *The combinatorics of Harish-Chandra bimodules*. J. Reine Angew. Math. **429**, 49–74 (1992).
- [34] W.Soergel, *Kategorie  $\mathcal{O}$ , perverse Garben und Moduln über den Koinvarianten zur Weyl-gruppe*. J. Amer. Math. Soc. **3** (1990), no. 2, 421–445.