ON FINITISTIC DIMENSION OF STRATIFIED ALGEBRAS

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1. Notation

$\Bbbk$ — algebraically closed field.

A — finite-dimensional associative $\Bbbk$-algebra.

A-mod — category of all finite-dimensional A-modules.

$\{e_1, \dots, e_n\}$ — a complete set of primitive idempotents.

$L(i)$, $P(i)$, $I(i)$, $i = 1, \dots, n$, — the corresponding simple, projective and injective modules.

$$L = \bigoplus_{i=1}^{n} L(i), \quad P = \bigoplus_{i=1}^{n} P(i), \quad I = \bigoplus_{i=1}^{n} I(i).$$

2. (Generalized) tilting module

Definition. $T \in A$-mod is called a (generalized) *tilting* module provided that

1. $\mathrm{Ext}^i_A(T,T) = 0$, $i > 0$;
2. p.d.$(T) < \infty$;
3. there exists a coresolution $0 \to P \to T_0 \to \cdots \to T_k \to 0$, where $T_i \in \mathrm{Add}(T)$ for all i.

Remark. Minimal k above equals p.d.(T).

3. Duality

Definition. The algebra A is said to have a (simple preserving) *duality*, if there exists a contravariant exact equivalence on A-mod, which preserves the iso-classes of simple modules.

Example. Any isomorphism $\varphi : A \cong A^{opp}$, such that $\varphi(e_i) = e_i$ for all i, gives rise to a duality.

3. Finitistic dimension

Global dimension of A:

$$\mathrm{gl.d.}(A) = \max_{M \in A-\mathrm{mod}} \mathrm{p.d.}(M).$$

$\mathcal{P}^{<\infty}(A)$ — the full subcategory of A-mod, consisting of all modules of finite projective dimension.

Projectively defined finitistic dimension of A:

$$\mathrm{fin.d.}(A) = \max_{M \in \mathcal{P}^{<\infty}(A)} \mathrm{p.d.}(M).$$

Finitistic dimension conjecture. $\mathrm{fin.d.}(A) < \infty$ for every A.

4. Finitistic dimension algebras with duality and self-dual tilting modules

Lemma. Assume that $\text{p.d.}(I), \text{fin.d.}(A) < \infty$. Then $\text{fin.d.}(A) = \text{p.d.}(I)$.

Proof. Let $M \in \mathcal{P}^{<\infty}(A)$ be such that $\text{p.d.}(M) = \text{fin.d.}(A) = m$. Choose $M \hookrightarrow \hat{I} \twoheadrightarrow K$ and apply $\text{Hom}_A(\_, S)$. One gets the exact sequence

$$\cdots \to \text{Ext}_A^m(\hat{I}, S) \to \text{Ext}_A^m(M, S) \neq 0 \to \text{Ext}_A^{m+1}(K, S) = 0.$$

Hence $\text{Ext}_A^m(\hat{I}, S) \neq 0$ and therefore $\text{p.d.}(I) = \text{p.d.}(\hat{I}) = m = \text{fin.d.}(A)$. **Q.E.D.**

Theorem A. [M.-Ovsienko] Assume that

(i) A has a duality, $\circ$.

(ii) There is a (generalized) tilting module, T, such that all indecomposable summands of T are self-dual with respect to $\circ$.

(iii) $\text{fin.d.}(A) < \infty$.

Then $\text{fin.d.}(A) = 2 \cdot \text{p.d.}(T)$

Proof. Let

$$0 \to P \to T_0 \to \cdots \to T_k \to 0 \qquad (1)$$

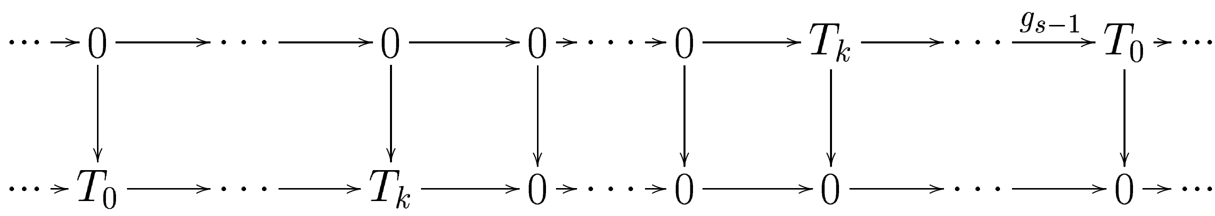
be a minimal tilting coresolution of P. Remark that $k = \text{p.d.}(T)$. Apply $\circ$ form (i) and use (ii) to obtain a tilting resolution for I:

$$0 \to T_k \to \cdots \to T_0 \to I \to 0. \qquad (2)$$

In particular, $\text{p.d.}(I) < \infty$. Hence Lemma implies that $\text{fin.d.}(A)$ equals the maximal m such that $\text{Ext}_A^m(I, P) \neq 0$. We calculate such m using (1) and (2).

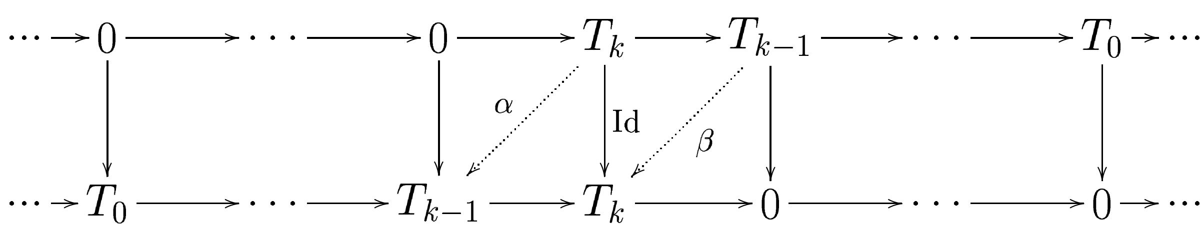
In $D^b(A)$ we can substitute P and I by tilting complexes $\mathcal{T}_1^\bullet$ and $\mathcal{T}_2^\bullet$ obtained from (1) and (2) respectively. Then the extensions can be calculated as the usual homomorphisms between the shifted complexes up to homotopy.

If $t > 2k$, we have the following picture for the homomorphisms from $\mathcal{T}_2^\bullet[-t]$ to $\mathcal{T}_1^\bullet$:



Hence $\text{Ext}_A^t(I, P) = 0$ for all $t > 2k$.

If $t > 2k$, we have the following non-trivial homomorphism:



Minimality of the resolution implies that it is not homotopic to zero, giving a non-trivial extension of degree $2k$ between I and P. **Q.E.D.**

5. Various stratified algebras

For $i = 1, \dots, n$ define:

standard modules $\Delta(i)$ as the maximal quotient of $P(i)$ such that $[\Delta(i) : L(j)] = 0$, $j > i$;

proper standard modules $\overline{\Delta}(i)$ as the maximal quotient of $\Delta(i)$ such that $[\overline{\Delta}(i) : L(i)] = 1$;

costandard modules $\nabla(i)$ as the maximal submodule of $I(i)$ such that $[\nabla(i) : L(j)] = 0$, $j > i$;

proper costandard modules $\overline{\nabla}(i)$ as the maximal submodule of $\nabla(i)$ such that $[\overline{\nabla}(i) : L(i)] = 1$.

Definition. A is called *strongly standardly stratified* provided that for every i the kernel of $P(i) \twoheadrightarrow \Delta(i)$ has a filtration with subquotients $\Delta(j)$, $j > i$.

Definition. A is called *properly stratified* provided that it is strongly standardly stratified and each $\Delta(i)$ has a filtration with subquotients $\overline{\Delta}(i)$.

Definition. A is called *quasi-hereditary* provided that it is properly stratified and $\Delta(i) = \overline{\Delta}(i)$ for all i.

6. Application of Theorem A to quasi-hereditary algebras

A — quasi-hereditary with duality.

$\mathcal{F}(\Delta)$ — category of modules having a standard filtration.

$\mathcal{F}(\nabla)$ — category of modules having a costandard filtration.

Fact. $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \mathrm{Add}(T)$, where T is a (generalized) tilting module with self-dual indecomposable summands.

Corollary. $\text{gl.dim.}(A) = 2 \cdot \text{p.d.}(T)$.

Corollary. $\text{gl.dim.}(A) = 2 \cdot \dim_\Delta(A) = 2 \cdot \text{gl.dim.}(B)$, where $\dim_\Delta(A)$ is the Δ-filtration dimension of A, and B is an exact Borel subalgebra of some $A' \simeq_{Morita} A$.

7. Application of Theorem A to properly stratified algebras

A — properly stratified with duality.

$\mathcal{F}(\Delta)$, $\mathcal{F}(\nabla)$ as above.

$\mathcal{F}(\overline{\Delta})$ — category of modules having a proper standard filtration.

$\mathcal{F}(\overline{\nabla})$ — category of modules having a proper costandard filtration.

Fact. $\mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\nabla}) = \mathrm{Add}(T)$, where T is a (generalized) tilting module.

Fact. $\mathcal{F}(\overline{\Delta}) \cap \mathcal{F}(\nabla) = \mathrm{Add}(C)$, where C is a (generalized) cotilting module.

Fact. If $T = C$ then all indecomposable summands of T are self-dual.

Corollary. Assume $T = C$. Then $\mathrm{fin.d.}(A) = 2 \cdot \mathrm{p.d.}(T)$.

Conjecture. [M.-Parker] $\mathrm{fin.d.}(A) = 2 \cdot \mathrm{p.d.}(T)$ for any properly stratified algebra A with duality.

7. New generalized tilting module for strongly stratified algebras

A — strongly stratified.

T — characteristic tilting module for A.

$R = \mathrm{End}_A(T)$ — the Ringel dual of A.

$F = \mathrm{Hom}_A(T, \_) : A - \mathrm{mod} \to R - \mathrm{mod}$ — the Ringel duality functor.

Fact. $F : \mathcal{F}(\overline{\nabla}^{(A)}) \to \mathcal{F}(\overline{\Delta}^{(R)})$ is an exact equivalence.

Theorem. [Frisk-M.] Assume that R is properly stratified, then $H = F^{-1}(T^{(R)})$ is a (generalized) tilting module for A.

Corollary. Assume that R is properly stratified. Then

$$\mathrm{fin.d.}(A) = \mathrm{p.d.}(H).$$

Corollary. Assume that R is properly stratified. Then $\mathcal{P}^{<\infty}(A)$ is contravariantly finite.

7. Two-step duality for strongly stratified algebras

A — strongly stratified.

Assume that R is properly stratified.

H — new (*two-step*) tilting module for A.

Theorem. [Frisk-M.]

1. $B = \mathrm{End}_A(H)^{opp}$ is strongly stratified.
2. The Ringel dual of B is properly stratified.
3. The two-step dual for B is Morita equivalent to A^{opp}.

$G = \mathrm{Hom}_{\Bbbk}(\mathrm{Hom}_A({}_-, H), \Bbbk) : A\text{−mod} \to B\text{−mod}$ — the two-step duality functor.

Corollary. $G : \mathcal{P}^{<\infty}(A) \to \mathcal{I}^{<\infty}(B)$ is an exact equivalence.

8. Finitistic dimension for strongly stratified algebras

Theorem. [Frisk-M.] Assume that both A and R are properly stratified with duality. Then

$$\mathrm{fin.d.}(A) = 2 \cdot \mathrm{p.d.}(T).$$

Theorem. [Frisk-M.] Assume that A is properly stratified with duality and R is properly stratified. Then

$$\mathrm{fin.d.}(A) = 2 \cdot \mathrm{p.d.}(T^{(R)}).$$