CATEGORIFICATION, KOSTANT'S PROBLEM AND GENERALIZED VERMA MODULES

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1. Motivation — generalized Verma modules

$\mathfrak{g}$ — **semi-simple finite-dimensional complex Lie algebra.**

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — **triangular decomposition.**

$\mathfrak{p} \supset \mathfrak{h} \oplus \mathfrak{n}_+$ — **parabolic subalgebra.**

$\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{n}$

$\mathfrak{n}$ — **nilpotent radical of** $\mathfrak{p}$

$\mathfrak{a}$ — **Levi factor**

V — **simple** $\mathfrak{a}$**-module**

$\mathfrak{n}V = 0$

$M(\mathfrak{p}, V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$ — **generalized Verma module**

`Question 1:` **What is the structure of** $M(\mathfrak{p}, V)$**?**

`Question 2:` **When is** $M(\mathfrak{p}, V)$ **irreducible?**

`Discouragement:` **No classification of simple $\mathfrak{a}$-modules.**

`Encouragement 1:` **Many partial cases are known, in particular, $\mathfrak{a} = \mathfrak{h}$, V finite-dimensional, V weight dense with f.d. weight spaces, V generic Gelfand-Zetlin, V Whittaker. (Names: Verma, BGG, Jantzen, McDowell, Futorny, M., Milicic, Soergel, Khomenko, Mathieu, Britten, Lemire, others)**

`Encouragement 2:` **Annihilators of V are classified via annihilators of simple highest weight modules.**

`Idea (following Milicic-Soergel's study of the case when` V `is a Whittaker module):`

- **Take a simple highest weight $\mathfrak{a}$-module V' with the same annihilator as V.**
- **Realize $M(\mathfrak{p}, V)$ and $M(\mathfrak{p}, V')$ as objects in some Coker-categories.**
- **Prove (using Harish-Chandra bimodules) that these categories are equivalent and that the equivalence sends $M(\mathfrak{p}, V)$ to $M(\mathfrak{p}, V')$.**
- **Deduce the structural properties of $M(\mathfrak{p}, V)$ from those of $M(\mathfrak{p}, V')$ and KL-type combinatorics.**

`Encouragement 1:` **Works for Whittaker and generic Gelfand-Zetlin modules.**

`Encouragement 2:` **The categories of Harish-Chandra bimodules which appear depend only on the annihilator of V.**

`Catch 1:` **Needs better understanding of the so-called Kostant's problem for V and some induced modules.**

`Catch 2:` **Answers the irreducibility question, but does not help to describe all subquotients of GVM as this description depends on more than the annihilator of V.**

`Example:` **The Verma module $M(s\cdot 0)$ over $\mathfrak{sl}_3$ is parabolically induced from a simple Verma $\mathfrak{sl}_3$-module, say X. The module $M(s\cdot 0)$ has simple subquotients**

$$L(s\cdot 0),\quad L(st\cdot 0),\quad L(ts\cdot 0),\quad L(sts\cdot 0).$$

Let X' be a simple dense $\mathfrak{sl}_3$-module with the same annihilator as X. Then (Futorny) $M(\mathfrak{p},X')$ has only three subquotients N_1, N_2 and N_3.

Mathieu's functor can be used to associate N_1, N_2 and N_3 with $L(s\cdot 0)$, $L(st\cdot 0)$ and $L(sts\cdot 0)$ respectively.

$L(ts\cdot 0)$ is induced from a module with the annihilator, which is `‘‘strictly bigger’’` **than that of X.**

2. Kostant's problem

M — $\mathfrak{g}$**-module.**

$\mathcal{L}(M,M) = \mathrm{Hom}_{\mathbb{C}}(M,M)^{ad-fin}$ — **locally** ad $U(\mathfrak{g})$**-finite** $\mathbb{C}$**-endomorphisms of** M**.**

`Kostant's problem`: **For which (simple)** M **is the natural injection**

$$U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M) \hookrightarrow \mathcal{L}(M,M)$$

surjective?

Answer is:

- **not known in general, not even for simple highest weight modules**

- **known to be** `positive` **for Verma modules and for simple highest weight modules of the form** $L(w_0^{\mathfrak{p}} w_0 \cdot \lambda)$**,** λ **is regular and dominant (Joseph, Gabber-Joseph).**

- **known to be** `negative` **for** $L(st \cdot 0)$ **in type** B_2 **(Joseph).**

`Theorem 1.`**(M.) Let s be a set of simple roots for $\mathfrak{p}$. Then the answer to Kostant's problem is positive for the simple highest weight module of the form $L(sw_0^{\mathfrak{p}}w_0 \cdot \lambda)$ where λ is regular and dominant.**

`Example:` **For the regular block in type B_2 the answer to Kostant's problem is thus** `positive` **for $L(0)$, $L(s \cdot 0)$, $L(t \cdot 0)$, $L(sts \cdot 0)$, $L(tst \cdot 0)$ and $L(tsts \cdot 0)$; and it is** `negative` **for $L(st \cdot 0)$ and $L(ts \cdot 0)$.**

`Theorem 2.`**(M.-Stroppel) Let $\mathfrak{g} = \mathfrak{sl}_n$. Then for simple highest weight modules of the form $L(x \cdot \lambda)$ where λ is regular and dominant the answer to Kostant's problem is a left cell invariant.**

3. Why? Twisting FUNCTORS

s — **simple reflection corresponding to simple root** α

$X_{-\alpha}$ — **some non-zero element in** $\mathfrak{g}_{-\alpha}$

U_α — **localization of** $U(\mathfrak{g})$ **with respect to** $X_{-\alpha}$

Θ_α — **an automorphism of** $\mathfrak{g}$ **corresponding to** s

`Twisting functor` **(Arkhipov):**

$$\mathrm{T}_s : M \mapsto \Theta_\alpha\big(U_\alpha/U(\mathfrak{g}) \otimes_{\mathfrak{g}} M\big).$$

`Properties` **(Andersen-Stroppel, Khomenko-M.):**

- T_s **commutes with projective functors.**
- $\mathcal{R}\mathrm{T}_s$ **is an autoequivalence of** $\mathcal{D}^b(\mathcal{O}_0)$**.**
- $\mathcal{R}\mathrm{T}_s$**'s satisfy braid relations and hence define an action of the braid group on** $\mathcal{D}^b(\mathcal{O}_0)$**.**
- **The action of** $\mathcal{R}\mathrm{T}_s$**'s on** $\mathcal{D}^b(\mathcal{O}_0)$ **categorifies the left regular representation of the Weyl group.**
- $\mathrm{T}_s M(x \cdot 0) \cong M(sx \cdot 0)$ **if** $sx > x$**.**
- T_s **is left adjoint to Joseph's completion functor.**

Kostant's problem can be reduced to numerical calculations using:

- $\mathrm{Hom}_{\mathfrak{g}}(V, \mathcal{L}(M, M)) = \mathrm{Hom}_{\mathfrak{g}}(M, M \otimes V^*)$, V — **simple finite-dimensional.**
- **Annihilators of simple highest weight modules correspond bijectively to left cells.**

`Need:` $\dim \mathrm{Hom}_{\mathfrak{g}}(L(x\cdot 0), L(x\cdot 0)\otimes V^*)$ **is a left cell invariant.**

Roughly speaking the left cell is a simple S_n-module, where S_n acts via twisting functors.

Twistings commute with projective functors $\_ \otimes V^*$.

$\mathrm{T}_s L(x \cdot 0)$ **is either** 0 **(if** $sx > s$**) or has simple top** $L(x \cdot 0)$ **and semisimple radical consisting of** $L(sx \cdot 0)$ **and some other modules** $L(y \cdot 0)$**, where** x **and** y **are in the same left cell (multiplicity is given by KL-combinatorics).**

Using the properties of (derived) twisting functors one can show that

$$\dim \mathrm{Hom}_{\mathfrak{g}}(L(x\cdot 0), L(x\cdot 0)\otimes V^*) \leq \dim \mathrm{Hom}_{\mathfrak{g}}(L(y\cdot 0), L(y\cdot 0)\otimes V^*)$$

for any x, y **in the same left cell.**

4. Structure of generalized Verma modules

V — **simple $\mathfrak{a}$-module**

$\mathrm{Coker}(V)$ — **category of all modules** X **which admit resolution** $M_2 \to M_1 \to X \to 0$**, where** M_2 **and** M_1 **are direct summands of some** $E \otimes V$**,** E **finite-dimensional (Milicic-Soergel).**

`Need:` V — **projective in** $\mathrm{Coker}(V)$

For $\mathfrak{sl}_n$ **we can always substitute** V **by some** $\tilde{V}$**, which will be projective in** $\mathrm{Coker}(\tilde{V})$ **by Irving-Shelton.**

Using "parabolic Harsh-Chandra homomorphism" (Drozd-Futorny-Ovsienko) we can assume that $M(\mathfrak{p}, \tilde{V})$ **is projective in** $\mathrm{Coker}(M(\mathfrak{p}, \tilde{V}))$**.**

From the above results on Kostant's problem it follows that Kostant's problem has a positive answer for $M(\mathfrak{p}, \tilde{V})$**.**

`Corollary:` $\mathrm{Coker}(M(\mathfrak{p}, \tilde{V}))$ **is equivalent to a certain category of Harish-Chandra bimodules.**

Blocks of $\mathrm{Coker}(M(\mathfrak{p},\tilde{V}))$ **are described by** `weakly properly stratified algebras` **in the sense of Cline-Parshall-Scott and Frisk.**

This means that projectives in these categories are filtered by the so-called standard and proper standard modules, both having a clear categorical interpretation (and thus preserved by "nice" equivalences). Generalized Vermas correspond to proper standard modules.

`Catch:` **Simple objects in these categories are not simple** $\mathfrak{g}$**-modules in general.**

`Example:` $\mathfrak{g}=\mathfrak{a}=\mathfrak{sl}_2$, $V=L(s\cdot 0)$.

The corresponding block of $\mathrm{Coker}(M(\mathfrak{p},\tilde{V}))$ **is equivalent to the category of modules over the algebra** $\mathbb{C}[x]/(x^2)$**. It contains two indecomposable objects: the projective object** $P(s\cdot 0)$ **and the simple object** $\hat{L}(s\cdot 0)$**, which have the following Loewy filtrations:**

$$P(s\cdot 0)=\begin{array}{c}L(s\cdot 0)\\ L(0)\\ L(s\cdot 0)\end{array},\qquad \hat{L}(s\cdot 0)=\begin{array}{c}L(s\cdot 0)\\ L(0)\end{array},$$

There is no projective module in $\mathrm{Coker}(M(\mathfrak{p},\tilde{V}))$ **with simple top** $L(0)$**.**

This is very similar to the classical realization of eAe-modules inside A-modules for an Artin algebra A.

`Conclusion:` **There is no hope to obtain a complete description of all composition factors of $M(\mathfrak{p}, V)$ in full generality using this approach.**

On can only describe the `rough structure` **of $M(\mathfrak{p}, V)$, that is multiplicities of those simples, for which there is a projective cover in $\mathrm{Coker}(M(\mathfrak{p}, \tilde{V}))$.**

Other simples correspond to "strictly bigger annihilators".

`Theorem 3:` **(M.-Stroppel) Let L be the simple top of some projective in $\mathrm{Coker}(M(\mathfrak{p}, \tilde{V}))$ then**

$$[M(\mathfrak{p}, V) : L] = [M(\mathfrak{p}, L(\lambda)) : L(\mu)]$$

where $L(\lambda)$ is a simple highest weight module with the same annihilator as V and the weight μ can be described explicitly (the right hand side is combinatorially understood).

`Corollary:` **$M(\mathfrak{p}, V)$ is irreducible if and only if so is $M(\mathfrak{p}, L(\lambda))$.**