

Higher representation theory

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Elementary example: the regular representation of S_2

$$S_2 = \{e, s\}, \quad s^2 = e$$

$\mathbf{H} := \{H_e, H_s\}$ — standard basis of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$

actions: $x \cdot H_y := H_{xy}$

$$\text{matrices: } [e]_{\mathbf{H}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [s]_{\mathbf{H}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\underline{H}_e := e, \quad \underline{H}_s = e + s, \quad \underline{H}_s^2 = 2\underline{H}_s$$

$\underline{\mathbf{H}} := \{\underline{H}_e, \underline{H}_s\}$ — new (Kazhdan-Lusztig) basis

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Elementary example: categorification of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$, part 1

$D := \mathbb{C}[x]/(x^2)$ — the algebra of dual numbers

$\mathcal{C} := D\text{-mod}$

$B := D \otimes_{\mathbb{C}} D \in D\text{-mod-}D$

$F_e := \text{Id} : \mathcal{C} \rightarrow \mathcal{C}, \quad F_s := B \otimes_D - : \mathcal{C} \rightarrow \mathcal{C}$

$F_s \circ F_s \cong F_s \oplus F_s$

$\mathcal{F} := \text{add}(F_e, F_s)$ — tensor category

$V := [\mathcal{C}]_{\oplus}$ — split Grothendieck group

basis \mathbf{Q} : $[\mathbb{C}]$ (class of simple) and $[D]$ (class of indec. projective)

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$A := [\mathcal{F}]_{\oplus}$ — split Grothendieck group

basis: $[F_e]$ and $[F_s]$

$\mathbb{C} \otimes_{\mathbb{Z}} A$ acts on $\mathbb{C} \otimes_{\mathbb{Z}} V$

$$[[F_e]]_{\mathbb{Q}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } [[F_s]]_{\mathbb{Q}} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

This is a **categorification** of ${}_{\mathbb{C}S_2}\mathbb{C}S_2$ in the KL basis

Note: This is not an action of S_2 and I do not know whether there is any reasonable action of S_2 around (but there is an action of the braid group B_2 on a certain derived category)

This generalizes to all finite Coxeter groups

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Categorification

What is **categorification**?

“**Upgrade**” of set-theoretic notions to category theoretic

What happened in the above example?

a **vector space** became a **category**

a **function** (linear map) became a **functor**

an **algebra** became a **tensor category** (or a **2-category**)

in the same spirit:

an **equality of functions** becomes an **isomorphism of functors**

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Advantage: stronger knot invariant

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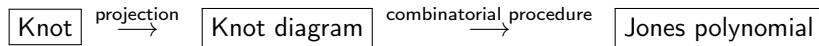
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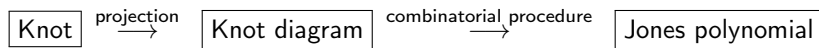
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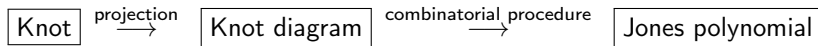
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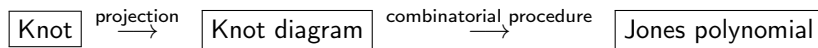
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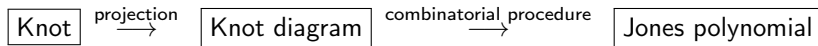
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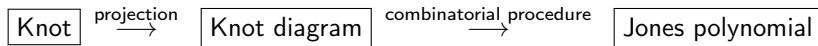
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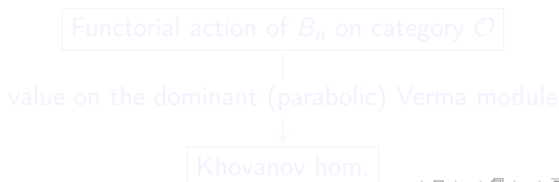
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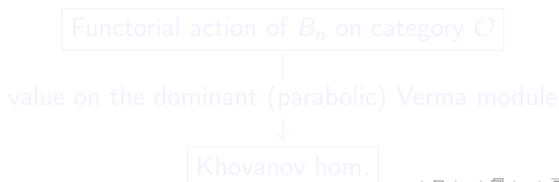
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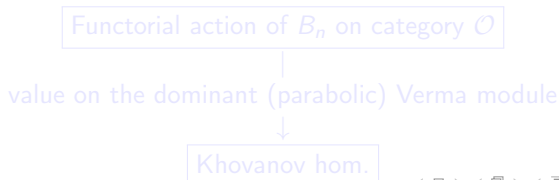
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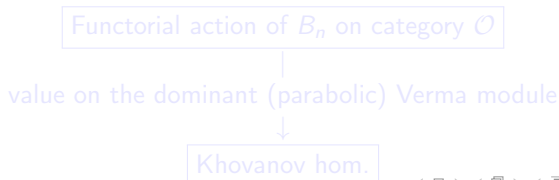
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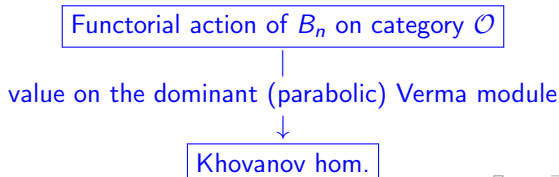
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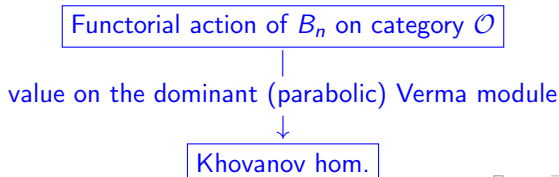
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Definition. A **2-category** is a category enriched over the monoidal category **Cat** of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category \mathcal{C} is given by the following data:

- ▶ objects of \mathcal{C} ;
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Principal example. The category **Cat** is a 2-category.

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2-functors and 2-representations

\mathcal{A} and \mathcal{C} — two 2-categories

Definition. A **2-functor** $F : \mathcal{A} \rightarrow \mathcal{C}$ is a functor which sends 1-morphisms to 1-morphisms and 2-morphisms to 2-morphisms in a way that is coordinated with all the categorical structures (domains, codomains, identities and compositions).

Definition. A **2-representation** of a 2-category \mathcal{C} is a 2-functor from \mathcal{C} to some “classical” 2-category.

A special case of “higher representation theory” is the
2-representation theory of 2-categories

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Khovanov homology \longrightarrow functorial action of B_n

functorial action of B_n = 2-representation of a certain 2-category

In the case of Khovanov homology this 2-cat. is Rouquier's 2-braid group

In general: A — algebra

Definition: A categorification of A is a 2-category \mathfrak{A} which decategorifies (e.g. via the “Grothendieck group” construction) to A

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Application to Broué's abelian defect group conjecture for S_n

Rough conjecture: Certain two algebras A and B (related to modular representations of S_n) are derived equivalent.

Approach of Chuang and Rouquier:

- ▶ Categorify $U(\mathfrak{sl}_2)$ to a 2-category \mathfrak{G}
- ▶ Construct "minimal" 2-representations of \mathfrak{G}
- ▶ Prove "uniqueness" of "minimal" 2-representations of \mathfrak{G} (this implies equivalence of certain module categories)
- ▶ Prove that "filtrations" by "minimal" 2-representations of \mathfrak{G} gives rise to derived equivalences between certain module categories
- ▶ Realize $A\text{-mod}$ and $B\text{-mod}$ in a "nice" way inside some 2-representation of \mathfrak{G} which is "filtered" by "minimal" 2-representations

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Application to Lie superalgebras

Want: Describe blocks of finite dimensional supermodules for $\mathfrak{sl}(m|n)$

Approach of Brundan and Stroppel:

- ▶ Define on such a block the structure of a 2-representation of a certain 2-category \mathfrak{G}
- ▶ Construct a combinatorially defined candidate A for the answer
- ▶ Define on $A\text{-mod}$ the structure of a 2-representation of \mathfrak{G}
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- ▶ Describe the 2-category $\mathcal{C}\text{-mod}$ of 2-representations of \mathcal{C}
- ▶ Compare $\mathcal{C}\text{-mod}$ and $\mathcal{D}\text{-mod}$ for some other 2-category \mathcal{D}

More concrete questions:

- ▶ What are “simple” 2-representations of \mathcal{C} ?
- ▶ What kind of uniqueness properties hold for 2-representations of \mathcal{C} ?
- ▶ Is there any kind of Jordan-Hölder property?
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2-analogues of finite dimensional algebras

\mathbb{k} — algebraically closed field

Definition: A 2-category \mathcal{C} is called **finitary** (over \mathbb{k}) provided that

- ▶ \mathcal{C} has finitely many objects;
- ▶ each $\mathcal{C}(i, j)$ is additive, \mathbb{k} -linear, idempotent split, with finitely many indecomposables (up to isomorphism);
- ▶ all spaces of 2-morphisms are finite dimensional (over \mathbb{k});
- ▶ the identity 1-morphisms are indecomposable.

2-analogues of finite dimensional algebras

\mathbb{k} — algebraically closed field

Definition: A 2-category \mathcal{C} is called *finitary* (over \mathbb{k}) provided that

- ▶ \mathcal{C} has finitely many objects;
- ▶ each $\mathcal{C}(i, j)$ is additive, \mathbb{k} -linear, idempotent split, with finitely many indecomposables (up to isomorphism);
- ▶ all spaces of 2-morphisms are finite dimensional (over \mathbb{k});
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2-analogues of finite dimensional algebras

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Example: projective functors on $A\text{-mod}$

\mathbb{k} — algebraically closed field

A — finite dimensional connected associative \mathbb{k} -algebra

Definition: $F : A\text{-mod} \rightarrow A\text{-mod}$ is **projective** if it is isomorphic to tensoring with a projective bimodule.

Definition: The 2-category \mathcal{C}_A is defined as follows:

- ▶ \mathcal{C} has one object \clubsuit (which is identified with $A\text{-mod}$);
- ▶ 1-morphisms in $\mathcal{C}(\clubsuit, \clubsuit)$ are functors isomorphic to direct sum of the identity and projective functors;
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Combinatorics of finitary 2-categories

\mathcal{C} — finitary 2-category

$\mathcal{S}[\mathcal{C}]$ — the set of isoclasses of indecomposable 1-morphisms in \mathcal{C} together with 0

Note: if F, G are indecomposable 1-morphisms, then $F \circ G$ usually decomposes

Hence: $\mathcal{S}[\mathcal{C}]$ has the natural structure of a **multisemigroup**, that is a “semigroup” with multivalued operation

$\mathcal{S}[\mathcal{C}]$ describes combinatorics of horizontal composition in \mathcal{C}

Note: $\mathcal{S}[\mathcal{C}]$ does not “remember” 2-morphisms in a straightforward way, however, “indecomposability”, used to define $\mathcal{S}[\mathcal{C}]$, is a property of the 2-endomorphism algebra

Example: $\mathcal{S}[\mathcal{C}_A]$ is, in fact, a semigroup (the operation is single valued)

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2-representation theory of finitary 2-categories

Combin of semigrps

Green's relations \longrightarrow

Simple reps of semigrps

Combinatorics of $\mathcal{S}[\mathcal{C}]$

Green's relations \longrightarrow

Cell 2-representations of \mathcal{C}

- ▶ cell 2-representations of \mathcal{C} are “most natural” candidates to be simple 2-representations
- ▶ they are “2-generated” by any simple object
- ▶ they have the “smallest possible” endomorphism category
- ▶ they have “maximal possible” annihilators
- ▶ in many cases they have appropriate “uniqueness” properties

There is a Morita theory for the additive 2-representation theory of finitary 2-categories.

Apart from that the 2-representation theory of finitary 2-categories is a big mystery

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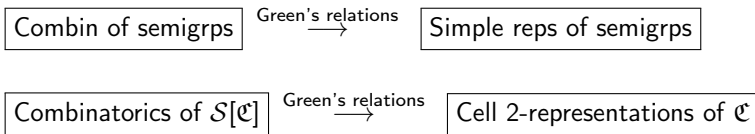
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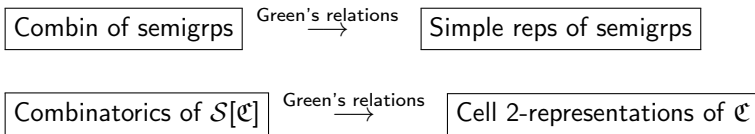
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