On the structure of Brauer semigroup and its partial analogue

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1 Introduction

The Brauer semigroup (or chip semigroup) $\mathfrak{B}_n$ was introduced by Brauer in [B] in order to study the representations of symplectyc and orthogonal groups. From that time the semigroup algebra of $\mathfrak{B}_n$ was studied by several authors (see for example [K] and references therein). Regretfully, it is a kind of tradition in semigroup theory that only few is known about the semigroup structure of arbitrary concrete semigroup, in particular, about $\mathfrak{B}_n$. For example, among other recent results on the structure of special semigroups one can find the description of special classes of subsemigroups in inverse symmetric semigroup ([GK]) or in factor power of finite symmetric group ([GM]).

The aim of this paper is to study the basic subsemigroup properties of $\mathfrak{B}_n$ and its partially defined analogue $\mathfrak{PB}_n$ (see section 2 for precise definitions). In particular, we provide a description for the idempotents and maximal subgroups in $\mathfrak{B}_n$ and $\mathfrak{PB}_n$, classify completely isolated subsemigroups, determine all automorphisms and describe Green relations on these semigroups.

The paper is organized in the following way: in section 2 we present all necessary definitions and preliminaries. In section 3 we study the idempotents and maximal subgroups. In section 4 we define a natural aniautomorphism on our semigroups and determine all regular and inverse elements. In section 5 we describe all completely isolated subgroups. In section 6 we prove that any automorphism of $\mathfrak{B}_n$ or $\mathfrak{PB}_n$ is inner. Finally, in section 7 we give a combinatoric description for the Green relations on $\mathfrak{B}_n$ and p_n.

2 The semigroups $\mathfrak{B}_n$ and $\mathfrak{PB}_n$

Fix some positive integer n. Let $M_n = \{1, 2, \dots, n\}$ and set $N_n = \{x, x^\circ \mid x \in M_n\}$. Elements $1, 2, \dots, n$ will be called initial and elements $1^\circ, 2^\circ, \dots, n^\circ$ will be called terminal. In general, for any subset $A \subset M_n$ we define $A^\circ = \{x^\circ \mid x \in A\}$. Clearly, $|N_n| = 2n$. Arbitrary decomposition of N_n into a disjoint union of subsets containing exactly 2 elements each is called a chip with $2n$ legs. If a chip a contains a set $\{x, y\}$ we will say that x and y

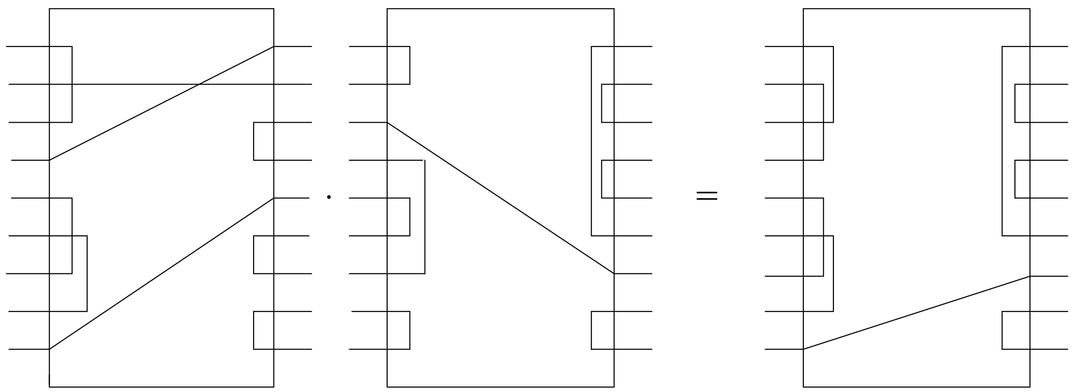


Figure 1: Chips and their multiplication.

are connected by a or that $\{x, y\}$ belongs to a. On the set $\mathfrak{B}_n$ of all chips with $2n$ legs a natural associative operation can be defined. In fact, having two chips $a, b \in \mathfrak{B}_n$ we form their product ab in the following way: First we identify all initial elements from b with the corresponding terminal elements from a and redenote them by $\hat{1}, \hat{2}, \dots, \hat{n}$. Then we construct the subsets of $N_n \cup \{\hat{1}, \hat{2}, \dots, \hat{n}\}$ by taking union of those subsets from both a and b that have a non-trivial intersection. Finally, we erase all the elements that do not belong to N_n obtaining new chip ab. One can easily calculate that $|\mathfrak{B}_n| = (2n-1)!!$.

It will be convenient to realize a chip as a "scheme" with n initial and n terminal legs. One can enumerate the initial (left-hand sided) legs and the corresponding terminal (right-hand sided) legs for example from up to down (see Figure 1). Two legs x and y of a chip a should be connected if and only if $\{x, y\}$ do belong to a. In this realization the multiplication of chips corresponds to their "concatenation" (see Figure 1) or simply to the gluing of terminal legs from the first chip with corresponding initial legs from the second one. In this way some "dead chains" can arise just like it happened for two lower legs on Figure 1. In this realization the associativity of the multiplication is obvious.

Let a be a chip from $\mathfrak{B}_n$. We will denote by $\mathrm{In}(a)$ the set of all initial elements of a and by $\mathrm{Ter}(a)$ the set of all terminal elements of a. The subset of $\mathrm{In}(a)$ ($\mathrm{Ter}(a)$) consisting of all elements those are connected with terminal (initial) elements will be called the initial (terminal) image of a and will be denoted by $\mathrm{Im}(a)$ ($\mathrm{Im}^\circ(a)$). The subset of $\mathrm{In}(a)$ ($\mathrm{Ter}(a)$) consisting of all elements those are connected with initial (terminal) elements will be called the initial (terminal) kernel of a and will be denoted by $\mathrm{Ker}(a)$ ($\mathrm{Ker}^\circ(a)$). Clearly, both $|\mathrm{Ker}(a)|$ and $|\mathrm{Ker}^\circ(a)|$ are even and $|\mathrm{Im}(a)| = |\mathrm{Im}^\circ(a)|$.

There exists a natural monomorphism from the symmetric group S_n into $\mathfrak{B}_n$. It maps any permutation on n symbols into the chip in which each initial leg is connected with the corresponding permuted terminal leg. Clearly, the image of this monomorphism coincides with the set of all chips a such that $|\mathrm{Im}(a)| = n$ or equivalently $\mathrm{Ker}(a) = \mathrm{Ker}^\circ(a) = \emptyset$. An image of the identity permutation under this monomorphism is the identity element in

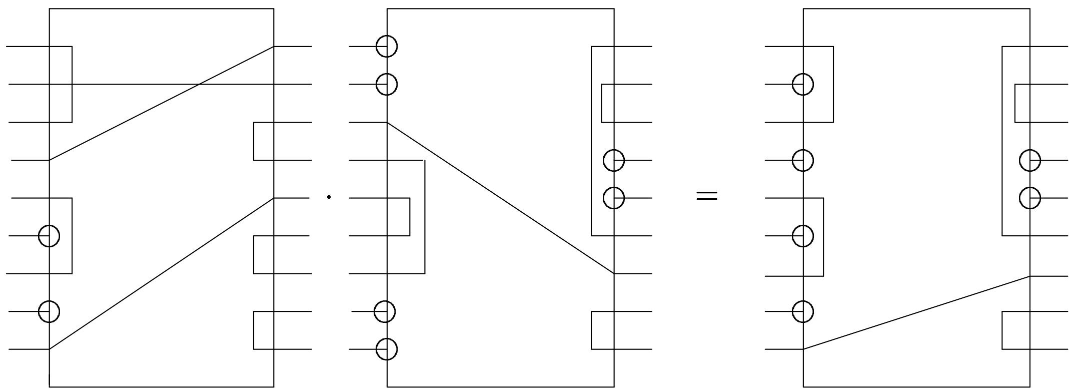


Figure 2: Partial chips and their multiplication.

$\mathfrak{B}_n$. We will also call this element the trivial chip and will denote it by e. Sometimes we will identify S_n with its image under this monomorphism.

Now we can define a partial analogue of $\mathfrak{B}_n$. Consider the set $\mathfrak{PB}_n$ of partially defined chips (partial chips), i.e. decompositions into pairs of subset in N_n containing even number of elements. If $x \in N_n$ does not belong to any subset of a fixed chip a we will say that a is undefined in x. It follows directly from the definition that

$$|\mathfrak{PB}_n| = \sum_{i=0}^{n} \binom{2n}{2i} (2i-1)!!.$$

Analogously to the chip situation a natural associative multiplication on $\mathfrak{PB}_n$ is defined. In fact, given two partial chips $a, b \in \mathfrak{PB}_n$ we form their product ab in the following way: First, for any $x \in N_n$ such that a (b) is undefined in x we will say that a (b) contains the subset $\{x, *\}$. Now we identify all initial elements from b with the corresponding terminal elements from a and redenote them by $\hat{1}, \hat{2}, \dots, \hat{n}$. Then we construct the subsets of $N_n \cup \{\hat{1}, \hat{2}, \dots, \hat{n}\}$ by taking union of those subsets from both a and b that have a non-trivial intersection and erase all the subsets that contains $*$ element. Finally, from the rest of subsets we erase all the elements that do not belong to N_n obtaining new partial chip ab.

At the same way as for chips, any partial chip can be realized as a "scheme" with $2n$ legs. Those legs of a chip in which it is undefined can be marked for example with circles. In this realization the multiplication defined above is again a concatenation and thus is associative. An example of the multiplication of two partial chips is given on the Figure 2.

For any partial chip a we retain the same notions of $\mathrm{In}(a)$, $\mathrm{Ter}(a)$, $\mathrm{Im}(a)$, $\mathrm{Im}^\circ(a)$, $\mathrm{Ker}(a)$ and $\mathrm{Ker}^\circ(a)$ as used for $\mathfrak{B}_n$.

Clearly, there exists a natural monomorphism from $\mathfrak{B}_n$ into $\mathfrak{PB}_n$ since any chip can be viewed as partial chip defined in all legs. Thus S_n is a subgroup of $\mathfrak{PB}_n$ and e is the identity element in $\mathfrak{PB}_n$. Moreover, there exists a natural monomorphism from the inverse symmetrical semigroup $\mathcal{IS}_n$ into $\mathfrak{PB}_n$ (see, for example, [GK] for definition). This monomorphism is coordinated with all mentioned above. Sometimes we will identify $\mathcal{IS}_n$ with its image under this monomorphism. Nevertheless, unlike any classical partial semigroup $\mathfrak{PB}_n$ does not contain any zero element. Really, the standard pretender for this role — partial chip u undefined in all legs is, in fact, idempotent but not zero. The product of any chip a with non-empty initial and terminal kernels with u does not equal to u since it inherits the non-emptiness either of initial or of terminal kernel of a. It the subsequent sections we will describe a lot of unusual properties of $\mathfrak{PB}_n$ more carefully.

3 Idempotents and maximal subgroups

One of the most important question related to an arbitrary semigroup is the structure of its idempotents. First of all we will study the structure of idempotents in $\mathfrak{B}_n$.

Let $1 < k \leqslant n$ be an odd number and $A \subset M_n$ such that $|A| = k$. Fix a linear order $<$ on A. Rewrite elements from A with respect to this order: $A = \{a_1, a_2, \dots, a_k\}$, $a_i < a_{i+1}$ for all i. Consider the chip $e_{A,<}$ defined as follows: $e_{A,<}$ contains all subsets $\{x, x^\circ\}$ for $x \in M_n \setminus A$, all subsets $\{a_i, a_{i+1}\}$ where $1 \leqslant i < k$ is odd, all subsets $\{a_i^\circ, a_{i+1}^\circ\}$ where $1 < i < k$ is even and the subset $\{a_k, a_1^\circ\}$. The same construction can be applied for $k = 1$ that gives one an identity idempotent e.

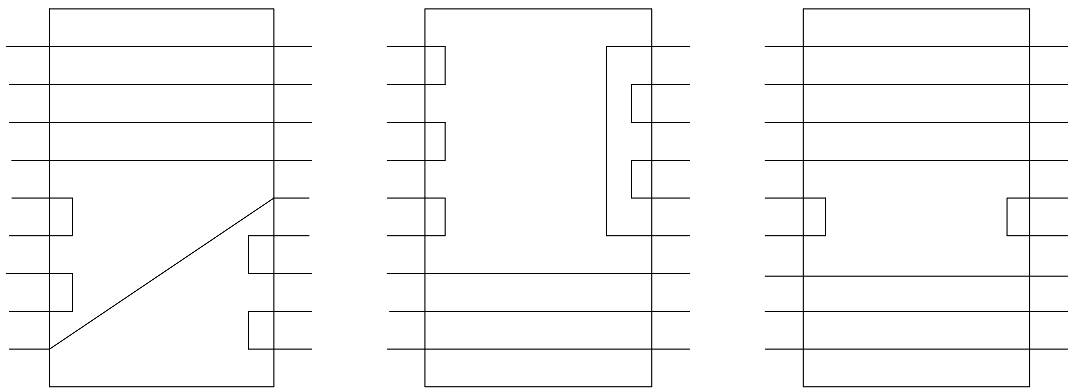
Let $1 < k \leqslant n$ be an even number and $A \subset M_n$ such that $|A| = k$. Fix a linear order $<$ on A. Rewrite elements from A with respect to this order: $A = \{a_1, a_2, \dots, a_k\}$, $a_i < a_{i+1}$ for all i. Consider the chip $e_{A,<}$ defined as follows: $e_{A,<}$ contains all subsets $\{x, x^\circ\}$ for $x \in M_n \setminus A$, all subsets $\{a_i, a_{i+1}\}$ where $1 \leqslant i < k$ is odd, all subsets $\{a_i^\circ, a_{i+1}^\circ\}$ where $1 < i < k$ is even and the subset $\{a_k^\circ, a_1^\circ\}$.

It is straightforward that $e_{A,<}$ is idempotent for any A and $<$ described above. We will call these idempotents elementary. In the case when $|A| = 2$ the idempotent $e_{A,<}$ will be called atom. The structure of the elementary idempotents in the "scheme" realization is quite clear. The examples are shown on Figure 3.

Let T be a finite linearly ordered set $T = \{t_1, t_2, \dots, t_l\}$, $t_i < t_{i+1}$. By primitive cyclic permutation of T we will mean the permutation $t_i \mapsto t_{i+1}$, $1 \leqslant i < l$, $t_l \mapsto t_1$. An order $<^\circ$ on T such that $t_i > t_{i+1}$ for all i will be called opposite to $<$.

Lemma 1. *$e_{A,<_A} = e_{B,<_B}$ if and only if one of the following holds:*

1. *$A = B$, $<_A = <_B$.*
2. *$A = B$, $|A|$ is even and $<_B$ is obtained from $<_A$ by an even power of the primitive cyclic permutation.*
3. *$A = B$, $|A|$ is even and $<_B$ is obtained from $<_A^\circ$ by an odd power of the primitive cyclic permutation.*



a) $k = 5$ b) $k = 6$ c) atom

Figure 3: Elementary idempotents and an atom.

Proof. The if part is obvious. To prove the only if part we have to consider two different cases of odd and even $|A|$. In the case when $|A|$ is odd the result follows from the uniqueness of the subset containing both initial and terminal elements from $A \cup A^\circ$. In the case of even $|A|$ one has to consider only $A = B$ situation with $<_B$ and $<_A$ $(<_A^\circ)$ that differs on a power of the primitive cyclic permutation. Then the result follows by trivial calculation. $\square$

By lemma 1 any atom depends only on the set A such that $|A| = 2$. In the sequal we will denote all atoms simply by e_A.

A product $e_{A_1,<_1} e_{A_2,<_2} \dots e_{A_l,<_l}$ will be called simple if the subsets A_i, $1 \leqslant i \leqslant l$ are pairwise disjoint.

Lemma 2. *Any idempotent $f \in \mathfrak{B}_n$, $f \neq e$ splits into a simple product of the elementary idempotents. Moreover, the shortest product of this kind is uniquely defined up to a permutation of the multiplicands.*

Proof. Let f be an idempotent in $\mathfrak{B}_n$. First we define an "orbits" of f on M_n. Two elements x and y from M_n is said to belong to the same orbit provided $\{x, y\}$ or $\{x, y^\circ\}$ or $\{x^\circ, y\}$ or $\{x^\circ, y^\circ\}$ belongs to f. In such way we split M_n into disjoint union of subsets. For any orbit A containing more that one element we define an element f_A as follows: f_A contains $\{x, x^\circ\}$ for any $x \in M_n \setminus A$, f_A contains $\{x, y\}$ $(\{x^\circ, y\})$ with $x \in A$, $y \in N_n$ if and only if f contains $\{x, y\}$ $(\{x^\circ, y\})$. Since f is idempotent and A is an orbit it follows immediately that f_A is idempotent. Moreover, one can see that each f_A is elementary. Further, it follows directly from the construction of f_A's that they commute and $f = \prod_A f_A$. The uniqueness of the shortest decomposition is trivial. $\square$

Lemma 3. *Any elementary idempotent in $\mathfrak{B}_n$ can be decomposed into a product of atoms.*

Proof. Let $e_{A,<}$ be a given elementary idempotent. We will proceed by induction in $|A|$. The base is trivial. Suppose that $|A|$ is odd. Let x be the minimal element of A with respect to $<$ and y be the minimal element of $B = A \setminus \{x\}$ with respect to the natural restriction of $<$ on B. Shift this restricted order on B by the primitive cyclic permutation obtaining $<'$. One can check that $e_{A,<} = e_{\{x,y\}} e_{B,<'}$. Now let $|A|$ be even. Using the same notations as above we have $e_{A,<} = e_{\{x,y\}} e_{B,<^{\circ}}$ again. We complete or proof using the inductive assumption. □

Corollary 1. *Any idempotent $f \in \mathfrak{B}_n$, $f \neq e$ can be decomposed into a product of atoms.*

Proof. Follows from lemmas 2,3. □

It is well-known that there is a natural one-to-one correspondence between idempotents in a semigroup S and maximal subgroups in S. Using the results above we can describe all maximal subgroups of $\mathfrak{B}_n$. Let f be an idempotent in $\mathfrak{B}_n$. Set $O(f)$ be the set of all orbits of f on M_n defined in the proof of lemma 2. Let $O_e(f)$ be the subset of $O(f)$ consisting of all $A \in O(f)$ such that $|A|$ is odd. One can easily remark that $|O_e(f)| = |\operatorname{Im}(f)|$.

Theorem 1. *The maximal subgroup $S(f)$ of $\mathfrak{B}_n$ with the unit f is isomorphic to the symmetric group $S_{|O_e(f)|}$. Moreover, $a \in S(f)$ if and only if the following two conditions are satisfied:*

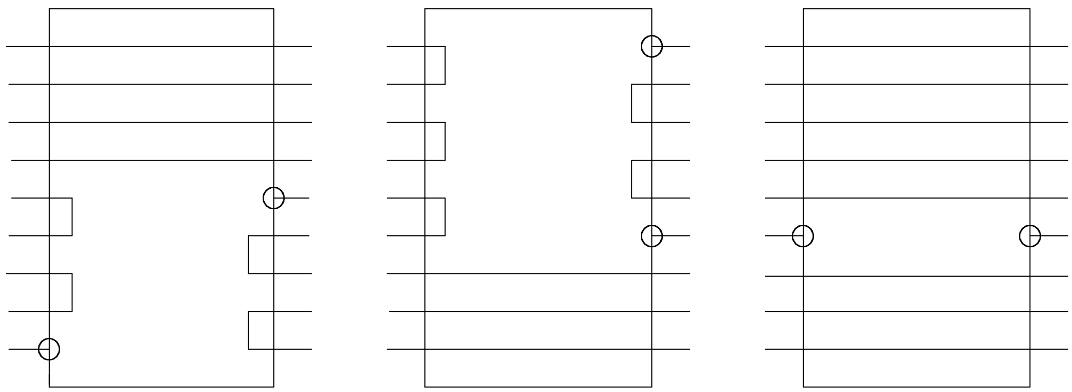
- *For any $x, y \in M_n$ the subset $\{x, y\}$ belongs to a if and only if $\{x, y\}$ belongs to f.*
- *For any $x, y \in M_n$ the subset $\{x^{\circ}, y^{\circ}\}$ belongs to a if and only if $\{x^{\circ}, y^{\circ}\}$ belongs to f.*

Proof. Let $a \in S(f)$. Since $fa = af = a$ one obtains that for any $x, y \in M_n$ the subset $\{x, y\}$ ($\{x^{\circ}, y^{\circ}\}$) belongs to a if and only if $\{x, y\}$ ($\{x^{\circ}, y^{\circ}\}$) belongs to f. Clearly, the set of all those a satisfying these conditions form a subgroup of $\mathfrak{B}_n$ isomorphic to $S_{|O_e(f)|}$. This completes the proof. □

An analogue of all the results above can be obtain for $\mathfrak{PB}_n$. As a first step we will construct partial analogues of elementary idempotents.

Let $1 < k \leqslant n$ be an odd number and $A \subset M_n$ such that $|A| = k$. Fix a linear order $<$ on A. Rewrite elements from A with respect to this order: $A = \{a_1, a_2, \ldots, a_k\}$, $a_i < a_{i+1}$ for all i. Consider the chip $e^p_{A,<}$ defined as follows: $e^p_{A,<}$ contains all subsets $\{x, x^{\circ}\}$ for $x \in M_n \setminus A$, all subsets $\{a_i, a_{i+1}\}$ where $1 \leqslant i < k$ is odd and all subsets $\{a_i^{\circ}, a_{i+1}^{\circ}\}$ where $1 < i < k$ is even. Set that $e^p_{A,<}$ is undefined in a_k and a_1°. The same construction can be applied for $k = 1$ that gives one the chip e^p_x, $x \in M_n$ which will be called p-atom.

Let $1 < k \leqslant n$ be an even number and $A \subset M_n$ such that $|A| = k$. Fix a linear order $<$ on A. Rewrite elements from A with respect to this order: $A = \{a_1, a_2, \ldots, a_k\}$, $a_i < a_{i+1}$ for all i. Consider the chip $e^{p^{\circ}}_{A,<}$ defined as follows: $e^{p^{\circ}}_{A,<}$ contains all subsets $\{x, x^{\circ}\}$ for $x \in M_n \setminus A$, all subsets $\{a_i^{\circ}, a_{i+1}^{\circ}\}$ where $1 \leqslant i < k$ is odd and all subsets $\{a_i, a_{i+1}\}$ where $1 < i < k$ is even. We set that $e^{p^{\circ}}_{A,<}$ is undefined in a_1 and a_k.



a) $k=5$ b) $k=6$ c) p-atom

Figure 4: p-elementary idempotents and a p-atom.

Let $1<k\leqslant n$ be an even number and $A\subset M_n$ such that $|A|=k$. Fix a linear order $<$ on A. Rewrite elements from A with respect to this order: $A=\{a_1,a_2,\ldots,a_k\}$, $a_i<a_{i+1}$ for all i. Consider the chip $e^p_{A,<}$ defined as follows: $e^p_{A,<}$ contains all subsets $\{x,x^\circ\}$ for $x\in M_n\setminus A$, all subsets $\{a_i,a_{i+1}\}$ where $1\leqslant i<k$ is odd number and all subsets $\{a_i^\circ,a_{i+1}^\circ\}$ where $1<i<k$ is even number. We set that $e^p_{A,<}$ is undefined in a_1° and a_k°.

It is straightforward that $e^p_{A,<}$, e^p_x and $e^{p^\circ}_{A,<}$ are idempotents for any A and $<$. We will call this idempotents p-elementary. The structure of the elementary idempotents in the "scheme" realization is shown on Figure 4.

Lemma 4. *$e^p_{A,<_A}=e^p_{B,<_B}$ ($e^{p^\circ}_{A,<_A}=e^{p^\circ}_{B,<_B}$) if and only if $A=B$, $<_A=<_B$.*

Proof. Follows immediately from the construction of idempotents. □

We retain the same notation of the simple product of idempotents as in the case of $\mathfrak{B}_n$. This allows us to formulate the following "partial" analogue for lemma 2.

Lemma 5. *Any idempotent $f\in\mathfrak{PB}_n$, $f\neq e$ splits into a simple product of the elementary and p-elementary idempotents. Moreover, the shortest product of this kind is uniquely defined up to a permutation of the multiplicands.*

Proof. Proof is essentially the same as that of lemma 2. □

Lemma 6. *Any p-elementary idempotent in $\mathfrak{PB}_n$ can be decomposed into a product of atoms and p-atoms.*

Proof. It is easy to verify that any p-elementary idempotent $e^p_{A,<}$ ($e^{p^\circ}_{A,<}$) can be obtained as a product of some p-atom with the elementary idempotent. Indeed, let x be the minimal element of A with respect to $<$ and let $<'$ be obtained from $<$ by the primitive cyclic permutation. For $|A|$ odd one has $e^p_{A,<}=e_{A,<}e^p_x$. For $|A|$ even one has $e^{p^\circ}_{A,<}=e^p_xe_{A,<'}$, $e^p_{A,<}=e_{A,<}e^p_x$. Now the statement follows from lemma 3. □

Corollary 2. *Any idempotent $f \in \mathfrak{PB}_n$, $f \neq e$ can be decomposed into a product of atoms and p-atoms.*

Proof. Follows from lemmas 5,6. ☐

The structure of maximal subgroups in $\mathfrak{PB}_n$ is analogous to that of $\mathfrak{B}_n$. Let f be an idempotent in $\mathfrak{PB}_n$. By a non-degenerate orbit of f we will mean a subset $A \subset M_n$ defined as follows: either $A = \{x\}$ such that $\{x, x^\circ\}$ belongs to f or $e_{A,<}$ occurs in a shortest decomposition of f into a simple product of elementary and p-elementary idempotents. One can see that this definition coincides (in $\mathfrak{B}_n$ case) with that given in the proof of lemma 2. Set $O(f)$ be the set of all non-degenerated orbits of f and let $O_e(f)$ be the subset of $O(f)$ consisting of all $A \in O(f)$ such that $|A|$ is odd. Clearly, $|O_e(f)| = |\operatorname{Im}(f)|$.

Theorem 2. *The maximal subgroup $S(f)$ of $\mathfrak{B}_n$ with the unit f is isomorphic to the symmetric group $S_{|O_e(f)|}$. Moreover, $a \in S(f)$ if and only if the following four conditions are satisfied:*

- *For any $x, y \in M_n$ the subset $\{x, y\}$ belongs to a if and only if $\{x, y\}$ belongs to f.*
- *For any $x, y \in M_n$ the subset $\{x^\circ, y^\circ\}$ belongs to a if and only if $\{x^\circ, y^\circ\}$ belongs to f.*
- *a is undefined in $x \in M_n$ if and only if f is undefined in x.*
- *a is undefined in x°, $x \in M_n$ if and only if f is undefined in x°.*

Proof. Proof is essentially the same as that of lemma 2. ☐

4 Anti-involution, regular and inverse elements

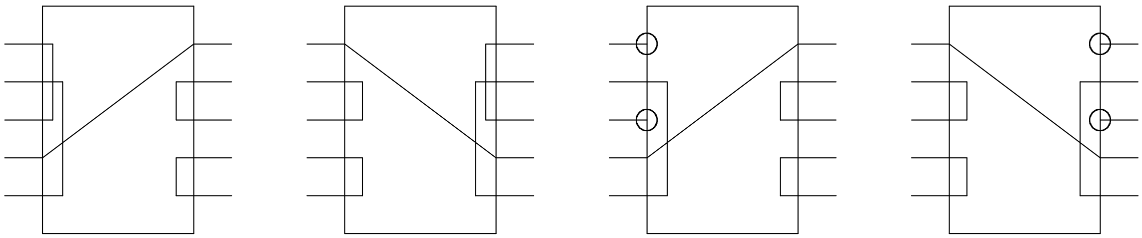
A natural anti-involution σ on $\mathfrak{B}_n$ ($\mathfrak{PB}_n$) can be defined. We set $\sigma(a)$, $a \in \mathfrak{B}_n$ ($a \in \mathfrak{PB}_n$) be the chip (partial chip) that is obtained from a by interchanging elements $x \leftrightarrow x^\circ$ for all $x \in M_n$. Geometrically, $\sigma(a)$ is the mirror reflection of a (see Figure 5).

We recall that an element a from a semigroup S is called regular provided there exist $b \in S$ such that $aba = a$. If this element b is uniquely defined, a is called inverse. It happened that both $\mathfrak{B}_n$ and $\mathfrak{PB}_n$ contain a lot of regular but only few inverse elements.

Theorem 3. *1. Any element in $\mathfrak{B}_n$ ($\mathfrak{PB}_n$) is regular.*

2. An element $a \in \mathfrak{B}_n$ is inverse if and only if $|\operatorname{Im}(a)| = n$ or $|\operatorname{Im}(a)| = n - 2$.

3. An element $a \in \mathfrak{PB}_n$ is inverse if and only if $|\operatorname{Im}(a)| = n$ or $|\operatorname{Im}(a)| = n - 1$.



a) chip t b) chip $\sigma(t)$ c) partial chip s d) partial chip $\sigma(s)$

Figure 5: Anti-involution σ.

Proof. The first part of the theorem follows from the obvious formula $a\sigma(a)a = a$ for all $a \in \mathfrak{B}_n$ ($a \in \mathfrak{PB}_n$). Clearly, any element $a \in \mathfrak{B}_n$ ($a \in \mathfrak{PB}_n$) satisfying $|\operatorname{Im}(a)| = n$ is inverse. It is straightforward to show that any $a \in \mathfrak{B}_n$ ($a \in \mathfrak{PB}_n$) satisfying $|\operatorname{Im}(a)| = n-2$ ($|\operatorname{Im}(a)| = n-1$) is inverse. Thus, it is enough to prove that any $a \in \mathfrak{B}_n$ ($a \in \mathfrak{PB}_n$) satisfying $|\operatorname{Im}(a)| < n-2$ ($|\operatorname{Im}(a)| < n-1$) is not inverse. Under the conditions above there exists $b \in \mathfrak{B}_n$ ($b \in \mathfrak{PB}_n$), $b \neq \sigma(a)$ such that for any $A \subset N_n$, $A \cap \operatorname{Ker}(\sigma(a)) = \emptyset$ chip b contains A if and only if $\sigma(a)$ contains A (and for any $x \in N_n \setminus \operatorname{Ker}(\sigma(a))$ chip b is undefined in x if and only if $\sigma(a)$ is undefined in x). It follows that $aba = a$. The last implies that a is not inverse. □

It follows from theorem 3 that $a \in \mathfrak{PB}_n$ is inverse only if a is contained in $\mathcal{IS}_n$ (see section 3).

5 Completely isolated subsemigroups

We recall that a subsemigroup T of a semigroup S is called completely isolated provided for all $a, b \in S$ holds: $ab \in T$ implies $a \in T$ or $b \in T$.

Lemma 7. *Let S be a finite semigroup with unit element 1 and S_1 be the maximal subgroup of S corresponding to 1. Suppose that S satisfies the following conditions:*

1. *There exists a subset P of idempotents of S that generates the set of all non-invertible idempotents.*
2. *For any $x, y \in P$ there exists $a, b \in S_1$ such that $axb = y$.*

Then the only completely isolated subsemigroups of S are S, S_1 and $S \setminus S_1$.

Proof. Since S has unit element it follows that all S, S_1 and $S \setminus S_1$ are completely isolated. Let T be a completely isolated subsemigroup. Since S is finite it follows that $T \cap S_1 \neq \emptyset$ implies $S_1 \subset T$. Suppose that T differs from S_1 and $S \setminus S_1$. Then there exists $t \in T \cap (S \setminus S_1)$.

Thus T contains an idempotent ($\neq 1$) of S. Since T is completely isolated it follows from the first condition of the lemma that T contains an element from P. If $S_1 \subset T$ it follows from the second condition of the lemma that T contains P since T is subsemigroup. If $T \cap S_1 = \emptyset$ it follows from the second condition of the lemma that T contains P since T is completely isolated. Finally, $P \subset T$ and hence $(S \setminus S_1) \subset T$. This implies $T = S$. □

Corollary 3. *The only completely isolated subsemigroups of* $\mathfrak{B}_n$ *are* $\mathfrak{B}_n$, S_n *and* $\mathfrak{B}_n \setminus S_n$.

Proof. We set P to be the set of all atoms. The first condition of lemma 7 is satisfied by corollary 1. The second condition is obvious. Thus the result follows from lemma 7. □

We remark that using lemma 7 one can describe completely isolated subsemigroups in inverse symmetrical semigroup $\mathcal{IS}_n$ ([GK]) and in factor power of a symmetrical group $\mathcal{FP}^+(S_n)$ ([GM, Theorem 3]).

Unlike the case of $\mathfrak{B}_n$, the description of completely isolated subsemigroups in $\mathfrak{PB}_n$ is quite non-trivial. To proceed we need the following notations: Let D_l (D_r) be the set of all non-invertible elements a from $\mathfrak{PB}_n$ that satisfy the following condition: a contains $\{x, y\}$ ($\{x^\circ, y^\circ\}$) for some $x, y \in M_n$. Clearly, both D_l and D_r are subsemigroups of $\mathfrak{PB}_n$. Let S_l (S_r) be the set of all non-invertible elements a from $\mathfrak{PB}_n$ that satisfy the following condition: a does not contain $\{x, y\}$ ($\{x^\circ, y^\circ\}$) for any $x, y \in M_n$. Clearly, both S_l and S_r are subsemigroups of $\mathfrak{PB}_n$. Set $P_1 = D_l \cap D_r$; $P_2 = D_r \cap S_l$, $P_3 = S_r \cap D_l$, $P_4 = S_l \cap S_r$. Clearly, all P_i, $i = 1, 2, 3, 4$ are subsemigroups.

Lemma 8. *Let* T *be a completely isolated subsemigroup of* $\mathfrak{PB}_n$. *Suppose that* $T \cap S_n$ *is non-empty. Then* $S_n \subset T$.

Proof. Let $x \in T \cap S_n$. Then $x^k \in T$ for all k and thus $1 \in T$. Since $y^{n!} = 1$ for any $y \in S_n$ it follows that $y^{n!} \in T$ and thus $y \in T$. □

Lemma 9. *Any idempotent in* P_3 (P_2) *can be decomposed into the product of idempotents of the form* e^p_A ($e^{p^\circ}_A$), $A \subset M_n$, $|A| = 2$.

Proof. For P_3 this follows by induction from the formula

$$e^p_{\{x,y\}} e^p_{\{y,z\}} = e^p_{\{x,y\}} e^p_z$$

that holds for any pairwise different elements $\{x, y, z\} \in M_n$. For P_2 one has to apply σ to the both sides of the above formula. □

An idempotent of the form e^p_A ($e^{p^\circ}_A$) will be called an l-atom (r-atom).

Theorem 4. *Any completely isolated subsemigroup of* $\mathfrak{PB}_n$ *is one of the following completely isolated subsemigroups:* $\mathfrak{PB}_n$, S_n, $\mathfrak{PB}_n \setminus S_n$, S_l, $S_n \cup S_l$, S_r, $S_n \cup S_r$, D_l, $S_n \cup D_l$, D_r, $S_n \cup D_r$.

Proof of theorem 4. First we note that D_l and S_l (D_r and S_r) are complements in $\mathfrak{PB}_n \setminus S_n$. Thus by lemma 8 to prove our theorem it is enough to show that the only completely isolated subsemigroups in $\mathfrak{PB}_n \setminus S_n$ are D_l, D_r, S_l and S_r.

Let T be a completely isolated subsemigroup in $\mathfrak{PB}_n \setminus S_n$. Clearly, T contains an idempotent and $S_nTS_n = T$. By corollary 2 any idempotent ($\neq 1$) in $\mathfrak{PB}_n$ can be decomposed into a product of atoms. Since $e_A = e_A^p e_A^{p^\circ}$ for any $A \subset M_n$ such that $|A| = 2$ it follows that any idempotent ($\neq 1$) in $\mathfrak{PB}_n$ can be decomposed into a product of p-atoms, l-atoms and r-atoms.

Note that as soon as T contains a p-atom, an l-atom and an r-atom it should contain all p-atoms, all l-atoms and all r-atoms since $S_nTS_n = T$. Thus T contains all non-invertible idempotents and hence, coincides with $\mathfrak{PB}_n \setminus S_n$. For the rest of the proof we suppose that T does not coincide with $\mathfrak{PB}_n \setminus S_n$.

It follows from the discussion above that T contains a p-atom, an l-atom or an r-atom. If T contains both l-atoms and r-atoms it follows from $e^p_{\{1,2\}} e^p_{\{1^\circ,2^\circ\}} = e^p_1 e^p_2$ that T contains a p-atom and thus coincides with $\mathfrak{PB}_n \setminus S_n$. Moreover, it follows from the same formula that T can not contain only p-atoms. Thus there can be exactly four possibilities: (a) T contains only an l-atoms (and thus all l-atoms), (b) T contains only r-atoms; (c) T contains both l-atoms and p-atoms but not r-atoms; (d) T contains both r-atoms and p-atoms but not l-atoms.

(a) By lemma 9 any idempotent in P_3 can be decomposed into a product of l-atoms. Thus T contains P_3. From

$$e^p_{\{1,2\}} = e_{\{1,2\}} e^p_1 e^p_2$$

it follows that T contains $e_{\{1,2\}}$ and thus contains all atoms. Hence, T contains all idempotents from P_1 and therefore T contains P_1. Since T does not contain any p-atom it does not contain any element from P_4. Finally, suppose that T contains an element $a \in P_2$. From lemma 9 it follows that T should contain an r-atom. The last is impossible. Thus $T = P_1 \cup P_3 = D_l$

(b) Applying the σ anti-involution to the case (a) we obtain $T = D_r$.

(c) Since T contains l-atoms it follows from (a) that T contains P_3. From the other hand T contains P_4 since it contains all p-atoms. Suppose that T contains an element $a \in P_1$. We prove that T contains a element from P_2. Indeed, let f be the partial chip undefined in all legs. Clearly, $f \in P_4 \subset T$ and thus $fa \in P_2 \cap T$. It follows immediately from lemma 9 that T contains an r-atom. And the last is trivially impossible. Hence $T = P_4 \cup P_3 = S_r$.

(d) Applying the σ anti-involution to the case (c) we obtain $T = S_l$. □

The list of all completely isolated subsemigroups of $\mathfrak{PB}_2 \setminus S_2$ is shown on the Figure 6

6 Automorphisms

The classical result about symmetric groups states that any automorphism of S_n is inner provided $n \neq 6$ ([KM, Theorem 5.3.1]). It is also known ([S]) that any automorphism of $\mathcal{IS}_n$ is inner. In the present section we will prove the same results for $\mathfrak{B}_n$ and $\mathfrak{PB}_n$.

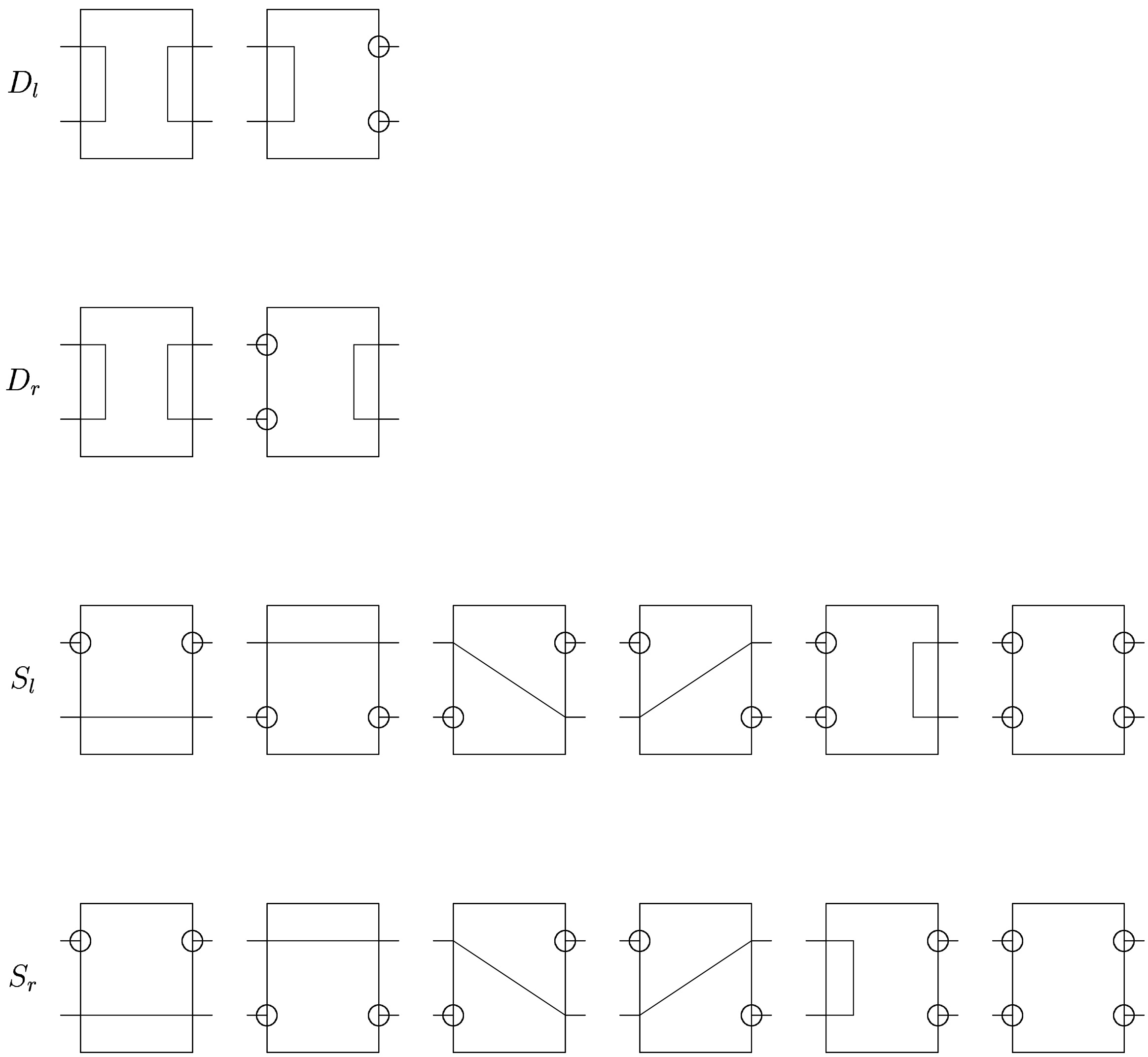


Figure 6: Completely isolated subsemigroups of $\mathfrak{PB}_2 \setminus S_2$.

Theorem 5. *Any automorphism of $\mathfrak{B}_n$ is inner.*

To prove this theorem we will need the following lemma:

Lemma 10. *$\mathfrak{B}_n$ is generated by S_n and any single atom e_A.*

Proof. Let S be the semigroup of $\mathfrak{B}_n$ containing S_n and e_A. First we note that S contains all atoms e_B cause $e_B = s^{-1}e_A s$ for some $s \in S_n$. It follows from corollary 1 that S contains all idempotents of $\mathfrak{B}_n$. For any $a \in \mathfrak{B}_n$ one can easily construct a permutation $s \in S_n$ such that as is an idempotent. Indeed, fix some bijection $\pi : \mathrm{Ker}(a) \to \mathrm{Ker}^\circ(a)$ such that $\{x, y\} \subset M_n$ belongs to a if and only if $\{\pi(x), \pi(y)\}$ belongs to a. Let s be the chip defined as follows: if a contains $\{x, y^\circ\}$ for some $x, y \in M_n$ then s contains $\{x^\circ, y\}$, s contains all $\{x, y^\circ\}$ such that $y \in \mathrm{Ker}\, a$, $x^\circ = \pi(y)$. One can see that $asas = as$. This implies that $\mathfrak{B}_n \subset S$ and the lemma follows. $\square$

Proof of theorem 5. First we suppose that $n = 6$ and consider a non-inner automorphism of S_6. It sends each elementary transposition into a product of three pairwise commuting transpositions. For each elementary transposition $s \in S_6$ there exists an atom e_A such that $se_A = e_A$. Clearly, any automorphism of $\mathfrak{B}_n$ sends any atom to atom. But for any product t of pairwise commuting transpositions and any atom e_B holds $te_B \neq e_B$. This implies that our automorphism of S_n can not be continued to an automorphism of $\mathfrak{B}_n$.

Now suppose that $\varphi : \mathfrak{B}_n \to \mathfrak{B}_n$ is an automorphism. From the remark above it follows that the restriction ψ of φ on S_n is some inner automorphism of S_n. Composing φ with the natural inner automorphism ψ^{-1} of S_n we can suppose that ψ is trivial. It follows from lemma 10 that it is enough to show that $\varphi(e_{\{1,2\}}) = e_{\{1,2\}}$. Clearly, $\varphi(e_{\{1,2\}})$ is an atom, say e_B. Let T_1 (T_2) be the set $\{s \in S_n \,|\, se_{\{1,2\}} = e_{\{1,2\}}s\}$ ($\{s \in S_n \,|\, se_B = e_B s\}$). Since φ is an automorphism it follows that $T_1 = T_2$. But T_2 contains all those permutations s under which B is invariant. This implies $B = \{1, 2\}$ and completes our proof. $\square$

Lemma 11. *$\mathfrak{PB}_n$ is generated by $\mathcal{IS}_n$ and any single atom e_A.*

Proof. Proof is analogous to that of lemma 10. $\square$

Theorem 6. *Any automorphism of $\mathfrak{PB}_n$ is inner.*

Proof. Let φ be an automorphism of $\mathfrak{PB}_n$. Since φ preserves inverse elements and $\mathcal{IS}_n$ is generated (as a subsemigroup in $\mathfrak{PB}_n$) by inverse elements of $\mathfrak{PB}_n$ it follows that the restriction of φ on $\mathcal{IS}_n$ is an automorphism of $\mathcal{IS}_n$ and thus is inner. Thus we can suppose that this restriction is trivial. Applying the arguments analogous to that used in the proof of theorem 5 one can show that φ preserves all atoms and thus is trivial by lemma 11. $\square$

7 Green relations

Since both $\mathfrak{B}_n$ and $\mathfrak{PB}_n$ are finite semigroups it is necessary to describes only four basic Green relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ and $\mathcal{H}$ on them. We recall that for arbitrary semigroup S the

Green relations are defined as follows: $s\mathcal{L}t$ if and only if $Ss = St$; $s\mathcal{R}t$ if and only if $sS = tS$; $\mathcal{D} = \mathcal{L} \cdot \mathcal{R}$ and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

The elements $a, b \in \mathfrak{B}_n$ is said to be left (right) neighbors provided a contains $\{x, y\}$ ($\{x^\circ, y^\circ\}$) if and only if b contains $\{x, y\}$ ($\{x^\circ, y^\circ\}$) for any $x, y \in M_n$. If elements a and b are both left and right neighbors we will call them neighbors.

Theorem 7. *Let a and b be two elements from $\mathfrak{B}_n$*

1. *$a\mathcal{L}b$ if and only if a and b are right neighbors;*

2. *$a\mathcal{R}b$ if and only if a and b are left neighbors;*

3. *$a\mathcal{H}b$ if and only if a and b are neighbors;*

4. *$a\mathcal{D}b$ if and only if $|\operatorname{Im}(a)| = |\operatorname{Im}(b)|$.*

Proof. The necessity of all conditions are obvious. Moreover, the sufficiency in the last two statements follows from that of the first and the second. We only will prove the sufficiency of the second one. The first will follow by applying the anti-involution σ described in section 4.

It is enough to show that $a \in b\mathfrak{B}_n$ for any left neighbors a and b in $\mathfrak{B}_n$. Clearly, there exist permutations s_1 and s_2 in S_n such that as_1 and bs_2 are idempotents. Since a and b are left neighbors, we can assume that $as_1 = bs_2$ and thus $a = bs_2s_1^{-1}$. Finally, $a \in bS_n \subset b\mathfrak{B}_n$. □

The elements $a, b \in \mathfrak{B}_n$ is said to be left (right) neighbors provided a contains $\{x, y\}$ ($\{x^\circ, y^\circ\}$) if and only if b contains $\{x, y\}$ ($\{x^\circ, y^\circ\}$) for any $x, y \in M_n$ and a is undefined in x (x°) if and only if b is undefined in x (x°) for any $x \in M_n$. If elements a and b are both left and right neighbors we will call them neighbors.

Theorem 8. *Let a and b be two elements from $\mathfrak{PB}_n$*

1. *$a\mathcal{L}b$ if and only if a and b are right neighbors;*

2. *$a\mathcal{R}b$ if and only if a and b are left neighbors;*

3. *$a\mathcal{H}b$ if and only if a and b are neighbors;*

4. *$a\mathcal{D}b$ if and only if $|\operatorname{Im}(a)| = |\operatorname{Im}(b)|$.*

Proof. Proof is analogous to that of theorem 7 □

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