TWISTING, COMPLETING AND APPROXIMATING CATEGORY $\mathcal{O}$

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1. Category $\mathcal{O}$

$\mathfrak{g}$ — simple finite-dimensional Lie algebra over $\mathbb{C}$

$U(\mathfrak{g})$ — the universal enveloping algebra of $\mathfrak{g}$

$Z(\mathfrak{g})$ — the center of $U(\mathfrak{g})$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — fixed triangular decomposition of $\mathfrak{g}$

Definition: (BGG) Category $\mathcal{O}$ is the full subcategory in $\mathfrak{g}$-mod that consists of all modules, which are

- finitely generated;
- $\mathfrak{h}$-diagonalizable;
- locally $U(\mathfrak{n}_+)$-finite.

With respect to the action of $Z(\mathfrak{g})$ the category $\mathcal{O}$ decomposes:

$$\mathcal{O} = \bigoplus_{\chi \in Z(\mathfrak{g})^*} \mathcal{O}_\chi.$$

Every $\mathcal{O}_\chi$ is equivalent to the module category of a finite-dimensional, associative, quasi-hereditary algebra.

Simple modules in $\mathcal{O}_\chi$ are indexed (sometimes non-bijectively) by the elements of the Weyl group W.

Example: Indecomposable modules in the regular block of $\mathcal{O}$ for $\mathfrak{sl}(2, \mathbb{C})$:

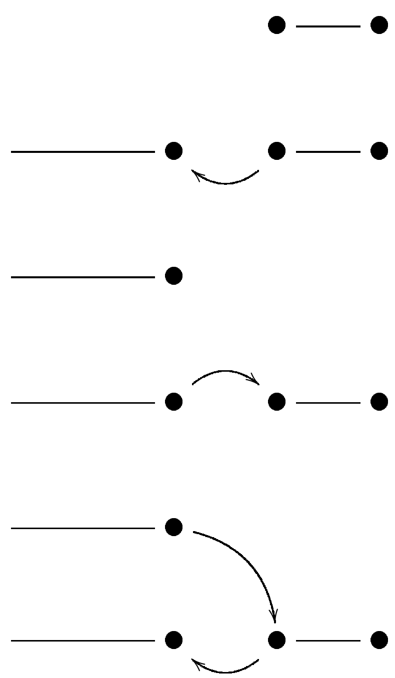
$L(\lambda)$:

$M(\lambda)$:

$M(s_\alpha \cdot \lambda)$:

$M(\lambda)^\star$:

$P(s_\alpha \cdot \lambda)$:



2. Twisting functors on $\mathcal{O}$

Let α be a simple root and $X_{-\alpha}$ be a non-zero root vector.

Let U_α denote the (Ore) localization of $(\mathfrak{g})$ with respect to $\{X_{-\alpha}^l : l \geq 0\}$.

$B_\alpha = U_\alpha / U(\mathfrak{g})$ is the twisting $U(\mathfrak{g})$-bimodule (Arkhipov).

Let Φ_α be the inner automorphism of $\mathfrak{g}$, corresponding to α.

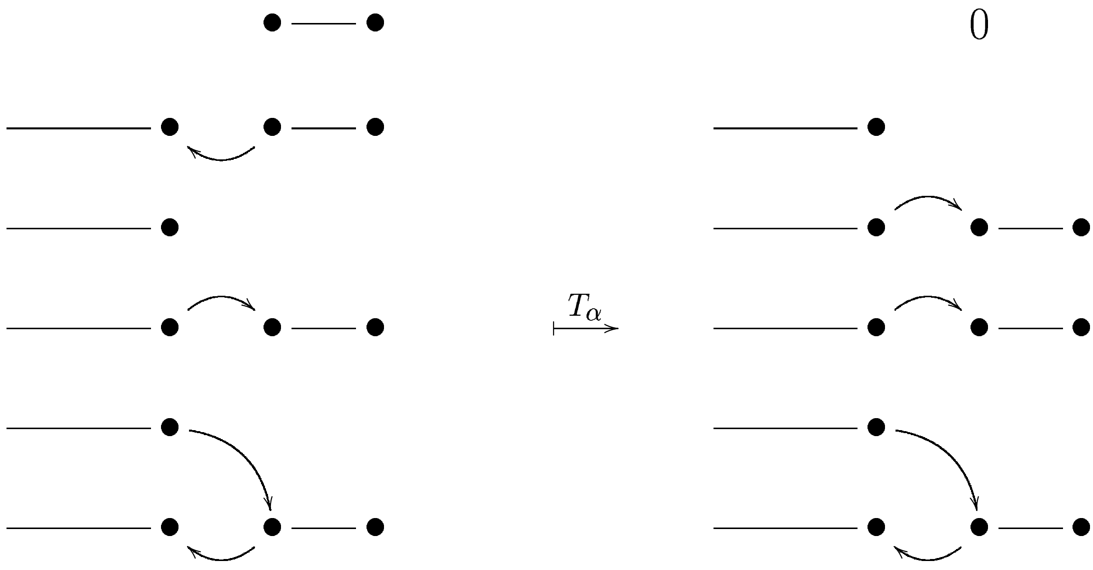
Definition: (Arkhipov) The twisting functor $T_\alpha : \mathfrak{g}\text{-mod} \to \mathfrak{g}\text{-mod}$ is defined as the functor $\Phi_\alpha\big(B_\alpha \otimes_{U(\mathfrak{g})} -\big)$.

T_α preserves all integral blocks of $\mathcal{O}$.

T_α is right exact.

Theorem. (Arkhipov?, Andersen?, Andersen-Lauritzen?, Khomenko-M.) Functors T_α, α simple, (weakly) satisfy braid relations on the integral blocks of $\mathcal{O}$.

Example: Action of T_α on the regular block of $\mathcal{O}$ for $\mathfrak{sl}(2,\mathbb{C})$:



3. Enright-Deodhar's completion functor on $\mathcal{O}$

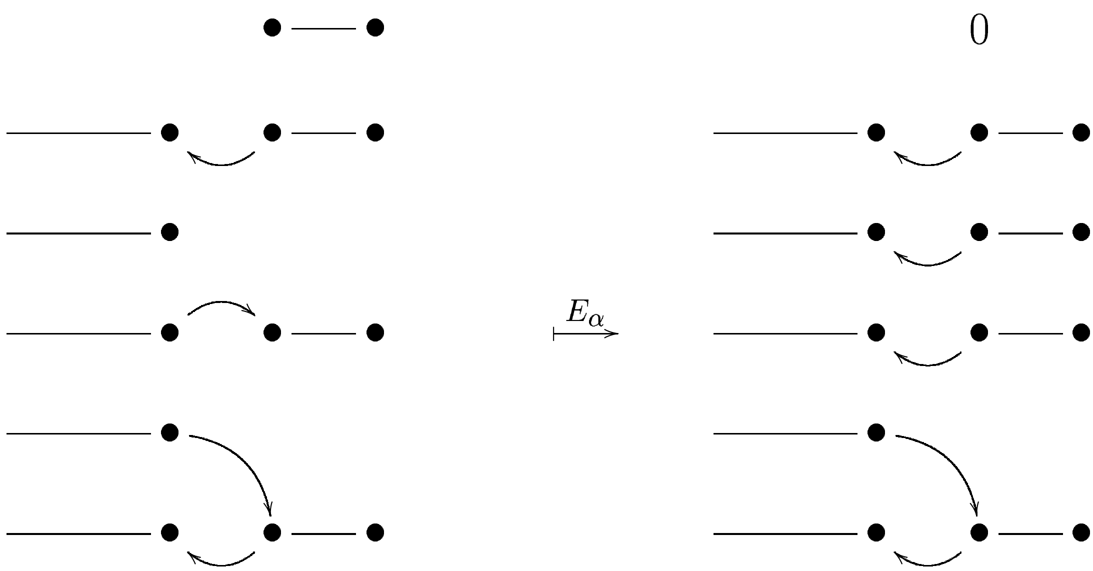
Let α and U_α be as above.

Definition: The Enright-Deodhar's completion functor $E_\alpha : \mathfrak{g}\text{-mod} \to \mathfrak{g}\text{-mod}$ is defined as the composition of the following functors:

1. $U_\alpha \otimes_{U(\mathfrak{g})} -$;
2. restriction to $U(\mathfrak{g})$;
3. taking $\mathfrak{g}_\alpha$-locally finite part.

E_α is left exact and idempotent.

Example: Action of E_α on the regular block of $\mathcal{O}$ for $\mathfrak{sl}(2, \mathbb{C})$:



4. Enright-Joseph's completion functor on $\mathcal{O}$

For $M, N \in U(\mathfrak{g})$-mod denote by $\mathcal{L}(M, N)$ the space of all locally ad-$\mathfrak{g}$-finite linear maps from M to N.

$M(\lambda)$ is the Verma module with highest weight $\lambda \in \mathfrak{h}^*$.

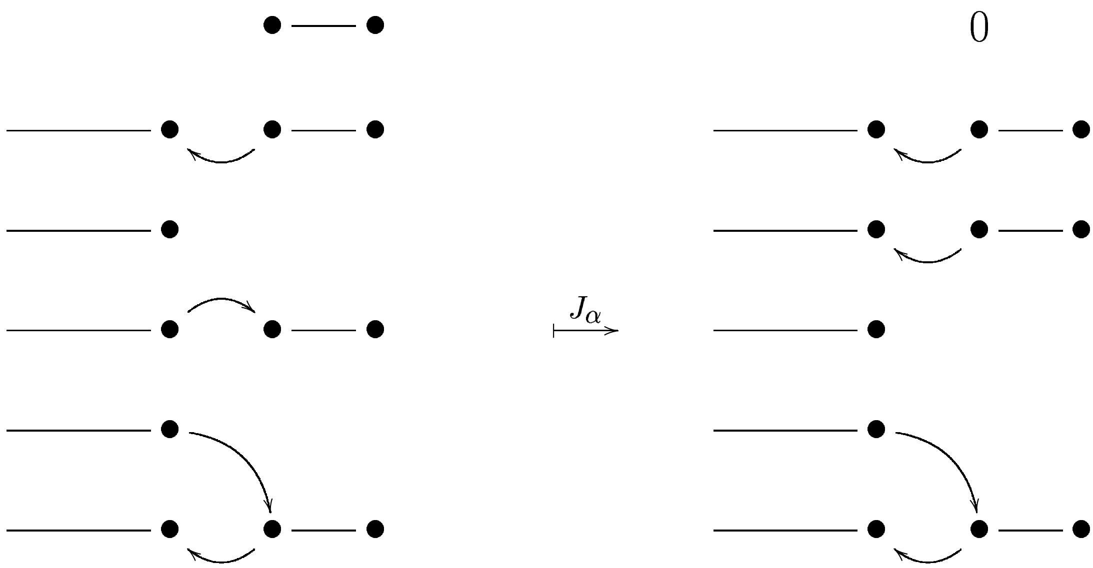
Definition: The Enright-Joseph's completion functor $J_\alpha : \mathfrak{g}\text{-mod} \to \mathfrak{g}\text{-mod}$ is defined as the functor

$$J_\alpha = \mathcal{L}(M(s_\alpha \cdot \lambda), \_) \otimes_{U(\mathfrak{g})} M(\lambda),$$

where $M(\lambda)$ is the dominant Verma module in $\mathcal{O}_\chi$.

J_α is left exact and $J_\alpha^3 \cong J_\alpha^2$.

Example: Action of J_α on the regular block of $\mathcal{O}$ for $\mathfrak{sl}(2, \mathbb{C})$:



5. Approximation functor

A — finite-dimensional associative algebra.

Υ — a set of primitive pairwise orthogonal idempotents.

$P(\Upsilon)$, $I(\Upsilon)$ — the corresponding projective and injective modules respectively.

Definition: (Auslander?) The approximation functor $\mathfrak{c}_\Upsilon : A\text{-mod} \to A\text{-mod}$ is defined as

$$\mathfrak{c}_\Upsilon = \mathrm{Hom}_{\mathrm{End}_A(P(\Upsilon))}\big(\mathrm{Hom}_A(P(\Upsilon), A), \mathrm{Hom}_A(P(\Upsilon), \_)\big).$$

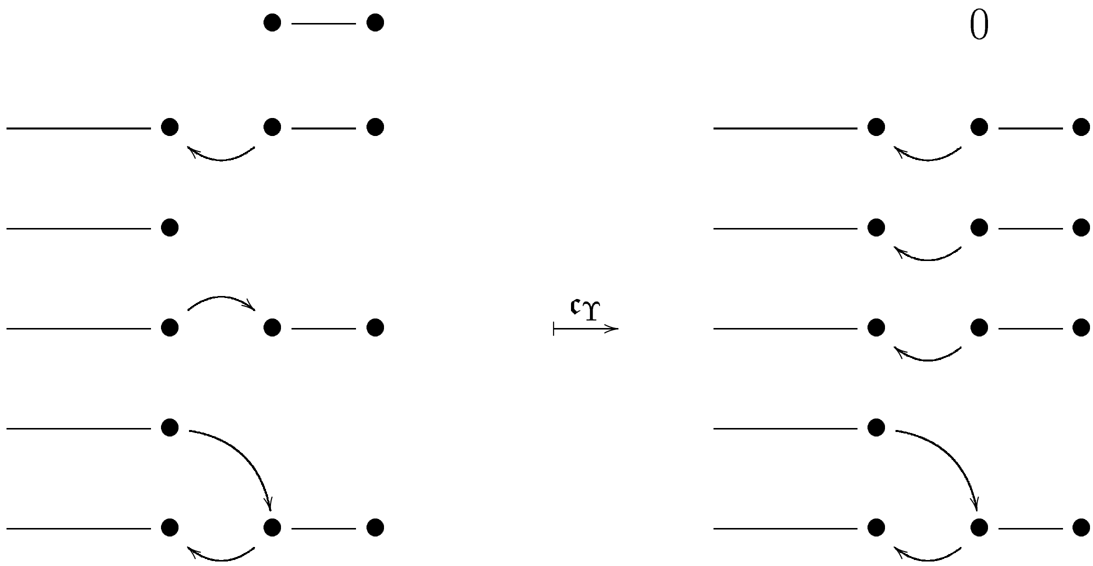
$\mathfrak{c}_\Upsilon$ is left exact and idempotent.

$\mathfrak{c}_\Upsilon$ can be viewed as the composition of the following two procedures. Start with $M \in A$-mod.

1. Take the maximal possible image M_1 of M in some $I(\Upsilon)^n$.
2. Make the maximal possible coextension of M_1 inside $I(\Upsilon)^n$ with non-Υ simples.

For a simple root α and a block $\mathcal{O}_\chi$ we let Υ denote the set of α-antidominant simples.

Example: Action of $\mathfrak{c}_\Upsilon$ on the regular block of $\mathcal{O}$ for $\mathfrak{sl}(2,\mathbb{C})$:



The coapproximation functor $\tilde{\mathfrak{c}}_\Upsilon$ is defined dually.

Theorem. (Auslander?) The functor $\tilde{\mathfrak{c}}_\Upsilon$ is left adjoint to $\mathfrak{c}_\Upsilon$.

6. Functor of partial approximation

$A, \Upsilon, P(\Upsilon), I(\Upsilon)$ — as above.

I — injective generator of A-mod.

Definition: (Khomenko-M.) The functor of partial approximation $\mathfrak{d}_\Upsilon : A\text{-mod} \to A\text{-mod}$ is defined as the composition of the following three procedures. Start with $M \in A$-mod.

1. Take a minimal injective envelope $M \subset I_M$ of M.
2. Make the maximal possible coextension of M inside I_M with non-Υ simples obtaining M_1.
3. Take the maximal possible image of M_1 in some $I(\Upsilon)^n$.

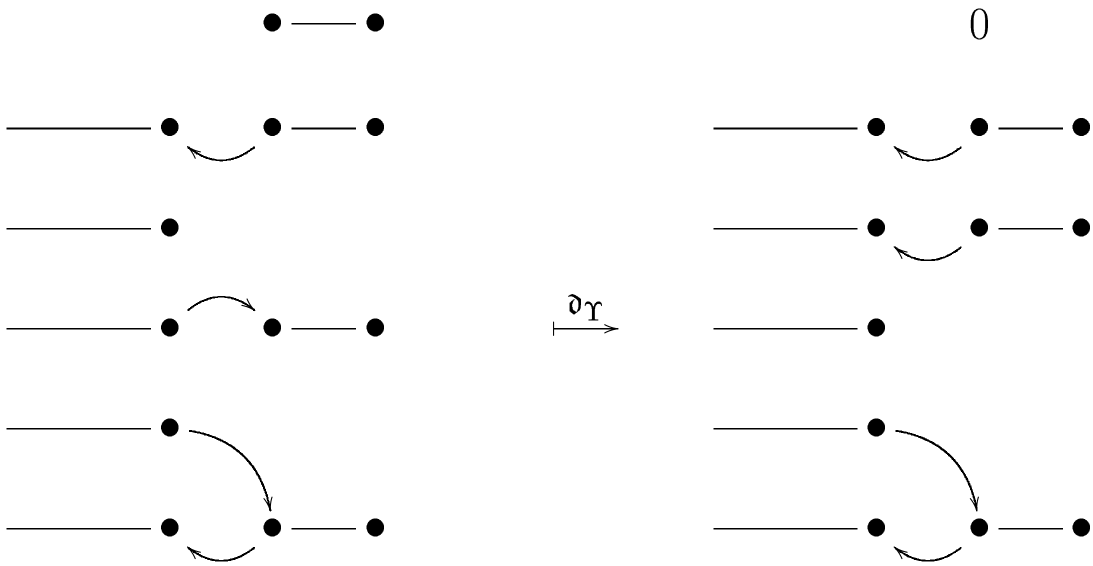
$\mathfrak{d}_\Upsilon$ is left exact and $\mathfrak{d}_\Upsilon^3 = \mathfrak{d}_\Upsilon^2$.

The coapproximation functor $\tilde{\mathfrak{d}}_\Upsilon$ is defined dually.

Theorem. (Khomenko-M.) The functor $\tilde{\mathfrak{d}}_\Upsilon$ is left adjoint to $\mathfrak{d}_\Upsilon$.

For a simple root α and a block $\mathcal{O}_\chi$ we let Υ denote the set of α-antidominant simples.

Example: Action of $\mathfrak{c}_\Upsilon$ on the regular block of $\mathcal{O}$ for $\mathfrak{sl}(2,\mathbb{C})$:



7. Relations between these functors (Khomenko-M.)

Let $\mathcal{O}_\chi$ be integral and regular.

Theorem: The functors E_α and $\mathfrak{c}_v$ are isomorphic.

Theorem: The functors J_α and $\mathfrak{d}_v$ are isomorphic.

Theorem: There is a non-trivial natural transformation from T_α to the identity functor.

Theorem: The functors T_α and $\tilde{\mathfrak{d}}_v$ are isomorphic.

Corollary: The functor T_α is left adjoint to the functor J_α.

Corollary: The functor T_α is left adjoint to the functor $\star \circ T_\alpha \circ \star$.

Corollary: The functor J_α is left adjoint to the functor $\star \circ J_\alpha \circ \star$.

Corollary: (Joseph) The functors J_α, α simple, satisfy braid relations.

Corollary: (Deodhar, Bouaziz) The functors E_α, α simple, satisfy braid relations on the full subcategory of $\mathcal{O}_\chi$, which consists of all modules, torsion free with respect to all $\mathfrak{g}_{-\beta}$, β positive.

8. T_α and the Kazhdan-Lusztig conjecture

Let $\mathcal{O}_\chi$ be integral and regular.

Let $L(\lambda) \in \mathcal{O}_\chi$ be a simple module such that $T_\alpha(L(\lambda)) \neq 0$ (that is λ is α-antidominant).

Theorem: (Andersen-Stroppel) The Kazhdan-Lusztig conjecture is equivalent to the following statement: The kernel of the natural morphism $T_\alpha(L(\lambda)) \to L(\lambda)$ is semi-simple for all α and λ as above.