CATEGORY $\mathcal{O}$ AS A SOURCE FOR CATEGORIFICATION

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$\mathfrak{g}$ — semi-simple finite-dimensional Lie algebra over $\mathbb{C}$

$\mathcal{O}$ — Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for $\mathfrak{g}$

$\mathcal{O}_0$ — the regular block of $\mathcal{O}$

$\mathcal{O}_0 \cong A$-mod, where

A — finite-dimensional associative algebra over $\mathbb{C}$

W — Weyl group of $\mathfrak{g}$

A-simples $\leftrightarrow$ elements in W

$K_0(\mathcal{O}_0)$ — Grothendieck group of $\mathcal{O}_0$

projective functors are exact and thus induce endomorphisms of $K_0(\mathcal{O}_0)$

This is a categorification of the regular right $\mathbb{Z}W$-module

twisting functors satisfy braid relations

they are **NOT** equivalences of $\mathcal{O}_0$

derived twisting functors **ARE** equivalences of $\mathcal{D}^b(\mathcal{O}_0)$

This gives a categorification of the regular left $\mathbb{Z}W$-module

twisting and projective functors commute

This gives a categorification of ${}_{\mathbb{Z}W}\mathbb{Z}W_{\mathbb{Z}W}$

A admits a $\mathbb{Z}$-grading

A-gmod — category of graded A-modules

all our functors admit graded lifts

$\mathcal{H}$ — Hecke algebra of W

effect on categorification: change $\mathbb{Z}W$ by $\mathcal{H}$

A is Koszul

Taking certain subcategories one produces categorifications of other modules

$\mathfrak{p}$ — some parabolic subalgebra of $\mathfrak{g}$

$W^{\mathfrak{p}}$ — the corresponding parabolic subgroup of W

$\mathcal{O}_0^{\mathfrak{p}}$ — the corresponding parabolic subcategory of $\mathcal{O}_0$

projective functors preserve $\mathcal{O}_0^{\mathfrak{p}}$

This gives a categorification of the $\mathbb{Z}W$-module, induced from the sign $W^{\mathfrak{p}}$-module

$\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$

λ — partition corresponding to $\mathfrak{p}$

λ' — the conjugate partition

Q — basic projective-injective module in $\mathcal{O}_0^{\mathfrak{p}}$

End(Q)-mod can be viewed as a subcategory of $\mathcal{O}_0^{\mathfrak{p}}$

projective functors preserve End(Q)-mod

$$\boxed{\begin{array}{c}\text{This gives a categorification of the}\\ \text{Specht module corresponding to } \lambda'\end{array}}$$

$W(\mathfrak{p})$ — longest coset representatives in $W^{\mathfrak{p}} \backslash W$

e_w — primitive idempotent of A corresponding to $w \in W$

$e_{\mathfrak{p}} = \sum_{w \in W(\mathfrak{p})} e_w$

$B = e_{\mathfrak{p}} A e_{\mathfrak{p}}$

B-mod can be realized as a subcategory of A-mod

projective functors preserve B-mod

This gives a categorification of the permutation module corresponding to W and $W^{\mathfrak{p}}$

$\mathtt{R}$ — right cell in W

for $w \in \mathtt{R}$ set $P^{\mathtt{R}}(w) = P(w)/X$, where

$X \subset P(w)$ is generated by all $L(v)$, $v \not\leq_{right} w$

$P^{\mathtt{R}} = \oplus_{w \in \mathtt{R}} P^{\mathtt{R}}(w)$

$C = \mathrm{End}(P^{\mathtt{R}})$

C-mod can be realized as a subcategory of A-mod

projective functors preserve C-mod

This gives a categorification of the cell module corresponding to $\mathtt{R}$