Category $\mathcal{O}$ for classical Lie superalgebras

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"Algebra seminar"
July 16, 2013, Köln, Germany

Classical BGG category $\mathcal{O}$ — definition

$\mathfrak{g}$ — semi-simple f.dim. Lie algebra over $\mathbb{C}$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — triangular decomposition

Definition. [Bernstein-S.Gelfand-I.Gelfand] Category $\mathcal{O}$ is the full subcategory of $\mathfrak{g}\text{-mod}$ containing all

- finitely generated,
- $\mathfrak{h}$-diagonalizable;
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Classical BGG category $\mathcal{O}$ — properties

for $\lambda \in \mathfrak{h}^*$ the Verma module $M(\lambda) = U(\mathfrak{g})/(\mathfrak{n}_+, h - \lambda(h))$ is in $\mathcal{O}$

simple tops $M(\lambda) \twoheadrightarrow L(\lambda)$, $\lambda \in \mathfrak{h}^*$, classify simples in $\mathcal{O}$

$$\mathcal{O} \cong \bigoplus_{\chi : Z(\mathfrak{g}) \to \mathbb{C}} \mathcal{O}_\chi$$

$\mathcal{O}_\chi \cong A_\chi\text{-mod}$ where A_χ is a f.dim. associative algebra

each $\mathcal{O}_\chi$ is equivalent to an integral block (maybe for other $\mathfrak{g}$)

A_χ is quasi-hereditary and Koszul

A_χ has the double centralizer property with respect to proj.-inj. modules

Cartan matrix of A_χ — Kazhdan-Lusztig combinatorics

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Classical Lie superalgebras

$\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$

$\mathfrak{g}_{\overline{0}}$ — finite dimensional reductive

$\mathfrak{g}_{\overline{1}}$ — finite dimensional and semi-simple over $\mathfrak{g}_{\overline{0}}$

Some examples:

- General linear Lie superalgebra $\mathfrak{gl}(m|n)$
- Queer Lie superalgebra $\mathfrak{q}(n)$
- Generalized Takiff Lie superalgebra $\mathfrak{g}_{\mathfrak{a},V}$ where $\mathfrak{g}_{\overline{0}} = \mathfrak{a}$, $\mathfrak{g}_{\overline{1}} = V \in \mathfrak{a}\text{-mod}$ and $[V, V] = 0$.

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Naive definition of category $\mathcal{O}$

$\mathfrak{g}$ — classical Lie superalgebra

Note: $U(\mathfrak{g})$ is free of finite rank over $U(\mathfrak{g}_{\bar{0}})$

$\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}$ — restriction

$\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}$ — induction

Π — parity change

Theorem. $(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}, \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}})$ and $(\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}, \Pi^{\dim \mathfrak{g}_{\bar{1}}} \circ \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}})$ are adjoint pairs.

Definition. $\mathcal{O} := \mathcal{O}_{\mathfrak{g}}$ is the full subcategory of $\mathfrak{g}\text{-smod}$ consisting of all supermodules M such that $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}(M) \in \mathcal{O}_{\mathfrak{g}_{\bar{0}}}$.

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First difficulties

Triangular decomposition from $\mathfrak{g}_{\overline{0}}$ does not extend uniquely to $\mathfrak{g}$.

Any extension is given in geometric terms by choosing a hyperplane in $\mathfrak{h}_{\overline{0}}^*$.

If $(\mathfrak{g}_{\overline{1}})_0 \neq 0$, then the "Cartan subalgebra" $\mathfrak{h}$ of $\mathfrak{g}$ might turn out to be non-commutative (this happens, for example, in the case of $\mathfrak{q}_n$).

The $\mathfrak{h}_{\overline{0}}$ –highest weight of a simple highest weight module depends on the choice of an extension of the triangular decomposition (the set of modules is independent of any such choice).

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Similarities between $\mathcal{O}_{\mathfrak{g}}$ and $\mathcal{O}_{\mathfrak{g}_{\overline{0}}}$

Fact. $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}$ and $\mathrm{Res}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}$ restrict to a pair of biadjoint (up to parity change) functors between $\mathcal{O}_{\mathfrak{g}}$ and $\mathcal{O}_{\mathfrak{g}_{\overline{0}}}$

Corollaries.

- Each object in $\mathcal{O}_{\mathfrak{g}}$ has finite length (already over $\mathfrak{g}_{\overline{0}}$)
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Fact. $\mathcal{O}_{\mathfrak{g}_{\overline{0}}}$ decomposes into a direct sum of indecomposable blocks, each having at most countably many isoclasses of simple modules

Corollary. Each block is equivalent to a module category over some strongly locally finite $\mathbb{C}$-linear category with at most contably many objects

Strongly locally finite means that both projectives and injectives are finite dimensional

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$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — triangular decomposition

V — simple $\mathfrak{h}_{\overline{0}}$-weight $\mathfrak{h}$-module

$\hat{V}$ — projective cover of V in $\mathfrak{h}_{\overline{0}}$-weight $\mathfrak{h}$-modules

$\Delta(V) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \hat{V}$ — standard module

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Fact. Standard modules have a proper standard filtration.

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stratification implies existence of tilting module

Tilting module — self dual (up to Π) module with standard filtration

tilting in general super-setup — Brundan

for classical superalgebras all tilting modules are direct summands of induced tilting modules

Corollary. All tilting modules have uniformly bounded projective dimension.

Corollary. $\mathrm{fin.dim.}\mathcal{O} = 2\mathrm{p.dim.}T$ (also blockwise).

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Double centralizer

Fact. Each projective P in $\mathcal{O}$ has a coresolution $0 \to P \to X_0 \to X_1$ where both X_0 and X_1 are projective-injective.

Proof. Use induction from $\mathcal{O}_{\mathfrak{g}_{\overline{0}}}$.

Drawback. A block of $\mathcal{O}$ may contain infinitely many pairwise non-isomorphic projective-injective modules.

In particular: The endomorphism category of projective-injective modules in a block of $\mathcal{O}$ is as complicated as the whole block.

To compare: For Lie algebras the endomorphism category of projective-injective modules is the coinvariant algebra, [Soergel], which is the basis of the combinatorial description of $\mathcal{O}$.

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In particular: The endomorphism category of projective-injective modules in a block of $\mathcal{O}$ is as complicated as the whole block.

To compare: For Lie algebras the endomorphism category of projective-injective modules is the coinvariant algebra, [Soergel], which is the basis of the combinatorial description of $\mathcal{O}$.

Irving type theorems

Theorem. Let V be a simple module in $\mathcal{O}$. TFAE

- The projective cover of V is injective.
- V appears in the socle of a projective-injective module.
- V appears in the socle of a tilting module.
- V appears in the socle of a standard modules.
- V appears in the socle of a proper standard modules.
- V has maximal Gelfand-Kirillov dimension.

Proof. Use, with some care, induction from $\mathcal{O}_{\mathfrak{g}_{\overline{0}}}$.

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Equivalence to integral blocks

Theorem. [Cheng-M.-Wang] Each block of $\mathcal{O}_{\mathfrak{gl}(m|n)}$ is equivalent to an integral block.

Soergel's approach does not work

Our alternative approach: Use twisting functors.

Another alternative approach (Mathieu-Kashiwara-Tanisaki): Use non-integral Enright's functors.

Note. None of the above works for $\mathfrak{q}_n$.

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Generic blocks

Theorem. "Generic" blocks of $\mathcal{O}$ are equivalent to certain blocks of $\mathcal{O}_{\mathfrak{g}_{\overline{0}}}$.

- Typical blocks far from the walls for basic: Penkov 1994
- Strongly typical blocks for basic: Gorelik 2002
- Regular strongly typical blocks for $\mathfrak{q}_n$: Frisk-M. 2009

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Integral blocks

Brundan's version of Kazhdan-Lusztig conjecture for $\mathfrak{gl}(m|n)$ has recently been proved by Cheng-Lam-Wang.

Together withthe above result on equivalence to integral blocks, it follows that we know Cartan matrices for all blocks of $\mathcal{O}_{\mathfrak{gl}(m|n)}$.

The quiver of block of the category of integral finite dimensional $\mathfrak{gl}(m|n)$-modules is combinatorially described by Brundan-Stroppel.

Nothing of the above is known even for $\mathfrak{q}_n$.

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THANK YOU!!!