Higher representation theory

Volodymyr Mazorchuk

(Uppsala University)

Third International Symposium on "Groups, Algebras and related topics"
June 10-16, 2013, Beijing, P. R. CHINA

Elementary example: the regular representation of S_2

$S_2 = \{e, s\}, \qquad s^2 = e$

$\mathbf{H} := \{H_e, H_s\}$ — standard basis of $_{\mathbb{C}S_2}\mathbb{C}S_2$

actions: $x \cdot H_y := H_{xy}$

matrices: $[e]_{\mathbf{H}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad [s]_{\mathbf{H}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\underline{H}_e := e, \qquad \underline{H}_s = e + s, \qquad \underline{H}_s^2 = 2\underline{H}_s$

$\underline{\mathbf{H}} := \{\underline{H}_e, \underline{H}_s\}$ — new (Kazhdan-Lusztig) basis

$[e]_{\underline{\mathbf{H}}} = [\underline{H}_e]_{\underline{\mathbf{H}}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, [s]_{\underline{\mathbf{H}}} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, [\underline{H}_s]_{\underline{\mathbf{H}}} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$

Elementary example: the regular representation of S_2

$$S_2 = \{e, s\}, \qquad s^2 = e$$

$\mathbf{H} := \{H_e, H_s\}$ — standard basis of $\mathbb{C}S_2\mathbb{C}S_2$

actions: $x \cdot H_y := H_{xy}$

matrices: $[e]_{\mathbf{H}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad [s]_{\mathbf{H}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\underline{H}_e := e, \qquad \underline{H}_s = e + s, \qquad \underline{H}_s^2 = 2\underline{H}_s$

$\underline{\mathbf{H}} := \{\underline{H}_e, \underline{H}_s\}$ — new (Kazhdan-Lusztig) basis

$[e]_{\underline{\mathbf{H}}} = [\underline{H}_e]_{\underline{\mathbf{H}}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, [s]_{\underline{\mathbf{H}}} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, [\underline{H}_s]_{\underline{\mathbf{H}}} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$

Elementary example: the regular representation of S_2

$S_2 = \{e, s\}, \qquad s^2 = e$

$\mathbf{H} := \{H_e, H_s\}$ — standard basis of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$

actions: $x \cdot H_y := H_{xy}$

matrices: $[e]_{\mathbf{H}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad [s]_{\mathbf{H}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\underline{H}_e := e, \qquad \underline{H}_s = e + s, \qquad \underline{H}_s^2 = 2\underline{H}_s$

$\underline{\mathbf{H}} := \{\underline{H}_e, \underline{H}_s\}$ — new (Kazhdan-Lusztig) basis

$[e]_{\underline{\mathbf{H}}} = [\underline{H}_e]_{\underline{\mathbf{H}}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, [s]_{\underline{\mathbf{H}}} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, [\underline{H}_s]_{\underline{\mathbf{H}}} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$

Elementary example: the regular representation of S_2

$S_2 = \{e, s\}, \qquad s^2 = e$

$\mathbf{H} := \{H_e, H_s\}$ — standard basis of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$

actions: $x \cdot H_y := H_{xy}$

matrices: $[e]_{\mathbf{H}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad [s]_{\mathbf{H}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\underline{H}_e := e, \qquad \underline{H}_s = e + s, \qquad \underline{H}_s^2 = 2\underline{H}_s$

$\underline{\mathbf{H}} := \{\underline{H}_e, \underline{H}_s\}$ — new (Kazhdan-Lusztig) basis

$[e]_{\underline{\mathbf{H}}} = [\underline{H}_e]_{\underline{\mathbf{H}}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, [s]_{\underline{\mathbf{H}}} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, [\underline{H}_s]_{\underline{\mathbf{H}}} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$

Elementary example: the regular representation of S_2

$S_2 = \{e, s\}, \qquad s^2 = e$

$\mathbf{H} := \{H_e, H_s\}$ — standard basis of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$

actions: $x \cdot H_y := H_{xy}$

matrices: $[e]_{\mathbf{H}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad [s]_{\mathbf{H}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\underline{H}_e := e, \qquad \underline{H}_s = e + s, \qquad \underline{H}_s^2 = 2\underline{H}_s$

$\underline{\mathbf{H}} := \{\underline{H}_e, \underline{H}_s\}$ — new (Kazhdan-Lusztig) basis

$[e]_{\underline{\mathbf{H}}} = [\underline{H}_e]_{\underline{\mathbf{H}}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, [s]_{\underline{\mathbf{H}}} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, [\underline{H}_s]_{\underline{\mathbf{H}}} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$

Elementary example: the regular representation of S_2

$S_2 = \{e, s\}, \qquad s^2 = e$

$\mathbf{H} := \{H_e, H_s\}$ — standard basis of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$

actions: $x \cdot H_y := H_{xy}$

matrices: $[e]_{\mathbf{H}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad [s]_{\mathbf{H}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\underline{H}_e := e, \qquad \underline{H}_s = e + s, \qquad \underline{H}_s^2 = 2\underline{H}_s$

$\underline{\mathbf{H}} := \{\underline{H}_e, \underline{H}_s\}$ — new (Kazhdan-Lusztig) basis

$[e]_{\underline{\mathbf{H}}} = [\underline{H}_e]_{\underline{\mathbf{H}}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, [s]_{\underline{\mathbf{H}}} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, [\underline{H}_s]_{\underline{\mathbf{H}}} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$

Elementary example: the regular representation of S_2

$S_2 = \{e, s\}, \qquad s^2 = e$

$\mathbf{H} := \{H_e, H_s\}$ — standard basis of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$

actions: $x \cdot H_y := H_{xy}$

matrices: $[e]_{\mathbf{H}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad [s]_{\mathbf{H}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\underline{H}_e := e, \qquad \underline{H}_s = e + s, \qquad \underline{H}_s^2 = 2\underline{H}_s$

$\underline{\mathbf{H}} := \{\underline{H}_e, \underline{H}_s\}$ — new (Kazhdan-Lusztig) basis

$[e]_{\underline{\mathbf{H}}} = [\underline{H}_e]_{\underline{\mathbf{H}}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, [s]_{\underline{\mathbf{H}}} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, [\underline{H}_s]_{\underline{\mathbf{H}}} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$

Elementary example: the regular representation of S_2

$S_2 = \{e, s\}, \qquad s^2 = e$

$\mathbf{H} := \{H_e, H_s\}$ — standard basis of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$

actions: $x \cdot H_y := H_{xy}$

matrices: $[e]_{\mathbf{H}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad [s]_{\mathbf{H}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\underline{H}_e := e, \qquad \underline{H}_s = e + s, \qquad \underline{H}_s^2 = 2\underline{H}_s$

$\underline{\mathbf{H}} := \{\underline{H}_e, \underline{H}_s\}$ — new (Kazhdan-Lusztig) basis

$$[e]_{\underline{\mathbf{H}}} = [\underline{H}_e]_{\underline{\mathbf{H}}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ [s]_{\underline{\mathbf{H}}} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \ [\underline{H}_s]_{\underline{\mathbf{H}}} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

Elementary example: the regular representation of S_2

$S_2 = \{e, s\}, \qquad s^2 = e$

$\mathbf{H} := \{H_e, H_s\}$ — standard basis of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$

actions: $x \cdot H_y := H_{xy}$

matrices: $[e]_{\mathbf{H}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad [s]_{\mathbf{H}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\underline{H}_e := e, \qquad \underline{H}_s = e + s, \qquad \underline{H}_s^2 = 2\underline{H}_s$

$\underline{\mathbf{H}} := \{\underline{H}_e, \underline{H}_s\}$ — new (Kazhdan-Lusztig) basis

$[e]_{\underline{\mathbf{H}}} = [\underline{H}_e]_{\underline{\mathbf{H}}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $[s]_{\underline{\mathbf{H}}} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, $[\underline{H}_s]_{\underline{\mathbf{H}}} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 1

$D := \mathbb{C}[x]/(x^2)$ — the algebra of dual numbers

$\mathcal{C} := D\text{-mod}$

$B := D \otimes_{\mathbb{C}} D \in D\text{-mod-}D$

$F_e := \mathrm{Id} : \mathcal{C} \to \mathcal{C}$, $\qquad$ $F_s := B \otimes_D {}_- : \mathcal{C} \to \mathcal{C}$

$F_s \circ F_s \cong F_s \oplus F_s$

$\mathcal{F} := \mathrm{add}(F_e, F_s)$ — tensor category

$V := [\mathcal{C}]_\oplus$ — split Grothendieck group

basis $\mathbf{Q}$: $[\mathbb{C}]$ (class of simple) and $[D]$ (class of indec. projective)

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 1

$D := \mathbb{C}[x]/(x^2)$ — the algebra of dual numbers

$\mathcal{C} := D\text{-mod}$

$B := D \otimes_{\mathbb{C}} D \in D\text{-mod-}D$

$F_e := \mathrm{Id} : \mathcal{C} \to \mathcal{C}, \qquad F_s := B \otimes_D {}_- : \mathcal{C} \to \mathcal{C}$

$F_s \circ F_s \cong F_s \oplus F_s$

$\mathcal{F} := \mathrm{add}(F_e, F_s)$ — tensor category

$V := [\mathcal{C}]_{\oplus}$ — split Grothendieck group

basis $\mathbf{Q}$: $[\mathbb{C}]$ (class of simple) and $[D]$ (class of indec. projective)

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 1

$D := \mathbb{C}[x]/(x^2)$ — the algebra of dual numbers

$\mathcal{C} := D\text{-mod}$

$B := D \otimes_{\mathbb{C}} D \in D\text{-mod-}D$

$F_e := \mathrm{Id} : \mathcal{C} \to \mathcal{C}$, $\qquad F_s := B \otimes_D - : \mathcal{C} \to \mathcal{C}$

$F_s \circ F_s \cong F_s \oplus F_s$

$\mathcal{F} := \mathrm{add}(F_e, F_s)$ — tensor category

$V := [\mathcal{C}]_{\oplus}$ — split Grothendieck group

basis $\mathbf{Q}$: $[\mathbb{C}]$ (class of simple) and $[D]$ (class of indec. projective)

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 1

$D := \mathbb{C}[x]/(x^2)$ — the algebra of dual numbers

$\mathcal{C} := D\text{-mod}$

$B := D \otimes_{\mathbb{C}} D \in D\text{-mod-}D$

$F_e := \mathrm{Id} : \mathcal{C} \to \mathcal{C}, \qquad F_s := B \otimes_D {}_- : \mathcal{C} \to \mathcal{C}$

$F_s \circ F_s \cong F_s \oplus F_s$

$\mathcal{F} := \mathrm{add}(F_e, F_s)$ — tensor category

$V := [\mathcal{C}]_\oplus$ — split Grothendieck group

basis $\mathbf{Q}$: $[\mathbb{C}]$ (class of simple) and $[D]$ (class of indec. projective)

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 1

$D := \mathbb{C}[x]/(x^2)$ — the algebra of dual numbers

$\mathcal{C} := D\text{-}\mathrm{mod}$

$B := D \otimes_{\mathbb{C}} D \in D\text{-}\mathrm{mod}\text{-}D$

$F_e := \mathrm{Id} : \mathcal{C} \to \mathcal{C}, \qquad F_s := B \otimes_D {}_- : \mathcal{C} \to \mathcal{C}$

$F_s \circ F_s \cong F_s \oplus F_s$

$\mathcal{F} := \mathrm{add}(F_e, F_s)$ — tensor category

$V := [\mathcal{C}]_\oplus$ — split Grothendieck group

basis $\mathbf{Q}$: $[\mathbb{C}]$ (class of simple) and $[D]$ (class of indec. projective)

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 1

$D := \mathbb{C}[x]/(x^2)$ — the algebra of dual numbers

$\mathcal{C} := D\text{-mod}$

$B := D \otimes_{\mathbb{C}} D \in D\text{-mod-}D$

$F_e := \mathrm{Id} : \mathcal{C} \to \mathcal{C}$, $\quad F_s := B \otimes_D {}_- : \mathcal{C} \to \mathcal{C}$

$F_s \circ F_s \cong F_s \oplus F_s$

$\mathcal{F} := \mathrm{add}(F_e, F_s)$ — tensor category

$V := [\mathcal{C}]_{\oplus}$ — split Grothendieck group

basis $\mathbf{Q}$: $[\mathbb{C}]$ (class of simple) and $[D]$ (class of indec. projective)

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 1

$D := \mathbb{C}[x]/(x^2)$ — the algebra of dual numbers

$\mathcal{C} := D\text{-mod}$

$B := D \otimes_{\mathbb{C}} D \in D\text{-mod-}D$

$F_e := \mathrm{Id} : \mathcal{C} \to \mathcal{C}$, $\qquad F_s := B \otimes_D {}_- : \mathcal{C} \to \mathcal{C}$

$F_s \circ F_s \cong F_s \oplus F_s$

$\mathcal{F} := \mathrm{add}(F_e, F_s)$ — tensor category

$V := [\mathcal{C}]_\oplus$ — split Grothendieck group

basis $\mathbf{Q}$: $[\mathbb{C}]$ (class of simple) and $[D]$ (class of indec. projective)

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 1

$D := \mathbb{C}[x]/(x^2)$ — the algebra of dual numbers

$\mathcal{C} := D\text{-mod}$

$B := D \otimes_{\mathbb{C}} D \in D\text{-mod-}D$

$F_e := \mathrm{Id} : \mathcal{C} \to \mathcal{C}$, $\qquad F_s := B \otimes_D {}_{-} : \mathcal{C} \to \mathcal{C}$

$F_s \circ F_s \cong F_s \oplus F_s$

$\mathcal{F} := \mathrm{add}(F_e, F_s)$ — tensor category

$V := [\mathcal{C}]_{\oplus}$ — split Grothendieck group

basis $\mathbf{Q}$: $[\mathbb{C}]$ (class of simple) and $[D]$ (class of indec. projective)

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 1

$D := \mathbb{C}[x]/(x^2)$ — the algebra of dual numbers

$\mathcal{C} := D\text{-mod}$

$B := D \otimes_{\mathbb{C}} D \in D\text{-mod-}D$

$F_e := \mathrm{Id} : \mathcal{C} \to \mathcal{C}$, $\qquad F_s := B \otimes_D {}_- : \mathcal{C} \to \mathcal{C}$

$F_s \circ F_s \cong F_s \oplus F_s$

$\mathcal{F} := \mathrm{add}(F_e, F_s)$ — tensor category

$V := [\mathcal{C}]_{\oplus}$ — split Grothendieck group

basis $\mathbf{Q}$: $[\mathbb{C}]$ (class of simple) and $[D]$ (class of indec. projective)

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 1

$D := \mathbb{C}[x]/(x^2)$ — the algebra of dual numbers

$\mathcal{C} := D\text{-mod}$

$B := D \otimes_{\mathbb{C}} D \in D\text{-mod-}D$

$F_e := \mathrm{Id} : \mathcal{C} \to \mathcal{C}, \qquad F_s := B \otimes_D {}_- : \mathcal{C} \to \mathcal{C}$

$F_s \circ F_s \cong F_s \oplus F_s$

$\mathcal{F} := \mathrm{add}(F_e, F_s)$ — tensor category

$V := [\mathcal{C}]_\oplus$ — split Grothendieck group

basis $\mathbf{Q}$: $[\mathbb{C}]$ (class of simple) and $[D]$ (class of indec. projective)

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 2

$A := [\mathcal{F}]_\oplus$ — split Grothendieck group

basis: $[F_e]$ and $[F_s]$

$\mathbb{C} \otimes_\mathbb{Z} A$ acts on $\mathbb{C} \otimes_\mathbb{Z} V$

$[[F_e]]_\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $[[F_s]]_\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$

This is a categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$ in the KL basis

Note: This is not an action of S_2 and I do not know whether there is any reasonable action of S_2 around (but there **is** an action of the braid group B_2 on a certain derived category)

This generalizes to all finite Coxeter groups

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 2

$A := [\mathcal{F}]_{\oplus}$ — split Grothendieck group

basis: $[F_e]$ and $[F_s]$

$\mathbb{C} \otimes_{\mathbb{Z}} A$ acts on $\mathbb{C} \otimes_{\mathbb{Z}} V$

$[[F_e]]_{\mathbf{Q}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $[[F_s]]_{\mathbf{Q}} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$

This is a categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$ in the KL basis

Note: This is not an action of S_2 and I do not know whether there is any reasonable action of S_2 around (but there **is** an action of the braid group B_2 on a certain derived category)

This generalizes to all finite Coxeter groups

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 2

$A := [\mathcal{F}]_{\oplus}$ — split Grothendieck group

basis: $[F_e]$ and $[F_s]$

$\mathbb{C} \otimes_{\mathbb{Z}} A$ acts on $\mathbb{C} \otimes_{\mathbb{Z}} V$

$[[F_e]]_{\mathbf{Q}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $[[F_s]]_{\mathbf{Q}} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$

This is a categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$ in the KL basis

Note: This is not an action of S_2 and I do not know whether there is any reasonable action of S_2 around (but there **is** an action of the braid group B_2 on a certain derived category)

This generalizes to all finite Coxeter groups

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 2

$A := [\mathcal{F}]_{\oplus}$ — split Grothendieck group

basis: $[F_e]$ and $[F_s]$

$\mathbb{C} \otimes_{\mathbb{Z}} A$ acts on $\mathbb{C} \otimes_{\mathbb{Z}} V$

$[[F_e]]_{\mathbf{Q}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $[[F_s]]_{\mathbf{Q}} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$

This is a categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$ in the KL basis

Note: This is not an action of S_2 and I do not know whether there is any reasonable action of S_2 around (but there **is** an action of the braid group B_2 on a certain derived category)

This generalizes to all finite Coxeter groups

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 2

$A := [\mathcal{F}]_\oplus$ — split Grothendieck group

basis: $[F_e]$ and $[F_s]$

$\mathbb{C} \otimes_\mathbb{Z} A$ acts on $\mathbb{C} \otimes_\mathbb{Z} V$

$$[[F_e]]_\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } [[F_s]]_\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

This is a categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$ in the KL basis

Note: This is not an action of S_2 and I do not know whether there is any reasonable action of S_2 around (but there **is** an action of the braid group B_2 on a certain derived category)

This generalizes to all finite Coxeter groups

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 2

$A := [\mathcal{F}]_{\oplus}$ — split Grothendieck group

basis: $[F_e]$ and $[F_s]$

$\mathbb{C} \otimes_{\mathbb{Z}} A$ acts on $\mathbb{C} \otimes_{\mathbb{Z}} V$

$$[[F_e]]_{\mathbf{Q}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } [[F_s]]_{\mathbf{Q}} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

This is a categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$ in the KL basis

Note: This is not an action of S_2 and I do not know whether there is any reasonable action of S_2 around (but there **is** an action of the braid group B_2 on a certain derived category)

This generalizes to all finite Coxeter groups

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 2

$A := [\mathcal{F}]_\oplus$ — split Grothendieck group

basis: $[F_e]$ and $[F_s]$

$\mathbb{C} \otimes_\mathbb{Z} A$ acts on $\mathbb{C} \otimes_\mathbb{Z} V$

$$[[F_e]]_\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } [[F_s]]_\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

This is a categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$ in the KL basis

Note: This is not an action of S_2 and I do not know whether there is any reasonable action of S_2 around (but there **is** an action of the braid group B_2 on a certain derived category)

This generalizes to all finite Coxeter groups

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 2

$A := [\mathcal{F}]_\oplus$ — split Grothendieck group

basis: $[F_e]$ and $[F_s]$

$\mathbb{C} \otimes_\mathbb{Z} A$ acts on $\mathbb{C} \otimes_\mathbb{Z} V$

$$[[F_e]]_\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } [[F_s]]_\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

This is a categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$ in the KL basis

Note: This is not an action of S_2 and I do not know whether there is any reasonable action of S_2 around (but there **is** an action of the braid group B_2 on a certain derived category)

This generalizes to all finite Coxeter groups

Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 2

$A := [\mathcal{F}]_{\oplus}$ — split Grothendieck group

basis: $[F_e]$ and $[F_s]$

$\mathbb{C} \otimes_{\mathbb{Z}} A$ acts on $\mathbb{C} \otimes_{\mathbb{Z}} V$

$$[[F_e]]_{\mathbf{Q}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } [[F_s]]_{\mathbf{Q}} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

This is a categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$ in the KL basis

Note: This is not an action of S_2 and I do not know whether there is any reasonable action of S_2 around (but there **is** an action of the braid group B_2 on a certain derived category)

This generalizes to all finite Coxeter groups

Categorification

What is categorification?

"Upgrade" of set-theoretic notions to category theoretic

What happened in the above example?

a vector space became a category

a function (linear map) became a functor

an algebra became a tensor category (or a 2-category)

in the same spirit:

an equality of functions becomes an isomorphism of functors

Categorification

What is categorification?

"Upgrade" of set-theoretic notions to category theoretic

What happened in the above example?

a vector space became a category

a function (linear map) became a functor

an algebra became a tensor category (or a 2-category)

in the same spirit:

an equality of functions becomes an isomorphism of functors

Categorification

What is categorification?

“Upgrade” of set-theoretic notions to category theoretic

What happened in the above example?

a vector space became a category

a function (linear map) became a functor

an algebra became a tensor category (or a 2-category)

in the same spirit:

an equality of functions becomes an isomorphism of functors

Categorification

What is categorification?

“Upgrade” of set-theoretic notions to category theoretic

What happened in the above example?

a vector space became a category

a function (linear map) became a functor

an algebra became a tensor category (or a 2-category)

in the same spirit:

an equality of functions becomes an isomorphism of functors

Categorification

What is categorification?

“Upgrade” of set-theoretic notions to category theoretic

What happened in the above example?

a vector space became a category

a function (linear map) became a functor

an algebra became a tensor category (or a 2-category)

in the same spirit:

an equality of functions becomes an isomorphism of functors

Categorification

What is categorification?

"Upgrade" of set-theoretic notions to category theoretic

What happened in the above example?

a vector space became a category

a function (linear map) became a functor

an algebra became a tensor category (or a 2-category)

in the same spirit:

an equality of functions becomes an isomorphism of functors

Categorification

What is categorification?

"Upgrade" of set-theoretic notions to category theoretic

What happened in the above example?

a vector space became a category

a function (linear map) became a functor

an algebra became a tensor category (or a 2-category)

in the same spirit:

an equality of functions becomes an isomorphism of functors

Categorification

What is categorification?

"Upgrade" of set-theoretic notions to category theoretic

What happened in the above example?

a vector space became a category

a function (linear map) became a functor

an algebra became a tensor category (or a 2-category)

in the same spirit:

an equality of functions becomes an isomorphism of functors

Categorification

What is categorification?

"Upgrade" of set-theoretic notions to category theoretic

What happened in the above example?

a vector space became a category

a function (linear map) became a functor

an algebra became a tensor category (or a 2-category)

in the same spirit:

an equality of functions becomes an isomorphism of functors

Categorification

What is categorification?

"Upgrade" of set-theoretic notions to category theoretic

What happened in the above example?

a vector space became a category

a function (linear map) became a functor

an algebra became a tensor category (or a 2-category)

in the same spirit:

an equality of functions becomes an isomorphism of functors

Example: Khovanov homology

Jones polynomial

$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{Jones polynomial}}$

Khovanov homology (Khovanov)

$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{chain complex}}$

$\boxed{\text{chain complex}} \xrightarrow{\text{homol.}} \boxed{\text{Khovanov hom.}} \xrightarrow{\text{Euler char.}} \boxed{\text{Jones polynomial}}$

Advantage: stronger knot invariant

Example: Khovanov homology

Jones polynomial

$$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{Jones polynomial}}$$

Khovanov homology (Khovanov)

$$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{chain complex}}$$

$$\boxed{\text{chain complex}} \xrightarrow{\text{homol.}} \boxed{\text{Khovanov hom.}} \xrightarrow{\text{Euler char.}} \boxed{\text{Jones polynomial}}$$

Advantage: stronger knot invariant

Example: Khovanov homology

Jones polynomial

$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{Jones polynomial}}$

Khovanov homology (Khovanov)

$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{chain complex}}$

$\boxed{\text{chain complex}} \xrightarrow{\text{homol.}} \boxed{\text{Khovanov hom.}} \xrightarrow{\text{Euler char.}} \boxed{\text{Jones polynomial}}$

Advantage: stronger knot invariant

Example: Khovanov homology

Jones polynomial

$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{Jones polynomial}}$

Khovanov homology (Khovanov)

$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{chain complex}}$

$\boxed{\text{chain complex}} \xrightarrow{\text{homol.}} \boxed{\text{Khovanov hom.}} \xrightarrow{\text{Euler char.}} \boxed{\text{Jones polynomial}}$

Advantage: stronger knot invariant

Example: Khovanov homology

Jones polynomial

$$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{Jones polynomial}}$$

Khovanov homology (Khovanov)

$$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{chain complex}}$$

$$\boxed{\text{chain complex}} \xrightarrow{\text{homol.}} \boxed{\text{Khovanov hom.}} \xrightarrow{\text{Euler char.}} \boxed{\text{Jones polynomial}}$$

Advantage: stronger knot invariant

Example: Khovanov homology

Jones polynomial

$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{Jones polynomial}}$

Khovanov homology (Khovanov)

$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{chain complex}}$

$\boxed{\text{chain complex}} \xrightarrow{\text{homol.}} \boxed{\text{Khovanov hom.}} \xrightarrow{\text{Euler char.}} \boxed{\text{Jones polynomial}}$

Advantage: stronger knot invariant

Example: Khovanov homology

Jones polynomial

$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{Jones polynomial}}$

Khovanov homology (Khovanov)

$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{chain complex}}$

$\boxed{\text{chain complex}} \xrightarrow{\text{homol.}} \boxed{\text{Khovanov hom.}} \xrightarrow{\text{Euler char.}} \boxed{\text{Jones polynomial}}$

Advantage: stronger knot invariant

Example: Khovanov homology

Jones polynomial

$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{Jones polynomial}}$

Khovanov homology (Khovanov)

$\boxed{\text{Knot}} \xrightarrow{\text{projection}} \boxed{\text{Knot diagram}} \xrightarrow{\text{combinatorial procedure}} \boxed{\text{chain complex}}$

$\boxed{\text{chain complex}} \xrightarrow{\text{homol.}} \boxed{\text{Khovanov hom.}} \xrightarrow{\text{Euler char.}} \boxed{\text{Jones polynomial}}$

Advantage: stronger knot invariant

Example: Khovanov homology - an alternative approach

A standard approach to knot invariants

$\boxed{\text{Knot}} \xrightarrow{\text{Alexander Theorem}} \boxed{\text{Closure of a braid}}$

$\boxed{\text{Braid invariants}} \xrightarrow{\text{"Markov moves"}} \boxed{\text{Knot invariants}}$

Invariants of reps. of the braid group B_n give rise to knot invariants

An alternative approach to Khovanov homology
(Bernstein, Frenkel, Khovanov, Stroppel, Brundan)

$\boxed{\text{Functorial action of } B_n \text{ on category } \mathcal{O}}$

|

value on the dominant (parabolic) Verma module

↓

$\boxed{\text{Khovanov hom.}}$

Example: Khovanov homology - an alternative approach

A standard approach to knot invariants

$\boxed{\text{Knot}} \xrightarrow{\text{Alexander Theorem}} \boxed{\text{Closure of a braid}}$

$\boxed{\text{Braid invariants}} \xrightarrow{\text{“Markov moves”}} \boxed{\text{Knot invariants}}$

Invariants of reps. of the braid group B_n give rise to knot invariants

An alternative approach to Khovanov homology
(Bernstein, Frenkel, Khovanov, Stroppel, Brundan)

$\boxed{\text{Functorial action of } B_n \text{ on category } \mathcal{O}}$

|

value on the dominant (parabolic) Verma module

$\downarrow$

$\boxed{\text{Khovanov hom.}}$

Example: Khovanov homology - an alternative approach

A standard approach to knot invariants

$$\boxed{\text{Knot}} \xrightarrow{\text{Alexander Theorem}} \boxed{\text{Closure of a braid}}$$

$$\boxed{\text{Braid invariants}} \xrightarrow{\text{“Markov moves”}} \boxed{\text{Knot invariants}}$$

Invariants of reps. of the braid group B_n give rise to knot invariants

An alternative approach to Khovanov homology
(Bernstein, Frenkel, Khovanov, Stroppel, Brundan)

$$\boxed{\text{Functorial action of } B_n \text{ on category } \mathcal{O}}$$
$$\downarrow$$
$$\text{value on the dominant (parabolic) Verma module}$$
$$\downarrow$$
$$\boxed{\text{Khovanov hom.}}$$

Example: Khovanov homology - an alternative approach

A standard approach to knot invariants

$\boxed{\text{Knot}} \xrightarrow{\text{Alexander Theorem}} \boxed{\text{Closure of a braid}}$

$\boxed{\text{Braid invariants}} \xrightarrow{\text{“Markov moves”}} \boxed{\text{Knot invariants}}$

Invariants of reps. of the braid group B_n give rise to knot invariants

An alternative approach to Khovanov homology
(Bernstein, Frenkel, Khovanov, Stroppel, Brundan)

$$\begin{array}{c}\boxed{\text{Functorial action of } B_n \text{ on category } \mathcal{O}}\\ \mid \\ \text{value on the dominant (parabolic) Verma module}\\ \downarrow \\ \boxed{\text{Khovanov hom.}}\end{array}$$

Example: Khovanov homology - an alternative approach

A standard approach to knot invariants

$$\boxed{\text{Knot}} \xrightarrow{\text{Alexander Theorem}} \boxed{\text{Closure of a braid}}$$

$$\boxed{\text{Braid invariants}} \xrightarrow{\text{``Markov moves''}} \boxed{\text{Knot invariants}}$$

Invariants of reps. of the braid group B_n give rise to knot invariants

An alternative approach to Khovanov homology
(Bernstein, Frenkel, Khovanov, Stroppel, Brundan)

$$\boxed{\text{Functorial action of } B_n \text{ on category } \mathcal{O}}$$

value on the dominant (parabolic) Verma module

$$\downarrow$$

$$\boxed{\text{Khovanov hom.}}$$

Example: Khovanov homology - an alternative approach

A standard approach to knot invariants

$\boxed{\text{Knot}} \xrightarrow{\text{Alexander Theorem}} \boxed{\text{Closure of a braid}}$

$\boxed{\text{Braid invariants}} \xrightarrow{\text{``Markov moves''}} \boxed{\text{Knot invariants}}$

Invariants of reps. of the braid group B_n give rise to knot invariants

An alternative approach to Khovanov homology
(Bernstein, Frenkel, Khovanov, Stroppel, Brundan)

$\boxed{\text{Functorial action of } B_n \text{ on category } \mathcal{O}}$

value on the dominant (parabolic) Verma module

$\downarrow$

$\boxed{\text{Khovanov hom.}}$

Example: Khovanov homology - an alternative approach

A standard approach to knot invariants

$\boxed{\text{Knot}} \xrightarrow{\text{Alexander Theorem}} \boxed{\text{Closure of a braid}}$

$\boxed{\text{Braid invariants}} \xrightarrow{\text{“Markov moves”}} \boxed{\text{Knot invariants}}$

Invariants of reps. of the braid group B_n give rise to knot invariants

An alternative approach to Khovanov homology
(Bernstein, Frenkel, Khovanov, Stroppel, Brundan)

$\boxed{\text{Functorial action of } B_n \text{ on category } \mathcal{O}}$

value on the dominant (parabolic) Verma module

$\downarrow$

$\boxed{\text{Khovanov hom.}}$

Example: Khovanov homology - an alternative approach

A standard approach to knot invariants

$\boxed{\text{Knot}} \xrightarrow{\text{Alexander Theorem}} \boxed{\text{Closure of a braid}}$

$\boxed{\text{Braid invariants}} \xrightarrow{\text{“Markov moves”}} \boxed{\text{Knot invariants}}$

Invariants of reps. of the braid group B_n give rise to knot invariants

An alternative approach to Khovanov homology
(Bernstein, Frenkel, Khovanov, Stroppel, Brundan)

$\boxed{\text{Functorial action of } B_n \text{ on category } \mathcal{O}}$

value on the dominant (parabolic) Verma module

$\downarrow$

$\boxed{\text{Khovanov hom.}}$

2-categories: definition

Definition. A 2-category is a category enriched over the monoidal category **Cat** of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category $\mathscr{C}$ is given by the following data:

- objects of $\mathscr{C}$;
- small categories $\mathscr{C}(\mathtt{i},\mathtt{j})$ of morphisms;
- bifunctorial composition $\mathscr{C}(\mathtt{j},\mathtt{k}) \times \mathscr{C}(\mathtt{i},\mathtt{j}) \to \mathscr{C}(\mathtt{i},\mathtt{k})$;
- identity objects $\mathbb{1}_{\mathtt{j}}$;

which are subject to the obvious set of (strict) axioms.

2-categories: definition

Definition. A 2-category is a category enriched over the monoidal category **Cat** of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category $\mathscr{C}$ is given by the following data:

- objects of $\mathscr{C}$;
- small categories $\mathscr{C}(\mathtt{i},\mathtt{j})$ of morphisms;
- bifunctorial composition $\mathscr{C}(\mathtt{j},\mathtt{k}) \times \mathscr{C}(\mathtt{i},\mathtt{j}) \to \mathscr{C}(\mathtt{i},\mathtt{k})$;
- identity objects $\mathbb{1}_{\mathtt{j}}$;

which are subject to the obvious set of (strict) axioms.

2-categories: definition

Definition. A 2-category is a category enriched over the monoidal category **Cat** of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category $\mathscr{C}$ is given by the following data:

- objects of $\mathscr{C}$;
- small categories $\mathscr{C}(\mathtt{i},\mathtt{j})$ of morphisms;
- bifunctorial composition $\mathscr{C}(\mathtt{j},\mathtt{k}) \times \mathscr{C}(\mathtt{i},\mathtt{j}) \to \mathscr{C}(\mathtt{i},\mathtt{k})$;
- identity objects $\mathbb{1}_{\mathtt{j}}$;

which are subject to the obvious set of (strict) axioms.

2-categories: definition

Definition. A 2-category is a category enriched over the monoidal category **Cat** of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category $\mathscr{C}$ is given by the following data:

- objects of $\mathscr{C}$;
- small categories $\mathscr{C}(\mathtt{i},\mathtt{j})$ of morphisms;
- bifunctorial composition $\mathscr{C}(\mathtt{j},\mathtt{k}) \times \mathscr{C}(\mathtt{i},\mathtt{j}) \to \mathscr{C}(\mathtt{i},\mathtt{k})$;
- identity objects $\mathbb{1}_{\mathtt{j}}$;

which are subject to the obvious set of (strict) axioms.

2-categories: definition

Definition. A 2-category is a category enriched over the monoidal category **Cat** of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category $\mathscr{C}$ is given by the following data:

- objects of $\mathscr{C}$;
- small categories $\mathscr{C}(\mathtt{i}, \mathtt{j})$ of morphisms;
- bifunctorial composition $\mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$;
- identity objects $\mathbb{1}_{\mathtt{j}}$;

which are subject to the obvious set of (strict) axioms.

2-categories: definition

Definition. A 2-category is a category enriched over the monoidal category **Cat** of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category $\mathscr{C}$ is given by the following data:

- objects of $\mathscr{C}$;
- small categories $\mathscr{C}(\mathtt{i}, \mathtt{j})$ of morphisms;
- bifunctorial composition $\mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$;
- identity objects $\mathbb{1}_{\mathtt{j}}$;

which are subject to the obvious set of (strict) axioms.

Definition. A 2-category is a category enriched over the monoidal category **Cat** of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category $\mathscr{C}$ is given by the following data:

- objects of $\mathscr{C}$;
- small categories $\mathscr{C}(\mathtt{i},\mathtt{j})$ of morphisms;
- bifunctorial composition $\mathscr{C}(\mathtt{j},\mathtt{k})\times\mathscr{C}(\mathtt{i},\mathtt{j})\to\mathscr{C}(\mathtt{i},\mathtt{k})$;
- identity objects $\mathbb{1}_{\mathtt{j}}$;

which are subject to the obvious set of (strict) axioms.

2-categories: definition

Definition. A 2-category is a category enriched over the monoidal category **Cat** of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category $\mathscr{C}$ is given by the following data:

- objects of $\mathscr{C}$;
- small categories $\mathscr{C}(\mathtt{i},\mathtt{j})$ of morphisms;
- bifunctorial composition $\mathscr{C}(\mathtt{j},\mathtt{k}) \times \mathscr{C}(\mathtt{i},\mathtt{j}) \to \mathscr{C}(\mathtt{i},\mathtt{k})$;
- identity objects $\mathbb{1}_{\mathtt{j}}$;

which are subject to the obvious set of (strict) axioms.

2-categories: definition

Definition. A 2-category is a category enriched over the monoidal category **Cat** of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category $\mathscr{C}$ is given by the following data:

- objects of $\mathscr{C}$;
- small categories $\mathscr{C}(\mathtt{i},\mathtt{j})$ of morphisms;
- bifunctorial composition $\mathscr{C}(\mathtt{j},\mathtt{k}) \times \mathscr{C}(\mathtt{i},\mathtt{j}) \to \mathscr{C}(\mathtt{i},\mathtt{k})$;
- identity objects $\mathbb{1}_{\mathtt{j}}$;

which are subject to the obvious set of (strict) axioms.

2-categories: definition

Definition. A 2-category is a category enriched over the monoidal category **Cat** of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category $\mathscr{C}$ is given by the following data:

- objects of $\mathscr{C}$;
- small categories $\mathscr{C}(\mathtt{i},\mathtt{j})$ of morphisms;
- bifunctorial composition $\mathscr{C}(\mathtt{j},\mathtt{k}) \times \mathscr{C}(\mathtt{i},\mathtt{j}) \to \mathscr{C}(\mathtt{i},\mathtt{k})$;
- identity objects $\mathbb{1}_{\mathtt{j}}$;

which are subject to the obvious set of (strict) axioms.

2-categories: terminology and the prinicpal example

Terminology.

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
- Identity 1-morphisms are the identity functors.

2-categories: terminology and the prinicpal example

Terminology.

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
- Identity 1-morphisms are the identity functors.

Terminology.

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
- Identity 1-morphisms are the identity functors.

2-categories: terminology and the prinicpal example

Terminology.

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
- Identity 1-morphisms are the identity functors.

Terminology.

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
- Identity 1-morphisms are the identity functors.

2-categories: terminology and the prinicpal example

Terminology.

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
- Identity 1-morphisms are the identity functors.

Terminology.

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
- Identity 1-morphisms are the identity functors.

2-categories: terminology and the prinicpal example

Terminology.

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
- Identity 1-morphisms are the identity functors.

2-categories: terminology and the prinicpal example

Terminology.

- An object in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
- Identity 1-morphisms are the identity functors.

2-categories: terminology and the prinicpal example

Terminology.

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
- Identity 1-morphisms are the identity functors.

2-categories: terminology and the prinicpal example

Terminology.

- An object in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
- Identity 1-morphisms are the identity functors.

2-categories: terminology and the prinicpal example

Terminology.

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
- Identity 1-morphisms are the identity functors.

2-categories: terminology and the prinicpal example

Terminology.

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
- Identity 1-morphisms are the identity functors.

2-categories: terminology and the prinicpal example

Terminology.

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
- Identity 1-morphisms are the identity functors.

2-categories: terminology and the prinicpal example

Terminology.

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 1-morphism of $\mathscr{C}$.
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 2-morphism of $\mathscr{C}$.
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called vertical and denoted $\circ_1$.
- Composition in $\mathscr{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories.
- 1-morphisms in **Cat** are functors.
- 2-morphisms in **Cat** are natural transformations.
- Composition is the usual composition.
- Identity 1-morphisms are the identity functors.

2-functors and 2-representations

$\mathscr{A}$ and $\mathscr{C}$ — two 2-categories

Definition. A 2-functor $\mathrm{F} : \mathscr{A} \to \mathscr{C}$ is a functor which sends 1-morphisms to 1-morphisms and 2-morphisms to 2-morphisms in a way that is coordinated with all the categorical structures (domains, codomains, identities and compositions).

Definition. A 2-representation of a 2-category $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to some "classical" 2-category.

A special case of "higher representation theory" is the

2-representation theory of 2-categories

2-functors and 2-representations

$\mathscr{A}$ and $\mathscr{C}$ — two 2-categories

Definition. A 2-functor $\mathrm{F} : \mathscr{A} \to \mathscr{C}$ is a functor which sends 1-morphisms to 1-morphisms and 2-morphisms to 2-morphisms in a way that is coordinated with all the categorical structures (domains, codomains, identities and compositions).

Definition. A 2-representation of a 2-category $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to some "classical" 2-category.

A special case of "higher representation theory" is the

$\boxed{\text{2-representation theory of 2-categories}}$

2-functors and 2-representations

$\mathscr{A}$ and $\mathscr{C}$ — two 2-categories

Definition. A 2-functor $\mathrm{F} : \mathscr{A} \to \mathscr{C}$ is a functor which sends 1-morphisms to 1-morphisms and 2-morphisms to 2-morphisms in a way that is coordinated with all the categorical structures (domains, codomains, identities and compositions).

Definition. A 2-representation of a 2-category $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to some "classical" 2-category.

A special case of "higher representation theory" is the

$\boxed{\text{2-representation theory of 2-categories}}$

2-functors and 2-representations

$\mathscr{A}$ and $\mathscr{C}$ — two 2-categories

Definition. A 2-functor $\mathrm{F} : \mathscr{A} \to \mathscr{C}$ is a functor which sends 1-morphisms to 1-morphisms and 2-morphisms to 2-morphisms in a way that is coordinated with all the categorical structures (domains, codomains, identities and compositions).

Definition. A 2-representation of a 2-category $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to some "classical" 2-category.

A special case of "higher representation theory" is the

2-representation theory of 2-categories

2-functors and 2-representations

$\mathscr{A}$ and $\mathscr{C}$ — two 2-categories

Definition. A 2-functor $\mathrm{F} : \mathscr{A} \to \mathscr{C}$ is a functor which sends 1-morphisms to 1-morphisms and 2-morphisms to 2-morphisms in a way that is coordinated with all the categorical structures (domains, codomains, identities and compositions).

Definition. A 2-representation of a 2-category $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to some "classical" 2-category.

A special case of "higher representation theory" is the

$\boxed{\text{2-representation theory of 2-categories}}$

2-functors and 2-representations

$\mathscr{A}$ and $\mathscr{C}$ — two 2-categories

Definition. A 2-functor $\mathrm{F} : \mathscr{A} \to \mathscr{C}$ is a functor which sends 1-morphisms to 1-morphisms and 2-morphisms to 2-morphisms in a way that is coordinated with all the categorical structures (domains, codomains, identities and compositions).

Definition. A 2-representation of a 2-category $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to some "classical" 2-category.

A special case of "higher representation theory" is the

$\boxed{\text{2-representation theory of 2-categories}}$

$\boxed{\text{Khovanov homology}} \longrightarrow \boxed{\text{functorial action of } B_n}$

$\boxed{\text{functorial action of } B_n} = \boxed{\text{2-representation of a certain 2-category}}$

In the case of Khovanov homology this 2-cat. is Rouquier's 2-braid group

In general: A — algebra

Definition: A categorification of A is a 2-category $\mathfrak{A}$ which decategorifies (e.g. via the "Grothendieck group" construction) to A

2-representations of $\mathfrak{A}$ decategorify to A-modules

$\boxed{\text{Khovanov homology}} \longrightarrow \boxed{\text{functorial action of } B_n}$

$\boxed{\text{functorial action of } B_n} = \boxed{\text{2-representation of a certain 2-category}}$

In the case of Khovanov homology this 2-cat. is Rouquier's 2-braid group

In general: A — algebra

Definition: A categorification of A is a 2-category $\mathfrak{A}$ which decategorifies (e.g. via the "Grothendieck group" construction) to A

2-representations of $\mathfrak{A}$ decategorify to A-modules

Categorification and 2-categories

$\boxed{\text{Khovanov homology}} \longrightarrow \boxed{\text{functorial action of } B_n}$

$\boxed{\text{functorial action of } B_n} = \boxed{\text{2-representation of a certain 2-category}}$

In the case of Khovanov homology this 2-cat. is Rouquier's 2-braid group

In general: A — algebra

Definition: A categorification of A is a 2-category $\mathfrak{A}$ which decategorifies (e.g. via the "Grothendieck group" construction) to A

2-representations of $\mathfrak{A}$ decategorify to A-modules

Categorification and 2-categories

$\boxed{\text{Khovanov homology}} \longrightarrow \boxed{\text{functorial action of } B_n}$

$\boxed{\text{functorial action of } B_n} = \boxed{\text{2-representation of a certain 2-category}}$

In the case of Khovanov homology this 2-cat. is Rouquier's 2-braid group

In general: A — algebra

Definition: A categorification of A is a 2-category $\mathfrak{A}$ which decategorifies (e.g. via the "Grothendieck group" construction) to A

2-representations of $\mathfrak{A}$ decategorify to A-modules

Categorification and 2-categories

$\boxed{\text{Khovanov homology}} \longrightarrow \boxed{\text{functorial action of } B_n}$

$\boxed{\text{functorial action of } B_n} = \boxed{\text{2-representation of a certain 2-category}}$

In the case of Khovanov homology this 2-cat. is Rouquier's 2-braid group

In general: A — algebra

Definition: A categorification of A is a 2-category $\mathfrak{A}$ which decategorifies (e.g. via the "Grothendieck group" construction) to A

2-representations of $\mathfrak{A}$ decategorify to A-modules

Categorification and 2-categories

$\boxed{\text{Khovanov homology}} \longrightarrow \boxed{\text{functorial action of } B_n}$

$\boxed{\text{functorial action of } B_n} = \boxed{\text{2-representation of a certain 2-category}}$

In the case of Khovanov homology this 2-cat. is Rouquier's 2-braid group

In general: A — algebra

Definition: A categorification of A is a 2-category $\mathfrak{A}$ which decategorifies (e.g. via the "Grothendieck group" construction) to A

2-representations of $\mathfrak{A}$ decategorify to A-modules

Categorification and 2-categories

$\boxed{\text{Khovanov homology}} \longrightarrow \boxed{\text{functorial action of } B_n}$

$\boxed{\text{functorial action of } B_n} = \boxed{\text{2-representation of a certain 2-category}}$

In the case of Khovanov homology this 2-cat. is Rouquier's 2-braid group

In general: A — algebra

Definition: A categorification of A is a 2-category $\mathfrak{A}$ which decategorifies (e.g. via the "Grothendieck group" construction) to A

2-representations of $\mathfrak{A}$ decategorify to A-modules

Categorification and 2-categories

$\boxed{\text{Khovanov homology}} \longrightarrow \boxed{\text{functorial action of } B_n}$

$\boxed{\text{functorial action of } B_n} = \boxed{\text{2-representation of a certain 2-category}}$

In the case of Khovanov homology this 2-cat. is Rouquier's 2-braid group

In general: A — algebra

Definition: A categorification of A is a 2-category $\mathfrak{A}$ which decategorifies (e.g. via the "Grothendieck group" construction) to A

2-representations of $\mathfrak{A}$ decategorify to A-modules

Application to Broué's abelian defect group conjecture for S_n

Rough conjecture: Certain two algebras A and B (related to modular representations of S_n) are derived equivalent.

Approach of Chuang and Rouquier:

- Categorify $U(\mathfrak{sl}_2)$ to a 2-category $\mathfrak{G}$
- Construct "minimal" 2-representations of $\mathfrak{G}$
- Prove "uniqueness" of "minimal" 2-representations of $\mathfrak{G}$ (this implies equivalence of certain module categories)
- Prove that "filtrations" by "minimal" 2-representations of $\mathfrak{G}$ gives rise to derived equivalences between certain module categories
- Realize A-mod and B-mod in a "nice" way inside some 2-representation of $\mathfrak{G}$ which is "filtered" by "minimal" 2-representations

Rough conjecture: Certain two algebras A and B (related to modular representations of S_n) are derived equivalent.

Approach of Chuang and Rouquier:

- Categorify $U(\mathfrak{sl}_2)$ to a 2-category $\mathfrak{G}$
- Construct "minimal" 2-representations of $\mathfrak{G}$
- Prove "uniqueness" of "minimal" 2-representations of $\mathfrak{G}$ (this implies equivalence of certain module categories)
- Prove that "filtrations" by "minimal" 2-representations of $\mathfrak{G}$ gives rise to derived equivalences between certain module categories
- Realize A-mod and B-mod in a "nice" way inside some 2-representation of $\mathfrak{G}$ which is "filtered" by "minimal" 2-representations

Application to Broué's abelian defect group conjecture for S_n

Rough conjecture: Certain two algebras A and B (related to modular representations of S_n) are derived equivalent.

Approach of Chuang and Rouquier:

- Categorify $U(\mathfrak{sl}_2)$ to a 2-category $\mathfrak{G}$
- Construct "minimal" 2-representations of $\mathfrak{G}$
- Prove "uniqueness" of "minimal" 2-representations of $\mathfrak{G}$ (this implies equivalence of certain module categories)
- Prove that "filtrations" by "minimal" 2-representations of $\mathfrak{G}$ gives rise to derived equivalences between certain module categories
- Realize A-mod and B-mod in a "nice" way inside some 2-representation of $\mathfrak{G}$ which is "filtered" by "minimal" 2-representations

Rough conjecture: Certain two algebras A and B (related to modular representations of S_n) are derived equivalent.

Approach of Chuang and Rouquier:

- Categorify $U(\mathfrak{sl}_2)$ to a 2-category $\mathfrak{G}$
- Construct "minimal" 2-representations of $\mathfrak{G}$
- Prove "uniqueness" of "minimal" 2-representations of $\mathfrak{G}$ (this implies equivalence of certain module categories)
- Prove that "filtrations" by "minimal" 2-representations of $\mathfrak{G}$ gives rise to derived equivalences between certain module categories
- Realize A-mod and B-mod in a "nice" way inside some 2-representation of $\mathfrak{G}$ which is "filtered" by "minimal" 2-representations

Application to Broué's abelian defect group conjecture for S_n

Rough conjecture: Certain two algebras A and B (related to modular representations of S_n) are derived equivalent.

Approach of Chuang and Rouquier:

- Categorify $U(\mathfrak{sl}_2)$ to a 2-category $\mathfrak{G}$
- Construct "minimal" 2-representations of $\mathfrak{G}$
- Prove "uniqueness" of "minimal" 2-representations of $\mathfrak{G}$ (this implies equivalence of certain module categories)
- Prove that "filtrations" by "minimal" 2-representations of $\mathfrak{G}$ gives rise to derived equivalences between certain module categories
- Realize A-mod and B-mod in a "nice" way inside some 2-representation of $\mathfrak{G}$ which is "filtered" by "minimal" 2-representations

Rough conjecture: Certain two algebras A and B (related to modular representations of S_n) are derived equivalent.

Approach of Chuang and Rouquier:

- Categorify $U(\mathfrak{sl}_2)$ to a 2-category $\mathfrak{G}$
- Construct "minimal" 2-representations of $\mathfrak{G}$
- Prove "uniqueness" of "minimal" 2-representations of $\mathfrak{G}$ (this implies equivalence of certain module categories)
- Prove that "filtrations" by "minimal" 2-representations of $\mathfrak{G}$ gives rise to derived equivalences between certain module categories
- Realize A-mod and B-mod in a "nice" way inside some 2-representation of $\mathfrak{G}$ which is "filtered" by "minimal" 2-representations

Application to Broué's abelian defect group conjecture for S_n

Rough conjecture: Certain two algebras A and B (related to modular representations of S_n) are derived equivalent.

Approach of Chuang and Rouquier:

- Categorify $U(\mathfrak{sl}_2)$ to a 2-category $\mathfrak{G}$
- Construct "minimal" 2-representations of $\mathfrak{G}$
- Prove "uniqueness" of "minimal" 2-representations of $\mathfrak{G}$ (this implies equivalence of certain module categories)
- Prove that "filtrations" by "minimal" 2-representations of $\mathfrak{G}$ gives rise to derived equivalences between certain module categories
- Realize A-mod and B-mod in a "nice" way inside some 2-representation of $\mathfrak{G}$ which is "filtered" by "minimal" 2-representations

Application to Broué's abelian defect group conjecture for S_n

Rough conjecture: Certain two algebras A and B (related to modular representations of S_n) are derived equivalent.

Approach of Chuang and Rouquier:

- Categorify $U(\mathfrak{sl}_2)$ to a 2-category $\mathfrak{G}$
- Construct "minimal" 2-representations of $\mathfrak{G}$
- Prove "uniqueness" of "minimal" 2-representations of $\mathfrak{G}$ (this implies equivalence of certain module categories)
- Prove that "filtrations" by "minimal" 2-representations of $\mathfrak{G}$ gives rise to derived equivalences between certain module categories
- Realize A-mod and B-mod in a "nice" way inside some 2-representation of $\mathfrak{G}$ which is "filtered" by "minimal" 2-representations

Application to Broué's abelian defect group conjecture for S_n

Rough conjecture: Certain two algebras A and B (related to modular representations of S_n) are derived equivalent.

Approach of Chuang and Rouquier:

- Categorify $U(\mathfrak{sl}_2)$ to a 2-category $\mathfrak{G}$
- Construct "minimal" 2-representations of $\mathfrak{G}$
- Prove "uniqueness" of "minimal" 2-representations of $\mathfrak{G}$ (this implies equivalence of certain module categories)
- Prove that "filtrations" by "minimal" 2-representations of $\mathfrak{G}$ gives rise to derived equivalences between certain module categories
- Realize A-mod and B-mod in a "nice" way inside some 2-representation of $\mathfrak{G}$ which is "filtered" by "minimal" 2-representations

Application to Lie superalgebras

Want: Describe blocks of finite dimensional supermodules for $\mathfrak{sl}(m|n)$

Approach of Brundan and Stroppel:

- Define on such a block the structure of a 2-representation of a certain 2-category $\mathfrak{G}$
- Construct a combinatorially defined candidate A for the answer
- Define on A-mod the structure of a 2-representation of $\mathfrak{G}$
- Prove that both 2-representations above are "minimal" and are given by the same parameters
- Prove that this minimal 2-representation of $\mathfrak{G}$ is unique

Application to Lie superalgebras

Want: Describe blocks of finite dimensional supermodules for $\mathfrak{sl}(m|n)$

Approach of Brundan and Stroppel:

- Define on such a block the structure of a 2-representation of a certain 2-category $\mathfrak{G}$
- Construct a combinatorially defined candidate A for the answer
- Define on A-mod the structure of a 2-representation of $\mathfrak{G}$
- Prove that both 2-representations above are "minimal" and are given by the same parameters
- Prove that this minimal 2-representation of $\mathfrak{G}$ is unique

Want: Describe blocks of finite dimensional supermodules for $\mathfrak{sl}(m|n)$

Approach of Brundan and Stroppel:

- Define on such a block the structure of a 2-representation of a certain 2-category $\mathfrak{G}$
- Construct a combinatorially defined candidate A for the answer
- Define on A-mod the structure of a 2-representation of $\mathfrak{G}$
- Prove that both 2-representations above are "minimal" and are given by the same parameters
- Prove that this minimal 2-representation of $\mathfrak{G}$ is unique

Application to Lie superalgebras

Want: Describe blocks of finite dimensional supermodules for $\mathfrak{sl}(m|n)$

Approach of Brundan and Stroppel:

- Define on such a block the structure of a 2-representation of a certain 2-category $\mathfrak{G}$
- Construct a combinatorially defined candidate A for the answer
- Define on A-mod the structure of a 2-representation of $\mathfrak{G}$
- Prove that both 2-representations above are "minimal" and are given by the same parameters
- Prove that this minimal 2-representation of $\mathfrak{G}$ is unique

Application to Lie superalgebras

Want: Describe blocks of finite dimensional supermodules for $\mathfrak{sl}(m|n)$

Approach of Brundan and Stroppel:

- Define on such a block the structure of a 2-representation of a certain 2-category $\mathfrak{G}$
- Construct a combinatorially defined candidate A for the answer
- Define on A-mod the structure of a 2-representation of $\mathfrak{G}$
- Prove that both 2-representations above are "minimal" and are given by the same parameters
- Prove that this minimal 2-representation of $\mathfrak{G}$ is unique

Application to Lie superalgebras

Want: Describe blocks of finite dimensional supermodules for $\mathfrak{sl}(m|n)$

Approach of Brundan and Stroppel:

- Define on such a block the structure of a 2-representation of a certain 2-category $\mathfrak{G}$
- Construct a combinatorially defined candidate A for the answer
- Define on A-mod the structure of a 2-representation of $\mathfrak{G}$
- Prove that both 2-representations above are "minimal" and are given by the same parameters
- Prove that this minimal 2-representation of $\mathfrak{G}$ is unique

Application to Lie superalgebras

Want: Describe blocks of finite dimensional supermodules for $\mathfrak{sl}(m|n)$

Approach of Brundan and Stroppel:

- Define on such a block the structure of a 2-representation of a certain 2-category $\mathfrak{G}$
- Construct a combinatorially defined candidate A for the answer
- Define on A-mod the structure of a 2-representation of $\mathfrak{G}$
- Prove that both 2-representations above are "minimal" and are given by the same parameters
- Prove that this minimal 2-representation of $\mathfrak{G}$ is unique

Application to Lie superalgebras

Want: Describe blocks of finite dimensional supermodules for $\mathfrak{sl}(m|n)$

Approach of Brundan and Stroppel:

- Define on such a block the structure of a 2-representation of a certain 2-category $\mathfrak{G}$
- Construct a combinatorially defined candidate A for the answer
- Define on A-mod the structure of a 2-representation of $\mathfrak{G}$
- Prove that both 2-representations above are "minimal" and are given by the same parameters
- Prove that this minimal 2-representation of $\mathfrak{G}$ is unique

Application to Lie superalgebras

Want: Describe blocks of finite dimensional supermodules for $\mathfrak{sl}(m|n)$

Approach of Brundan and Stroppel:

- Define on such a block the structure of a 2-representation of a certain 2-category $\mathfrak{G}$
- Construct a combinatorially defined candidate A for the answer
- Define on A-mod the structure of a 2-representation of $\mathfrak{G}$
- Prove that both 2-representations above are "minimal" and are given by the same parameters
- Prove that this minimal 2-representation of $\mathfrak{G}$ is unique

Abstract 2-representation theory

Given a 2-category $\mathfrak{C}$

- Construct 2-representations of $\mathfrak{C}$
- Identify a given 2-representations of $\mathfrak{C}$
- Describe the 2-category $\mathfrak{C}\text{-mod}$ of 2-representations of $\mathfrak{C}$
- Compare $\mathfrak{C}\text{-mod}$ and $\mathfrak{D}\text{-mod}$ for some other 2-category $\mathfrak{D}$

More concrete questions:

- What are "simple" 2-representations of $\mathfrak{C}$?
- What kind of uniqueness properties hold for 2-representations of $\mathfrak{C}$?
- Is there any kind of Jordan-Hölder property?
- Homological algebra for 2-representations?

Given a 2-category $\mathfrak{C}$

- Construct 2-representations of $\mathfrak{C}$
- Identify a given 2-representations of $\mathfrak{C}$
- Describe the 2-category $\mathfrak{C}\text{-mod}$ of 2-representations of $\mathfrak{C}$
- Compare $\mathfrak{C}\text{-mod}$ and $\mathfrak{D}\text{-mod}$ for some other 2-category $\mathfrak{D}$

More concrete questions:

- What are "simple" 2-representations of $\mathfrak{C}$?
- What kind of uniqueness properties hold for 2-representations of $\mathfrak{C}$?
- Is there any kind of Jordan-Hölder property?
- Homological algebra for 2-representations?

Abstract 2-representation theory

Given a 2-category $\mathfrak{C}$

- Construct 2-representations of $\mathfrak{C}$
- Identify a given 2-representations of $\mathfrak{C}$
- Describe the 2-category $\mathfrak{C}\text{-mod}$ of 2-representations of $\mathfrak{C}$
- Compare $\mathfrak{C}\text{-mod}$ and $\mathfrak{D}\text{-mod}$ for some other 2-category $\mathfrak{D}$

More concrete questions:

- What are "simple" 2-representations of $\mathfrak{C}$?
- What kind of uniqueness properties hold for 2-representations of $\mathfrak{C}$?
- Is there any kind of Jordan-Hölder property?
- Homological algebra for 2-representations?

Abstract 2-representation theory

Given a 2-category $\mathfrak{C}$

- Construct 2-representations of $\mathfrak{C}$
- Identify a given 2-representations of $\mathfrak{C}$
- Describe the 2-category $\mathfrak{C}\text{-mod}$ of 2-representations of $\mathfrak{C}$
- Compare $\mathfrak{C}\text{-mod}$ and $\mathfrak{D}\text{-mod}$ for some other 2-category $\mathfrak{D}$

More concrete questions:

- What are "simple" 2-representations of $\mathfrak{C}$?
- What kind of uniqueness properties hold for 2-representations of $\mathfrak{C}$?
- Is there any kind of Jordan-Hölder property?
- Homological algebra for 2-representations?

Given a 2-category $\mathfrak{C}$

- Construct 2-representations of $\mathfrak{C}$
- Identify a given 2-representations of $\mathfrak{C}$
- Describe the 2-category $\mathfrak{C}\text{-mod}$ of 2-representations of $\mathfrak{C}$
- Compare $\mathfrak{C}\text{-mod}$ and $\mathfrak{D}\text{-mod}$ for some other 2-category $\mathfrak{D}$

More concrete questions:

- What are "simple" 2-representations of $\mathfrak{C}$?
- What kind of uniqueness properties hold for 2-representations of $\mathfrak{C}$?
- Is there any kind of Jordan-Hölder property?
- Homological algebra for 2-representations?

Abstract 2-representation theory

Given a 2-category $\mathfrak{C}$

- Construct 2-representations of $\mathfrak{C}$
- Identify a given 2-representations of $\mathfrak{C}$
- Describe the 2-category $\mathfrak{C}\text{-mod}$ of 2-representations of $\mathfrak{C}$
- Compare $\mathfrak{C}\text{-mod}$ and $\mathfrak{D}\text{-mod}$ for some other 2-category $\mathfrak{D}$

More concrete questions:

- What are “simple” 2-representations of $\mathfrak{C}$?
- What kind of uniqueness properties hold for 2-representations of $\mathfrak{C}$?
- Is there any kind of Jordan-Hölder property?
- Homological algebra for 2-representations?

Abstract 2-representation theory

Given a 2-category $\mathfrak{C}$

- Construct 2-representations of $\mathfrak{C}$
- Identify a given 2-representations of $\mathfrak{C}$
- Describe the 2-category $\mathfrak{C}\text{-mod}$ of 2-representations of $\mathfrak{C}$
- Compare $\mathfrak{C}\text{-mod}$ and $\mathfrak{D}\text{-mod}$ for some other 2-category $\mathfrak{D}$

More concrete questions:

- What are "simple" 2-representations of $\mathfrak{C}$?
- What kind of uniqueness properties hold for 2-representations of $\mathfrak{C}$?
- Is there any kind of Jordan-Hölder property?
- Homological algebra for 2-representations?

Abstract 2-representation theory

Given a 2-category $\mathfrak{C}$

- Construct 2-representations of $\mathfrak{C}$
- Identify a given 2-representations of $\mathfrak{C}$
- Describe the 2-category $\mathfrak{C}\text{-mod}$ of 2-representations of $\mathfrak{C}$
- Compare $\mathfrak{C}\text{-mod}$ and $\mathfrak{D}\text{-mod}$ for some other 2-category $\mathfrak{D}$

More concrete questions:

- What are "simple" 2-representations of $\mathfrak{C}$?
- What kind of uniqueness properties hold for 2-representations of $\mathfrak{C}$?
- Is there any kind of Jordan-Hölder property?
- Homological algebra for 2-representations?

Abstract 2-representation theory

Given a 2-category $\mathfrak{C}$

- Construct 2-representations of $\mathfrak{C}$
- Identify a given 2-representations of $\mathfrak{C}$
- Describe the 2-category $\mathfrak{C}\text{-mod}$ of 2-representations of $\mathfrak{C}$
- Compare $\mathfrak{C}\text{-mod}$ and $\mathfrak{D}\text{-mod}$ for some other 2-category $\mathfrak{D}$

More concrete questions:

- What are "simple" 2-representations of $\mathfrak{C}$?
- What kind of uniqueness properties hold for 2-representations of $\mathfrak{C}$?
- Is there any kind of Jordan-Hölder property?
- Homological algebra for 2-representations?

Abstract 2-representation theory

Given a 2-category $\mathfrak{C}$

- Construct 2-representations of $\mathfrak{C}$
- Identify a given 2-representations of $\mathfrak{C}$
- Describe the 2-category $\mathfrak{C}\text{-mod}$ of 2-representations of $\mathfrak{C}$
- Compare $\mathfrak{C}\text{-mod}$ and $\mathfrak{D}\text{-mod}$ for some other 2-category $\mathfrak{D}$

More concrete questions:

- What are "simple" 2-representations of $\mathfrak{C}$?
- What kind of uniqueness properties hold for 2-representations of $\mathfrak{C}$?
- Is there any kind of Jordan-Hölder property?
- Homological algebra for 2-representations?

Abstract 2-representation theory

Given a 2-category $\mathfrak{C}$

- Construct 2-representations of $\mathfrak{C}$
- Identify a given 2-representations of $\mathfrak{C}$
- Describe the 2-category $\mathfrak{C}\text{-mod}$ of 2-representations of $\mathfrak{C}$
- Compare $\mathfrak{C}\text{-mod}$ and $\mathfrak{D}\text{-mod}$ for some other 2-category $\mathfrak{D}$

More concrete questions:

- What are "simple" 2-representations of $\mathfrak{C}$?
- What kind of uniqueness properties hold for 2-representations of $\mathfrak{C}$?
- Is there any kind of Jordan-Hölder property?
- Homological algebra for 2-representations?

Abstract 2-representation theory

Given a 2-category $\mathfrak{C}$

- Construct 2-representations of $\mathfrak{C}$
- Identify a given 2-representations of $\mathfrak{C}$
- Describe the 2-category $\mathfrak{C}\text{-mod}$ of 2-representations of $\mathfrak{C}$
- Compare $\mathfrak{C}\text{-mod}$ and $\mathfrak{D}\text{-mod}$ for some other 2-category $\mathfrak{D}$

More concrete questions:

- What are "simple" 2-representations of $\mathfrak{C}$?
- What kind of uniqueness properties hold for 2-representations of $\mathfrak{C}$?
- Is there any kind of Jordan-Hölder property?
- Homological algebra for 2-representations?

2-analogues of finite dimensional algebras

$\Bbbk$ — algebraically closed field

Definition: A 2-category $\mathscr{C}$ is called finitary (over $\Bbbk$) provided that

- $\mathscr{C}$ has finitely many objects;
- each $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is additive, $\Bbbk$-linear, idempotent split, with finitely many indecomposables (up to isomorphism);
- all spaces of 2-morphisms are finite dimensional (over $\Bbbk$);
- the identity 1-morphisms are indecomposable.

2-analogues of finite dimensional algebras

$\Bbbk$ — algebraically closed field

Definition: A 2-category $\mathfrak{C}$ is called finitary (over $\Bbbk$) provided that

- $\mathfrak{C}$ has finitely many objects;
- each $\mathfrak{C}(\mathtt{i}, \mathtt{j})$ is additive, $\Bbbk$-linear, idempotent split, with finitely many indecomposables (up to isomorphism);
- all spaces of 2-morphisms are finite dimensional (over $\Bbbk$);
- the identity 1-morphisms are indecomposable.

2-analogues of finite dimensional algebras

$\Bbbk$ — algebraically closed field

Definition: A 2-category $\mathfrak{C}$ is called finitary (over $\Bbbk$) provided that

- $\mathfrak{C}$ has finitely many objects;
- each $\mathfrak{C}(\mathtt{i}, \mathtt{j})$ is additive, $\Bbbk$-linear, idempotent split, with finitely many indecomposables (up to isomorphism);
- all spaces of 2-morphisms are finite dimensional (over $\Bbbk$);
- the identity 1-morphisms are indecomposable.

2-analogues of finite dimensional algebras

$\Bbbk$ — algebraically closed field

Definition: A 2-category $\mathfrak{C}$ is called finitary (over $\Bbbk$) provided that

- $\mathfrak{C}$ has finitely many objects;
- each $\mathfrak{C}(\mathtt{i},\mathtt{j})$ is additive, $\Bbbk$-linear, idempotent split, with finitely many indecomposables (up to isomorphism);
- all spaces of 2-morphisms are finite dimensional (over $\Bbbk$);
- the identity 1-morphisms are indecomposable.

2-analogues of finite dimensional algebras

$\Bbbk$ — algebraically closed field

Definition: A 2-category $\mathfrak{C}$ is called finitary (over $\Bbbk$) provided that

- $\mathfrak{C}$ has finitely many objects;
- each $\mathfrak{C}(\mathtt{i}, \mathtt{j})$ is additive, $\Bbbk$-linear, idempotent split, with finitely many indecomposables (up to isomorphism);
- all spaces of 2-morphisms are finite dimensional (over $\Bbbk$);
- the identity 1-morphisms are indecomposable.

2-analogues of finite dimensional algebras

$\Bbbk$ — algebraically closed field

Definition: A 2-category $\mathfrak{C}$ is called finitary (over $\Bbbk$) provided that

- $\mathfrak{C}$ has finitely many objects;
- each $\mathfrak{C}(\mathtt{i}, \mathtt{j})$ is additive, $\Bbbk$-linear, idempotent split, with finitely many indecomposables (up to isomorphism);
- all spaces of 2-morphisms are finite dimensional (over $\Bbbk$);
- the identity 1-morphisms are indecomposable.

2-analogues of finite dimensional algebras

$\Bbbk$ — algebraically closed field

Definition: A 2-category $\mathfrak{C}$ is called finitary (over $\Bbbk$) provided that

- $\mathfrak{C}$ has finitely many objects;
- each $\mathfrak{C}(\mathtt{i}, \mathtt{j})$ is additive, $\Bbbk$-linear, idempotent split, with finitely many indecomposables (up to isomorphism);
- all spaces of 2-morphisms are finite dimensional (over $\Bbbk$);
- the identity 1-morphisms are indecomposable.

$\Bbbk$ — algebraically closed field

Definition: A 2-category $\mathfrak{C}$ is called finitary (over $\Bbbk$) provided that

- $\mathfrak{C}$ has finitely many objects;
- each $\mathfrak{C}(\mathtt{i}, \mathtt{j})$ is additive, $\Bbbk$-linear, idempotent split, with finitely many indecomposables (up to isomorphism);
- all spaces of 2-morphisms are finite dimensional (over $\Bbbk$);
- the identity 1-morphisms are indecomposable.

Example: projective functors on A-mod

$\Bbbk$ — algebraically closed field

A — finite dimensional connected associative $\Bbbk$-algebra

Definition: $F : A\text{-mod} \to A\text{-mod}$ is projective is it is isomorphic to tensoring with a projective bimodule.

Definition: The 2-category $\mathfrak{C}_A$ is defined as follows:

- $\mathfrak{C}$ has one object $\clubsuit$ (which is identified with $A\text{-mod}$);
- 1-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are functors isomorphic to direct sum of the identity and projective functors;
- 2-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are natural transformations of functors.

$\mathfrak{C}_A$ is a "simple" finitary 2-category

Example: projective functors on A-mod

$\Bbbk$ — algebraically closed field

A — finite dimensional connected associative $\Bbbk$-algebra

Definition: $F : A\text{-mod} \to A\text{-mod}$ is projective is it is isomorphic to tensoring with a projective bimodule.

Definition: The 2-category $\mathfrak{C}_A$ is defined as follows:

- $\mathfrak{C}$ has one object $\clubsuit$ (which is identified with $A\text{-mod}$);
- 1-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are functors isomorphic to direct sum of the identity and projective functors;
- 2-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are natural transformations of functors.

$\mathfrak{C}_A$ is a "simple" finitary 2-category

Example: projective functors on A-mod

$\Bbbk$ — algebraically closed field

A — finite dimensional connected associative $\Bbbk$-algebra

Definition: $F : A\text{-mod} \to A\text{-mod}$ is projective is it is isomorphic to tensoring with a projective bimodule.

Definition: The 2-category $\mathfrak{C}_A$ is defined as follows:

- $\mathfrak{C}$ has one object $\clubsuit$ (which is identified with $A\text{-mod}$);
- 1-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are functors isomorphic to direct sum of the identity and projective functors;
- 2-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are natural transformations of functors.

$\mathfrak{C}_A$ is a "simple" finitary 2-category

Example: projective functors on A-mod

$\Bbbk$ — algebraically closed field

A — finite dimensional connected associative $\Bbbk$-algebra

Definition: $F : A\text{-mod} \to A\text{-mod}$ is projective is it is isomorphic to tensoring with a projective bimodule.

Definition: The 2-category $\mathfrak{C}_A$ is defined as follows:

- $\mathfrak{C}$ has one object $\clubsuit$ (which is identified with $A\text{-mod}$);
- 1-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are functors isomorphic to direct sum of the identity and projective functors;
- 2-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are natural transformations of functors.

$\mathfrak{C}_A$ is a "simple" finitary 2-category

Example: projective functors on A-mod

$\Bbbk$ — algebraically closed field

A — finite dimensional connected associative $\Bbbk$-algebra

Definition: $F : A\text{-mod} \to A\text{-mod}$ is projective is it is isomorphic to tensoring with a projective bimodule.

Definition: The 2-category $\mathfrak{C}_A$ is defined as follows:

- $\mathfrak{C}$ has one object $\clubsuit$ (which is identified with $A\text{-mod}$);
- 1-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are functors isomorphic to direct sum of the identity and projective functors;
- 2-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are natural transformations of functors.

$\mathfrak{C}_A$ is a "simple" finitary 2-category

Example: projective functors on A-mod

$\Bbbk$ — algebraically closed field

A — finite dimensional connected associative $\Bbbk$-algebra

Definition: $F : A\text{-mod} \to A\text{-mod}$ is projective is it is isomorphic to tensoring with a projective bimodule.

Definition: The 2-category $\mathfrak{C}_A$ is defined as follows:

- $\mathfrak{C}$ has one object $\clubsuit$ (which is identified with $A\text{-mod}$);
- 1-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are functors isomorphic to direct sum of the identity and projective functors;
- 2-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are natural transformations of functors.

$\mathfrak{C}_A$ is a "simple" finitary 2-category

Example: projective functors on A-mod

$\mathbb{k}$ — algebraically closed field

A — finite dimensional connected associative $\mathbb{k}$-algebra

Definition: $F : A\text{-mod} \to A\text{-mod}$ is projective is it is isomorphic to tensoring with a projective bimodule.

Definition: The 2-category $\mathfrak{C}_A$ is defined as follows:

- $\mathfrak{C}$ has one object $\clubsuit$ (which is identified with $A\text{-mod}$);
- 1-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are functors isomorphic to direct sum of the identity and projective functors;
- 2-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are natural transformations of functors.

$\mathfrak{C}_A$ is a "simple" finitary 2-category

Example: projective functors on A-mod

$\Bbbk$ — algebraically closed field

A — finite dimensional connected associative $\Bbbk$-algebra

Definition: $F : A\text{-mod} \to A\text{-mod}$ is projective is it is isomorphic to tensoring with a projective bimodule.

Definition: The 2-category $\mathfrak{C}_A$ is defined as follows:

- $\mathfrak{C}$ has one object $\clubsuit$ (which is identified with $A\text{-mod}$);
- 1-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are functors isomorphic to direct sum of the identity and projective functors;
- 2-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are natural transformations of functors.

$\mathfrak{C}_A$ is a "simple" finitary 2-category

Example: projective functors on A-mod

$\Bbbk$ — algebraically closed field

A — finite dimensional connected associative $\Bbbk$-algebra

Definition: $F : A\text{-mod} \to A\text{-mod}$ is projective is it is isomorphic to tensoring with a projective bimodule.

Definition: The 2-category $\mathfrak{C}_A$ is defined as follows:

- $\mathfrak{C}$ has one object $\clubsuit$ (which is identified with $A\text{-mod}$);
- 1-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are functors isomorphic to direct sum of the identity and projective functors;
- 2-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are natural transformations of functors.

$\mathfrak{C}_A$ is a "simple" finitary 2-category

Example: projective functors on A-mod

$\mathbb{k}$ — algebraically closed field

A — finite dimensional connected associative $\mathbb{k}$-algebra

Definition: $F : A\text{-}\mathrm{mod} \to A\text{-}\mathrm{mod}$ is projective is it is isomorphic to tensoring with a projective bimodule.

Definition: The 2-category $\mathfrak{C}_A$ is defined as follows:

- $\mathfrak{C}$ has one object $\clubsuit$ (which is identified with $A\text{-}\mathrm{mod}$);
- 1-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are functors isomorphic to direct sum of the identity and projective functors;
- 2-morphisms in $\mathfrak{C}(\clubsuit, \clubsuit)$ are natural transformations of functors.

$\mathfrak{C}_A$ is a "simple" finitary 2-category

Combinatorics of finitary 2-categories

$\mathfrak{C}$ — finitary 2-category

$\mathcal{S}[\mathfrak{C}]$ — the set of isoclasses of indecomposable 1-morphisms in $\mathfrak{C}$ together with 0

Note: if F, G are indecomposable 1-morphisms, then $F \circ G$ usually decomposes

Hence: $\mathcal{S}[\mathfrak{C}]$ has the natural structure of a multisemigroup, that is a "semigroup" with multivalued operation

$\mathcal{S}[\mathfrak{C}]$ describes combinatorics of horizontal composition in $\mathfrak{C}$

Note: $\mathcal{S}[\mathfrak{C}]$ does not "remember" 2-morphisms in a straightforward way, however, "indecomposability", used to define $\mathcal{S}[\mathfrak{C}]$, is a property of the 2-endomorphism algebra

Example: $\mathcal{S}[\mathfrak{C}_A]$ is, in fact, a semigroup (the operation is single valued)

Combinatorics of finitary 2-categories

$\mathfrak{C}$ — finitary 2-category

$\mathcal{S}[\mathfrak{C}]$ — the set of isoclasses of indecomposable 1-morphisms in $\mathfrak{C}$ together with 0

Note: if F, G are indecomposable 1-morphisms, then $F \circ G$ usually decomposes

Hence: $\mathcal{S}[\mathfrak{C}]$ has the natural structure of a multisemigroup, that is a "semigroup" with multivalued operation

$\mathcal{S}[\mathfrak{C}]$ describes combinatorics of horizontal composition in $\mathfrak{C}$

Note: $\mathcal{S}[\mathfrak{C}]$ does not "remember" 2-morphisms in a straightforward way, however, "indecomposability", used to define $\mathcal{S}[\mathfrak{C}]$, is a property of the 2-endomorphism algebra

Example: $\mathcal{S}[\mathfrak{C}_A]$ is, in fact, a semigroup (the operation is single valued)

Combinatorics of finitary 2-categories

$\mathfrak{C}$ — finitary 2-category

$\mathcal{S}[\mathfrak{C}]$ — the set of isoclasses of indecomposable 1-morphisms in $\mathfrak{C}$ together with 0

Note: if F, G are indecomposable 1-morphisms, then $F \circ G$ usually decomposes

Hence: $\mathcal{S}[\mathfrak{C}]$ has the natural structure of a multisemigroup, that is a "semigroup" with multivalued operation

$\mathcal{S}[\mathfrak{C}]$ describes combinatorics of horizontal composition in $\mathfrak{C}$

Note: $\mathcal{S}[\mathfrak{C}]$ does not "remember" 2-morphisms in a straightforward way, however, "indecomposability", used to define $\mathcal{S}[\mathfrak{C}]$, is a property of the 2-endomorphism algebra

Example: $\mathcal{S}[\mathfrak{C}_A]$ is, in fact, a semigroup (the operation is single valued)

Combinatorics of finitary 2-categories

$\mathfrak{C}$ — finitary 2-category

$\mathcal{S}[\mathfrak{C}]$ — the set of isoclasses of indecomposable 1-morphisms in $\mathfrak{C}$ together with 0

Note: if F, G are indecomposable 1-morphisms, then $F \circ G$ usually decomposes

Hence: $\mathcal{S}[\mathfrak{C}]$ has the natural structure of a multisemigroup, that is a "semigroup" with multivalued operation

$\mathcal{S}[\mathfrak{C}]$ describes combinatorics of horizontal composition in $\mathfrak{C}$

Note: $\mathcal{S}[\mathfrak{C}]$ does not "remember" 2-morphisms in a straightforward way, however, "indecomposability", used to define $\mathcal{S}[\mathfrak{C}]$, is a property of the 2-endomorphism algebra

Example: $\mathcal{S}[\mathfrak{C}_A]$ is, in fact, a semigroup (the operation is single valued)

Combinatorics of finitary 2-categories

$\mathfrak{C}$ — finitary 2-category

$\mathcal{S}[\mathfrak{C}]$ — the set of isoclasses of indecomposable 1-morphisms in $\mathfrak{C}$ together with 0

Note: if F, G are indecomposable 1-morphisms, then $F \circ G$ usually decomposes

Hence: $\mathcal{S}[\mathfrak{C}]$ has the natural structure of a multisemigroup, that is a "semigroup" with multivalued operation

$\mathcal{S}[\mathfrak{C}]$ describes combinatorics of horizontal composition in $\mathfrak{C}$

Note: $\mathcal{S}[\mathfrak{C}]$ does not "remember" 2-morphisms in a straightforward way, however, "indecomposability", used to define $\mathcal{S}[\mathfrak{C}]$, is a property of the 2-endomorphism algebra

Example: $\mathcal{S}[\mathfrak{C}_A]$ is, in fact, a semigroup (the operation is single valued)

Combinatorics of finitary 2-categories

$\mathfrak{C}$ — finitary 2-category

$\mathcal{S}[\mathfrak{C}]$ — the set of isoclasses of indecomposable 1-morphisms in $\mathfrak{C}$ together with 0

Note: if F, G are indecomposable 1-morphisms, then $F \circ G$ usually decomposes

Hence: $\mathcal{S}[\mathfrak{C}]$ has the natural structure of a multisemigroup, that is a "semigroup" with multivalued operation

$\mathcal{S}[\mathfrak{C}]$ describes combinatorics of horizontal composition in $\mathfrak{C}$

Note: $\mathcal{S}[\mathfrak{C}]$ does not "remember" 2-morphisms in a straightforward way, however, "indecomposability", used to define $\mathcal{S}[\mathfrak{C}]$, is a property of the 2-endomorphism algebra

Example: $\mathcal{S}[\mathfrak{C}_A]$ is, in fact, a semigroup (the operation is single valued)

Combinatorics of finitary 2-categories

$\mathfrak{C}$ — finitary 2-category

$\mathcal{S}[\mathfrak{C}]$ — the set of isoclasses of indecomposable 1-morphisms in $\mathfrak{C}$ together with 0

Note: if F, G are indecomposable 1-morphisms, then $F \circ G$ usually decomposes

Hence: $\mathcal{S}[\mathfrak{C}]$ has the natural structure of a multisemigroup, that is a "semigroup" with multivalued operation

$\mathcal{S}[\mathfrak{C}]$ describes combinatorics of horizontal composition in $\mathfrak{C}$

Note: $\mathcal{S}[\mathfrak{C}]$ does not "remember" 2-morphisms in a straightforward way, however, "indecomposability", used to define $\mathcal{S}[\mathfrak{C}]$, is a property of the 2-endomorphism algebra

Example: $\mathcal{S}[\mathfrak{C}_A]$ is, in fact, a semigroup (the operation is single valued)

Combinatorics of finitary 2-categories

$\mathfrak{C}$ — finitary 2-category

$\mathcal{S}[\mathfrak{C}]$ — the set of isoclasses of indecomposable 1-morphisms in $\mathfrak{C}$ together with 0

Note: if F, G are indecomposable 1-morphisms, then $F \circ G$ usually decomposes

Hence: $\mathcal{S}[\mathfrak{C}]$ has the natural structure of a multisemigroup, that is a "semigroup" with multivalued operation

$\mathcal{S}[\mathfrak{C}]$ describes combinatorics of horizontal composition in $\mathfrak{C}$

Note: $\mathcal{S}[\mathfrak{C}]$ does not "remember" 2-morphisms in a straightforward way, however, "indecomposability", used to define $\mathcal{S}[\mathfrak{C}]$, is a property of the 2-endomorphism algebra

Example: $\mathcal{S}[\mathfrak{C}_A]$ is, in fact, a semigroup (the operation is single valued)

Combinatorics of finitary 2-categories

$\mathfrak{C}$ — finitary 2-category

$\mathcal{S}[\mathfrak{C}]$ — the set of isoclasses of indecomposable 1-morphisms in $\mathfrak{C}$ together with 0

Note: if F, G are indecomposable 1-morphisms, then $F \circ G$ usually decomposes

Hence: $\mathcal{S}[\mathfrak{C}]$ has the natural structure of a multisemigroup, that is a "semigroup" with multivalued operation

$\mathcal{S}[\mathfrak{C}]$ describes combinatorics of horizontal composition in $\mathfrak{C}$

Note: $\mathcal{S}[\mathfrak{C}]$ does not "remember" 2-morphisms in a straightforward way, however, "indecomposability", used to define $\mathcal{S}[\mathfrak{C}]$, is a property of the 2-endomorphism algebra

Example: $\mathcal{S}[\mathfrak{C}_A]$ is, in fact, a semigroup (the operation is single valued)

2-representation theory of finitary 2-categories

$$\boxed{\text{Combin of semigrps}} \xrightarrow{\text{Green's relations}} \boxed{\text{Simple reps of semigrps}}$$

$$\boxed{\text{Combinatorics of } \mathcal{S}[\mathfrak{C}]} \xrightarrow{\text{Green's relations}} \boxed{\text{Cell 2-representations of } \mathfrak{C}}$$

- cell 2-representations of $\mathfrak{C}$ are "most natural" candidates to be simple 2-representations
- they are "2-generated" by any simple object
- they have the "smallest possible" endomorphism category
- they have "maximal possible" annihilators
- in many cases they have appropriate "uniqueness" properties

There is a Morita theory for the additive 2-representation theory of finitary 2-categories.

Apart from that the 2-representation theory of finitary 2-categories is a big mystery

2-representation theory of finitary 2-categories

$\boxed{\text{Combin of semigrps}} \xrightarrow{\text{Green's relations}} \boxed{\text{Simple reps of semigrps}}$

$\boxed{\text{Combinatorics of } \mathcal{S}[\mathfrak{C}]} \xrightarrow{\text{Green's relations}} \boxed{\text{Cell 2-representations of } \mathfrak{C}}$

- cell 2-representations of $\mathfrak{C}$ are "most natural" candidates to be simple 2-representations
- they are "2-generated" by any simple object
- they have the "smallest possible" endomorphism category
- they have "maximal possible" annihilators
- in many cases they have appropriate "uniqueness" properties

There is a Morita theory for the additive 2-representation theory of finitary 2-categories.

Apart from that the 2-representation theory of finitary 2-categories is a big mystery

2-representation theory of finitary 2-categories

$\boxed{\text{Combin of semigrps}} \xrightarrow{\text{Green's relations}} \boxed{\text{Simple reps of semigrps}}$

$\boxed{\text{Combinatorics of } \mathcal{S}[\mathfrak{C}]} \xrightarrow{\text{Green's relations}} \boxed{\text{Cell 2-representations of } \mathfrak{C}}$

- cell 2-representations of $\mathfrak{C}$ are "most natural" candidates to be simple 2-representations
- they are "2-generated" by any simple object
- they have the "smallest possible" endomorphism category
- they have "maximal possible" annihilators
- in many cases they have appropriate "uniqueness" properties

There is a Morita theory for the additive 2-representation theory of finitary 2-categories.

Apart from that the 2-representation theory of finitary 2-categories is a big mystery

2-representation theory of finitary 2-categories

$\boxed{\text{Combin of semigrps}} \xrightarrow{\text{Green's relations}} \boxed{\text{Simple reps of semigrps}}$

$\boxed{\text{Combinatorics of } \mathcal{S}[\mathfrak{C}]} \xrightarrow{\text{Green's relations}} \boxed{\text{Cell 2-representations of } \mathfrak{C}}$

- cell 2-representations of $\mathfrak{C}$ are "most natural" candidates to be simple 2-representations
- they are "2-generated" by any simple object
- they have the "smallest possible" endomorphism category
- they have "maximal possible" annihilators
- in many cases they have appropriate "uniqueness" properties

There is a Morita theory for the additive 2-representation theory of finitary 2-categories.

Apart from that the 2-representation theory of finitary 2-categories is a big mystery

2-representation theory of finitary 2-categories

Combin of semigrps $\xrightarrow{\text{Green's relations}}$ Simple reps of semigrps

Combinatorics of $\mathcal{S}[\mathfrak{C}]$ $\xrightarrow{\text{Green's relations}}$ Cell 2-representations of $\mathfrak{C}$

- cell 2-representations of $\mathfrak{C}$ are "most natural" candidates to be simple 2-representations
- they are "2-generated" by any simple object
- they have the "smallest possible" endomorphism category
- they have "maximal possible" annihilators
- in many cases they have appropriate "uniqueness" properties

There is a Morita theory for the additive 2-representation theory of finitary 2-categories.

Apart from that the 2-representation theory of finitary 2-categories is a big mystery

2-representation theory of finitary 2-categories

$\boxed{\text{Combin of semigrps}} \xrightarrow{\text{Green's relations}} \boxed{\text{Simple reps of semigrps}}$

$\boxed{\text{Combinatorics of } \mathcal{S}[\mathfrak{C}]} \xrightarrow{\text{Green's relations}} \boxed{\text{Cell 2-representations of } \mathfrak{C}}$

- cell 2-representations of $\mathfrak{C}$ are "most natural" candidates to be simple 2-representations
- they are "2-generated" by any simple object
- they have the "smallest possible" endomorphism category
- they have "maximal possible" annihilators
- in many cases they have appropriate "uniqueness" properties

There is a Morita theory for the additive 2-representation theory of finitary 2-categories.

Apart from that the 2-representation theory of finitary 2-categories is a big mystery

2-representation theory of finitary 2-categories

$$\boxed{\text{Combin of semigrps}} \xrightarrow{\text{Green's relations}} \boxed{\text{Simple reps of semigrps}}$$

$$\boxed{\text{Combinatorics of } \mathcal{S}[\mathfrak{C}]} \xrightarrow{\text{Green's relations}} \boxed{\text{Cell 2-representations of } \mathfrak{C}}$$

- cell 2-representations of $\mathfrak{C}$ are "most natural" candidates to be simple 2-representations
- they are "2-generated" by any simple object
- they have the "smallest possible" endomorphism category
- they have "maximal possible" annihilators
- in many cases they have appropriate "uniqueness" properties

There is a Morita theory for the additive 2-representation theory of finitary 2-categories.

Apart from that the 2-representation theory of finitary 2-categories is a big mystery

$\boxed{\text{Combin of semigrps}} \xrightarrow{\text{Green's relations}} \boxed{\text{Simple reps of semigrps}}$

$\boxed{\text{Combinatorics of } \mathcal{S}[\mathfrak{C}]} \xrightarrow{\text{Green's relations}} \boxed{\text{Cell 2-representations of } \mathfrak{C}}$

- cell 2-representations of $\mathfrak{C}$ are "most natural" candidates to be simple 2-representations
- they are "2-generated" by any simple object
- they have the "smallest possible" endomorphism category
- they have "maximal possible" annihilators
- in many cases they have appropriate "uniqueness" properties

There is a Morita theory for the additive 2-representation theory of finitary 2-categories.

Apart from that the 2-representation theory of finitary 2-categories is a big mystery

2-representation theory of finitary 2-categories

$\boxed{\text{Combin of semigrps}} \xrightarrow{\text{Green's relations}} \boxed{\text{Simple reps of semigrps}}$

$\boxed{\text{Combinatorics of } \mathcal{S}[\mathfrak{C}]} \xrightarrow{\text{Green's relations}} \boxed{\text{Cell 2-representations of } \mathfrak{C}}$

- cell 2-representations of $\mathfrak{C}$ are "most natural" candidates to be simple 2-representations
- they are "2-generated" by any simple object
- they have the "smallest possible" endomorphism category
- they have "maximal possible" annihilators
- in many cases they have appropriate "uniqueness" properties

There is a Morita theory for the additive 2-representation theory of finitary 2-categories.

Apart from that the 2-representation theory of finitary 2-categories is a big mystery

2-representation theory of finitary 2-categories

$$\boxed{\text{Combin of semigrps}} \xrightarrow{\text{Green's relations}} \boxed{\text{Simple reps of semigrps}}$$

$$\boxed{\text{Combinatorics of } \mathcal{S}[\mathfrak{C}]} \xrightarrow{\text{Green's relations}} \boxed{\text{Cell 2-representations of } \mathfrak{C}}$$

- cell 2-representations of $\mathfrak{C}$ are "most natural" candidates to be simple 2-representations
- they are "2-generated" by any simple object
- they have the "smallest possible" endomorphism category
- they have "maximal possible" annihilators
- in many cases they have appropriate "uniqueness" properties

There is a Morita theory for the additive 2-representation theory of finitary 2-categories.

Apart from that the 2-representation theory of finitary 2-categories is a big mystery

2-representation theory of finitary 2-categories

$$\boxed{\text{Combin of semigrps}} \xrightarrow{\text{Green's relations}} \boxed{\text{Simple reps of semigrps}}$$

$$\boxed{\text{Combinatorics of } \mathcal{S}[\mathfrak{C}]} \xrightarrow{\text{Green's relations}} \boxed{\text{Cell 2-representations of } \mathfrak{C}}$$

- cell 2-representations of $\mathfrak{C}$ are "most natural" candidates to be simple 2-representations
- they are "2-generated" by any simple object
- they have the "smallest possible" endomorphism category
- they have "maximal possible" annihilators
- in many cases they have appropriate "uniqueness" properties

There is a Morita theory for the additive 2-representation theory of finitary 2-categories.

Apart from that the 2-representation theory of finitary 2-categories is a big mystery

THANK YOU!!!