STRATIFIED ALGEBRAS ARISING IN LIE THEORY

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1. Quasi-hereditary algebras

Definition. (Cline, Parshall, Scott) Let A be a finite-dimensional algebra over a field, k, and $\leq$ be a partial order on the set Λ, indexing the isomorphism classes $\{L(\lambda)\}$ of simple A-modules. We say that $(A, \leq)$ is a quasi-hereditary algebra if and only if for all $\lambda \in \Lambda$ there exists an A-module, $\Delta(\lambda)$, called a *standard module*, such that

- $\Delta(\lambda)$ surjects onto $L(\lambda)$ and the composition factors, $L(\mu)$, of the kernel satisfy $\mu < \lambda$;
- the indecomposable projective cover $P(\lambda)$ of $L(\lambda)$ surjects onto $\Delta(\lambda)$ such that the kernel of this surjection is filtered by modules $\Delta(\mu)$ with $\lambda < \mu$.

2. Basic example: The category $\mathcal{O}$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — semi-simple finite-dimensional complex Lie algebra with a fixed triangular decomposition

Definition. (Bernstein, I.Gelfand, S.Gelfand) The category $\mathcal{O}$ is the full subcategory in $\mathfrak{g}$-mod, consisting of all finitely generated, $\mathfrak{h}$-diagonalizable and $U(\mathfrak{n}_+)$-locally finite modules.

Verma modules. $\lambda \in \mathfrak{h}^*$, $\mathbb{C}_\lambda = \mathbb{C}$ is an $\mathfrak{h} \oplus \mathfrak{n}_+$-module via $(h+n)(c) = \lambda(h)c$, $n \in \mathfrak{n}$, $h \in \mathfrak{h}$, $c \in \mathbb{C}$.

$$\mathcal{O} \ni M(\lambda) = U(\mathfrak{g}) \bigotimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda.$$

Simple highest weight modules. $M(\lambda)$ has the unique simple quotient $L(\lambda)$ and $\{L(\lambda), \lambda \in \mathfrak{h}^*\}$ is a complete set of simple modules in $\mathcal{O}$.

Theorem. (Bernstein, I.Gelfand, S.Gelfand) Category $\mathcal{O}$ has enough projective modules and every projective module is filtered by Verma modules.

Let $P(\lambda)$ be the indecomposable projective cover of $L(\lambda)$.

Theorem. (BGG-reciprocity)

$$[P(\lambda) : M(\mu)] = (M(\mu) : L(\lambda)).$$

$\lambda \leq \mu$ if and only if $\mu - \lambda$ is a linear combination of simple roots with non-negative integral coefficients.

$(M(\lambda) : L(\mu)) \neq 0$ implies $\mu \leq \lambda$.

From the BGG-reciprocity it now follows that $[P(\lambda) : M(\mu)] \neq 0$ implies $\lambda \leq \mu$.

$Z(\mathfrak{g})$ – the center of $U(\mathfrak{g})$. For $\theta \in Z(\mathfrak{g})^*$ set

$$\mathcal{O}^\theta = \{M \in \mathcal{O} : (z - \theta(z))^k M = 0, z \in Z(\mathfrak{g}), \text{ for some } k \in \mathbb{N}\}.$$

$\mathcal{O} = \oplus_{\theta \in Z(\mathfrak{g})^*} \mathcal{O}^\theta$ and all $\mathcal{O}^\theta$ have finitely many simples.

Corollary. Every $\mathcal{O}^\theta$ is equivalent to the module category of a quasi-hereditary algebra. (Equivalently: Let λ be dominant and W be the Weyl group of $\mathfrak{g}$. Then the algebra $\mathrm{End}_{\mathfrak{g}}(\oplus_{w \in W} P(w \cdot \lambda))$ is quasi hereditary.)

3. Parabolic generalizations of $\mathcal{O}$

$\mathfrak{p} \supset \mathfrak{h} \oplus \mathfrak{n}_+$ — a parabolic subalgebra, $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{h}_\mathfrak{a} \oplus \mathfrak{n}$, $\mathfrak{n}$ – the radical of $\mathfrak{p}$, $\tilde{\mathfrak{a}} = \mathfrak{a} \oplus \mathfrak{h}_\mathfrak{a}$ – reductive Levi factor, $\mathfrak{a}$ – semi-simple, $\mathfrak{h}_\mathfrak{a}$ – center of $\tilde{\mathfrak{a}}$.

Let Λ be a “reasonable” category of $\mathfrak{a}$-modules.

Definition. The category $\mathcal{O}(\mathfrak{p}, \Lambda)$ is the full subcategory in the category $\mathfrak{g}$-mod, consisting of all finitely generated, $\mathfrak{h}_\mathfrak{a}$-diagonalizable and $U(\mathfrak{n})$-locally finite modules, which decompose into a direct sum of modules from Λ, when viewed as $\mathfrak{a}$-modules.

Example. (Rocha-Caridi) $\Lambda = Fin.Dim.(\mathfrak{a})$ — the category of finite-dimensional $\mathfrak{a}$-modules (is “reasonable”).

Theorem. The category $\mathcal{O}(\mathfrak{p}, Fin.Dim.(\mathfrak{a}))$ decomposes into a direct sum of full subcategories, each of which is equivalent to the module category of a quasi-hereditary algebra.

Remark. Standard module in the category $\mathcal{O}(\mathfrak{p}, Fin.Dim.(\mathfrak{a}))$ are *generalized Verma modules* $M(\lambda, V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$, where V is a simple finite-dimensional $\mathfrak{a}$-module, $\lambda \in \mathfrak{h}_\mathfrak{a}^*$, $\mathfrak{n}V = 0$, and $\mathfrak{h}_\mathfrak{a}$ acts on V via λ.

Theorem. (Futorny-M.) Assume that

1. Λ is a semi-simple category;
2. Λ is stable under taking submodules and under tensoring with finite-dimensional $\mathfrak{a}$-modules;
3. every module in Λ is finitely generated and the action of $Z(\mathfrak{a})$ on it is diagonalizable;

Then the category $\mathcal{O}(\mathfrak{p}, \Lambda)$ decomposes into a direct sum of full subcategories, each of which is equivalent to the module category of a quasi-hereditary algebra.

Remark. The conditions above describe what does "reasonable" mean in the case, when one looks for quasi-hereditary algebras. To produce examples one has to construct Λ.

Basic Example. $\mathfrak{a} = \mathfrak{sl}(2, \mathbb{C})$ and λ is the category of simple weight $\mathfrak{a}$-modules without highest and lowest weight such that their central character (the eigenvalue of the quadratic Casimir) is not the square of an integer.

Other examples: Certain categories of Harish-Chandra modules for $\mathfrak{a} = \mathfrak{sl}(2, \mathbb{C})$ and generic Gelfand-Zetlin modules for $\mathfrak{a} = \mathfrak{sl}(n, \mathbb{C})$. Also categories, generatied (with respect to tensoring with finite-dimensional modules) by a simple module with a generic (with the trivial integral Weyl group) central character.

4. Examples of $\mathcal{O}(\mathfrak{p}, \Lambda)$, which do not lead to quasi-hereditary algebras

Basic Example. Let $\mathfrak{a} = \mathfrak{sl}(2, \mathbb{C})$ and V be a simple weight $\mathfrak{a}$-modules without highest and lowest weight, whose central character is the square of an integer. Then Λ consists of all subquotients of modules $F \otimes V$, where F is finite-dimensional. In this case Λ contains indecomposable modules, which are not simple. Actually, Λ decomposes into a direct sum of full subcategories, each of which is equivalent to the category of modules over either $\mathbb{C}$ or $\mathbb{C}[x]/(x^2)$.

General Example. Λ is the category of modules with integral support from the category $\mathcal{O}$ (for $\mathfrak{a}$), complete in the sense of Enright. This category has a "strange" abelian structure, which is not inherited from $\mathfrak{a}$-mod. In particular, simple objects in the category are projective Verma modules, which are usually non-simple. This Λ decomposes into a direct sum of full subcategories, each of which is equivalent to the module category over a commutative local algebra (in fact this is either the coinvariant algebra or the subalgebra of some invariants in it).

Theorem. (Futorny-König-M.) For both examples above the category $\mathcal{O}(\mathfrak{p}, \Lambda)$ decomposes into a direct sum of full subcategories, each of which is equivalent to the module category over a finite-dimensional associative algebra. These algebras are not quasi-hereditary in general because they usually have infinite global dimension.

5. Properly stratified algebras and $\mathcal{O}(\mathfrak{p}, \Lambda)$

Definition. (Dlab) Let A be a finite-dimensional algebra over a field, k, and $\leq$ be a partial order on the set Λ, indexing the isomorphism classes $\{L(\lambda)\}$ of simple A-modules. We say that $(A, \leq)$ is a quasi-hereditary algebra if and only if for all $\lambda \in \Lambda$ there exists an A-module, $\Delta(\lambda)$, called a *standard module*, and an A-module, $\overline{\Delta}(\lambda)$, called a *proper standard module*,such that

- $\overline{\Delta}(\lambda)$ surjects onto $L(\lambda)$ and the composition factors, $L(\mu)$, of the kernel satisfy $\mu < \lambda$;
- the indecomposable projective cover $P(\lambda)$ of $L(\lambda)$ surjects onto $\Delta(\lambda)$ such that the kernel of this surjection is filtered by modules $\Delta(\mu)$ with $\lambda < \mu$;
- for every λ the module $\Delta(\lambda)$ is filtered by $\overline{\Delta}(\lambda)$.

Remarks.

1. A properly stratified algebra is quasi-hereditary if and only if it has finite global dimension if and only if one can take $\Delta(\lambda) = \overline{\Delta}(\lambda)$ for all λ.
2. Properly stratified algebras is a subclass in the class of stratified algebras, introduced earlier (than properly stratified algebras) by Cline, Parshall and Scott. Stratified algebras will appear later in this talk.

Theorem. (Futorny-König-M.) For both examples above the category $\mathcal{O}(\mathfrak{p}, \Lambda)$ decomposes into a direct sum of full subcategories, each of which is equivalent to the module category over a properly stratified algebra. The proper standard modules in the natural proper stratification are generalized Verma modules $M(\lambda, V)$ (i.e. modules, parabolically induced from simple objects V in Λ). The standard modules in the natural proper stratification are modules $M(\lambda, \tilde{V})$, parabolically induced from indecomposable projective objects $\tilde{V}$ in Λ.

Example. $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$, $\mathfrak{a} = \mathfrak{sl}(2, \mathbb{C})$, principal block.

Simple modules: $1 = L(1) \quad < \quad 2 = L(2) \quad < \quad 3 = L(3)$.

Projective, standard and proper standard modules:

$$P(1) = \begin{array}{ccccccc} & 1 & & & & & \\ & & & & 2 & & \\ 1 & & 1 & & & & 3 \\ & & & 2 & & 2 & \\ 1 & & 1 & & & & 3 \\ & & & & 2 & & \\ & 1 & & & & & \end{array}, \quad P(2) = \begin{array}{ccccccc} & & & & 2 & & \\ & 1 & & & & & 3 \\ & & & 2 & & 2 & \\ 1 & & 1 & & & & 3 \\ & & & & 2 & & \\ & 1 & & & & & \end{array},$$

$$\Delta(3) = P(3) = \begin{array}{ccc} & & 3 \\ & 2 & \\ 1 & & 3 \\ & 2 & \\ 1 & & \end{array}, \quad \Delta(2) = \begin{array}{cc} & 2 \\ 1 & \\ & 2 \\ 1 & \end{array}, \quad \Delta(1) = \begin{array}{c} 1 \\ 1 \end{array}.$$

Theorem. (BGG-reciprocity for $\mathcal{O}(\mathfrak{p}, \Lambda)$)

$$[P(\lambda, V) : \Delta(\mu, W)] = (\overline{\Delta}(\mu, W) : L(\lambda, V)).$$

Reformulation 1.

$$[P(\lambda, V) : \overline{\Delta}(\mu, W)] = (\Delta(\mu, W) : L(\lambda, V)).$$

Reformulation 2.

$$[P(\lambda, V) : \Delta(\mu, W)][\Delta(\mu, W) : \overline{\Delta}(\mu, W)] = (\Delta(\mu, W) : L(\lambda, V)).$$

Reformulation 3.

$$[P(\lambda, V) : \overline{\Delta}(\mu, W)] = [\Delta(\mu, W) : \overline{\Delta}(\mu, W)](\overline{\Delta}(\mu, W) : L(\lambda, V)).$$

Remark. $[\Delta(\mu, W) : \overline{\Delta}(\mu, W)] = (\tilde{W} : W)$, where $\tilde{W}$ is the projective cover of W in Λ.

6. Harish-Chandra bimodules and properly stratified algebras

Definition. (Bernstein-S.Gelfand-Joseph-Enright) Let $\mathfrak{g}$ be a semi-simple Lie algebra. A $\mathfrak{g}$-bimodule, V, is called a *Harish-Chandra bimodule*, provided that V is finitely generated and, as a $\mathfrak{g}$-module under the canonical diagonal action (twisted on the right by the Chevalley involution), is a direct sum of finite-dimensional modules.

Denote by $\mathcal{H}^1$ the category of all Harish-Chandra bimodules, the right action of $Z(\mathfrak{g})$ on which is diagonalizable.

Theorem. (König-M.) $\mathcal{H}^1$ decomposes into a direct sum of full subcategories each of which is equivalent to the module category of a properly stratified algebra. Moreover, these algebras are the same as properly stratified algebras associated with $\mathcal{O}(\mathfrak{p}, \Lambda)$ in the case, when Λ is the category of Enright-complete modules in $\mathcal{O}$ (for the algebra $\mathfrak{a}$) with integral support.

Remark. Properly stratified algebra also appear in the blocks of the "thick" category $\mathcal{O}$ (the action of $\mathfrak{h}$ is no longer diagonalizable but locally finite), studied by Soergel in connection with Harish-Chandra bimodules.

7. Beyond the properly stratified algebras

Theorem. (Futorny-König-M.) Assume that

1. Λ decomposes into a direct sum of full subcategories, each of which is equivalent to the module category of a self-injective local algebra;
2. tensoring with finite-dimensional $\mathfrak{a}$-modules is an exact functor on Λ;
3. every module in Λ is finitely generated and the action of $Z(\mathfrak{a})$ on it is locally finite;
4. for every central character Λ has only finitely many simple objects (up to isomorphism), corresponding to this central character.

Then the category $\mathcal{O}(\mathfrak{p}, \Lambda)$ decomposes into a direct sum of full subcategories, each of which is equivalent to the module category of a properly stratified algebra.

Remark and Example. It is very easy to construct an example of a "reasonable" Λ, which does not satisfy conditions of the above theorem. Indeed, let $\mathfrak{a} = \mathfrak{sl}(3, \mathbb{C})$, α, β are simple roots, $L(\lambda)$ is a simple highest weight module, such that λ is integral and $(\lambda, \alpha \in \mathbb{N})$, $(\lambda, \beta) \notin \mathbb{N}$. Define Λ as the category of all modules V, which have presentation $V_2 \to V_1 \to V \to 0$, where both V_i are direct summands in some $F_i \otimes L(\lambda)$, F_i finite dimensional. Then Λ decomposes into a direct sum of full subcategories each of which is equivalent to the module category over an associative algebra, however these algebras are not local in general.

Remark. (Khomenko-M.) The above effect does not appear if one starts with a simple $\mathfrak{a}$-module, whose annihilator is a minimal primitive ideal (this can be for example a simple generic Gelfand-Zetlin module). This situation is **always** described by properly stratified algebras (of $\mathcal{O}(\mathfrak{p}, \Lambda)$ associated with Enright-complete modules or, equivalently, with Harish-Chandra bimodules).

Theorem. (general BGG-reciprocity, Gomez-M.) Assume that Λ decomposes into a direct sum of full subcategories, each of which is equivalent to the module category of a self-injective finite-dimensional algebra and tensoring with finite-dimensional $\mathfrak{a}$-modules is an exact functor on Λ. Further assume that $\mathcal{O}(\mathfrak{p}, \Lambda)$ decomposes into a direct sum of full subcategories, each of which is equivalent to the module category of a finite-dimensional associative algebra. Then for all simple $V, W \in \Lambda$ and all $\lambda, \nu \in \mathfrak{h}_{\mathfrak{a}}^*$ one has

$$[P(\lambda, V) : M(\mu, \tilde{W})] = (M(\mu, W) : L(\lambda, V)),$$

where $\tilde{W}$ is the indecomposable projective cover of W.

Corollary. The Cartan matrix A of the corresponding block of the category $\mathcal{O}(\mathfrak{p}, \Lambda)$ has the form $A = B^t C B$, where B is the matrix of multiplicities of simples in generalized Verma modules (those induced from simples in Λ) and C is block-diagonal with blocks being the Cartan matrices of those blocks of Λ, which contribute to our block of $\mathcal{O}(\mathfrak{p}, \Lambda)$. In particular, for properly stratified algebras the matrix C is diagonal.

7. Stratified algebras as blocks of $\mathcal{O}(\mathfrak{p}, \Lambda)$

Definition. (Cline, Parshall, Scott) Let A be a finite-dimensional algebra over a field, k, and $\leq$ be a partial pre-order on the set Λ, indexing the isomorphism classes $\{L(\lambda)\}$ of simple A-modules. We say that $(A, \leq)$ is (standardly) stratified if and only if for all $\lambda \in \Lambda$ there exists an A-module, $\Delta(\lambda)$, called a *standard module*, such that

- $(\Delta(\lambda) : L(\mu)) \neq 0$ implies $\mu \leq \lambda$;
- the indecomposable projective cover $P(\lambda)$ of $L(\lambda)$ surjects onto $\Delta(\lambda)$ such that the kernel of this surjection is filtered by modules $\Delta(\mu)$ with $\lambda < \mu$.

Remark. $\leq$ is only assumed to be a pre-order. Hence every finite dimensional algebra is stratified if one takes $\leq$ to be the full relation (all elements are equal). This is an example of a trivial stratification. In general the "stratification" goes along the partial order induced by $\leq$ on the set $\Lambda/\sim$, where $\sim$ is the equivalence relation, generated by $\leq$. Then the "local" part of the stratification has the form eAe, where e is the direct sum of all idempotents, equivalent with respect to $\sim$.

Idea. For the category $\mathcal{O}(\mathfrak{p}, \Lambda)$ the equivalence relation $\sim$ will represent the blocks of Λ (which are not local algebras in general and hence will involve more than 1 point), which contribute to our block of $\mathcal{O}(\mathfrak{p}, \Lambda)$. The corresponding algebras for Λ will be the endomorphism algebras of the standard modules.

Theorem. (Futorny-Gomez-König-M.) Assume that

1. Λ decomposes into a direct sum of full subcategories, each of which is equivalent to the module category of a self-injective algebra;
2. tensoring with finite-dimensional $\mathfrak{a}$-modules is an exact functor on Λ;
3. every module in Λ is finitely generated and the action of $Z(\mathfrak{a})$ on it is locally finite;
4. for every central character Λ has only finitely many simple objects (up to isomorphism), corresponding to this central character.

Then the category $\mathcal{O}(\mathfrak{p}, \Lambda)$ decomposes into a direct sum of full subcategories, each of which is equivalent to the module category of a (standardly) stratified algebra.

Problem. Let V be a simple $\mathfrak{a}$-module and Λ be the category of all modules, which have presentation $V_2 \to V_1 \to V \to 0$, where V_i are direct summands in $F_i \otimes V$, F_i finite-dimensional. Then Λ satisfies conditions 2), 3), and 4) above. Does Λ satisfy 1)? If the answer is "yes" (which I do believe), then what kind of algebras do appear in such Λ? The answer is known for finite-dimensional modules and simple modules with minimal annihilators.