FINITISTIC DIMENSION OF PROPERLY STRATIFIED ALGEBRAS

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1. Properly stratified algebras

$\Bbbk$ — algebraically closed field.

A — finite-dimensional associative $\Bbbk$-algebra.

A-mod — category of all finite-dimensional A-modules.

$\{e_1, \dots, e_n\}$ — a complete set of primitive idempotents.

$L(i)$, $P(i)$, $I(i)$, $i = 1, \dots, n$, — corresponding simple, projective and injective module.

For $i = 1, \ldots, n$ define:

standard modules $\Delta(i)$ as the maximal quotient of $P(i)$ such that $[\Delta(i) : L(j)] = 0$, $j > i$;

proper standard modules $\overline{\Delta}(i)$ as the maximal quotient of $\Delta(i)$ such that $[\overline{\Delta}(i) : L(i)] = 1$;

costandard modules $\nabla(i)$ as the maximal submodule of $I(i)$ such that $[\nabla(i) : L(j)] = 0$, $j > i$;

proper costandard modules $\overline{\nabla}(i)$ as the maximal submodule of $\nabla(i)$ such that $[\overline{\nabla}(i) : L(i)] = 1$.

Definition. [Dlab] A is called *properly stratified* provided that

1. The kernel of $P(i) \twoheadrightarrow \Delta$ has a filtration with subquotients $\Delta(j)$, $j > i$;
2. Each $\Delta(i)$ has a filtration with subquotients $\overline{\Delta}(i)$.

Theorem. [Dlab] A is properly stratified if and only if A^{opp} is. In particular, A is properly stratified if and only if

1. The cokernel of $\nabla(i) \hookrightarrow I(i)$ has a filtration with subquotients $\nabla(j)$, $j > i$;
2. Each $\nabla(i)$ has a filtration with subquotients $\overline{\nabla}(i)$.

2. Finitistic dimension

Global dimension of A:

$$\text{gl.d.}(A) = \max_{M \in A-\text{mod}} \text{p.d.}(M).$$

$\mathcal{P}^{<\infty}(A)$ — the full subcategory of A-mod, consisting of all modules of finite projective dimension.

Projectively defined finitistic dimension of A:

$$\text{fin.d.}(A) = \max_{M \in \mathcal{P}^{<\infty}(A)} \text{p.d.}(M).$$

Finitistic dimension conjecture. fin.d.$(A) < \infty$ for every A.

Theorem. [Ágoston-Happel-Lukács-Unger] Let A be properly stratified. Then fin.d.$(A) \leq 2n - 2$.

The bound is exact in the sense that there are properly stratified algebras having finitistic dimension exactly $2n - 2$.

3. Tilting and cotilting modules

$\mathcal{F}(\Delta)$ — the full subcategory in A-mod, consisting of all modules, which have a filtration, whose subquotients are standard modules.

$\mathcal{F}(\overline{\Delta})$ — $\quad - - // - - \quad$ proper standard modules.

$\mathcal{F}(\nabla)$ — $\quad - - // - - \quad$ standard modules.

$\mathcal{F}(\overline{\nabla})$ — $\quad - - // - - \quad$ proper standard modules.

Tilting modules are modules in $\mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\nabla})$.

Cotilting modules are modules in $\mathcal{F}(\overline{\Delta}) \cap \mathcal{F}(\nabla)$.

Theorem. [Ágoston-Happel-Lukács-Unger]

1. Both $\mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\nabla})$ and $\mathcal{F}(\overline{\Delta}) \cap \mathcal{F}(\nabla)$ are closed under taking direct summands.
2. Isoclasses of indecomposable modules in both $\mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\nabla})$ and $\mathcal{F}(\overline{\Delta}) \cap \mathcal{F}(\nabla)$ are in a natural bijection with isoclasses of simple modules.

For $i = 1, \ldots, n$ set:

$T(i)$ — indecomposable tilting module whose any standard filtration starts with $\Delta(i)$;

$C(i)$ — indecomposable cotilting module whose any costandard filtration ends with $\nabla(i)$.

$$T = \oplus_{i=1}^{n} T(i), \quad C = \oplus_{i=1}^{n} C(i).$$

Theorem. [Mazorchuk-Parker] Let A be properly stratified. Then fin.d.$(A) \leq$ p.d.$(T) +$ i.d.(C).

Definition. A is called *quasi-hereditary* if A is properly stratified and $\Delta(i) = \overline{\Delta}(i)$ for all i.

A quasi-hereditary implies $T = C$.

A quasi-hereditary implies gl.d.$(A) < \infty$.

Corollary. Let A be quasi-hereditary. Then gl.d.$(A) \leq 2\cdot$p.d.(T).

4. Exact results for the finitistic dimension

A has a simple preserving duality if there is a contravariant involutive exact self-equivalence on A-mod.

Conjecture. [Mazorchuk-Parker] Let A be properly stratified with duality. Then $\mathrm{fin.d.}(A) = 2 \cdot \mathrm{p.d.}(T)$.

Theorem. [Mazorchuk-Ovsienko] Assume that

1. A is properly stratified;
2. A has a duality;
3. $T = C$.

Then $\mathrm{fin.d.}(A) = 2 \cdot \mathrm{p.d.}(T)$.

Corollary. Let A be quasi-hereditary with duality. Then $\mathrm{gl.d.}(A) = 2 \cdot \mathrm{p.d.}(T)$.

Examples of such algebras:

1. Schur algebras;
2. algebras of blocks of the category $\mathcal{O}$,
3. algebras of blocks of the $\mathcal{S}$-subcategories in $\mathcal{O}$.

Theorem. [Mazorchuk-Frisk] Assume that

1. A is properly stratified;
2. A has a duality;
3. $\mathrm{End}_A(T)$ is properly stratified;
4. $\mathrm{End}_A(T)$ has a duality.

Then $\mathrm{fin.d.}(A) = 2 \cdot \mathrm{p.d.}(T)$.

Examples of such algebras:

1. certain tensor products of quasi-hereditary and local algebras;
2. algebras of blocks of the thick category $\mathcal{O}$.

Theorem. [Mazorchuk-Frisk] Assume that

1. A is properly stratified;
2. A has a duality;
3. $R = \mathrm{End}_A(T)$ is properly stratified.

Then $\mathrm{fin.d.}(A) = 2 \cdot \mathrm{p.d.}(T^{(R)})$.

5. Finitistic dimension via integral algebras

Theorem. [Khomenko-König-Mazorchuk]

- R — local, complete, commutative algebra with the maximal ideal $\mathfrak{m}$;
- A — a quasi-hereditary algebra over R;
- I — an ideal of R of finite codimension.

Then:

1. A/AI is properly stratified.
2. A/AI is quasi-hereditary if and only if $I = \mathfrak{m}$.
3. $\mathrm{gl.d.}(A/A\mathfrak{m}) = 2 \cdot \mathrm{p.d.}(T^{(A/A\mathfrak{m})})$ implies the equality $\mathrm{fin.d.}(A/AI) = 2 \cdot \mathrm{p.d.}(T^{(A/AI)})$.

Examples of such algebras:

1. all tensor products of quasi-hereditary and local algebras;
2. algebras of blocks of Harish-Chandra bimodules.

6. Connection with various filtration dimensions

$\mathcal{M}$ — a family of A-modules.

For $N \in A$-mod the *$\mathcal{M}$-filtration dimension* $\dim_{\mathcal{M}}(N)$ of N is the length of a shortest resolution of N by modules from $\mathcal{M}$ if such a resolution exists.

Assume that $\mathcal{M}$ is such that every $N \in A$-mod has a resolution by modules from $\mathcal{M}$. Define:

$$\dim_{\mathcal{M}}(A) = \max_{N \in A-\mathrm{mod}} \dim_{\mathcal{M}}(N);$$

Fact. A — properly stratified, $\Delta = \{\Delta(1), \dots, \Delta(n)\}$. Then $\dim_{\Delta}(N)$ is well-defined (possibly infinite) for all $N \in A$-mod.

Corollary. Let A be as in any theorem above. Then
$\mathrm{fin.d.}(A) = 2 \cdot \dim_{\Delta}(A)$.

7. Conjecture of Erdmann and Parker

Conjecture. [Erdmann-Parker] A-quasi-hereditary with duality $\circ$ and $N \in A$-mod is such that $\dim_{\Delta}(N) = t$. Then $\mathrm{Ext}_A^{2t}(N, N^\circ) \neq 0$.

Theorem. [Mazorchuk-Ovsienko] A-properly stratified with duality $\circ$ and $T = C$. $N \in A$-mod is such that $\dim_{\Delta}(N) = t$. Then $\mathrm{Ext}_A^{2t}(N, N^\circ) \neq 0$.

8. Connection with exact Borel subalgebras

Corollary. Let A be quasi-hereditary with duality and B be an exact Borel subalgebra of A. Then fin.d.$(A) = 2 \cdot$ fin.d.(B).