Category $\mathcal{O}$ for quantum groups

(joint work with Henning Haahr Andersen)

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with both X and Y projective injective.

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