SERRE FUNCTORS AND SYMMETRIC ALGEBRAS

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1. Serre functors

$\Bbbk$ — algebraically closed field.

$\mathcal{C}$ — $\Bbbk$-linear additive category with finite-dimensional morphism spaces.

Definition. (Bondal-Kapranov) An additive functor, $F : \mathcal{C} \to \mathcal{C}$, is called a *Serre functor* if F is an auto-equivalence of $\mathcal{C}$ and there are isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X, FY) \cong \mathrm{Hom}_{\mathcal{C}}(Y, X)^*,$$

natural in X and Y.

Example 1. Let X be a smooth projective variety, $n = \dim X$, $\mathcal{A} = \mathcal{D}^b_{coh}(X)$ be the derived category of coherent sheaves on X, $K_X = \Omega^n_X$ be the canonical sheaf. Then $(\_) \otimes K_X[n]$ is a Serre functor on $\mathcal{A}$ because of the Serre duality.

Example 2. Let A be a finite-dimensional $\Bbbk$-algebra of finite global dimension, $\mathcal{A} = \mathcal{D}^b(A)$ be the derived category of finite-dimensional A-modules. Then the left derived of the Nakayama functor $A^* \otimes_A \_$ is a Serre functor on $\mathcal{A}$.

Some properties:

- If a Serre functor exists, it is unique up to an isomorphism.
- Let A be a finite-dimensional $\Bbbk$-algebra. Then $\mathcal{D}^b(A)$ has a Serre functor if and only if $\text{gl.dim}(A) < \infty$.
- Let $\mathcal{C}$ be a category, which is equivalent to A-mod for some finite-dimensional $\Bbbk$-algebra of finite global dimension. If A is not given explicitly, then the Serre functor on $\mathcal{D}^b(\mathcal{C})$ can be very hard to compute.

Main theorem for detection of Serre functors. Let A be a finite dimensional $\Bbbk$-algebra of finite global dimension. Assume that a basic projective-injective A-module has isomorphic socle and top and that both, A and A^{opp}, have the double centralizer property with respect to a projective-injective module. Let $F : A\text{-mod} \to A\text{-mod}$ be a right exact functor. Then $\mathcal{L}F$ is a Serre functor of $\mathcal{D}^b(A)$ if and only if the following conditions are satisfied:

1. Its left derived functor $\mathcal{L}F : \mathcal{D}^b(A) \to \mathcal{D}^b(A)$ is an auto-equivalence.
2. F maps projective A-modules to injective A-modules.
3. F preserves the full subcategory $\mathcal{PI}$ of $A\text{-mod}$, consisting of all projective-injective modules, and the restrictions of F and the Nakayama functor to $\mathcal{PI}$ are isomorphic.

2. Category $\mathcal{O}$

$\mathfrak{g}$ — semi-simple complex finite-dimensional Lie algebra

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — fixed triangular decomposition.

$U(\mathfrak{g})$— the universal enveloping algebra of $\mathfrak{g}$.

$\mathcal{O}$ — full subcategory of $\mathfrak{g}$-mod, consisting of those modules, which are

- finitely generated;
- $\mathfrak{h}$-diagonalizable;
- $U(\mathfrak{n}_+)$-finite.

$\mathcal{O}_0$ — the principal block of $\mathcal{O}$, that is the indecomposable block, containing the trivial module.

W — the Weyl group of $\mathfrak{g}$.

$\mathcal{O}_0 \cong A-$mod for some A, which is not explicitly given.

s — simple reflection in W.

$U(\mathfrak{g})_s$ — the localization of $U(\mathfrak{g})$ with respect to Y_α, where α is the root corresponding to s.

$F_1^s = U(\mathfrak{g})_s \otimes_{U(\mathfrak{g})} -.$

$F_2^s = \mathrm{Coker}(\mathrm{ID} \to F_1)$ induced by $U(\mathfrak{g}) \hookrightarrow U(\mathfrak{g})_s$.

$T_s : \mathcal{O}_0 \to \mathcal{O}_0$ is the composition of F_2^s and the inner automorphism of $\mathfrak{g}$, corresponding to s.

T_s— Arkhipov's twisting functor.

T_s's satisfy braid relations.

$w \in W$, $w = s_1 s_2 \cdots s_k$.

$T_w = T_{s_1} T_{s_2} \cdots T_{s_k}$.

w_0 — the longest element of W.

Theorem. $\mathcal{L}T_{w_0}T_{w_0}$ is the Serre functor on $\mathcal{D}^b(\mathcal{O}_0)$.

Note that w_0^2 generates the center of the braid group (at least for type A).

Idea of the proof: Use our main theorem and known properties of Arkhipov's functors.

3. Parabolic category $\mathcal{O}$

$\mathfrak{p}$ — a parabolic subalgebra of $\mathfrak{g}$, containing $\mathfrak{h} \oplus \mathfrak{n}_+$.

$\mathcal{O}^{\mathfrak{p}}$ the subcategory of $\mathcal{O}$, consisting of $U(\mathfrak{p})$-locally finite modules (the parabolic category of Rocha-Caridi).

$W^{\mathfrak{p}}$— the Weyl group of $\mathfrak{p}$.

$w_0^{\mathfrak{p}}$— the longest element in $W^{\mathfrak{p}}$.

Theorem. $\mathcal{L}(T_{w_0}^2)[-2l(w_0^{\mathfrak{p}})]$ is the Serre functor on $\mathcal{D}^b(\mathcal{O}_0^{\mathfrak{p}})$.

A finite-dimensional algebra, A, is called *symmetric* if $A \cong A^*$ as A-bimodules.

Corollary. (The positive answer to a conjecture of Khovanov) The endomorphism algebra of a basic projective-injective module in $\mathcal{O}_0^{\mathfrak{p}}$ is symmetric.

Explanation:

Let B be the endomorphism algebra of a basic projective-injective module in $\mathcal{O}_0^{\mathfrak{p}}$.

We know the Serre functor for $\mathcal{D}^b(\mathcal{O}_0^{\mathfrak{p}})$.

A direct compotation shows that it induces the identity functor on the category of projective-injective modules.

However, it also induces a Serre functor for the perfect part of the bounded derived category of B-mod.

Hence for $\mathcal{K}^{perf}(B)$ the Serre functor is the identity functor.

This means that $B \cong B^*$, namely that B is symmetric.

Using our main theorem one can compute Serre functors for various categories of Harish-Chandra bimodules and for several rational Cherednik algebras.