QUIVERS, REPRESENTATIONS, ROOTS AND LIE ALGEBRAS

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1. QUIVERS

Definition: **A quiver is a quadruple $Q = (V, A, t, h)$, where**

- V **is a non-empty set;**
- A **is a set;**
- t **and** h **are maps from** A **to** V**.**

- **The elements of** V **are called** ***vertices.***
- **The elements from** A **are called** ***arrows.***
- **An arrow** $\alpha \in A$ **has** ***tail*** $t(\alpha)$ **and** ***head*** $h(\alpha)$**, which means that it goes from** $t(\alpha)$ **to** $h(\alpha)$**.**

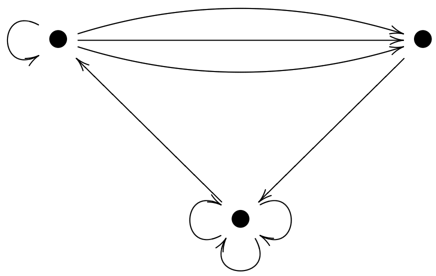
Detailed example: **For the quiver**

$$1 \xrightarrow{\alpha} 2$$

we have

- $V = \{1, 2\}$**.**
- $A = \{\alpha\}$**.**
- $t(\alpha) = 1$**,** $h(\alpha) = 2$**.**

More complicated picture:



2. REPRESENTATIONS OF QUIVERS

- **To each vertex $v \in V$ associate a (complex) vectorspace M_v.**
- **To each arrow $\alpha \in A$ associate a linear map $M(\alpha) : M_{t(\alpha)} \to M_{h(\alpha)}$.**

$$\mathbb{C} \xrightarrow{(x) \mapsto \begin{pmatrix} 2x \\ -3x \end{pmatrix}} \mathbb{C}^2$$

is a representation of

$$1 \xrightarrow{\alpha} 2$$

Definition. **Tow representations M and N of some quiver Q are said to be *isomorphic* provided that for each $v \in V$ there exists an isomorphism $F_v : M_v \to N_v$ such that for every $\alpha \in A$ the following diagram is commutative (i.e. $F_{h(\alpha)}M(\alpha) = N(\alpha)F_{t(\alpha)}$):**

$$\begin{tikzcd}
M_{t(\alpha)} \arrow[r, "M(\alpha)"] \arrow[d, "F_{t(\alpha)}"'] & M_{h(\alpha)} \arrow[d, "F_{h(\alpha)}"] \\
N_{t(\alpha)} \arrow[r, "N(\alpha)"] & N_{h(\alpha)}
\end{tikzcd}$$

Example:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "(x)\mapsto(x)"] \arrow[d, "(x)\mapsto(x)"'] & \mathbb{C} \arrow[d, "(x)\mapsto(2x)"] \\
\mathbb{C} \arrow[r, "(x)\mapsto(2x)"] & \mathbb{C}
\end{tikzcd}$$

is an isomorphism of representations of

$$\begin{tikzcd}
1 \arrow[r, "\alpha"] & 2
\end{tikzcd}$$

3. INDECOMPOSABLE REPRESENTATIONS

Definition. **The *trivial* representation of Q is the representation M for which $M_v = 0$ for all $v \in V$. Automatically $M(\alpha) = 0$ for all $\alpha \in A$.**

Definition. **If M and N are representations of Q then the *direct sum of* M *and* N is the representation $M \oplus N$ defined as follows:**

- $(M \oplus N)_v = M_v \oplus N_v$ **for all** $v \in V$**;**
- $(M \oplus N)(\alpha) = M(\alpha) \oplus N(\alpha)$ **for all** $\alpha \in A$**.**

Definition. **A representation is called *indecomposable* if it is not isomorphic to a direct sum of two non-trivial representations.**

Example:

$$\mathbb{C} \xrightarrow{(x)\mapsto(x)} \mathbb{C}$$

is an indecomposable representation of

$$1 \xrightarrow{\alpha} 2$$

Example:

$$\mathbb{C} \xrightarrow{(x)\mapsto(0)} \mathbb{C}$$

is a decomposable representation of

$$1 \xrightarrow{\alpha} 2$$

because it is isomorphic to
the direct sum of representations

$$\mathbb{C} \xrightarrow{(x)\mapsto(0)} 0$$

and

$$0 \xrightarrow{(x)\mapsto(0)} \mathbb{C}$$

Krull-Schmidt Theorem: **Each representation is isomorphic to a direct sum of indecomposable summands, which are uniquely determined up to isomorphism and permutation.**

4. GABRIEL'S THEOREM

BASIC QUESTION: **Which quivers have only finitely many isomorphism classes of indecomposable representation (i.e. are or finite type)?**

Example:

$$1 \xrightarrow{\alpha} 2$$

has *THREE* indecomposable representations, namely:

$$\mathbb{C} \xrightarrow{(x)\mapsto(x)} \mathbb{C}$$

$$\mathbb{C} \xrightarrow{(x)\mapsto(0)} 0$$

$$0 \xrightarrow{(x)\mapsto(0)} \mathbb{C}$$

Example:

$$\bullet \circlearrowright \alpha$$

has *INFINITELY MANY* indecomposable representations, namely:

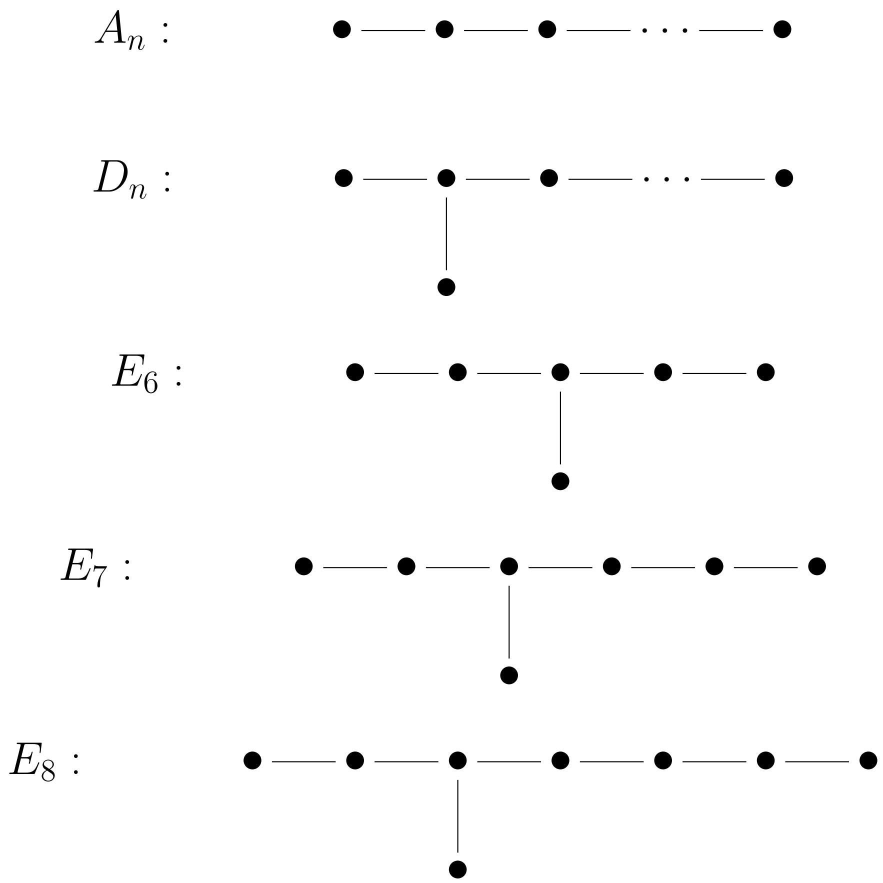
$$\mathbb{C}^n \circlearrowright J_n(\lambda)$$

where $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \lambda \end{pmatrix}$$

is the *Jordan cell* of size n and eigenvalue λ.

GABRIEL'S THEOREM: **A quiver is of finite type if and only if after forgetting the orientation of arrows it reduces to one of the following graphs:**



The above graphs are known as

simply laced Dynkin diagrams.

5. LIE ALGEBRAS

Dynkin diagrams (not only simply laced) are known to classify
simple finite-dimensional complex Lie algebras.

If D is a simply laced Dynkin diagram then the universal enveloping algebra of the "positive part" of the corresponding Lie algebra is defined as follows:

- **generators: X_v, where v is a vertex;**
- **relation $[X_v, X_w] = 0$ if v and w are not connected;**
- **$[X_v, [X_v, X_w]] = 0$ if v and w are connected.**

For example, starting from the quiver of type A_n the above relations define the universal enveloping algebra of the Lie algebra of all *strictly upper triangular* complex $n \times n$ matrices.

6. HALL ALGEBRA OF A QUIVER

Q — **quiver;**

M — **representation of** Q**;**

Definition: **A *subrepresentation*** $N \subset M$ **of** M **is a collection of** $N_v \subset M_v$**, closed with respect to the action of all** $M(\alpha)$**,** $\alpha \in A$**. The restriction of** $M(\alpha)$ **to** N **is then** $N(\alpha)$**.**

Definition: **If** $N \subset M$ **are representation of** Q**, then their *quotient*** M/N **is defined in the natural way.**

$\mathbb{F}_q$ — **finite field with** q **elements;**

$\mathcal{H}(Q)_{\mathbb{F}_q}$ — $\mathbb{F}_q$**-Hall algebra of** Q**:**

1. **basis: isomorphism classes** $[N]$ **of (finite-dimensional) representation of** Q**;**
2. **multiplication:** $[X] * [Y] = \sum_{[M]} C(\mathbb{F}_q)^M_{X,Y}[M]$**, where** $C(\mathbb{F}_q)^M_{X,Y}$ **is the number of subrepresentation of** M **such that the subrepresentation itself is isomorphic to** X**, while the quotient is isomorphic to** Y**.**

Proposition: **Let Q be a Dynkin quiver, and X, Y, M be finite-dimensional representations of Q. Then there exist polynomials $f_{X,Y}^M(t) \in \mathbb{Z}[t]$ such that $C(\mathbb{F}_q)_{X,Y}^M = f_{X,Y}^M(q)$.**

Q — **Dynkin quiver**

$\mathcal{H}(Q)$ — **(specialized) Hall algebra of Q:**

1. **$\mathbb{C}$-basis: isomorphism classes $[N]$ of finite-dimensional representation of Q;**
2. **multiplication: $[X] * [Y] = \sum_{[M]} f_{X,Y}^M(1)[M]$.**

Ringel's Theorem: **Let Q be Dynkin. Then $\mathcal{H}(Q)$ is isomorphic to the universal enveloping algebra of the positive part of the Lie algebra, associated to Q.**

Everything extends even to quantum groups!