SCHUR–WEYL DUALITIES FOR SYMMETRIC INVERSE SEMIGROUPS

Volodymyr Mazorchuk

(Uppsala University)

joint work with Ganna Kudryavtseva (Kyiv University)

1. CLASSICAL SCHUR-WEYL DUALITY

$V = \mathbb{C}^n$ — n**-dimensional complex vector space**

$$V^{\otimes k} = \underbrace{V \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} V}_{k \text{ factors}}$$

$\mathbf{GL}(n)$ **— group of all non-degenerated complex** $n \times n$ **matrices**

$\mathbb{C}\mathbf{GL}(n)$ **— the group algebra**

$\mathbf{GL}(n)$ **acts on** $V^{\otimes k}$ **diagonally**

S_k **— the symmetric group on** $\{1, 2, \ldots, k\}$

$\mathbb{C}S_k$ **— the group algebra**

S_k **acts on** $V^{\otimes k}$ **by permuting the factors**

Schur-Weyl Duality. **The actions of** $\mathbb{C}\mathbf{GL}(n)$ **and** $\mathbb{C}S_k$ **on** $V^{\otimes k}$ **are centralizers of each other.**

2. SOME OTHER CLASSICAL SCHUR-WEYL TYPE DUALITIES

One can restrict the action of $\mathbf{GL}(n)$ on V to a subgroup $G \subset \mathbf{GL}(n)$.

If one is lucky, one could obtain a Schur-Weyl type of duality on $V^{\otimes k}$ for some algebra $X = X(G)$, which is "bigger" than $\mathbb{C}S_k$:

$$_G\left(V^{\otimes k}\right)_X$$

X — is the centralizer of the G-action;
$\mathbb{C}G$ — is the centralizer of the X-action.

Some known cases:

- **$G = \mathbf{O}(n)$, the orthogonal group, then X is the so-called *Brauer algebra* (R. Brauer).**
- **$G = S_n$, then X is the so-called *partition algebra* (V. Jones, P. Martin)**

3. SYMMETRIC INVERSE SEMIGROUP $\mathcal{IS}_n$

S_n **is the group of all bijections on** $\{1, 2, \ldots, n\}$

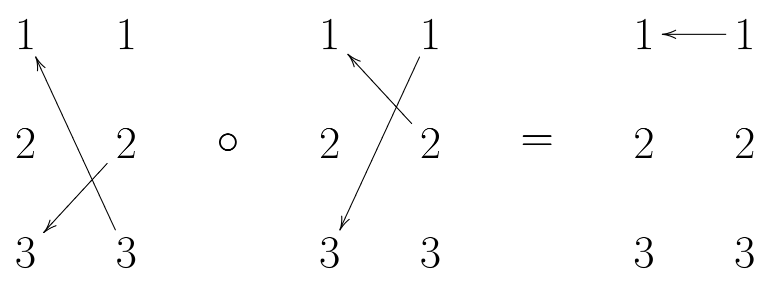
$\mathcal{IS}_n$ **is the monoid of all bijections between** ***subsets of*** $\{1, 2, \ldots, n\}$

$A, B, X, Y \subset \{1, 2, \ldots, n\}$

$f : A \to B$ **and** $g : X \to Y$ **are bijection**

$g \circ f$ **is a bijection from** $f^{-1}(B \cap X)$ **to** $g(B \cap X)$ **given by the composition of** f **and** g **whenever it makes sense to compose these maps**

Example:



$\mathcal{IS}_n$ has the *natural* faithful representation by $n \times n$ matrices over $\{0, 1\}$:

If $f \in \mathcal{IS}_n$ is a bijection from A to B then the corresponding matrix $M_f = (m_{i,j})_{i,j=1,\dots,n}$ is defined as follows:

$$m_{i,j} = \begin{cases} 1, & f(j) = i; \\ 0, & \textbf{otherwise}. \end{cases}$$

Hence $\mathcal{IS}_n$ is also called the *rook monoid*

Example:

$$\mathcal{IS}_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

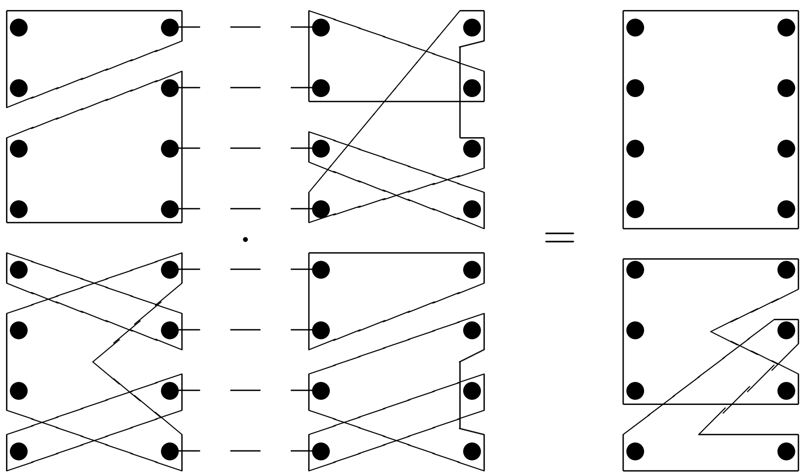
This defines on V the *natural* structure of a module over the semigroup algebra $\mathbb{C}\mathcal{IS}_n$.

4. DUAL SYMMETRIC INVERSE SEMIGROUP $\mathcal{I}_n^*$

S_n is the group of all bijections on $\{1, 2, \ldots, n\}$

$\mathcal{I}_n^*$ is the monoid of all bijections between *quotient sets of* $\{1, 2, \ldots, n\}$ (D. FitzGerald, J. Leech 1998)

Example of elements from $\mathcal{I}_8^*$ and their multiplication:



The action of $\mathcal{I}_k^*$ on $V^{\otimes k}$ is defined as follows:

$f \in \mathcal{I}_k^*$**, that is** f **is a bijection between some decompositions (disjoint unions of non-empty subsets)** $\{1, \dots, n\} = A_1 \cup A_2 \cup \cdots \cup A_k$ **and** $\{1, \dots, n\} = B_1 \cup B_2 \cup \cdots \cup B_k$

$\{e_1, e_2, \dots, e_n\}$ **— the standard basis of** V

$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ **— a basis vector of** $V^{\otimes k}$

$f(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k})$ **is defined as follows:**

- 0 **if there exists** $x \neq y$ **in some** A_s **such that** $e_{i_x} \neq e_{i_y}$
- $e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k}$**, where** $j_x = i_y$ **for all** $x \in B_s$ **and** $y \in A_t$ **such that** $f(A_t) = B_s$**.**

$\mathbb{C}\mathcal{I}_k^*$ **is a subalgebra of the Jones-Martin partition algebra and the above action is just the restriction of the action of partition algebra on** $V^{\otimes k}$

5. SCHUR-WEYL DUALITY INVOLVING $\mathcal{IS}_n$ and $\mathcal{I}_k^*$

MAIN THEOREM.

- **The action of $\mathbb{C}\mathcal{I}_k^*$ on $V^{\otimes k}$ gives the centralizer of the action of $\mathbb{C}\mathcal{IS}_n$.**
- **The action of $\mathbb{C}\mathcal{IS}_n$ on $V^{\otimes k}$ gives the centralizer of the action of $\mathbb{C}\mathcal{I}_k^*$.**
- **The representation of $\mathcal{IS}_n$ on $V^{\otimes k}$ is faithful.**
- **The representation of $\mathcal{I}_k^*$ on $V^{\otimes k}$ is faithful if and only if $n \geq 2$ or $k = 1$.**
- **The representation of $\overline{\mathbb{C}\mathcal{IS}_n}$ (the quotient modulo the zero element) on $V^{\otimes k}$ is faithful if and only if $k \geq n$.**
- **The representation of $\mathbb{C}\mathcal{I}_k^*$ on $V^{\otimes k}$ is faithful if and only if $k \leq n$.**

6. GENERALIZATIONS

$\mathbb{C}$ — **trivial** $\mathcal{IS}_n$**-module (all elements including zero act as the identity)**

$U = V \oplus \mathbb{C}$ — $\mathcal{IS}_n$**-module (follows Solomon's Schur-Weyl type dualities for** $\mathcal{IS}_n$**)**

There is a Schur-Weyl type duality for $U^{\otimes k}$

The object, which centralizes the action of $\mathcal{IS}_n$ **on** $U^{\otimes k}$ **is the *partial dual symmetric inverse semigroup*** $\mathcal{PI}_k^*$ **(G. Kudryavtseva, V. Maltcev 2006)**

Partial dual symmetric inverse semigroup consists of bijections between quotients sets of *SUBSETS* of $\{1, 2, \ldots, k\}$

Multiplication in $\mathcal{PI}_k^*$ **can be deformed to accommodated Vernitskii's semigroup defined in 2005.**